Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 14757–14783.
DOI: 10.3934/math.2025664
Received: 07 February 2025
Revised: 09 June 2025
Accepted: 12 June 2025
Published: 27 June 2025

Research article

The classification and representations of ternary quadratic forms with level $8N$

Yifan Luo[1,2] **and Haigang Zhou**[1,3,*]

[1] School of Mathematical Sciences, Tongji University, Shanghai, China, 200092
[2] School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, China, 450046
[3] Institute of Mathematics, Henan Academy of Sciences, Zhengzhou, China, 450046

* **Correspondence:** Email: haigangz@tongji.edu.cn.

Abstract: In this paper, we investigated positive definite ternary quadratic forms of level $8N$, where N is an odd positive squarefree integer. Our study makes two main contributions. First, we provided an explicit classification of positive definite ternary quadratic forms of level $8N$. Second, we derived exact formulas for the weighted sum of representations over each class within every genus of ternary quadratic forms of level $8N$, which involved modified Hurwitz class numbers. The proof of our main results leverages the relations between ternary quadratic forms, quaternion algebras, and weight $3/2$ modular forms of level $8N$. As applications, we obtained exact formulas for the class number of positive ternary quadratic forms of level $8N$.

Keywords: ternary quadratic forms; quaternion algebras; type number
Mathematics Subject Classification: 11E20, 11R52, 11F37, 11E41

1. Introduction

Let f be a ternary quadratic form with integer coefficients, given by the equation

$$f(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy.$$

We say f is primitive if $\gcd(a,b,c,r,s,t) = 1$. The associated matrix of f is

$$M = M_f = \begin{pmatrix} 2a & t & s \\ t & 2b & r \\ s & r & 2c \end{pmatrix}$$

with discriminant

$$d = d_f = \frac{\det(M_f)}{2} = 4abc + rst - ar^2 - bs^2 - ct^2.$$

The level of f is defined to be the smallest positive integer N such that NM_f^{-1} is an even matrix.

Let N be a product of s distinct odd primes. In our prior work [3], we classified positive definite ternary quadratic forms of level $4N$ and analyzed their representations. This paper extends these results to forms of level $8N$, and makes several original contributions that significantly extend the existing literature. First, we investigate levels of the ternary quadratic forms of $8N$, which go well beyond the levels previously considered, both in scope and arithmetic complexity. Second, we study a new and structurally distinct class of quaternion orders, differing fundamentally from the classical Eichler orders: all elements in these orders have even trace, a property not addressed in earlier works. Most importantly, we establish, for the first time, explicit formulas for the type numbers of these orders. To the best of our knowledge, there have been no explicit formulas for the type numbers of these orders before.

Let N^{odd} (resp. N^{even} or N_r) denote divisors of N containing odd (resp. even or exactly $r \geq 0$) number of prime factors. We denote $G_{8N,d,N^{\text{odd}}}$ (resp. $G_{8N,d,2N^{\text{even}}}$) for the genus of primitive positive definite ternary quadratic forms of level $8N$, discriminant d, anisotropy at primes dividing N^{odd} (resp. $2N^{\text{even}}$).

Theorem 1.1. *Let N be a product of s distinct odd primes. The primitive positive definite ternary quadratic forms of level $8N$ partition into 2^{2s+2} genera:*

$$G_{8N,2N^2/N_r,N^{\text{odd}}},\ G_{8N,2N^2/N_r,2N^{\text{even}}},\ G_{8N,64N^2/N_r,N^{\text{odd}}},\ G_{8N,64N^2/N_r,2N^{\text{even}}},$$
$$G_{8N,8N^2/N_r,N^{\text{odd}}},\ G_{8N,8N^2/N_r,2N^{\text{even}}},\ G_{8N,16N^2/N_r,N^{\text{odd}}},\ G_{8N,16N^2/N_r,2N^{\text{even}}},$$

where N^{odd} (resp. N^{even} or N_r) runs over all positive divisors of N containing odd (resp. even or exactly r) numbers of prime factors.

For coprime squarefree integers (N_1, N_2) and negative discriminant $-D$, following [1], Hurwitz class numbers $H(D)$ can be modified as follows:

$$H^{(N_1,N_2)}(D) = H(D/f_{N_1,N_2}^2) \prod_{p|N_1} \left(1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\right) \prod_{p|N_2} \frac{2pf_p - p - 1 - \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)\left(2f_p - p - 1\right)}{p-1},$$

where f_{N_1,N_2} is the maximal integer containing only prime factors of N_1N_2 whose square divides D such that $-D/f_{N_1,N_2}^2$ remains a negative discriminant; the products run through all primes p dividing N_1 and N_2, respectively. We use f_p for the exact p-power dividing f_{N_1,N_2}. In particular, when $f_p = 1$, the above fraction containing f_p becomes $1 + \left(\frac{-D/f_{N_1,N_2}^2}{p}\right)$. The symbol $(\frac{\cdot}{p})$ is the Kronecker symbol. Define

$$H^{(N_1,N_2)}(0) = -\frac{1}{12} \prod_{p|N_1} (1-p) \prod_{p|N_2} (1+p),$$

and $H^{(N_1,N_2)}(D) = 0$ for every positive integer $D \equiv 1, 2 \pmod 4$.

To give the classification of ternary quadratic forms, we introduce two notations. Set

$$\widetilde{H}^{(N^{\text{odd}},2N/N^{\text{odd}})}(n) = 4H^{(N^{\text{odd}},2N/N^{\text{odd}})}(n) - H^{(N^{\text{odd}},2N/N^{\text{odd}})}(4n) + 3H^{(2N^{\text{odd}},N/N^{\text{odd}})}(4n),$$

and

$$\widetilde{H}^{(2N^{\text{even}},N/N^{\text{even}})}(n) = 2H^{(2N^{\text{even}},N/N^{\text{even}})}(4n) + 4H^{(2N^{\text{even}},N/N^{\text{even}})}(n).$$

Theorem 1.2. *For any squarefree positive integer N and any divisors N^{odd} and N^{even} of N, and for any nonnegative integer n, one has*

$$\sum_{f\in G_{8N,2N^2/N_r,N^{\text{odd}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-2}H^{(N^{\text{odd}},2N/N^{\text{odd}})}(8N_rn),$$

$$\sum_{f\in G_{8N,2N^2/N_r,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-2}H^{(2N^{\text{even}},N/N^{\text{even}})}(8N_rn),$$

$$\sum_{f\in G_{8N,8N^2/N_r,N^{\text{odd}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-3}\widetilde{H}^{(N^{\text{odd}},2N/N^{\text{odd}})}(2N_rn),$$

$$\sum_{f\in G_{8N,8N^2/N_r,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-3}\widetilde{H}^{(2N^{\text{even}},N/N^{\text{even}})}(2N_rn),$$

$$\sum_{f\in G_{8N,16N^2/N_r,N^{\text{odd}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-3}\widetilde{H}^{(N^{\text{odd}},2N/N^{\text{odd}})}(N_rn),$$

$$\sum_{f\in G_{8N,16N^2/N_r,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-3}\widetilde{H}^{(2N^{\text{even}},N/N^{\text{even}})}(N_rn),$$

$$\sum_{f\in G_{8N,64N^2/N_r,N^{\text{odd}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-2}H^{(N^{\text{odd}},2N/N^{\text{odd}})}(N_rn),$$

and

$$\sum_{f\in G_{8N,64N^2/N_r,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = 2^{-s-2}H^{(2N^{\text{even}},N/N^{\text{even}})}(N_rn).$$

Here, as above $R_f(n)$ denotes the number of representations of n by the form f, and we denote $\text{Aut}(f)$ *for the number of automorphs of f. The sums are taken over a complete set of equivalent classes in the given genus classes.*

As applications, we will give explicit formulas for the number of classes in a genus and that of primitive definite ternary quadratic forms of level $8N$.

Theorem 1.3. *Let $|G|$ denote the number of classes in the genus G. Then*

$$|G_{8N,64N^2/N_r,N^{\text{odd}}}| = |G_{8N,2N^2/N_r,N^{\text{odd}}}| = 2^{-s-2}\sum_{n|2N}\sum_{\substack{n|r\\ r^2\le 4n}} H^{(N^{\text{odd}},2N/N^{\text{odd}})}(4n-r^2),$$

$$|G_{8N,64N^2/N_r,2N^{\text{even}}}| = |G_{8N,2N^2/N_r,2N^{\text{even}}}| = 2^{-s-2}\sum_{n|2N}\sum_{\substack{n|r\\ r^2\le 4n}} H^{(2N^{\text{even}},N/N^{\text{even}})}(4n-r^2),$$

$$|G_{8N,16N^2/N_r,N^{\text{odd}}}| = |G_{8N,8N^2/N_r,N^{\text{odd}}}| = 2^{-s-3}\sum_{\substack{n|4N\\ 2|n}}\sum_{\substack{n|r\\ 2|r}} \widetilde{H}^{(N^{\text{odd}},2N/N^{\text{odd}})}(n-\frac{r^2}{4}),$$

and

$$|G_{8N,16N^2/N_r,2N^{\text{even}}}| = |G_{8N,8N^2/N_r,2N^{\text{even}}}| = 2^{-s-3}\sum_{\substack{n|4N\\ 2|n}}\sum_{\substack{n|r\\ 2|r}} \widetilde{H}^{(2N^{\text{even}},N/N^{\text{even}})}(n-\frac{r^2}{4}).$$

Let $|C(8N)|$ denote the number of classes of primitive positive definite ternary quadratic forms of level $8N$. Then we have

$$|C(8N)| = 2^s\Bigg(\frac{5N}{12} + 3 - \frac{1}{4}\left(\frac{-4}{N}\right) - \frac{1}{3}\left(\frac{-3}{N}\right) + \frac{1}{3}\left(1-\left(\frac{N}{3}\right)^2\right) + \sum_{\substack{d|N\\ d\neq 1}}\left(H(8d) + \frac{1}{16}H(4d)\left(21 - \left(\frac{-d}{2}\right) + 3\left(\frac{-4}{d}\right) + \left(\frac{-d}{2}\right)\left(\frac{-4}{d}\right)\right)\right)\Bigg).$$

In Section 2, we review foundations of definite quaternion algebras. In Section 3, we establish commutative diagrams linking ternary quadratic forms to trace-even orders. In Section 4, we will present detailed proofs of the main theorems. In Appendices A and B, we provide tabulated data on representation numbers and class numbers for ternary quadratic forms, with computational support from SageMath.

2. Quaternion algebras

2.1. Trace-even order

In this section we will begin with basic knowledge about quaternion algebras. For comprehensive details, we refer to [5].

Let F be a field of characteristic 0 and let $a, b \in F^\times$. The quaternion algebra $Q = \left(\frac{a,b}{F}\right)$ is the F-algebra with a basis $\{1, i, j, k\}$, satisfying the relation $i^2 = a$, $j^2 = b$, and $k = ij = -ji$. Denote Q_p to be the localization of Q at prime p. In this paper, F specifically represents either $\mathbb{Q}$ or $\mathbb{Q}_p$ for prime p.

Let Q be a quadration algebra ramified at 2. Define the ramified order at 2 as:

$$\mathcal{O}_2^{\mathrm{r}} = \left\{\begin{pmatrix}\alpha & \beta\\ 2\bar{\beta} & \bar{\alpha}\end{pmatrix} : \alpha,\beta \in \mathbb{Z}_2 + \frac{1+\sqrt{\epsilon}}{2}\mathbb{Z}_2, \mathrm{tr}(\alpha), \mathrm{tr}(\beta) \in 2\mathbb{Z}_2\right\},$$

where ϵ is any positive integer such that $\left(\frac{\epsilon}{2}\right) = -1$. This is an order in Q_2, and the unique maximal order $\mathcal{O}_{2,1}^{\mathrm{r}}$ satisfies $[\mathcal{O}_{2,1}^{\mathrm{r}} : \mathcal{O}_2^{\mathrm{r}}] = 4$.

When $Q_p \cong M_2(\mathbb{Q}_p)$, the maximal order is isomorphic (over $\mathbb{Z}_p$) to $M_2(\mathbb{Z}_2)$. Define the split order at 2 as:

$$\mathcal{O}_2^{\mathrm{s}} = \left\{\begin{pmatrix}a & b\\ 2c & d\end{pmatrix} : a,b,c,d \in \mathbb{Z}_2, a+d, b+c \in 2\mathbb{Z}_2\right\}.$$

One can check that $\mathcal{O}_2^{\mathrm{r}}$ is an order in $M_2(\mathbb{Q}_p)$, $[M_2(\mathbb{Z}_2) : \mathcal{O}_2^{\mathrm{r}}] = 8$.

Definition 2.1. Let N be a product of s distinct odd primes. The order $\mathcal{O}$ is called a trace-even order of discriminant $8N$ with level $(N^{\rm odd}, 8N/N^{\rm odd})$ if it satisfies the following:

(1) $\mathcal{O}_p$ is maximal when Q ramifies at prime $p \mid N^{\rm odd}$,

(2) $\mathcal{O}_p$ is isomorphic (over $\mathbb{Z}_p$) to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ when Q splits at prime $p \mid N/N^{\rm odd}$,

(3) $\mathcal{O}_2 = \mathcal{O}_2^{\rm s}$.

The order $\mathcal{O}$ is called a trace-even order of discriminant $8N$ with level $(8N^{\rm even}, N/N^{\rm even})$ if it satisfies the following:

(1) $\mathcal{O}_p$ is maximal when Q ramifies at prime $p \mid N^{\rm even}$,

(2) $\mathcal{O}_p$ is isomorphic (over $\mathbb{Z}_p$) to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$, when Q splits at $p \mid N/N^{\rm even}$,

(3) $\mathcal{O}_2 = \mathcal{O}_2^{\rm r}$.

Example 2.2. *(1) Let* $Q = \left(\frac{-1,-6}{\mathbb{Q}}\right)$*, and then* $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ *is a trace-even order of discriminant* 24 *with level* $(3, 8)$.
(2) Let $Q = \left(\frac{-1,-1}{\mathbb{Q}}\right)$*, and then* $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}(j+k) + \mathbb{Z}(j-k)$ *is a trace-even order of discriminant* 8 *with level* $(8, 3)$.

2.2. Normalizer

Define the normalizer of $\mathcal{O}_p$ in $\mathbb{Q}_p \otimes_{\mathbb{Q}} Q$ as

$$N(\mathcal{O}_p) = \{x_p \in (\mathbb{Q}_p \otimes_{\mathbb{Q}} Q)^\times | x_p^{-1}\mathcal{O}_p x_p = \mathcal{O}_p\}.$$

Before we give the formula of $N(\mathcal{O}_p)$, we introduce the following lemma concerning $\mathcal{O}_2^\times$.

Lemma 2.3. *Let*

$$\mathcal{O}_{2,1}^r = \left\{ \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{Z}_2 + \frac{1+\sqrt{\epsilon}}{2}\mathbb{Z}_2 \right\},$$

which is maximal order in Q_2*. Let*

$$\mathcal{O}_{2,2}^r = \left\{ \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{Z}_2 + \frac{1+\sqrt{\epsilon}}{2}\mathbb{Z}_2, \mathrm{tr}(\alpha) \in 2\mathbb{Z}_2 \right\},$$

and then

$$(\mathcal{O}_{2,1}^r)^\times = (\mathcal{O}_{2,2}^r)^\times \cup \begin{pmatrix} \frac{1+\sqrt{\epsilon}}{2} & 1 \\ 2 & \frac{1-\sqrt{\epsilon}}{2} \end{pmatrix} (\mathcal{O}_{2,2}^r)^\times \cup \begin{pmatrix} \frac{1-\sqrt{\epsilon}}{2} & 1 \\ 2 & \frac{1+\sqrt{\epsilon}}{2} \end{pmatrix} (\mathcal{O}_{2,2}^r)^\times,$$

and

$$(\mathcal{O}_{2,2}^r)^\times = (\mathcal{O}_2^r)^\times \cup \begin{pmatrix} 1 & \frac{1+\sqrt{\epsilon}}{2} \\ 1-\sqrt{\epsilon} & 1 \end{pmatrix} (\mathcal{O}_2^r)^\times,$$

that is, $[(\mathcal{O}_{2,1}^r)^\times : (\mathcal{O}_2^r)^\times] = 6$*. Similarly, let*

$$\mathcal{O}_{2,1}^s = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2 \right\},$$

$$O^s_{2,2} = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2, a + d \in 2\mathbb{Z}_2 \right\},$$

and then

$$(O^s_{2,1})^\times = (O^s_{2,2})^\times$$

and

$$(O^s_{2,2})^\times = (O^s_2)^\times \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (O^s_2)^\times,$$

that is, $[(O^s_{2,1})^\times : (O^s_2)^\times] = 2$.

Proof. Set $\epsilon = -3$ and $\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in (O^{\mathrm{r}}_{2,1})^\times$, and we have $\mathrm{n}(\alpha) \in \mathbb{Z}_2^\times$. Let $\alpha = x + \frac{1+\sqrt{-3}}{2}y$, and then $x \in \mathbb{Z}_2^\times$ or $y \in \mathbb{Z}_2^\times$. If $x \in \mathbb{Z}_2^\times$ but $y \in 2\mathbb{Z}_2$, let $y = 2y'$, and we have

$$\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} x + y' + \sqrt{-3}y' & \beta \\ 2\bar{\beta} & x + y' - \sqrt{-3}y' \end{pmatrix}$$

and its trace is even. If $y \in \mathbb{Z}_2^\times$ but $x \in 2\mathbb{Z}_2$, let $x = 2x'$, and then we have

$$\begin{pmatrix} \frac{1+\sqrt{-3}}{2} & 1 \\ 2 & \frac{1-\sqrt{-3}}{2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{-3}}{2}\alpha + 2\bar{\beta} & \bar{\alpha} + \frac{-1+\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 - \sqrt{-3})\beta & \frac{-1-\sqrt{-3}}{2}\bar{\alpha} + 2\beta \end{pmatrix},$$

that is,

$$\begin{pmatrix} \frac{-1+\sqrt{-3}}{2}\alpha + 2\bar{\beta} & \bar{\alpha} + \frac{-1+\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 - \sqrt{-3})\beta & \frac{-1-\sqrt{-3}}{2}\bar{\alpha} + 2\beta \end{pmatrix} = \begin{pmatrix} -x' - y + \sqrt{-3}y + 2\bar{\beta} & \bar{\alpha} + \frac{-1+\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 - \sqrt{-3})\beta & -x' - y - \sqrt{-3}y + 2\beta \end{pmatrix}$$

and its trace is even. If $y \in \mathbb{Z}_2^\times$ and $x \in \mathbb{Z}_2^\times$, then we have

$$\begin{pmatrix} \frac{1-\sqrt{-3}}{2} & 1 \\ 2 & \frac{1+\sqrt{-3}}{2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{-1-\sqrt{-3}}{2}\alpha + 2\bar{\beta} & \bar{\alpha} + \frac{-1-\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 + \sqrt{-3})\beta & \frac{-1+\sqrt{-3}}{2}\bar{\alpha} + 2\beta \end{pmatrix},$$

that is,

$$\begin{pmatrix} \frac{-1+\sqrt{-3}}{2}\alpha + 2\bar{\beta} & \bar{\alpha} + \frac{-1+\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 - \sqrt{-3})\beta & \frac{-1-\sqrt{-3}}{2}\bar{\alpha} + 2\beta \end{pmatrix} = \begin{pmatrix} y - \frac{1+\sqrt{-3}}{2}(x + y) + 2\bar{\beta} & \bar{\alpha} + \frac{-1+\sqrt{-3}}{2}\bar{\beta} \\ 2\alpha + (-1 - \sqrt{-3})\beta & y - \frac{1-\sqrt{-3}}{2}(x + y) + 2\beta \end{pmatrix}$$

and its trace is even.

Let $\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in (O^{\mathrm{r}}_2)^\times$. If $\mathrm{tr}(\beta) \notin 2\mathbb{Z}_2$, then

$$\begin{pmatrix} 1 & \frac{1+\sqrt{-3}}{2} \\ 1 - \sqrt{-3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha - (1 + \sqrt{-3})\bar{\beta} & \beta - \frac{1+\sqrt{-3}}{2}\bar{\alpha} \\ -(1 - \sqrt{-3})\alpha + 2\bar{\beta} & \bar{\alpha} - (1 - \sqrt{-3})\beta \end{pmatrix}.$$

Let $\alpha = x + \sqrt{-3}y$, and then $x \in \mathbb{Z}_2^\times, y \in 2\mathbb{Z}_2$ or $y \in \mathbb{Z}_2^\times, x \in 2\mathbb{Z}_2$. Then

$$\mathrm{tr}(\beta - \frac{1 + \sqrt{-3}}{2}\bar{\alpha}) = \mathrm{tr}(\beta) - \mathrm{tr}(-2y + \frac{1 + \sqrt{-3}}{2}(x + y)) = \mathrm{tr}(\beta) - x + 3y,$$

and $-x+3y \notin 2\mathbb{Z}_2$, and we have $\mathrm{tr}(\beta - \frac{1+\sqrt{-3}}{2}\bar{\alpha}) \in 2\mathbb{Z}_2$.

Analogously, let

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in (\mathcal{O}^{\mathrm{s}}_{2,1})^{\times}.$$

Since $ad-2bc \in \mathbb{Z}_2^{\times}$, we have $ad \in \mathbb{Z}_2^{\times}$, that is, $a+d \in 2\mathbb{Z}_2$.

Let

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in (\mathcal{O}^{\mathrm{s}}_{2,2})^{\times},$$

where $a+d \in 2\mathbb{Z}_2$. If $b+c \notin \mathbb{Z}_2^{\times}$, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} = \begin{pmatrix} a-2c & b-d \\ 2c & d \end{pmatrix},$$

and $ad \in \mathbb{Z}_2^{\times}$, hence, $b+c-d \in \mathbb{Z}_2^{\times}$. □

Now we give the formulas of $N(\mathcal{O}_p)$ as follows.

Proposition 2.4. *If $\mathcal{O}$ is an order, such that*

$$\mathcal{O}_p \simeq \begin{cases} M_2(\mathbb{Z}_p) & \text{if} \quad p \nmid 2N, \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} & \text{if } Q \text{ ramifies at an odd prime } p, \\ \left\{ \begin{pmatrix} \alpha & \beta \\ p\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in R_p \right\} & \text{if } Q \text{ splits at an odd prime } p, \\ \left\{ \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{Z}_2 + \frac{1+\sqrt{\epsilon}}{2}\mathbb{Z}_2, \mathrm{tr}(\alpha), \mathrm{tr}(\beta) \in 2\mathbb{Z}_2 \right\} & \text{if } Q \text{ ramifies at } 2, \\ \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}_2, a+d \in 2\mathbb{Z}_2, b+c \in 2\mathbb{Z}_2 \right\} & \text{if } Q \text{ splits at } 2, \end{cases}$$

where $R_p = \mathbb{Z}_p + \sqrt{\epsilon}\mathbb{Z}_p$ and $\left(\frac{\epsilon}{p}\right) = -1$, then

$$N(\mathcal{O}_p) = \begin{cases} \mathcal{O}_p^{\times}\mathbb{Q}_p^{\times} & \text{if} \quad p \nmid 2N, \\ \mathcal{O}_p^{\times}\mathbb{Q}_p^{\times} \cup \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \mathcal{O}_p^{\times}\mathbb{Q}_p^{\times} & \text{if} \quad p \mid N. \end{cases}$$

If Q ramifies at 2, then

$$N(\mathcal{O}_2) = \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 1 & \frac{1+\sqrt{\epsilon}}{2} \\ 1-\sqrt{\epsilon} & 1 \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 1-\sqrt{\epsilon} & 1 \\ 2 & 1+\sqrt{\epsilon} \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times}.$$

If Q splits at 2, then

$$N(\mathcal{O}_2) = \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times} \cup \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \mathcal{O}_2^{\times}\mathbb{Q}_2^{\times}.$$

Proof. If Q ramifies at 2, let $\epsilon = -3$. Since $\mathrm{tr}(\alpha^{-1}\beta\alpha) = \mathrm{tr}(\beta)$ for $\alpha \in Q^\times$, we have

$$N(\mathcal{O}^{\mathrm{r}}_{2,2}) = N(\mathcal{O}^{\mathrm{r}}_{2,1}) = (\mathcal{O}^{\mathrm{r}}_{2,1})^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} (\mathcal{O}^{\mathrm{r}}_{2,1})^\times \mathbb{Q}_2^\times.$$

It is not hard to check that

$$N(\mathcal{O}^{\mathrm{r}}_2) \subset N(\mathcal{O}^{\mathrm{r}}_{2,2}),$$

and $\begin{pmatrix} \frac{1+\sqrt{-3}}{2} & 1 \\ 2 & \frac{1-\sqrt{-3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1-\sqrt{-3}}{2} & 1 \\ 2 & \frac{1+\sqrt{-3}}{2} \end{pmatrix}$ are not in $N(\mathcal{O}^{\mathrm{r}}_2)$, and

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \mathcal{O}_2 \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \mathcal{O}_2, \begin{pmatrix} 1 & \frac{1+\sqrt{-3}}{2} \\ 1-\sqrt{-3} & 1 \end{pmatrix}^{-1} \mathcal{O}_2 \begin{pmatrix} 1 & \frac{1+\sqrt{-3}}{2} \\ 1-\sqrt{-3} & 1 \end{pmatrix} = \mathcal{O}_2,$$

that is,

$$N(\mathcal{O}_2) = \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 1 & \frac{1+\sqrt{\epsilon}}{2} \\ 1-\sqrt{\epsilon} & 1 \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 1-\sqrt{\epsilon} & 1 \\ 2 & 1+\sqrt{\epsilon} \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times.$$

When Q splits at 2, since $\mathrm{tr}(\alpha^{-1}\beta\alpha) = \mathrm{tr}(\beta)$ for $\alpha \in Q^\times$, we have

$$N(\mathcal{O}^{\mathrm{s}}_{2,2}) = N(\mathcal{O}^{\mathrm{s}}_{2,1}).$$

It is not hard to check that

$$N(\mathcal{O}^{\mathrm{s}}_2) \subset N(\mathcal{O}^{\mathrm{s}}_{2,2}),$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \mathcal{O}_2 \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \mathcal{O}_2,$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \mathcal{O}_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathcal{O}_2.$$

Hence,

$$N(\mathcal{O}_2) = \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \mathcal{O}_2^\times \mathbb{Q}_2^\times.$$

□

We present an equivalent condition for two-sided principal $\mathcal{O}$-ideals.

Lemma 2.5. *Let $\mathcal{O}$ be a trace-even order of level $(N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}})$ (resp. $(8N^{\mathrm{even}}, N/N^{\mathrm{even}})$). For any element $x \in \mathcal{O}$, $\mathcal{O}x$ is a two-sided principal $\mathcal{O}$-ideal if and only if* $\mathrm{n}(x) \mid 4N$, $2 \mid \mathrm{n}(x)$, *and* $\mathrm{n}(x) \mid \mathrm{tr}(x)$.

Proof. Let $\mathcal{O}x$ be a two-sided principal $\mathcal{O}$-ideal. We have

$$x \in \begin{cases} \mathcal{O}_p^\times & \text{if} \quad p \nmid 2N, \\ \mathcal{O}_p^\times \cup \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \mathcal{O}_p^\times & \text{if} \quad p \mid N, \\ 2\mathcal{O}_2^\times \cup 2 \cdot \begin{pmatrix} 1 & \frac{1+\sqrt{\epsilon}}{2} \\ 1-\sqrt{\epsilon} & 1 \end{pmatrix} \mathcal{O}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^\times \cup \begin{pmatrix} 1-\sqrt{\epsilon} & 1 \\ 2 & 1+\sqrt{\epsilon} \end{pmatrix} \mathcal{O}_2^\times & \text{if } Q \text{ ramifies at 2}, \\ 2\mathcal{O}_2^\times \cup 2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathcal{O}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \mathcal{O}_2^\times \cup \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \mathcal{O}_2^\times & \text{if } Q \text{ splits at 2}. \end{cases}$$

When p is an odd prime, it is analogous in the proof of [1, Lemma 2.13].

When $x \in 2\mathcal{O}_2^\times$, since $\mathrm{n}(x) \in 4\mathbb{Z}_2^\times$ and $\mathrm{tr}(x) \in 4\mathbb{Z}_2$, we have $\mathrm{n}(x) \mid \mathrm{tr}(x)$.

Let $\epsilon = -3$. When $x \in \begin{pmatrix} 2 & 1+\sqrt{-3} \\ 2-2\sqrt{-3} & 2 \end{pmatrix}\mathcal{O}_2^\times$, we have $\mathrm{n}(x) \in 4\mathbb{Z}_2^\times$. There exists $\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_2^\times, \mathrm{tr}(\alpha) \in 2\mathbb{Z}_2, \mathrm{tr}(\beta) \in 2\mathbb{Z}_2$ such that

$$\mathrm{tr}(x) = \mathrm{tr}\left(\begin{pmatrix} 2 & 1+\sqrt{-3} \\ 2-2\sqrt{-3} & 2 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) = \mathrm{tr}(2\alpha + 2(1+\sqrt{-3})\bar{\beta}) \in 4\mathbb{Z}_2.$$

Hence, $\mathrm{n}(x) \mid \mathrm{tr}(x)$.

When $x \in \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\mathcal{O}_2^\times$, we have $\mathrm{n}(x) \in 2\mathbb{Z}_2^\times$. There exists $\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_2^\times, \mathrm{tr}(\alpha) \in 2\mathbb{Z}_2, \mathrm{tr}(\beta) \in 2\mathbb{Z}_2$ such that

$$\mathrm{tr}(x) = \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) = 2\,\mathrm{tr}(\bar{\beta}) \in 4\mathbb{Z}_2.$$

Hence, $\mathrm{n}(x) \mid \mathrm{tr}(x)$.

When $x \in \begin{pmatrix} 1-\sqrt{-3} & 1 \\ 2 & 1+\sqrt{-3} \end{pmatrix}\mathcal{O}_2^\times$, we have $\mathrm{n}(x) \in 2\mathbb{Z}_2^\times$. There exists $\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_2^\times, \mathrm{tr}(\alpha) \in 2\mathbb{Z}_2, \mathrm{tr}(\beta) \in 2\mathbb{Z}_2$ such that

$$\mathrm{tr}(x) = \mathrm{tr}\left(\begin{pmatrix} 1-\sqrt{-3} & 1 \\ 2 & 1+\sqrt{-3} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) = \mathrm{tr}((1-\sqrt{-3})\alpha + 2\beta) \in 2\mathbb{Z}_2.$$

Hence, $\mathrm{n}(x) \mid \mathrm{tr}(x)$.

Conversely, let $x \in \mathcal{O}$, $\mathrm{n}(x) \mid 4N$, $2 \mid \mathrm{n}(x)$, $\mathrm{n}(x) \mid \mathrm{tr}(x)$. Since $\mathcal{O}x$ is a left $\mathcal{O}$-ideal, we will prove $\mathcal{O}x$ is a two-sided principal $\mathcal{O}$-ideal, that is, for all p, we have $x \in N(\mathcal{O}_p)$. When p is an odd prime, it is analogous.

When $2 \parallel \mathrm{n}(x)$, let $x_2 = \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_2$, where $\mathrm{tr}(\alpha), \mathrm{tr}(\beta) \in 2\mathbb{Z}_2$, $\mathrm{n}(\alpha) - 2\,\mathrm{n}(\beta) \in 2\mathbb{Z}_2^\times$. Let $\alpha = u_1 + v_1\sqrt{-3}$. If $\mathrm{tr}(\alpha) \in 4\mathbb{Z}_2$, then $u_1 \in 2\mathbb{Z}_2$, $\mathrm{n}(\alpha) \in 2\mathbb{Z}_2$, and $v_1 \in 2\mathbb{Z}_2$. We have $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \beta & \bar{\alpha}/2 \\ \alpha & \bar{\beta} \end{pmatrix}$, and $\mathrm{tr}(\beta) \in 2\mathbb{Z}$, $\mathrm{tr}(\alpha/2) = u_1 \in 2\mathbb{Z}_2$, that is, $x \in N(\mathcal{O}_p)$. If $\mathrm{tr}(\alpha) \notin 4\mathbb{Z}_2$, that is, $u_1 \in \mathbb{Z}_2^\times$, then $\mathrm{n}(\alpha) \in 2\mathbb{Z}_2$ implies $v_1 \in \mathbb{Z}_2^\times$. Hence, $\mathrm{n}(\alpha) \in 4\mathbb{Z}_2$, and $2\,\mathrm{n}(\beta) \in 2\mathbb{Z}_2^\times$. Let $\beta = u_2 + v_2\sqrt{-3}$, that is, $u_2 + v_2 \in \mathbb{Z}_2^\times$. We have $\begin{pmatrix} 1-\sqrt{-3} & 1 \\ 2 & 1+\sqrt{-3} \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{-3}}{2}\alpha - \bar{\beta} & \frac{1+\sqrt{-3}}{2}\beta - \frac{1}{2}\bar{\alpha} \\ (1+\sqrt{-3})\bar{\beta} - \alpha & \frac{1+\alpha-3}{2}\bar{\alpha} - \beta \end{pmatrix}$, and $\mathrm{tr}(\frac{1+\sqrt{-3}}{2}\alpha - \bar{\beta}) = u_1 - 3v_1 - \mathrm{tr}(\beta) \in 2\mathbb{Z}_2$, $\mathrm{tr}(\frac{1+\sqrt{-3}}{2}\beta - \frac{1}{2}\bar{\alpha}) = u_2 - 3v_2 + u_1 \in 2\mathbb{Z}_2$, that is, $x \in N(\mathcal{O}_p)$.

When $4 \parallel \mathrm{n}(x)$, let $x_p = \begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathcal{O}_2$, where $\mathrm{tr}(\alpha) \in 4\mathbb{Z}_2$, $\mathrm{tr}(\beta) \in 2\mathbb{Z}_2$, $\mathrm{n}(\alpha) - 2\,\mathrm{n}(\beta) \in 4\mathbb{Z}_2^\times$. Let $\alpha = u_1 + v_1\sqrt{-3}$, and then $u_1 \in 2\mathbb{Z}_2$ implies $v_1 \in 2\mathbb{Z}_2$, that is, $\mathrm{n}(\alpha) \in 4\mathbb{Z}_2$. Let $\beta = u_2 + v_2\sqrt{-3}$. If $u_2 \in 2\mathbb{Z}_2$, then $2\,\mathrm{n}(\beta) \in 2\mathbb{Z}_2$ implies $v_2 \in 2\mathbb{Z}_2$, that is, $x/2 \in \mathcal{O}_2^\times$. Otherwise $x/2 \notin \mathcal{O}_2^\times$, that is, $u_2 \notin 2\mathbb{Z}_2$, and we have $u_2, v_2 \in \mathbb{Z}_2^\times$. Then $2\,\mathrm{n}(\beta) \in 8\mathbb{Z}_2$, which implies $\mathrm{n}(\alpha) \in 4\mathbb{Z}_2^\times$. Hence, $u_1 \in 4\mathbb{Z}_2$, $v_1 \in 2\mathbb{Z}_2^\times$ or $v_1 \in 4\mathbb{Z}_2$, $u_1 \in 2\mathbb{Z}_2^\times$. We have $\begin{pmatrix} 2 & 1+\sqrt{-3} \\ 2-2\sqrt{-3} & 2 \end{pmatrix}^{-1}\begin{pmatrix} \alpha & \beta \\ 2\bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\alpha + \frac{1-\sqrt{-3}}{2}\bar{\beta} & \frac{1-\sqrt{-3}}{4}\bar{\alpha} - \frac{1}{2}\beta \\ \frac{1+\sqrt{-3}}{4}\alpha - \frac{1}{2}\bar{\beta} & -\frac{1}{2}\bar{\alpha} + \frac{1+\sqrt{-3}}{2}\beta \end{pmatrix}$, and $\mathrm{tr}(-\frac{1}{2}\alpha + \frac{1-\sqrt{-3}}{2}\bar{\beta}) = -u_1 + u_2 + 3v_2 \in 2\mathbb{Z}_2$, $\mathrm{tr}(\frac{1-\sqrt{-3}}{4}\bar{\alpha} - \frac{1}{2}) = (u_1 - 3v_1)/2 - u_2 \in 2\mathbb{Z}_2$, that is, $x \in N(O_p)$.

The case when Q splits at 2 is analogous. □

2.3. Type number formula

Fix an order O of level (N_1, N_2), and then all two-sided O-ideals form a group, which is denoted by $\mathfrak{I}$. The number of elements of group $\mathfrak{I}(O)/\mathbb{Q}^\times$ is $2^{e(N+2)}$. All two-sided principal O-ideals form a subgroup, which is denoted by $\mathfrak{B}(O)$. We have the following proposition for $\mathrm{card}(\mathfrak{B}(O)/\mathbb{Q}^\times)$ and $\mathrm{Aut}(O)$ which will be used in proving the type number formula.

Theorem 2.6. *Let O be a trace-even order of level $(N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}})$ (resp. $(8N^{\mathrm{even}}, N/N^{\mathrm{even}})$), and $m(O)$ be the number of left O-ideal classes containing a two-sided O-ideal. Then we have*

$$\mathrm{card}(\mathfrak{B}(O)/\mathbb{Q}^\times) = \frac{2^{e(N)+2}}{m(O)}. \tag{2.1}$$

Let $\rho_O(n, r)$ be the number of $x \in O$ where $\mathrm{n}(x) = n$ and $\mathrm{tr}(x) = r$, and then

$$\mathrm{card}(\mathfrak{B}(O)/\mathbb{Q}^\times) = \sum_{\substack{n|4N \\ 2|n}} \sum_{\substack{n|r \\ r^2 \le 4n \\ 2|r}} \frac{\rho_O(n, r)}{\mathrm{card}(O^\times)} \tag{2.2}$$

and

$$\mathrm{card}(\mathrm{Aut}(O)) = \frac{2^{e(N)+2}}{\mathrm{card}(O^\times)} 2m(O). \tag{2.3}$$

Proof. See [1, Proposition 2.14]. □

By (2.1)–(2.3), we have

$$\begin{aligned}
T'_{N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}}} &= 2^{-e(N)+2} \sum_{\mu=1}^{T'_{N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}}}} 2^{e(N)+2} \\
&= 2^{-e(N)+2} \sum_{\mu=1}^{T'_{N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}}}} m(O_\mu)\, \mathrm{card}(\mathfrak{B}(O_\mu)/\mathbb{Q}^\times) \\
&= 2^{-e(N)+2} \sum_{\mu=1}^{T'_{N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}}}} m(O_\mu) \sum_{\substack{n|4N \\ 2|n}} \sum_{\substack{n|r \\ r^2 \le 4n}} \frac{\rho_{O_\mu}(n, r)}{\mathrm{card}(O_\mu^\times)} \\
&= 2^{-1} \sum_{\substack{n|4N \\ 2|n}} \sum_{\substack{n|r \\ r^2 \le 4n \\ 2|r}} \sum_{\mu=1}^{T'_{N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}}}} \frac{\rho_{O_\mu}(n, r)}{\mathrm{card}(\mathrm{Aut}(O_\mu))}.
\end{aligned}$$

It is analogous for $8N^{\text{even}}$ as follows:

$$T'_{8N^{\text{even}},N/N^{\text{even}}} = 2^{-1}\sum_{\substack{n|4N\\2|n}}\sum_{\substack{n|r\\r^2\leq 4n\\2|r}}\sum_{\mu=1}^{T'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{\rho_{O_\mu}(n,r)}{\text{card}(\text{Aut}(O_\mu))}.$$

Corollary 2.7. *Let O be a trace-even order of level $(N^{\text{odd}}, 8N/N^{\text{odd}})$ (resp. $(8N^{\text{even}}, N/N^{\text{even}})$), and we have*

$$\sum_{\mu=1}^{T'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O_\mu))} = 2^{-e(N)+1}\frac{1}{12}\prod_{p|N^{\text{odd}}}(p-1)\prod_{p|2N/N^{\text{odd}}}(p+1),$$

and

$$\sum_{\mu=1}^{T'_{8N^{\text{even}},N/N^{\text{even}}}}\frac{1}{\text{card}(\text{Aut}(O_\mu))} = 2^{-e(N)+1}\frac{1}{4}\prod_{p|2N^{\text{even}}}(p-1)\prod_{p|N/N^{\text{even}}}(p+1).$$

Proof. We will use the Eichler mass formula [5, Proposition 26.6.4]. Let O' be an Eichler order in $Q_{N^{\text{odd}}}$ with level $(N^{\text{odd}}, 2N/N^{\text{odd}})$, and then

$$\sum_{i=1}^{h_{N^{\text{odd}},2N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O'_i))} = \frac{1}{24}\prod_{p|N^{\text{odd}}}(p-1)\prod_{p|2N/N^{\text{odd}}}(p+1).$$

It is not hard to check that $O' \subset O$, and $O'_p = O_p$ for all odd primes p. Then we have

$$\sum_{i=1}^{h'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O_i))} = [O'^{\times}_2 : O^{\times}_2]\sum_{i=1}^{h_{N^{\text{odd}},2N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O'_i))}.$$

Since $[O'^{\times}_2 : O^{\times}_2] = 2$, and that is

$$\sum_{i=1}^{h'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O_i))} = \frac{1}{12}\prod_{p|N^{\text{odd}}}(p-1)\prod_{p|2N/N^{\text{odd}}}(p+1).$$

Hence,

$$\begin{aligned}\sum_{\mu=1}^{T'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O_\mu))} &= \sum_{\mu=1}^{T'_{N^{\text{odd}},2N/N^{\text{odd}}}}\frac{2m(O_\mu)}{2^{e(N)+2}\text{card}(\text{Aut}(O_\mu))}\\ &= 2^{-e(N)-2}\sum_{i=1}^{h'_{N^{\text{odd}},8N/N^{\text{odd}}}}\frac{1}{\text{card}(\text{Aut}(O_i))}\\ &= 2^{-e(N)+1}\frac{1}{12}\prod_{p|N^{\text{odd}}}(p-1)\prod_{p|2N/N^{\text{odd}}}(p+1).\end{aligned}$$

The rest of the proof is analogous. □

3. Ternary quadratic forms and quaternion orders

In this section, we will discuss two maps between trace-even orders and ternary quadratic forms.

3.1. Two bijections between even orders and ternary quadratic forms

Since the trace of all elements in trace-even orders are even, it is not hard to check that exists $\mathrm{tr}(\alpha_1) = \mathrm{tr}(\alpha_2) = \mathrm{tr}(\alpha_3) = 0$, such that

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3.$$

For a trace-even order, $\mathcal{O}^0 = \mathcal{O} \cap Q^0 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}(\alpha_3)$, where $Q^0 = \{\alpha \in Q : \mathrm{tr}(\alpha) = 0\}$, is an even integral lattice, when equipped with the bilinear form $(x, y) \mapsto \mathrm{tr}(x\bar{y})$. When the quaternion algebra is positive definite (ramifies at ∞), it is positive definite. We have the following ternary quadratic form:

$$\begin{aligned} f_{\mathcal{O}^0} &= \mathrm{n}(x\alpha_1 + y\alpha_2 + z\alpha_3) \\ &= \mathrm{n}(\alpha_1)x^2 + \mathrm{n}(\alpha_2)y^2 + (\mathrm{n}(\alpha_3))z^2 + \mathrm{tr}(\alpha_2\overline{\alpha_3})yz + \mathrm{tr}(\alpha_1\overline{\alpha_3})xz + \mathrm{tr}(\alpha_1\overline{\alpha_2})xy, \end{aligned}$$

and the Gram matrix of $f_{\mathcal{O}^0}$,

$$M_{f_{\mathcal{O}^0}} = \begin{pmatrix} \mathrm{tr}(\alpha_1\overline{\alpha_1}) & \mathrm{tr}(\alpha_1\overline{\alpha_2}) & \mathrm{tr}(\alpha_1\overline{\alpha_3}) \\ \mathrm{tr}(\alpha_2\overline{\alpha_1}) & \mathrm{tr}(\alpha_2\overline{\alpha_2}) & \mathrm{tr}(\alpha_2\overline{\alpha_3}) \\ \mathrm{tr}(\alpha_3\overline{\alpha_1}) & \mathrm{tr}(\alpha_3\overline{\alpha_2}) & \mathrm{tr}(\alpha_3\overline{\alpha_3}) \end{pmatrix}.$$

For a quaternion algebra over $\mathbb{Q}_p$, we can give the following characterization of the genus which $f_{\mathcal{O}^0}$ belongs to.

Proposition 3.1. *Let N be a product of s distinct odd primes, and for an order $\mathcal{O}$, we denote $f_{\mathcal{O}^0}$ by f.*

(1) Let $\mathcal{O} \subset Q_{N^{\mathrm{odd}}}$ be a trace-even order of level $(N^{\mathrm{odd}}, 8N/N^{\mathrm{odd}})$, and then $d_f = 16N^2, N_f = 8N$, and f is anisotropic only in the p-adic field for $p \mid N^{\mathrm{odd}}$. The genus which f belongs to is denoted by $G_{8N,16N^2,N^{\mathrm{odd}}}$.

(2) Let $\mathcal{O} \subset Q_{2N^{\mathrm{even}}}$ be a trace-even order of level $(8N^{\mathrm{even}}, N/N^{\mathrm{even}})$, and then $d_f = 16N^2, N_f = 8N$, and f is anisotropic only in the p-adic field for $p \mid 2N^{\mathrm{even}}$. The genus which f belongs to is denoted by $G_{8N,16N^2,2N^{\mathrm{even}}}$.

Proof. For an odd prime p, one can see [3, Proposition 4.3]. We only give f_2.

(1)

$$f \underset{2}{\sim} x^2 - 2y^2 - 2z^2.$$

(2)

$$f \underset{2}{\sim} 3x^2 + 2y^2 + 6z^2.$$

□

If f is a non-degenerate ternary quadratic form integral over $\mathbb{Z}$, we define $C_0(f)$ to be the even Clifford algebras over $\mathbb{Z}$ associated with f. Subsequently, $C_0(f)$ is an order in a quaternion algebra over $\mathbb{Q}$. For a more-detailed definition of Clifford algebras, refer to [5, Chapter 22]. Conversely, consider that $\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ is an order in a quaternion algebra Q. Then $\Lambda = \mathcal{O}^\sharp \cap Q^0$ represents a 3-dimensional $\mathbb{Z}$-lattice on Q^0. If $\mathcal{O}^\sharp = \langle e_0', e_1', e_2', e_3' \rangle$ where e_i' is the dual basis of e_i, then $\Lambda = \langle e_1', e_2', e_3' \rangle$. Define a ternary quadratic form $f_{\mathcal{O}}$ associated to $\mathcal{O}$ by

$$f_{\mathcal{O}} = \mathrm{discrd}(\mathcal{O}) \cdot n(xe_1' + ye_2' + ze_3').$$

Theorem 3.2. *[5, Main Theorem 22.1.1] Let R be a principal ideal domain. The maps $f \mapsto C_0(f)$ and $\mathcal{O} \mapsto f_{\mathcal{O}}$ are inverses to each other and the discriminants satisfy* $\mathrm{discrd}(\mathcal{O}) = d(f_{\mathcal{O}})$. *Furthermore, the maps give a bijection between analogousity classes of non-degenerate ternary quadratic forms integral over R and isomorphism classes of quaternion R-orders.*

Assume that $f = (a, b, c, r, s, t), d_f = d$. We have

$$e_1^2 = re_1 - bc, e_2e_3 = a\overline{e_1},$$

$$e_2^2 = se_2 - ac, e_3e_1 = b\overline{e_2},$$

$$e_3^2 = te_3 - ab, e_1e_2 = c\overline{e_3}.$$

By the definition of a dual basis, it follows that $\mathcal{O}^\sharp \supseteq \mathcal{O}$ and $\mathrm{tr}(e_0') = 1, \mathrm{tr}(e_1') = \mathrm{tr}(e_2') = \mathrm{tr}(e_3') = 0$, where

$$de_0' = d - 2(abc + rst) + (ar + st)e_1 + (bs + rt)e_2 + (ct + rs)e_3,$$

$$de_1' = ar + st - 2ae_1 - te_2 - se_3,$$

$$de_2' = bs + rt - te_1 - 2be_2 - re_3,$$

$$de_3' = ct + rs - se_1 - re_2 - 2ce_3,$$

and

$$\mathrm{n}(Ne_1') = Na, \mathrm{tr}(Ne_2'\overline{Ne_3'}) = Nr,$$

$$\mathrm{n}(Ne_2') = Nb, \mathrm{tr}(Ne_3'\overline{Ne_1'}) = Ns,$$

$$\mathrm{n}(Ne_3') = Nc, \mathrm{tr}(Ne_1'\overline{Ne_2'}) = Nt.$$

Recalling that $\mathcal{O}$ contains a basis of Q over $\mathbb{Q}$, it follows that $\langle e_1', e_2', e_3' \rangle$ form a $\mathbb{Q}$–basis for the trace-zero elements of Q (that is, $\langle i, j, k \rangle$ can be represented by $\langle e_1', e_2', e_3' \rangle$ over $\mathbb{Q}$). Now we have some properties of even Clifford algebras.

Theorem 3.3. *The following statements hold.*

(1) $f_{\mathcal{O}}$ is positive definite if and only if $\mathcal{O}$ is positive definite.
(2) $2\mathrm{card}(\mathrm{Aut}(\mathcal{O})) = |\mathrm{Aut}(f_{\mathcal{O}})|$.
(3) $\mathcal{O}$ ramifies at p if and only if $f_{\mathcal{O}}$ is anisotropic at p.
(4) $f_{\mathcal{O}}$ is primitive if and only if $\mathcal{O}$ is Gorenstein.

For more-detailed definition of Gorenstein orders, one can see [5, Chapter 24.2].

Proposition 3.4. *Let N be a product of s distinct odd primes, and for an order $\mathcal{O}$, we denote $f_{\mathcal{O}}$ by f.*

(1) *Let* $\mathcal{O} \subset Q_{N^{odd}}$ *be a trace-even order of level* $(N^{odd}, 8N/N^{odd})$*, and then* $d_f = 8N, N_f = 8N$*, and* f *is anisotropic only in the* p*-adic field for* $p \mid N^{odd}$*. The genus which* f *belongs to is denoted by* $G_{8N,8N,N^{odd}}$.
(2) *Let* $\mathcal{O} \subset Q_{N^{odd}}$ *be a trace-even order of level* $(8N^{even}, N/N^{even})$*, and then* $d_f = 8N, N_f = 8N$*, and* f *is anisotropic only in the* p*-adic field for* $p \mid 2N^{even}$*. The genus which* f *belongs to is denoted by* $G_{8N,8N,2N^{even}}$.

Proof. For an odd prime p, one can see [4, Proposition 4.8]. We only give f_2.

(1)

$$f \underset{2}{\sim} x^2 - y^2 - 2z^2.$$

(2)

$$f \underset{2}{\sim} 3x^2 + y^2 + 6z^2.$$

□

3.2. Commutative diagram

Recall $\phi_p : C(N, p^2d) \to C(N, pd)$ for an odd prime p, $p \nmid N, d$ and $\phi_2 : C(8N, 2^4d) \to C(8N, 2^3d)$ is a bijection defined by Lehman [2].

Proposition 3.5. *Let* $\mathcal{O}$ *be a trace-even order with level* $(N^{odd}, 8N/N^{odd})$ *or* $(8N^{even}, N/N^{even})$*, where* $N = p_1...p_s$*, and then* $\phi_{p_1} \circ ... \circ \phi_{p_s} \circ \phi_2(f_{\mathcal{O}^0}) = f_{\mathcal{O}}$*.*

Proof. For discrd$(\mathcal{O}) = p$, see [3, Proposition 6.1]. Let $f = (2a, b, c, 2r, 4s, 4t)$ be a positive definite ternary quadratic form of level 8 and discriminant 8. Recalling ϕ_2:

$$\phi_2^{-1}((2a, b, c, 2r, 4s, 4t)) = (a, 2b, 2c, 4r, 4s, 4t), 2 \nmid ac,$$

and then disc$(\mathcal{O}) = 8$, and

$$\mathcal{O} = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3,$$

with

$$e_1^2 = 2re_1 - bc, \qquad e_2e_3 = 2a\overline{e_1},$$
$$e_2^2 = 4se_2 - 2ac, \qquad e_3e_1 = b\overline{e_2},$$
$$e_3^2 = 4te_3 - 2ab, \qquad e_1e_2 = c\overline{e_3}.$$

Then

$$8e_0' = 8 - 2(2abc + 32rst) + (4ar + 16st)e_1 + (4bs + 8rt)e_2 + (4ct + 8rs)e_3,$$
$$8e_1' = 4ar + 16st - 4ae_1 - 4te_2 - 4se_3,$$
$$8e_2' = 4bs + 8rt - 4te_1 - 2be_2 - 2re_3,$$
$$8e_3' = 4ct + 8rs - 4se_1 - 2re_2 - 2ce_3.$$

We have $\mathrm{tr}(2e_1') = \mathrm{tr}(4e_2') = \mathrm{tr}(4e_3') = 0$, and

$$n(2e_1') = a, \mathrm{tr}(4e_2'\overline{4e_3'}) = 4r,$$

$$n(4e_2') = 2b, \mathrm{tr}(4e_3'\overline{2e_1'}) = 4s,$$

$$n(4e_3') = 2c, \mathrm{tr}(2e_1'\overline{4e_2'}) = 4t.$$

We will show that

$$O = \mathbb{Z}2e_0' + \mathbb{Z}2e_1' + \mathbb{Z}4e_2' + \mathbb{Z}4e_3'.$$

We have

$$\begin{pmatrix} 2e_0' \\ 2e_1' \\ 4e_2' \\ 4e_3' \end{pmatrix} = \begin{pmatrix} 2 - abc - 16rst & ar + 4st & bs + 2rt & ct + 2rs \\ ar + 4st & -a & -t & -s \\ 2bs + 4rt & -2t & -b & -r \\ 2ct + 4rs & -2s & -r & -c \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = M_p \begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Since $d_f = 8$, we have

$$abc + 4rst - ar^2 - 4bs^2 - 4ct^2 = 1,$$

and

$$2e_0' + 2re_1' + 4se_2' + 4te_3' = 2 - abc - 4rst + ar^2 + 2bs^2 + 2ct^2 = 1.$$

It is not hard to check that

$$\det(M_p) = -(abc + 4rst - ar^2 - 4bs^2 - 4ct^2) = -1.$$

Hence,

$$O = \mathbb{Z}2e_0' + \mathbb{Z}2e_1' + \mathbb{Z}4e_2' + \mathbb{Z}4e_3',$$

where $\mathrm{tr}(2e_1') = \mathrm{tr}(4e_2') = \mathrm{tr}(4e_3') = 0$, and

$$\begin{aligned} f_{O^0} &= n(x2e_1' + y4e_2' + z4e_3') \\ &= n(2e_1')x^2 + n(4e_2')y^2 + n(4e_3')z^2 + \mathrm{tr}(4e_2'\overline{4e_3'})yz + \mathrm{tr}(2e_1'\overline{4e_3'})xz + \mathrm{tr}(2e_1'\overline{4e_2'})xy \\ &= ax^2 + 2by^2 + 2cz^2 + 4ryz + 4sxz + 4txy. \end{aligned}$$

The rest of the proof is analogous.

□

Theorem 3.6. *Let $N = p_1...p_s$, and $\phi_{2N} = \phi_{p_1} \circ ... \circ \phi_{p_s} \circ \phi_2$. In a positive definite quaternion algebra $Q_{N^{odd}}$ (resp. $Q_{2N^{even}}$), we choose a complete set of representatives $\{O_\mu\}_{\mu=1,2...T'_{N^{odd},8N/N^{odd}}}$ (resp. $\{O_\mu\}'_{\mu=1,2...T_{8N^{even},N/N^{even}}}$) for these types of trace-even orders. We give the commutative diagrams as follows.*

$$\begin{array}{ccc} \{O_\mu\}_{\mu=1,2...T'_{N^{odd},8N/N^{odd}}} & \xrightarrow{C_0} & G_{8N,16N^2,N^{odd}} \\ & \searrow^{M_0} & \downarrow \phi_{2N}^{-1} \\ & & G_{8N,8N,N^{odd}} \end{array}$$

$$\begin{array}{ccc} \{O_\mu\}_{\mu=1,2...T'_{8N^{even},N/N^{even}}} & \xrightarrow{C_0} & G_{8N,16N^2,2N^{even}} \\ & \searrow^{M_0} & \downarrow \phi_{2N}^{-1} \\ & & G_{8N,8N,2N^{even}} \end{array}$$

Since C_0 is a bijection, $2\mathrm{card}(\mathrm{Aut}(O)) = |\,\mathrm{Aut}(f_O)|$ and ϕ are bijections of equivalence classes which preserve automorphism counts, abd it is not hard to prove the following corollary.

Corollary 3.7. *M_0 is a bijection, and*

$$2\mathrm{card}(\mathrm{Aut}(O)) = |\,\mathrm{Aut}(f_{O^0})|.$$

Corollary 3.8.

$$|G_{8N,16N^2,N^{\mathrm{odd}}}| = |G_{8N,8N,N^{\mathrm{odd}}}| = T'_{N^{\mathrm{odd}},8N/N^{\mathrm{odd}}}.$$
$$|G_{8N,16N^2,2N^{\mathrm{even}}}| = |G_{8N,8N,2N^{\mathrm{even}}}| = T'_{8N^{\mathrm{even}},N/N^{\mathrm{even}}}.$$

Proposition 3.9. *Let O be a trace-even order and $\rho_O(n,r)$ be the number of zeros of $x^2 - rx + n$ in O_μ, and then we have*

$$R_{f_{O^0}}(n - \frac{r^2}{4}) = \rho_O(n,r).$$

Proof. It is analogous as [3, Proposition 6.5]. □

4. Proof

Proof of Theorem 1.1. Note that there are 2^{s+1} genera in $C(4N, 64N^2)$, $C(4N, 16N^2)$, $C(4N, 8N^2)$, and $C(4N, 2N^2)$ [2, Lemma 3], and there are 2^{2s+1} genera for all primitive positive definite ternary quadratic forms of level $8N$. It is not hard to check that there are 2^{2s+1} genera which are

$$G_{8N,64N^2/N_r,N^{\mathrm{odd}}}, G_{8N,64N^2/N_r,2N^{\mathrm{even}}}, G_{8N,2N^2/N_r,N^{\mathrm{odd}}}, G_{8N,2N^2/N_r,2N^{\mathrm{even}}}$$

(see [3, Proposition 4.4, Proposition 4.8]) and

$$G_{8N,16N^2/N_r,N^{\mathrm{odd}}}, G_{8N,16N^2/N_r,2N^{\mathrm{even}}}, G_{8N,8N^2/N_r,N^{\mathrm{odd}}}, G_{8N,8N^2/N_r,2N^{\mathrm{even}}}$$

when $N^{\mathrm{odd}}, N^{\mathrm{even}}$, and N_r run over all factors that divide N. □

Before we give the proof of Theorem 1.2, we first give a new base of the Eisenstein space $\mathscr{E}(8N, \frac{3}{2}, \chi_l)$ where $\mathscr{E}(8N, \frac{3}{2}, \chi_l)$ is the orthogonal complement of the subspace of cusp forms in the complex linear space of modular forms of weight $3/2$, level $8N$, and character χ_l with respect to the Petersson inner product. For a more-detailed definition one can see [6].

Theorem 4.1. *Let I denote the set of all positive divisors of $2N$ except 1, $d \in I$, and I^{odd} denotes the set of all positive divisors of $2N$ with an odd number of primes except 1. Set*

$$\theta_{d,2N/d}(z) := \sum_{n=0}^{\infty} H^{(d,2N/d)}(4n) q^n,$$

where one can see [3, Theorem 7.2] and

$$\theta'_{d,2N/d}(z) := 2^{s+2} \sum_{n=0}^{\infty} \left(\sum_{f \in G_{8N,64N^2,d}} \frac{R_f(n)}{|\,\mathrm{Aut}(f)|} \right) q^n = \sum_{n=0}^{\infty} H^{(d,2N/d)}(n) q^n.$$

We have the following bases of spaces of Eisenstein series of weight $3/2$:

(1) The set $\{\theta_{d,2N/d}(z)\}_I$ *is a basis of* $\mathscr{E}(4N, \frac{3}{2}, id)$*, and the set* $\{\theta_{d,2N/d}(z)\}_I \cup \{\theta'_{d,2N/d}(z)\}_{I^{odd}}$ *is a basis of* $\mathscr{E}(8N, \frac{3}{2}, id)$*.*

(2) Let l *be the divisor of* N*, and* χ_l *denotes the primitive characters such that* $\chi_l(k) = \left(\frac{l}{k}\right)$ *for* $(k, 4l) = 1$*. The set* $\{\theta_{d,2N/d}(lz)\}_I$ *is a basis of* $\mathscr{E}(4N, \frac{3}{2}, \chi_l)$*, and the set* $\{\theta_{d,2N/d}(lz)\}_I \cup \{\theta'_{d,2N/d}(lz)\}_{I^{odd}}$ *is a basis of* $\mathscr{E}(8N, \frac{3}{2}, \chi_l)$*.*

Proof. It is well-known that these functions are in the space of Eisenstein series of weight 3/2. Since N is squarefree with s distinct odd prime factors, it is well-known that $\dim\mathscr{E}(8N, \frac{3}{2}, \chi_l) = 3 \cdot 2^s - 1$. To give a basis of $\mathscr{E}(8N, \frac{3}{2}, \chi_l)$, it suffices to find $3 \cdot 2^s - 1$ linear independent elements in the space $\mathscr{E}(8N, \frac{3}{2}, \chi_l)$.

We only prove the case $\chi_l = \text{id}$, that is, $l = 1$, and the others can be proved similarly.

We will show that the theta series $\theta_{d,2N/d}(z), d \in I$, and $\theta'_{d,2N/d}(z), d \in I^{\text{odd}}$, are linearly independent. Suppose

$$\sum_{d|2N, d\neq 1} c(d)\theta_{d,2N/d}(z) + \sum_{d|2N, d\neq 1} c'(d)\theta'_{d,2N/d}(z) = 0.$$

We will show that the coefficients $c(d), c'(d)$ vanish by induction. Assume that the divisor $d = p$ is prime. By the Chinese remainder theorem, choose a $-n_p < 0$ such that $\left(\frac{-n_p}{p}\right) = -1$ and $\left(\frac{-n_q}{q}\right) = 1$, where $q \mid 2N/p$, and if $p = 2$, set $n_2 \equiv 5 \pmod 8$, and then we have

$$H^{(2,N)}(4n_2) \neq 0,$$

while the others equal zero. Ortherwise set $n_p \equiv 1 \pmod 8$,

$$H^{(p,2N/p)}(4n_p) \neq 0, H^{(2p,N/p)}(4n_p) \neq 0,$$
$$H^{(p,2N/p)}(4n_p) = H^{(2p,N/p)}(4n_p),$$

and the others equal zero. Set $n_p \equiv 3 \pmod 8$,

$$H^{(p,2N/p)}(4n_p) \neq 0, H^{(2p,N/p)}(4n_p) \neq 0,$$
$$H^{(p,2N/p)}(4n_p) = 3H^{(2p,N/p)}(4n_p),$$

and the others equal zero. It follows that if d is a prime, $c(d) = 0$. By induction, for all $d \in I$, the coefficients $c(d)$ vanish. Assume that the divisor $d = p$ is prime. By the Chinese remainder theorem, choose a discriminant $-D_p < 0$ such that $\left(\frac{-D_p}{p}\right) = -1$ and $\left(\frac{-D_q}{q}\right) = 1$, where $q \mid 2N/p$, and we have

$$H^{(p,2N/p)}(D_p) \neq 0, H^{(p,2N/p)}(4D_p) \neq 0,$$

and the others equal zero. It follows that if d is a prime, $c'(d) = 0$. Similarly, by induction for all $d \in I^{\text{odd}}$, the coefficients $c'(d)$ vanish. □

Lemma 4.2. *Let* $n \equiv 1 \pmod 4$*, and we have*

$$\sum_{f\in G_{8N,16N^2,N^{\text{odd}}}} \frac{R_f(4n)}{|\text{Aut}(f)|} = \sum_{f\in G_{8N,16N^2,N^{\text{odd}}}} \frac{R_f(n)}{|\text{Aut}(f)|},$$

and

$$3 \cdot \sum_{f\in G_{8N,16N^2,2N^{\text{even}}}} \frac{R_f(4n)}{|\text{Aut}(f)|} = \sum_{f\in G_{8N,16N^2,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|}.$$

Proof. We will use the Siegel-Weil formula (see [4, Section 5]). We only prove that

$$d_{x^2-2y^2-2z^2,2}(n) = 2d_{x^2-2y^2-2z^2,2}(4n)$$

and

$$3d_{3x^2+2y^2+6z^2,2}(n) = 2d_{3x^2+2y^2+6z^2,2}(4n).$$

It is not hard to check that

$$\begin{aligned} d_{x^2-2y^2-2z^2,2}(4n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : x^2-2y^2-2z^2 \equiv 4n \pmod{2^t}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le y,z < 2^t : 2x^2-y^2-z^2 \equiv 2n \pmod{2^{t-1}}\}|, \end{aligned}$$

while $2x^2-2n \equiv 0,-2 \pmod 8$, that is, $y^2+z^2 \equiv 0 \pmod 8$. Hence, $2 \mid y$ and $2 \mid z$. We have

$$\begin{aligned} d_{x^2-2y^2-2z^2,2}(4n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^{t-1} : 2x^2-4y^2-4z^2 \equiv 2n \pmod{2^{t-1}}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^{t-1} : x^2-2y^2-2z^2 \equiv n \pmod{2^{t-2}}\}| \\ &= \frac{1}{2^{2t-3}}|\{0 \le x,y,z < 2^{t-2} : x^2-2y^2-2z^2 \equiv n \pmod{2^{t-2}}\}| \\ &= \frac{1}{2}d_{x^2-2y^2-2z^2,2}(n). \end{aligned}$$

Similarly,

$$\begin{aligned} d_{3x^2+2y^2+6z^2,2}(n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : 3x^2+2y^2+6z^2 \equiv n \pmod{2^t}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : 3x^2 \equiv n-2y^2-6z^2 \pmod{2^t}\}| \\ &= 4 \cdot \frac{1}{2^{2t}}|\{0 \le yz < 2^t, 2 \mid y, 2 \mid z\}| \\ &= 1, \end{aligned}$$

$$\begin{aligned} d_{3x^2+2y^2+6z^2,2}(4n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^t : 3x^2+2y^2+6z^2 \equiv 4n \pmod{2^t}\}| \\ &= \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le y,z < 2^t : 6x^2+y^2+3z^2 \equiv 2n \pmod{2^{t-1}}\}|, \end{aligned}$$

while $2n-6x^2 \equiv 2,4 \pmod 8$, that is, $y^2+3z^2 \equiv 4 \pmod 8$. Hence, $2 \mid y$, $2 \mid z$ or $2 \nmid yz$. We have

$$\begin{aligned} d_{3x^2+2y^2+6z^2,2}(4n) &= \frac{1}{2^{2t}}|\{0 \le x,y,z < 2^{t-1} : 3x^2+2y^2+6z^2 \equiv n \pmod{2^{t-2}}\}| \\ &\quad + \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le y,z < 2^t, 2 \nmid yz : 6x^2+y^2+3z^2 \equiv 2n \pmod{2^{t-1}}\}| \\ &= 4 \cdot \frac{1}{2^{2t}}|\{0 \le x < 2^{t-1}, 0 \le z < 2^t, 2 \nmid xz\}| + 4 \cdot \frac{1}{2^{2t}}|\{0 \le yz < 2^t, 2 \mid y, 2 \mid z\}| = \frac{3}{2}. \end{aligned}$$

□

Proof of Theorem 1.2. For the proof of the genera $G_{8N,64N^2/N_r,N^{\mathrm{odd}}}$, $G_{8N,64N^2/N_r,2N^{\mathrm{even}}}$, $G_{8N,2N^2/N_r,N^{\mathrm{odd}}}$ and $G_{8N,2N^2/N_r,2N^{\mathrm{even}}}$, one can see [4, Section 4, Section 6].

We have

$$\sum_{n=0}^{\infty}\left(\sum_{f\in G_{8N,16N^2,N^{\mathrm{odd}}}}\frac{R_f(n)}{|\mathrm{Aut}(f)|}\right)q^n \in \mathscr{E}(8N,\frac{3}{2},\mathrm{id}).$$

Let

$$\sum_{f\in G_{8N,16N^2,N^{\mathrm{odd}}}}\frac{R_f(n)}{|\mathrm{Aut}(f)|}=\sum_{d\in I}c(d)\theta_{d,2N/d}(z)+\sum_{d\in I^{\mathrm{odd}}}c'(d)\theta'_{d,N/d}(z).$$

We will prove that when $d \neq N^{\mathrm{odd}}$ or $d \neq 2N^{\mathrm{odd}}$, we will have

$$c(d)=0,$$

and when $d \neq N^{\mathrm{odd}}$, we will have

$$c'(d)=0.$$

Since $d \neq N^{\mathrm{odd}}$ and $d \neq 2N^{\mathrm{odd}}$, there exists an odd prime p such that $p \nmid N^{\mathrm{odd}}$, $p \mid d$ or $p \mid N^{\mathrm{odd}}$, $p \nmid d$. By the Chinese remainder theorem, there exists $-n_d < 0$ such that when $q \mid d$, we have $\left(\frac{-n_d}{q}\right)=-1$ and when $q \mid N/d$, we have $\left(\frac{-n_d}{q}\right)=1$, $n_d \equiv 5 \pmod 8$. It is not hard to check that

$$H^{(d,2d/N)}(4n_d)\neq 0, H^{(2d,d/N)}(4n_d)\neq 0$$

and

$$H^{(d,2d/N)}(4n_d)=H^{(2d,d/N)}(4n_d).$$

When $p \nmid N^{\mathrm{odd}}$, $p \mid d$, we have $\left(\frac{-n_d}{p}\right)=-1$, and when $p \mid N^{\mathrm{odd}}$, $p \nmid d$, we have $\left(\frac{-n_d}{p}\right)=1$. By [1, Proposition 3.5], we have $R_f(n_d)=0$. Hence, when $d \neq N^{\mathrm{odd}}$ or $d \neq 2N^{\mathrm{odd}}$, we have $c(d)=0$. By the Chinese remainder theorem, there exists $-n_d<0$ such that when $q \mid d$, we have $\left(\frac{-n_d}{q}\right)=-1$; when $q \mid N/d$, we have $\left(\frac{-n_d}{q}\right)=1$ and $n_d \equiv 7 \pmod 8$. It is not hard to check that

$$H^{(d,2d/N)}(4n_d)\neq 0, H^{(2d,d/N)}(n_d)\neq 0.$$

Similarly we have $R_f(n_d)=0$. Hence, when $d \neq N^{\mathrm{odd}}$ or $d \neq 2N^{\mathrm{odd}}$, we have $c'(d)=0$.

By Corollary 2.7, we have

$$c(N^{\mathrm{odd}})H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(0)+c(2N^{\mathrm{odd}})H^{(2N^{\mathrm{odd}},N/N^{\mathrm{odd}})}(0)+c'(N^{\mathrm{odd}})H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(0)$$
$$=2^{-s-2}H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(0).$$

By the Chinese remainder theorem, there exists $n' \equiv 3 \pmod 8$ such that for all odd prime $p \mid N^{\mathrm{odd}}$, we have $\left(\frac{-n'}{p}\right)=-1$ and for all odd prime $p \mid N/N^{\mathrm{odd}}$, we have $\left(\frac{-n'}{p}\right)=1$. Then

$$\sum_{f\in G_{8N,16N^2,N^{\mathrm{odd}}}}\frac{R_f(n')}{|\mathrm{Aut}(f)|}=0, H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(n')=0.$$

We have

$$c(N^{\text{odd}})H^{(N^{\text{odd}},2N/N^{\text{odd}})}(4n') + c(2N^{\text{odd}})H^{(2N^{\text{odd}},N/N^{\text{odd}})}(4n') = 0.$$

Let $D' \equiv 1 \pmod 4$, by Lemma 4.2, and we have

$$\sum_{f\in G_{8N,16N^2,N^{\text{odd}}}} \frac{R_f(4D')}{|\text{Aut}(f)|} = \sum_{f\in G_{8N,16N^2,N^{\text{odd}}}} \frac{R_f(D')}{|\text{Aut}(f)|}.$$

Hence,

$$c(N^{\text{odd}})H^{(N^{\text{odd}},2N/N^{\text{odd}})}(16D') + c(2N^{\text{odd}})H^{(2N^{\text{odd}},N/N^{\text{odd}})}(16D') + c'(N^{\text{odd}})H^{(N^{\text{odd}},2N/N^{\text{odd}})}(4D')$$
$$= c(N^{\text{odd}})H^{(N^{\text{odd}},2N/N^{\text{odd}})}(4D') + c(2N^{\text{odd}})H^{(2N^{\text{odd}},N/N^{\text{odd}})}(4D') + c'(N^{\text{odd}})H^{(N^{\text{odd}},2N/N^{\text{odd}})}(D').$$

That is

$$3c(N^{\text{odd}}) - c(2N^{\text{odd}}) + 3c'(N^{\text{odd}}) = 3\cdot 2^{-s-2},$$
$$3c(N^{\text{odd}}) + c(2N^{\text{odd}}) = 0,$$
$$4c(N^{\text{odd}}) + c'(N^{\text{odd}}) = 0.$$

We have

$$c(N^{\text{odd}}) = -2^{-s-3}, c(2N^{\text{odd}}) = 3\cdot 2^{-s-3}, c'(N^{\text{odd}}) = 4\cdot 2^{-s-3}.$$

Similarly, let

$$\sum_{f\in G_{8N,16N^2,2N^{\text{even}}}} \frac{R_f(n)}{|\text{Aut}(f)|} = \sum_{d\in I} c(d)\theta'_{d,2N/d}(z) + \sum_{d\in I^{\text{odd}}} c'(d)\theta_{d,N/d}(z),$$

and when $d \neq 2N^{\text{even}}$, we have

$$c(d) = 0, c'(d) = 0.$$

By Lemma 2.7, we have

$$c(2N^{\text{even}})H^{(2N^{\text{even}},N/N^{\text{even}})}(0) + c'(2N^{\text{even}})H^{(2N^{\text{even}},N/N^{\text{even}})}(0) = 3\cdot 2^{-s-2}H^{(2N^{\text{even}},N/N^{\text{even}})}(0),$$

that is,

$$c(2N^{\text{even}}) + c'(2N^{\text{even}}) = 3\cdot 2^{-s-2}.$$

Let $D' \equiv 1 \pmod 4$, by Lemma 4.2, and we have

$$3\cdot \sum_{f\in G_{8N,16N^2,2N^{\text{even}}}} \frac{R_f(4D')}{|\text{Aut}(f)|} = \sum_{f\in G_{8N,16N^2,2N^{\text{even}}}} \frac{R_f(D')}{|\text{Aut}(f)|}.$$

This implies

$$3c'(2N^{\text{even}}) = c'(2N^{\text{even}}) + c(2N^{\text{even}}).$$

Hence,

$$c(2N^{\text{even}}) = 2^{-s-2}, c'(2N^{\text{even}}) = 2^{-s-1}.$$

It is similar to [3, Proposition 6.6] to check if $f \in C(8N, p^2d)$ where p is an odd prime, $p \parallel N$, and $p \nmid d$, and then we have

$$R_f(pn) = R_{\phi_p(f)}(n).$$

If $f \in C(8N, 16d)$ where p is an odd prime, and $2 \nmid Nd$, then we have

$$R_f(2n) = R_{\phi_2(f)}(n).$$

□

Proof of Theorem 1.3. By [1, Corollary 1.2], we have

$$T_{N^{\mathrm{odd}},2N/N^{\mathrm{odd}}} = 2^{-s-2} \sum_{n|2N} \sum_{\substack{n|r \\ r^2 \le 4n}} H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(4n - r^2),$$

and

$$T_{2N^{\mathrm{even}},N/N^{\mathrm{even}}} = 2^{-s-2} \sum_{n|2N} \sum_{\substack{n|r \\ r^2 \le 4n}} H^{(2N^{\mathrm{even}},N/N^{\mathrm{even}})}(4n - r^2).$$

By Theorem 2.6, we have

$$\sum_{\mu=1}^{T'_{N^{\mathrm{odd}},8N/N^{\mathrm{odd}}}} \frac{\rho_{O_\mu}(n,r)}{\mathrm{card}(\mathrm{Aut}(O_\mu))} = 2^{-s-3} \widetilde{H}^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(n - \frac{r^2}{4}),$$

$$\sum_{\mu=1}^{T'_{8N^{\mathrm{even}},N/N^{\mathrm{even}}}} \frac{\rho_{O_\mu}(n,r)}{\mathrm{card}(\mathrm{Aut}(O_\mu))} = 2^{-s-3} \widetilde{H}^{(2N^{\mathrm{even}},N/N^{\mathrm{even}})}(n - \frac{r^2}{4}).$$

We have

$$|C(8N)| = 2^{s+1}\Bigg(\sum_{N^{\mathrm{odd}}|N} T_{N^{\mathrm{odd}},2N/N^{\mathrm{odd}}} + \sum_{N^{\mathrm{even}}|N} T_{2N^{\mathrm{even}},N/N^{\mathrm{even}}} + \sum_{N^{\mathrm{odd}}|N} T'_{N^{\mathrm{odd}},8N/N^{\mathrm{odd}}} + \sum_{N^{\mathrm{even}}|N} T'_{8N^{\mathrm{even}},N/N^{\mathrm{even}}} \Bigg).$$

Let

$$C(n) = 2^{-s-2} \left(\sum_{N^{\mathrm{odd}}|N} H^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(n) + \sum_{N^{\mathrm{even}}|N} H^{(2N^{\mathrm{even}},N/N^{\mathrm{even}})}(n) \right),$$

and

$$D(n) = 2^{-s-3} \left(\sum_{N^{\mathrm{odd}}|N} \widetilde{H}^{(N^{\mathrm{odd}},2N/N^{\mathrm{odd}})}(n) + \sum_{N^{\mathrm{even}}|N} \widetilde{H}^{(2N^{\mathrm{even}},N/N^{\mathrm{even}})}(n) \right).$$

If $3 \nmid N$, then

$$|C(8N)| = 2^{s+1}\Bigg(C(4) + 2C(3) + 2C(0) + C(8) + D(2) + 2D(1) + 2C(4) + D(4) + 2D(0) + \sum_{\substack{d|N \\ d \neq 1}} C(4d) + C(8d) + D(2d) + D(4d) \Bigg).$$

If $3 \mid N$, then

$$|C(8N)| = 2^{s+1}\Big(C(4) + 4C(3) + 2C(0) + C(8) + D(2) + 2D(1) + 2C(4) + D(4) + 2D(0) + \sum_{\substack{d\mid N\\ d\neq 1}} C(4d) + C(8d) + D(2d) + D(4d)\Big).$$

It is not hard to check that

$$C(0) = \frac{2N-1}{48}, C(3) = \frac{1+\left(\frac{-3}{N}\right)}{12}, C(4) = \frac{1}{8}, C(8) = \frac{1}{4}, C(4d) = \frac{1}{4}H(4d), C(8d) = \frac{1}{4}H(8d),$$

$$D(0) = \frac{N}{16}, D(1) = \frac{1}{8}, D(2) = \frac{1}{4}, D(4) = \frac{1}{4} + \frac{1}{8}\left(\frac{-4}{N}\right), D(2d) = \frac{1}{4}H(8d),$$

$$D(4d) = \frac{1}{2}H(4d) - \frac{1}{4}H(-\Delta(-4d))\left(1 - \left(\frac{-4}{d}\right)\right).$$

We have

$$H(-\Delta(-4d)) = \frac{1}{8}H(4d)\left(2 + \frac{1}{2}\left(\frac{-d}{2}\right) - \left(\frac{-4}{d}\right) - \frac{1}{2}\left(\frac{-4}{d}\right)\left(\frac{-d}{2}\right)\right),$$

$$D(4d) = \frac{1}{32}H(4d)\left(13 - \left(\frac{-d}{2}\right) + 3\left(\frac{-4}{d}\right) + \left(\frac{-d}{2}\right)\left(\frac{-4}{d}\right)\right).$$

□

5. Conclusions

This paper established a comprehensive framework for studying positive definite ternary quadratic forms of level $8N$ when N is an odd squarefree integer. Our main contributions were:

(1) Complete Classification: We provided an explicit classification of primitive positive definite ternary quadratic forms of level $8N$ into 2^{2s+2} distinct genera by level, discriminant and anisotropy.

(2) Representation Formulas: We derived exact formulas for the weighted sum of representations over each genus, expressed in terms of $H^{(N_1,N_2)}(D)$, where (N_1, N_2) depend on the discriminant and anisotropy of the genus.

(3) Class number Formulas: As a direct consequence, we obtained explicit formulas for the class number within each genus and the total number $|C(8N)|$ of classes of primitive positive definite ternary quadratic forms of level $8N$.

(4) Methodological Innovations: Our proofs established novel connections between ternary quadratic forms, quaternion algebras, modular forms of weight $3/2$ and Jacobi forms.

This work provided a foundation for deeper arithmetic investigations of ternary quadratic forms with complex level structure, potential applications in representation numbers, and the arithmetic of modular forms.

Appendix: Tables

In this appendix, we will give some examples of representations of ternary quadratic forms. If the class number of a genus is one, we can give an exact formula of the representation number of n by the ternary quadratic forms. For the genera attached to Eichler orders, one can see [4, Appendix B]. Let $\mathcal{O}$ be a trace-even order. Its type number equals 1 if its level is one of the following: (8,1), (8,3), (8,7), (3,8), (5,8), (8,7), (120,1), and we get 26 genera with one class (see Table 1). We give the explicit formulas for the representation number of ternary quadratic forms as follows (see Table 2).

Table 1. Genera with one class.

Genus	N_f	d_f	$R_f(n)$
$G_{8,64,2}$	8	64	$R_{(3,3,3,-2,-2,-2)}(n) = 12H^{(2,1)}(n)$
$G_{8,2,2}$	8	2	$R_{(1,1,1,1,1,1)}(n) = 12H^{(2,1)}(8n)$
$G_{8,16,2}$	8	16	$R_{(1,2,2,0,0,0)}(n) = 2\widetilde{H}^{(2,1)}(n)$
$G_{8,8,2}$	8	8	$R_{(1,1,2,0,0,0)}(n) = 2\widetilde{H}^{(2,1)}(2n)$
$G_{24,576,2}$	$8 \cdot 3$	$64 \cdot 3^2$	$R_{(3,8,8,-8,0,0)}(n) = 3H^{(2,3)}(n)$
$G_{24,192,2}$	$8 \cdot 3$	$64 \cdot 3$	$R_{(1,8,8,-8,0,0)}(n) = 3H^{(2,3)}(3n)$
$G_{24,18,2}$	$8 \cdot 3$	$2 \cdot 3^2$	$R_{(1,1,6,0,0,-1)}(n) = 3H^{(2,3)}(8n)$
$G_{24,6,2}$	$8 \cdot 3$	$2 \cdot 3$	$R_{(1,1,2,0,0,-1)}(n) = 3H^{(2,3)}(24n)$
$G_{24,576,3}$	$8 \cdot 3$	$64 \cdot 3^2$	$R_{(4,7,7,2,4,4)}(n) = 2H^{(3,2)}(n)$
$G_{24,192,3}$	$8 \cdot 3$	$64 \cdot 3$	$R_{(4,4,5,-4,-4,0)}(n) = 2H^{(3,2)}(3n)$
$G_{24,18,3}$	$8 \cdot 3$	$2 \cdot 3^2$	$R_{(2,2,2,1,2,2)}(n) = 2H^{(3,2)}(8n)$
$G_{24,6,3}$	$8 \cdot 3$	$2 \cdot 3$	$R_{1,1,2,-1,-1,0)}(n) = 2H^{(3,2)}(24n)$
$G_{24,144,2}$	$8 \cdot 3$	$16 \cdot 3^2$	$2R_{(2,3,6,0,0,0)}(n) = \widetilde{H}^{(2,3)}(n)$
$G_{24,72,2}$	$8 \cdot 3$	$8 \cdot 3^2$	$2R_{(1,3,6,0,0,0)}(n) = \widetilde{H}^{(2,3)}(2n)$
$G_{24,48,2}$	$8 \cdot 3$	$16 \cdot 3$	$2R_{(1,2,6,0,0,0)}(n) = \widetilde{H}^{(2,3)}(3n)$
$G_{24,24,2}$	$8 \cdot 3$	$8 \cdot 3$	$2R_{(1,2,3,0,0,0)}(n) = \widetilde{H}^{(2,3)}(6n)$
$G_{24,144,3}$	$8 \cdot 3$	$16 \cdot 3^2$	$R_{(1,6,6,0,0,0)}(n) = \widetilde{H}^{(3,2)}(n)$
$G_{24,72,3}$	$8 \cdot 3$	$8 \cdot 3^2$	$R_{(2,3,3,0,0,0)}(n) = \widetilde{H}^{(3,2)}(2n)$
$G_{24,48,3}$	$8 \cdot 3$	$16 \cdot 3$	$R_{(2,2,3,0,0,0)}(n) = \widetilde{H}^{(3,2)}(3n)$
$G_{24,24,3}$	$8 \cdot 3$	$8 \cdot 3$	$R_{(1,1,6,0,0,0)}(n) = \widetilde{H}^{(3,2)}(6n)$
$G_{40,1600,2}$	$8 \cdot 5$	$64 \cdot 5^2$	$R_{(4,11,11,2,4,4)}(n) = 2H^{(2,5)}(n)$
$G_{40,320,2}$	$8 \cdot 5$	$64 \cdot 5$	$R_{(4,4,7,-4,-4,0)}(n) = 2H^{(2,5)}(5n)$
$G_{40,50,2}$	$8 \cdot 5$	$2 \cdot 5^2$	$R_{(2,3,3,1,2,2)}(n) = 2H^{(2,5)}(8n)$
$G_{40,10,2}$	$8 \cdot 5$	$2 \cdot 5$	$R_{(1,1,3,-1,-1,0)}(n) = 2H^{(2,5)}(40n)$
$G_{40,1600,5}$	$8 \cdot 5$	$64 \cdot 5^2$	$R_{(7,7,12,-4,-4,-6)}(n) = H^{(5,2)}(n)$
$G_{40,320,5}$	$8 \cdot 5$	$64 \cdot 5$	$R_{(3,3,11,-2,-2,-2)}(n) = H^{(5,2)}(20n)$
$G_{40,50,5}$	$8 \cdot 5$	$2 \cdot 5^2$	$R_{(1,4,4,3,1,1)}(n) = H^{(5,2)}(8n)$
$G_{40,25,5}$	$8 \cdot 5$	$2 \cdot 5$	$R_{(1,2,2,2,1,1)}(n) = H^{(5,2)}(40n)$

$G_{40,400,5}$	$8\cdot 5$	$16\cdot 5^2$	$2R_{(2,5,10,0,0,0)}(n)=\widetilde{H}^{(5,2)}(n)$
$G_{40,200,5}$	$8\cdot 5$	$8\cdot 5^2$	$2R_{(1,5,10,0,0,0)}(n)=\widetilde{H}^{(5,2)}(2n)$
$G_{40,80,5}$	$8\cdot 5$	$16\cdot 5$	$2R_{(1,2,10,0,0,0)}(n)=\widetilde{H}^{(5,2)}(5n)$
$G_{40,40,5}$	$8\cdot 5$	$8\cdot 5$	$2R_{(1,2,5,0,0,0)}(n)=\widetilde{H}^{(5,2)}(10n)$
$G_{56,3136,2}$	$8\cdot 7$	$64\cdot 7^2$	$2R_{(3,19,19,-18,-2,-2)}(n)=3H^{(2,7)}(n)$
$G_{56,448,2}$	$8\cdot 7$	$64^2\cdot 7$	$2R_{(5,5,5,2,2,2)}(n)=3H^{(2,7)}(7n)$
$G_{56,98,2}$	$8\cdot 7$	$2\cdot 7^2$	$2R_{(3,3,3,-1,-1,-1)}(n)=3H^{(2,7)}(8n)$
$G_{56,14,2}$	$8\cdot 7$	$2\cdot 7$	$2R_{(1,1,5,1,1,1)}(n)=3H^{(2,7)}(56n)$
$G_{56,784,2}$	$8\cdot 7$	$16\cdot 7^2$	$4R_{(3,5,14,0,0,-2)}(n)=\widetilde{H}^{(2,7)}(n)$
$G_{56,392,2}$	$8\cdot 7$	$8\cdot 7^2$	$4R_{(3,5,7,0,0,-2)}(n)=\widetilde{H}^{(2,7)}(2n)$
$G_{56,112,2}$	$8\cdot 7$	$16\cdot 7$	$4R_{(2,3,5,-2,0,0)}(n)=\widetilde{H}^{(2,7)}(7n)$
$G_{56,56,2}$	$8\cdot 7$	$8\cdot 7$	$4R_{(1,3,5,-2,0,0)}(n)=\widetilde{H}^{(2,7)}(14n)$
$G_{88,7744,2}$	$8\cdot 11$	$64\cdot 11^2$	$R_{(8,11,24,0,-8,0)}(n)=H^{(2,11)}(n)$
$G_{88,704,2}$	$8\cdot 11$	$64\cdot 11$	$R_{(1,8,24,-8,0,0)}(n)=H^{(2,11)}(11n)$
$G_{88,242,2}$	$8\cdot 11$	$2\cdot 11^2$	$R_{(1,3,22,0,0,-1)}(n)=H^{(2,11)}(8n)$
$G_{88,22,2}$	$8\cdot 11$	$2\cdot 11$	$R_{(1,2,3,0,-1,0)}(n)=H^{(2,11)}(88n)$
$G_{120,14400,2}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5^2$	$2R_{(11,11,35,-10,-10,2)}(n)=H^{(2,15)}(n)$
$G_{120,4800,2}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5^2$	$2R_{(8,12,17,4,8,8)}(n)=H^{(2,15)}(3n)$
$G_{120,2880,2}$	$8\cdot 3\cdot 5$	$64\cdot 3^2\cdot 5$	$2R_{(4,7,31,2,4,4)}(n)=H^{(2,15)}(5n)$
$G_{120,960,2}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5$	$2R_{(5,5,12,-4,-4,-2)}(n)=H^{(2,15)}(15n)$
$G_{120,450,2}$	$8\cdot 3\cdot 5$	$2\cdot (3\cdot 5)^2$	$2R_{(3,7,7,4,3,3)}(n)=H^{(2,15)}(8n)$
$G_{120,150,2}$	$8\cdot 3\cdot 5$	$2\cdot 3\cdot 5^2$	$2R_{(1,5,9,-5,1,0)}(n)=H^{(2,15)}(24n)$
$G_{120,90,2}$	$8\cdot 3\cdot 5$	$2\cdot 3^2\cdot 5$	$2R_{(2,3,5,-3,2,0)}(n)=H^{(2,15)}(40n)$
$G_{120,30,2}$	$8\cdot 3\cdot 5$	$2\cdot 3\cdot 5$	$2R_{(1,3,3,1,1,1)}(n)=H^{(2,15)}(120n)$
$G_{120,3600,30}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5^2$	$2R_{(3,40,40,-40,0,0)}(n)=3H^{(30,1)}(n)$
$G_{120,1200,30}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5^2$	$2R_{(1,40,40,-40,0,0)}(n)=3H^{(30,1)}(3n)$
$G_{120,720,30}$	$8\cdot 3\cdot 5$	$64\cdot 3^2\cdot 5$	$2R_{(8,8,15,0,0,-8)}(n)=3H^{(30,1)}(5n)$
$G_{120,240,30}$	$8\cdot 3\cdot 5$	$64\cdot 3\cdot 5$	$2R_{(5,8,8,-8,0,0)}(n)=3H^{(30,1)}(15n)$
$G_{120,450,30}$	$8\cdot 3\cdot 5$	$2\cdot (3\cdot 5)^2$	$2R_{(5,6,6,0,0,-5)}(n)=3H^{(30,1)}(8n)$
$G_{120,150,30}$	$8\cdot 3\cdot 5$	$2\cdot 3\cdot 5^2$	$2R_{(2,5,5,-5,0,0)}(n)=3H^{(30,1)}(24n)$
$G_{120,90,30}$	$8\cdot 3\cdot 5$	$2\cdot 3^2\cdot 5$	$2R_{(1,1,30,0,0,-1)}(n)=3H^{(30,1)}(40n)$
$G_{120,30,30}$	$8\cdot 3\cdot 5$	$2\cdot 3\cdot 5$	$2R_{(1,1,10,0,0,-1)}(n)=3H^{(30,1)}(120n)$
$G_{120,3600,30}$	$8\cdot 3\cdot 5$	$16\cdot 3^2\cdot 5^2$	$4R_{(3,10,30,0,0,0)}(n)=\widetilde{H}^{(30,1)}(n)$
$G_{120,1800,30}$	$8\cdot 3\cdot 5$	$8\cdot 3^2\cdot 5^2$	$4R_{(5,6,15,0,0,0)}(n)=\widetilde{H}^{(30,1)}(2n)$
$G_{120,1200,30}$	$8\cdot 3\cdot 5$	$16\cdot 3\cdot 5^2$	$4R_{(1,10,30,0,0,0)}(n)=\widetilde{H}^{(30,1)}(3n)$
$G_{120,720,30}$	$8\cdot 3\cdot 5$	$16\cdot 3^2\cdot 5$	$4R_{(2,6,15,0,0,0)}(n)=\widetilde{H}^{(30,1)}(5n)$
$G_{120,600,30}$	$8\cdot 3\cdot 5$	$8\cdot 3\cdot 5^2$	$4R_{(2,5,15,0,0,0)}(n)=\widetilde{H}^{(30,1)}(6n)$
$G_{120,360,30}$	$8\cdot 3\cdot 5$	$8\cdot 3^2\cdot 5$	$4R_{(1,3,30,0,0,0)}(n)=\widetilde{H}^{(30,1)}(10n)$
$G_{120,240,30}$	$8\cdot 3\cdot 5$	$16\cdot 3\cdot 5$	$4R_{(2,5,6,0,0,0)}(n)=\widetilde{H}^{(30,1)}(15n)$
$G_{120,120,30}$	$8\cdot 3\cdot 5$	$8\cdot 3\cdot 5$	$4R_{(1,3,10,0,0,0)}(n)=\widetilde{H}^{(30,1)}(30n)$
$G_{184,33856,2}$	$8\cdot 23$	$64\cdot 23^2$	$2R_{(11,19,51,-14,-6,-10)}(n)=H^{(2,23)}(n)$
$G_{184,1472,2}$	$8\cdot 23$	$64\cdot 23$	$2R_{(5,8,12,-8,-4,0)}(n)=H^{(2,23)}(23n)$

$G_{184,1058,2}$	$8 \cdot 23$	$2 \cdot 23^2$	$2R_{(5,7,10,2,4,5)}(n) = H^{(2,23)}(8n)$
$G_{184,46,2}$	$8 \cdot 23$	$2 \cdot 23$	$2R_{(1,3,5,3,1,1)}(n) = H^{(2,23)}(184n)$
$G_{168,28224,42}$	$8 \cdot 3 \cdot 7$	$64 \cdot 3 \cdot 7^2$	$R_{(4,43,43,2,4,4)}(n) = H^{(42,1)}(n)$
$G_{168,9408,42}$	$8 \cdot 3 \cdot 7$	$64 \cdot 3 \cdot 7^2$	$R_{(12,17,17,6,12,12)}(n) = H^{(42,1)}(3n)$
$G_{168,4032,42}$	$8 \cdot 3 \cdot 7$	$64 \cdot 3^2 \cdot 7$	$R_{(12,12,13,-12,-12,0)}(n) = H^{(42,1)}(7n)$
$G_{168,1344,42}$	$8 \cdot 3 \cdot 7$	$64 \cdot 3 \cdot 7$	$R_{(4,4,23,-4,-4,0)}(n) = H^{(42,1)}(21n)$
$G_{168,882,42}$	$8 \cdot 3 \cdot 7$	$2 \cdot (3 \cdot 7)^2$	$R_{(2,11,11,1,2,2)}(n) = H^{(42,1)}(8n)$
$G_{168,294,42}$	$8 \cdot 3 \cdot 7$	$2 \cdot 3 \cdot 7^2$	$R_{(5,5,5,-3,-3,-4)}(n) = H^{(42,1)}(24n)$
$G_{168,126,42}$	$8 \cdot 3 \cdot 7$	$2 \cdot 3^2 \cdot 7$	$R_{(3,3,5,-3,-3,0)}(n) = H^{(42,1)}(56n)$
$G_{168,42,42}$	$8 \cdot 3 \cdot 7$	$2 \cdot 3 \cdot 7$	$R_{(1,1,11,-1,-1,0)}(n) = H^{(42,1)}(168n)$
$G_{280,78400,70}$	$8 \cdot 5 \cdot 7$	$64 \cdot 5 \cdot 7^2$	$2R_{(8,35,72,0,-8,0)}(n) = H^{(70,1)}(n)$
$G_{280,15680,70}$	$8 \cdot 5 \cdot 7$	$64 \cdot 5 \cdot 7^2$	$2R_{(7,24,24,-8,0,0)}(n) = H^{(70,1)}(5n)$
$G_{280,11200,70}$	$8 \cdot 5 \cdot 7$	$64 \cdot 5^2 \cdot 7$	$2R_{(5,24,24,-8,0,0)}(n) = H^{(70,1)}(7n)$
$G_{280,2240,70}$	$8 \cdot 5 \cdot 7$	$64 \cdot 5 \cdot 7$	$2R_{(1,8,72,-8,0,0)}(n) = H^{(70,1)}(35n)$
$G_{280,2450,70}$	$8 \cdot 5 \cdot 7$	$2 \cdot (5 \cdot 7)^2$	$2R_{(1,9,70,0,0,-1)}(n) = H^{(70,1)}(8n)$
$G_{280,490,70}$	$8 \cdot 5 \cdot 7$	$2 \cdot 5 \cdot 7^2$	$2R_{(3,3,14,0,0,-1)}(n) = H^{(70,1)}(40n)$
$G_{280,350,70}$	$8 \cdot 5 \cdot 7$	$2 \cdot 5^2 \cdot 7$	$2R_{(3,3,10,0,0,-1)}(n) = H^{(70,1)}(56n)$
$G_{280,70,70}$	$8 \cdot 5 \cdot 7$	$2 \cdot 5 \cdot 7$	$2R_{(1,2,9,0,-1,0)}(n) = H^{(70,1)}(280n)$
$G_{312,97344,78}$	$8 \cdot 3 \cdot 13$	$64 \cdot 3 \cdot 13^2$	$2R_{(19,19,84,-12,-12,-14)}(n) = H^{(78,1)}(n)$
$G_{312,32448,78}$	$8 \cdot 3 \cdot 13$	$64 \cdot 3 \cdot 13^2$	$2R_{(8,28,41,4,8,8)}(n) = H^{(78,1)}(3n)$
$G_{312,7488,78}$	$8 \cdot 3 \cdot 13$	$64 \cdot 3^2 \cdot 13$	$2R_{(4,7,79,2,4,4)}(n) = H^{(78,1)}(13n)$
$G_{312,2496,78}$	$8 \cdot 3 \cdot 13$	$64 \cdot 3 \cdot 13$	$2R_{(5,5,28,-4,-4,-2)}(n) = H^{(78,1)}(39n)$
$G_{312,3042,78}$	$8 \cdot 3 \cdot 13$	$2 \cdot (3 \cdot 13)^2$	$2R_{(3,17,17,8,3,3)}(n) = H^{(78,1)}(8n)$
$G_{312,1014,78}$	$8 \cdot 3 \cdot 13$	$2 \cdot 3 \cdot 13^2$	$2R_{(1,13,23,-13,-1,0)}(n) = H^{(78,1)}(24n)$
$G_{312,234,78}$	$8 \cdot 3 \cdot 13$	$2 \cdot 3^2 \cdot 13$	$2R_{(2,3,11,-3,-2,0)}(n) = H^{(78,1)}(104n)$
$G_{312,78,78}$	$8 \cdot 3 \cdot 13$	$2 \cdot 3^2 \cdot 13^2$	$2R_{(1,5,5,4,1,1)}(n) = H^{(78,1)}(312n)$

Table 2. Class number of primitive positive definite ternary quadratic forms of level $8N$.

N	$\|C(8N)\|$	N	$\|C(8N)\|$	N	$\|C(8N)\|$
1	4	35	168	69	264
3	16	37	64	71	100
5	20	39	176	73	112
7	24	41	72	77	272
11	28	43	72	79	108
13	36	47	80	83	124
15	104	51	208	85	288
17	40	53	80	87	312
19	44	55	232	89	132
21	128	57	224	91	312
23	44	59	96	93	312
29	52	61	96	95	312
31	60	65	240	97	140
33	168	67	100		

Author contributions

Yifan Luo: Conceptualization, data curation, investigation, software, visualization and writing – original draft; Haigang Zhou: Methodology, funding acquisition, methodology, supervision, validation and writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is partially supported National Natural Science Foundation of China (No.12271405). The first author is partially supported by the Joint Fund of Henan Province Science and Technology R & D Program (No.225200810032) and High-level Talent Research Start-up Project Funding of Henan Academy of Sciences (Nos. 252019083 and 242019104). The second author is partially supported by Shanghai Engineering Research Center for Blockchain Applications and Services (No.19DZ2255100). The authors thank the anonymous referee whose detailed comments have greatly improved the exposition of this paper.

Conflict of interest

The authors declare no conflict of interest.

References

1. Y. B. Li, N. P. Skoruppa, H. G. Zhou, Eichler orders and Jacobi forms of squarefree level, *J. Number Theory*, **236** (2022), 349–387. https://doi.org/10.1016/j.jnt.2021.07.025
2. J. L. Lehman, Levels of positive definite ternary quadratic forms, *Math. Comp.*, **58** (1992), 399–417. https://doi.org/10.2307/2153043
3. Y. Luo, H. Zhou, The classification and representations of positive definite ternary quadratic forms of level 4N, *arXiv:2402.17443*, 2024. Available from: https://arxiv.org/abs/2402.17443.
4. Y. Luo, H. Zhou, Type number for orders of level (N_1, N_2), *arXiv:2410.10882*, 2024. Available from: https://arxiv.org/abs/2410.10882.
5. J. Voight, *Quaternion algebras*, Graduate Texts in Mathematics, Springer, Cham, **288** (2021). https://doi.org/10.1007/978-3-030-56694-4
6. X. L. Wang, D. Y. Pei, *Modular forms with integral and half-integral weights*, Science Press Beijing, Beijing; Springer, Heidelberg, 2012. https://doi.org/10.1007/978-3-642-29302-3