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Review

Decreasing ratio between two normalized remainders of Maclaurin series expansion of exponential function

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Abstract: In the study, the authors concisely review the normalized remainders of the Maclaurin series and, by establishing an inequality for specific Maclaurin series, show that the ratio between two normalized remainders of the Maclaurin series of the exponential function is decreasing on the whole real axis. This decreasing property confirms a guess in Remark 5 of the paper "F. Qi, Absolute monotonicity of normalized tail of power series expansion of exponential function, *Mathematics*, **12** (2024), 2859, available online at https://doi.org/10.3390/math12182859".

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1. A concise review on normalized remainders

Let h be an infinitely differentiable function throughout a neighborhood I of the origin 0. Then the function h has the Maclaurin series

$$h(s) = h(0) + \sum_{\ell=1}^{\infty} h^{(\ell)}(0)\frac{s^\ell}{\ell!}, \quad |s| < r, \tag{1.1}$$

where r is a nonnegative number or ∞, and the error

$$R_n[h(s)] = h(s) - \sum_{\ell=0}^{n} h^{(\ell)}(0)\frac{s^\ell}{\ell!}, \quad s \in I,$$

after $n+1$ terms for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is called the nth remainder or the nth tail of the Maclaurin series (1.1).

Initially motivated by the problems and solutions in [7, 14], implicitly started in the papers [6, 8], explicitly considered in [18, 21, 23] and another two works in this paper, and especially in the review and research article [13, Section 1], Qi et al. introduced the notion and designed the notation of the normalized remainder of the Maclaurin series (1.1) as follows.

Definition 1. *([2, Section 5], [12, p. 4], and [13, Definition 1]) Let $I \supset \{0\}$ be an interval and let h have derivatives of all orders throughout a neighborhood of the point $s = 0 \in I$. If $h^{(n+1)}(0) \ne 0$ for some $n \in \mathbb{N}_0$, then we call the function*

$$T_n[h(s)] = \begin{cases} \dfrac{1}{h^{(n+1)}(0)}\dfrac{(n+1)!}{s^{n+1}}R_n[h(s)], & s \ne 0,\\ 1, & s = 0, \end{cases}$$
$$= \begin{cases} \dfrac{1}{h^{(n+1)}(0)}\dfrac{(n+1)!}{s^{n+1}}\left[h(s) - \displaystyle\sum_{\ell=0}^{n} h^{(\ell)}(0)\frac{s^\ell}{\ell!}\right], & s \ne 0,\\ 1, & s = 0, \end{cases}$$

for $s \in I$ the nth normalized remainder or the nth normalized tail of the Maclaurin series (1.1).

In [13, Theorem 1], some general properties for $T_n[h(s)]$ were examined. In [13], Qi connected central factorial numbers, the Stirling numbers, the inverse of the Vandermonde matrix, the partial Bell polynomials, and the normalized remainder $T_{2n+1}[(\frac{\arcsin s}{s})^q]$ for $n, q \in \mathbb{N}$ and $s \in (-1, 1)$.

Since 2023, Qi et al. researched the normalized remainder $T_{2n-1}[\cos s]$ (see [13, Section 1.6]), studied the normalized remainder $T_{2n}[\sin s]$ (see [13, Section 1.5]), investigated the normalized remainders $T_{2n}[\tan s]$, $T_{2n-1}[\tan^2 s]$, and $T_{2n-1}[\sec^2 s]$ (see [9] and [13, Section 1.4]), and discussed the normalized remainder $T_{2n-1}[\frac{s}{e^s-1}]$ (see [24] and [13, Section 1.8]). In [11], Pei and Guo investigated the normalized remainder for the Maclaurin power series expansion of the logarithm

$$\ln T_1[\cos s] = \begin{cases} \ln \dfrac{2(1-\cos s)}{s^2}, & 0 < |s| < 2\pi,\\ 0, & s = 0, \end{cases}$$

see also [13, Section 1.6]. In [21], Zhang et al. considered the normalized remainder $T_{2n-1}[\ln \sec s] = T_{2n-1}[-\ln\cos s]$, see also [13, Section 1.6]. In [2, 12], Qi et al. inquired into the normalized remainder $T_n[e^s]$. In [12, Section 1], [13, Section 1], and [23, Section 1], Qi et al. reviewed, described, depicted, and reexamined the motivations, ideas, and thoughts to introduce and invent the concept of normalized remainders $T_n[h(s)]$ systematically.

Since 2023, Qi et al. mainly explored the following properties of the normalized remainder $T_n[h(s)]$:

(1) Positivity of the normalized remainder $T_n[h(s)]$; see [12, 21], for example.
(2) Monotonicity of the normalized remainder $T_n[h(s)]$; see [2, 11], for example.

(3) Convexity of the normalized remainder $T_n[h(s)]$; see [21], for example.
(4) Logarithmic convexity of the normalized remainder $T_n[h(s)]$; see [21–23], for example.
(5) Absolute monotonicity of the normalized remainder $T_n[h(s)]$; see [12] and [17, Remarks 2 and 4], for example.
(6) Maclaurin series of the logarithm $\ln T_n[h(s)]$; see [2, 8, 21], for example.
(7) Monotonicity of the ratio $\frac{T_{n+1}[h(s)]}{T_n[h(s)]}$; see [9, 21, 22], for example.
(8) Monotonicity of the ratio $\frac{\ln T_{n+1}[h(s)]}{\ln T_n[h(s)]}$; see [2, 8, 18], for example.
(9) Some connections between the normalized remainder $T_n[h(s)]$ with several hypergeometric functions; see [11, 18] and [13, Section 1.10], for example.
(10) Inequalities for the function $h(s)$ and related ones; see [2, 12, 21], for example.
(11) Relations between $T_n[h(s)]$ and the Bernoulli numbers and polynomials, the Stirling numbers of the second kind, and other quantities in combinatorial number theory; see [17, Remark 2], for example.

In [13, Section 1], Qi deliberately surveyed and summed up almost all results developed between April 2023 and January 2025 on normalized remainders. Moreover, the survey article [13] was cited in [20], which is dedicated to several problems in combinatorial number theory. This citation implies that the normalized remainders are significant and meaningful in mathematics and engineering.

In [12], among other findings, Qi presented that the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ is decreasing on the positive half axis $(0, \infty)$.

In this study, we will prove that the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ is decreasing on the whole real axis $\mathbb{R}$.

2. Motivations of this paper

It is common knowledge that

$$e^s = \sum_{\ell=0}^{\infty} \frac{s^\ell}{\ell!} = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \frac{s^5}{5!} + \frac{s^6}{6!} + \cdots, \quad s \in \mathbb{R}, \tag{2.1}$$

and that the quantity

$$R_n[e^s] = e^s - \sum_{\ell=0}^{n} \frac{s^\ell}{\ell!} = \sum_{\ell=n+1}^{\infty} \frac{s^\ell}{\ell!},$$

for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$ is called the nth remainder or the nth tail of the Maclaurin series (2.1).

In [2], Qi et al. constructed the normalized remainder

$$T_n[e^s] = \begin{cases} \dfrac{(n+1)!}{s^{n+1}} R_n[e^s], & s \neq 0, \\ 1, & s = 0, \end{cases} \tag{2.2}$$

for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$. This is equivalent to Definition 1 applied to $h(s) = e^s$. The main results of the papers [2, 12] include the following seven conclusions:

(1) The normalized remainder $T_n[e^s]$ for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$ is positive; see [12, Section 2].
(2) The normalized remainder $T_n[e^s]$ for $n \in \mathbb{N}_0$ is an increasing and logarithmically convex function of $s \in \mathbb{R}$; see [2, Corollary 1].

(3) The normalized remainder $T_n[e^s]$ for $n \in \mathbb{N}$ is an absolutely monotonic function in $s \in \mathbb{R}$; see [12, Theorem 2].
(4) The logarithm $\ln T_n[e^s]$ for $n \in \mathbb{N}_0$ was expanded into a Maclaurin series; see [2, Theorem 1].
(5) The function

$$\begin{cases} \dfrac{\ln T_n[e^s]}{s}, & s \neq 0, \\ \dfrac{1}{n+2}, & s = 0, \end{cases}$$

for $n \in \mathbb{N}_0$ is increasing in $s \in \mathbb{R}$; see [2, Theorem 2].
(6) The inequality

$$\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+3}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} \geq \left[\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}\right]^2 \tag{2.3}$$

is true for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$; see [2, Corollary 2].
(7) The ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ is decreasing in $s \in (0, \infty)$; see [12, Theorem 1].

In this paper, we will present that the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ is decreasing in $s \in \mathbb{R}$. This result is a positive and confirmative answer to a guess posed in [12, Remark 5].

3. An inequality for specific Maclaurin series

Before verifying that the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ is decreasing in $s \in \mathbb{R}$, we prove the following inequality for specific Maclaurin series.

Theorem 1. *For $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$, the inequality*

$$\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n}{n}} \frac{s^\ell}{\ell!} < \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n}} \frac{s^\ell}{\ell!} \tag{3.1}$$

is true. Consequently, for given $s \in \mathbb{R}$, the sequence

$$\frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n}{n}} \frac{s^\ell}{\ell!}} = \frac{\sum_{\ell=0}^{\infty} \frac{(\ell+1)s^\ell}{(\ell+n+1)!}}{\sum_{\ell=0}^{\infty} \frac{s^\ell}{(\ell+n)!}} \tag{3.2}$$

is decreasing in $n \in \mathbb{N}_0$.

For proving the inequality (3.1), we present three lemmas below.

Lemma 1. *For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we have the integral representation*

$$\sum_{\ell=0}^{\infty} \frac{\ell+1}{(\ell+n+1)!} s^\ell = \frac{1}{(n-1)!} \int_0^1 v^{n-1}(1-v)\, e^{s(1-v)}\, dv. \tag{3.3}$$

Proof. For $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$, denote

$$q_n(s) = \sum_{\ell=0}^{\infty} \frac{\ell+1}{(\ell+n+1)!} s^\ell. \tag{3.4}$$

For $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$, we have the integral representation

$$R_n[\mathrm{e}^s] = \frac{s^{n+1}}{n!} \int_0^1 v^n \mathrm{e}^{s(1-v)} \,\mathrm{d}v, \tag{3.5}$$

see [2, Lemma 4], [12, Lemma 2], and [1, p. 502]. Making use of the integral representation (3.5), we acquire

$$q_n(s) = \left(\frac{R_{n-1}[\mathrm{e}^s]}{s^n}\right)' = \left[\frac{1}{(n-1)!} \int_0^1 v^{n-1} \mathrm{e}^{s(1-v)} \,\mathrm{d}v\right]' = \frac{1}{(n-1)!} \int_0^1 v^{n-1}(1-v) \mathrm{e}^{s(1-v)} \,\mathrm{d}v,$$

for $n \in \mathbb{N}$ and $s \in \mathbb{R}$. The integral representation (3.3) is acquired. The proof of Lemma 1 is complete. □

Lemma 2. *For $n = 2, 3, \dots$ and $s \geq 0$, we have*

$$\frac{n-1}{n} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+n-1)(\ell+n)} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+n+1)(\ell+n+2)} \frac{s^\ell}{\ell!} < \left[\sum_{\ell=0}^{\infty} \frac{1}{(\ell+n)(\ell+n+1)} \frac{s^\ell}{\ell!}\right]^2. \tag{3.6}$$

Proof. When $s = 0$, the inequality (3.6) is equivalent to $\frac{1}{n+2} < \frac{1}{n+1}$ for $n = 2, 3, \dots$.

It is clear that, for given $n = 2, 3, \dots$, the sequence

$$\frac{(\ell+n-1)(\ell+n+2)}{(\ell+n)(\ell+n+1)} = 1 - \frac{2}{(\ell+n)(\ell+n+1)}$$

is increasing in $\ell \in \mathbb{N}_0$. Hence, the inequality

$$\frac{(\ell+n-1)(\ell+n+2)}{(\ell+n)(\ell+n+1)} \geq \frac{(n-1)(n+2)}{n(n+1)}$$

is valid for $\ell \in \mathbb{N}_0$ and $n = 2, 3, \dots$. Therefore, we have

$$\frac{n-1}{n} \frac{1}{(\ell+n-1)(\ell+n+2)} \leq \frac{n+1}{n+2} \frac{1}{(\ell+n)(\ell+n+1)} < \frac{1}{(\ell+n)(\ell+n+1)}, \tag{3.7}$$

for $\ell \in \mathbb{N}_0$ and $n = 2, 3, \dots$.

For $n = 2, 3, \dots$ and $i, \ell \in \mathbb{N}_0$ such that $i < \ell$, let

$$h_n(i, \ell) = \frac{n-1}{n}\left[\frac{(i+n+1)(\ell+n)}{(i+n-1)(\ell+n+2)} + \frac{(i+n)(\ell+n+1)}{(i+n+2)(\ell+n-1)}\right] - 2.$$

It is obvious that

$$h_2(0, \ell) = \frac{1}{2}\left[\frac{3(\ell+2)}{\ell+4} + \frac{2(\ell+3)}{4(\ell+1)}\right] - 2 = -\frac{\ell^2 + 15\ell + 8}{4(\ell+1)(\ell+4)} < 0.$$

For $n = 2$ and $\ell > i \geq 1$, it follows that

$$h_2(i, \ell) = \frac{1}{2}\left[\frac{(i+3)(\ell+2)}{(i+1)(\ell+4)} + \frac{(i+2)(\ell+3)}{(\ell+1)(i+4)}\right] - 2$$

$$
\begin{aligned}
&= \frac{(i+3)(i+4)(\ell+1)(\ell+2)+(i+1)(i+2)(\ell+3)(\ell+4)}{2(i+1)(i+4)(\ell+1)(\ell+4)} - 2\\
&= \frac{(i+3)(i+4)(\ell+1)(\ell+2)-2(i+1)(i+4)(\ell+1)(\ell+4)}{2(i+1)(i+4)(\ell+1)(\ell+4)}\\
&\quad + \frac{(i+1)(i+2)(\ell+3)(\ell+4)-2(i+1)(i+4)(\ell+1)(\ell+4)}{2(i+1)(i+4)(\ell+1)(\ell+4)}\\
&= \frac{(i+4)(\ell+1)[\ell(1-i)-6i-2]}{2(i+1)(i+4)(\ell+1)(\ell+4)} + \frac{(i+1)(\ell+4)[i(1-\ell)-6\ell-2]}{2(i+1)(i+4)(\ell+1)(\ell+4)}\\
&< 0.
\end{aligned}
$$

For $n \geq 3$ and $0 \leq i < \ell$, we have

$$
\begin{aligned}
\frac{(i+n+1)(\ell+n)}{(i+n-1)(\ell+n+2)} &= 1 + \frac{2}{i+n-1} - \frac{2}{\ell+n+2} - \frac{4}{(i+n-1)(\ell+n+2)}\\
&< 1 + \frac{2}{i+n-1} - \frac{2}{\ell+n+2},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{(i+n)(\ell+n+1)}{(i+n+2)(\ell+n-1)} &= 1 + \frac{2}{\ell+n-1} - \frac{2}{i+n+2} - \frac{4}{(i+n+2)(\ell+n-1)}\\
&< 1 + \frac{2}{\ell+n-1} - \frac{2}{i+n+2}.
\end{aligned}
$$

Accordingly, by virtue of the inequality (3.7), we obtain

$$
\begin{aligned}
h_n(i,\ell) &= \frac{n-1}{n}\left[\frac{(i+n+1)(\ell+n)}{(i+n-1)(\ell+n+2)} + \frac{(i+n)(\ell+n+1)}{(i+n+2)(\ell+n-1)}\right] - 2\\
&\leq \frac{n-1}{n}\left(1 + \frac{2}{i+n-1} - \frac{2}{\ell+n+2} + 1 + \frac{2}{\ell+n-1} - \frac{2}{i+n+2}\right) - 2\\
&= \frac{n-1}{n}\left[\frac{6}{(i+n-1)(i+n+2)} + \frac{6}{(\ell+n-1)(\ell+n+2)}\right] - \frac{2}{n}\\
&\leq \frac{6(n+1)}{n+2}\left[\frac{1}{(i+n)(i+n+1)} + \frac{1}{(\ell+n)(\ell+n+1)}\right] - \frac{2}{n}\\
&\leq \frac{6(n+1)}{n+2}\left[\frac{1}{n(n+1)} + \frac{1}{(1+n)(n+2)}\right] - \frac{2}{n}\\
&= -\frac{2(n^2-2n-2)}{n(n+2)^2}\\
&< 0,
\end{aligned}
$$

for $n \geq 3$ and $0 \leq i < \ell$. In conclusion, for $n = 2, 3, \dots$ and $i, \ell \in \mathbb{N}_0$ such that $i < \ell$, we have

$$
\frac{(i+n+1)(\ell+n)}{(i+n-1)(\ell+n+2)} + \frac{(i+n)(\ell+n+1)}{(i+n+2)(\ell+n-1)} < \frac{2n}{n-1}. \tag{3.8}
$$

Using the inequalities (3.7) and (3.8), we obtain

$$
\frac{n-1}{n}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n-1)(\ell+n)}\frac{s^\ell}{\ell!}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n+1)(\ell+n+2)}\frac{s^\ell}{\ell!}
$$

$$
\begin{aligned}
&= \frac{n-1}{n}\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{1}{(i+n-1)(i+n)(\ell+n+1)(\ell+n+2)}\frac{s^{i+\ell}}{i!\ell!}\\
&= \frac{n-1}{n}\sum_{i=0}^{\infty}\sum_{\ell>i}\bigg[\frac{1}{(i+n-1)(i+n)(\ell+n+1)(\ell+n+2)}\\
&\quad+\frac{1}{(\ell+n-1)(\ell+n)(i+n+1)(i+n+2)}\bigg]\frac{s^{i+\ell}}{i!\ell!}\\
&\quad+\frac{n-1}{n}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n-1)(\ell+n)(\ell+n+1)(\ell+n+2)}\frac{s^{2\ell}}{(\ell!)^2}\\
&< \sum_{i=0}^{\infty}\sum_{\ell>i}\frac{2}{(i+n)(i+n+1)(\ell+n)(\ell+n+1)}\frac{s^{i+\ell}}{i!\ell!}\\
&\quad+\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n)^2(\ell+n+1)^2}\frac{s^{2\ell}}{(\ell!)^2}\\
&= \sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{1}{(i+n)(i+n+1)(\ell+n)(\ell+n+1)}\frac{s^{i+\ell}}{i!\ell!}\\
&= \left[\sum_{\ell=0}^{\infty}\frac{s^{\ell}}{\ell!(\ell+n)(\ell+n+1)}\right]^2.
\end{aligned}
$$

Lemma 2 is thus proved. □

Lemma 3. *For $s \in \mathbb{R}$ and $n \in \mathbb{N}$, we have*

$$
\sum_{\ell=0}^{\infty}\frac{\ell+1}{(\ell+n)!}s^{\ell}\sum_{\ell=0}^{\infty}\frac{\ell+1}{(\ell+n+2)!}s^{\ell} < \left[\sum_{\ell=0}^{\infty}\frac{\ell+1}{(\ell+n+1)!}s^{\ell}\right]^2. \tag{3.9}
$$

Proof. The inequality (3.9) is equivalent to

$$
q_{n-1}(s)q_{n+1}(s) < q_n^2(s), \tag{3.10}
$$

where $q_n(s)$ for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$ is defined by (3.4).

It is obvious that

$$
s^{n+1}q_n(s) = \sum_{\ell=0}^{\infty}\frac{(\ell+1)s^{\ell+n+1}}{(\ell+n+1)!} = \sum_{\ell=n+1}^{\infty}(\ell-n)r_{\ell}(s),
$$

and

$$
\sum_{\ell=n+2}^{\infty}(\ell-n-1)r_{\ell}(s) = \sum_{\ell=n+1}^{\infty}(\ell-n-1)r_{\ell}(s),
$$

for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where $r_{\ell}(s) = \frac{s^{\ell}}{\ell!}$.

A straightforward operation gives

$$
s^{2n+2}(q_{n-1}(s)q_{n+1}(s) - [q_n(s)]^2)
$$

$$
\begin{aligned}
&= [s^n q_{n-1}(s)][s^{n+2} q_{n+1}(s)] - [s^{n+1} q_n(s)]^2 \\
&= \sum_{\ell=n}^{\infty} (\ell - n + 1) r_\ell(s) \sum_{\ell=n+2}^{\infty} (\ell - n - 1) r_\ell(s) - [\sum_{\ell=n+1}^{\infty} (\ell - n) r_\ell(s)]^2 \\
&= \left[r_n(s) + \sum_{\ell=n+1}^{\infty} (\ell - n + 1) r_\ell(s) \right] \sum_{\ell=n+1}^{\infty} (\ell - n - 1) r_\ell(s) - \left[\sum_{\ell=n+1}^{\infty} (\ell - n) r_\ell(s) \right]^2 \\
&= \sum_{i=n+1}^{\infty} (i - n + 1) r_i(s) \sum_{\ell=n+1}^{\infty} (\ell - n - 1) r_\ell(s) \\
&\quad - \sum_{i=n+1}^{\infty} (i - n) r_i(s) \sum_{\ell=n+1}^{\infty} (\ell - n) r_\ell(s) + r_n(s) \sum_{\ell=n+1}^{\infty} (\ell - n - 1) r_\ell(s) \\
&= \sum_{i=n+1}^{\infty} \sum_{\ell=n+1}^{\infty} [(i - n + 1)(\ell - n - 1) - (i - n)(\ell - n)] r_i(s) r_\ell(s) + r_n(s) \sum_{\ell=n+2}^{\infty} (\ell - n - 1) r_\ell(s) \\
&= \sum_{i=n+1}^{\infty} \sum_{\ell=n+1}^{\infty} (\ell - i - 1) r_i(s) r_\ell(s) + r_n(s) \sum_{\ell=n+2}^{\infty} (\ell - n - 1) r_\ell(s) \\
&= \sum_{i=n+1}^{\infty} \sum_{\ell>i}^{\infty} [(\ell - i - 1) r_i(s) r_\ell(s) + (i - \ell - 1) r_\ell(s) r_i(s)] - \sum_{i=n+1}^{\infty} [r_i(s)]^2 + r_n(s) \sum_{\ell=n+2}^{\infty} (\ell - n - 1) r_\ell(s) \\
&= -2 \sum_{i=n+1}^{\infty} \sum_{\ell>i}^{\infty} r_i(s) r_\ell(s) - \sum_{i=n+1}^{\infty} [r_i(s)]^2 + r_n(s) \sum_{\ell=2n+2}^{\infty} (\ell - 2n - 1) r_{\ell-n}(s) \\
&= - \sum_{i=n+1}^{\infty} \sum_{\ell=n+1}^{\infty} r_i(s) r_\ell(s) + r_n(s) \sum_{\ell=2n+2}^{\infty} (\ell - 2n - 1) r_{\ell-n}(s) \\
&= - \sum_{\ell=2n+2}^{\infty} \sum_{i=n+1}^{\ell-n-1} r_i(s) r_{\ell-i}(s) + \sum_{\ell=2n+2}^{\infty} \sum_{i=n+1}^{\ell-n-1} r_{\ell-n}(s) r_n(s) \\
&= \sum_{\ell=2n+2}^{\infty} \sum_{i=n+1}^{\ell-n-1} [r_{\ell-n}(s) r_n(s) - r_i(s) r_{\ell-i}(s)] \\
&= \sum_{\ell=2n+2}^{\infty} \frac{s^\ell}{\ell!} \sum_{i=n+1}^{\ell-n-1} \left[\binom{\ell}{n} - \binom{\ell}{i} \right] \\
&< 0,
\end{aligned}
$$

for $s > 0$ and $n \in \mathbb{N}$, where we used in the last line the combinatorial inequality

$$
\binom{\ell}{n} < \binom{\ell}{i}, \quad n < i < \ell - n.
$$

The inequalities (3.9) and (3.10) are thus true for $s > 0$ and $n \in \mathbb{N}$.

Since $q_0(s) = \mathrm{e}^s$,

$$
q_1(s) = \left[\frac{1}{s} \sum_{\ell=0}^{\infty} \frac{s^{\ell+2}}{(\ell+2)!} \right]' = \left(\frac{\mathrm{e}^s - 1 - s}{s} \right)' = \left(\frac{1}{s} - \frac{1}{s^2} \right) \mathrm{e}^s + \frac{1}{s^2},
$$

and

$$q_2(s)=\left[\frac{1}{s^2}\sum_{\ell=0}^{\infty}\frac{s^{\ell+3}}{(\ell+3)!}\right]'=\left[\frac{\mathrm{e}^s-1-s-s^2/2}{s^2}\right]'=\left(\frac{1}{s^2}-\frac{2}{s^3}\right)\mathrm{e}^s+\frac{2}{s^3}+\frac{1}{s^2},$$

it follows that

$$q_1^2(s)-q_0(s)q_2(s)=\frac{\mathrm{e}^s}{s^4}[\mathrm{e}^s+\mathrm{e}^{-s}-(s^2+2)]=2\,\mathrm{e}^s\sum_{\ell=0}^{\infty}\frac{s^{2\ell}}{(2\ell+4)!}>0,$$

for $s\in\mathbb{R}$. The inequalities (3.9) and (3.10) are thus true for $s\in\mathbb{R}$ and $n=1$.

By virtue of the integral representation (3.3), we arrive at

$$\begin{aligned}
&q_{n-1}(s)q_{n+1}(s)-q_n^2(s)\\
=&\left[\frac{1}{(n-2)!}\int_0^1 v^{n-2}(1-v)\,\mathrm{e}^{s(1-v)}\,\mathrm{d}v\right]\left[\frac{1}{n!}\int_0^1 v^{n}(1-v)\,\mathrm{e}^{s(1-v)}\,\mathrm{d}v\right]-\left[\frac{1}{(n-1)!}\int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{s(1-v)}\,\mathrm{d}v\right]^2\\
=&\frac{\mathrm{e}^{2s}}{[(n-1)!]^2}\left(\frac{n-1}{n}\int_0^1 v^{n-2}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v\int_0^1 v^{n}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v-\left[\int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v\right]^2\right),
\end{aligned}$$

for $s\in\mathbb{R}$ and $n=2,3,\dotsc$. Since

$$\begin{aligned}
\int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v&=\int_0^1 v^{n-1}(1-v)\sum_{\ell=0}^{\infty}\frac{(-sv)^\ell}{\ell!}\,\mathrm{d}v\\
&=\sum_{\ell=0}^{\infty}\frac{(-s)^\ell}{\ell!}\int_0^1 v^{n+\ell-1}(1-v)\,\mathrm{d}v\\
&=\sum_{\ell=0}^{\infty}\frac{(-s)^\ell}{\ell!}\frac{1}{(n+\ell)(n+\ell+1)},\\
\int_0^1 v^{n-2}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v&=\sum_{\ell=0}^{\infty}\frac{(-s)^\ell}{\ell!}\frac{1}{(n+\ell-1)(n+\ell)},\\
\int_0^1 v^{n}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v&=\sum_{\ell=0}^{\infty}\frac{(-s)^\ell}{\ell!}\frac{1}{(n+\ell+1)(n+\ell+2)},
\end{aligned}$$

we derive

$$\begin{aligned}
&\frac{n-1}{n}\int_0^1 v^{n-2}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v\int_0^1 v^{n}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v-\left[\int_0^1 v^{n-1}(1-v)\,\mathrm{e}^{-sv}\,\mathrm{d}v\right]^2\\
=&\frac{n-1}{n}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n-1)(\ell+n)}\frac{(-s)^\ell}{\ell!}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n+1)(\ell+n+2)}\frac{(-s)^\ell}{\ell!}-\left[\sum_{\ell=0}^{\infty}\frac{1}{(\ell+n)(\ell+n+1)}\frac{(-s)^\ell}{\ell!}\right]^2\\
&<0,
\end{aligned}$$

for $s<0$ and $n=2,3,\dotsc$, where the last inequality follows from applying the inequality (3.6) in Lemma 2. The inequalities (3.9) and (3.10) are thus true for $s<0$ and $n=2,3,\dotsc$.

When $s=0$, the inequality (3.9) is $\frac{1}{n!(n+2)!}<\frac{1}{[(n+1)!]^2}$, which is still strictly valid for $n\in\mathbb{N}$. The proof of Lemma 3 is complete. □

We are now in a position to prove the main result in this section, the inequality (3.1) for specific series in Theorem 1.

Proof of Theorem 1. The inequality (3.1) in Theorem 1 is equivalent to

$$a_n(s)b_{n+1}(s) < a_{n+1}(s)b_n(s), \tag{3.11}$$

for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$, where $a_n(s)$ and $b_n(s)$ are defined by

$$a_n(s) = \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n}{n}} \frac{s^\ell}{\ell!} \quad \text{and} \quad b_n(s) = \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n}} \frac{s^\ell}{\ell!},$$

respectively.

When $n = 0$, the inequalities (3.1) and (3.11) are

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell+2} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{s^\ell}{\ell!} < \sum_{\ell=0}^{\infty} \frac{1}{\ell+1} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{s^\ell}{\ell!}, \quad s \in \mathbb{R},$$

that is,

$$\frac{e^s(s-1)+1}{s^2} e^s < \frac{e^s - 1}{s} e^s, \quad s \neq 0,$$

and $\frac{1}{2} < 1$ for $s = 0$. These inequalities are apparently true. Therefore, the strict inequalities (3.1) and (3.11) are valid for $n = 0$ and $s \in \mathbb{R}$.

For $s \in \mathbb{R}$ and $n \in \mathbb{N}$, denote

$$P_n(s) = sR_{n-1}[e^s] - nR_n[e^s],$$

and

$$F_n(s) = R_{n-1}[e^s]P_{n+1}(s) - R_n[e^s]P_n(s).$$

Then, for $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned} P_{n+1}(s) &= sR_n[e^s] - (n+1)R_{n+1}[e^s] \\ &= s(R_{n-1}[e^s] - r_n(s)) - n(R_n[e^s] - r_{n+1}(s)) - R_{n+1}[e^s] \\ &= sR_{n-1}[e^s] - nR_n[e^s] - ur_n(s) + nr_{n+1}(s) - R_{n+1}[e^s] \\ &= sR_{n-1}[e^s] - nR_n[e^s] - (n+1)r_{n+1}(s) + nr_{n+1}(s) - R_{n+1}[e^s] \\ &= P_n(s) - R_n[e^s], \end{aligned}$$

where $r_n(s) = \frac{s^n}{n!}$. This means that the equality

$$P_{n+1}(s) - P_n(s) = -R_n[e^s] \tag{3.12}$$

is true for $n \in \mathbb{N}$ and $s \in \mathbb{R}$. For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, we have

$$s^{n+1}q_n(s)=\sum_{\ell=0}^{\infty}\frac{\ell+1}{(\ell+n+1)!}s^{\ell+n+1}=\sum_{\ell=n+1}^{\infty}\frac{\ell-n}{\ell!}s^{\ell}=\sum_{\ell=n+1}^{\infty}\frac{\ell}{\ell!}s^{\ell}-n\sum_{\ell=n+1}^{\infty}\frac{1}{\ell!}s^{\ell}=sR_{n-1}[\mathrm{e}^s]-nR_n[\mathrm{e}^s]=P_n(s). \quad (3.13)$$

Then, in light of Lemma 3, we acquire

$$P_{n-1}(s)P_{n+1}(s)-P_n^2(s)=s^{2n+2}[q_{n-1}(s)q_{n+1}(s)-q_n^2(s)]<0,$$

that is,

$$P_{n-1}(s)P_{n+1}(s)<P_n^2(s), \quad (3.14)$$

for $n\in\mathbb{N}$ and $s\ne 0$. By the equality (3.12) and the inequality (3.14), we acquire

$$\begin{aligned}F_n(s)&=[P_{n-1}(s)-P_n(s)]P_{n+1}(s)-[P_n(s)-P_{n+1}(s)]P_n(s)\\&=P_{n-1}(s)P_{n+1}(s)-P_n^2(s)\\&<0,\end{aligned}$$

for $n\in\mathbb{N}$ and $s\ne 0$. In summary, the inequality

$$F_n(s)<0 \quad (3.15)$$

is true for $s\ne 0$ and $n\in\mathbb{N}_0$.

It is standard to obtain that

$$s^n a_n(s)=n!R_{n-1}[\mathrm{e}^s],$$

and

$$s^{n+1}b_n(s)=n!s^{n+1}q_n(s)=n!P_n(s),$$

for $s\in\mathbb{R}$ and $n\in\mathbb{N}_0$, where the last equality followed from the relation $s^{n+1}q_n(s)=P_n(s)$ derived in (3.13). Further, in view of the inequality (3.15), we obtain

$$\begin{aligned}s^{2n+2}[a_n(s)b_{n+1}(s)-a_{n+1}(s)b_n(s)]&=n!R_{n-1}[\mathrm{e}^s](n+1)!P_{n+1}(s)-(n+1)!R_n[\mathrm{e}^s]n!P_n(s)\\&=n!(n+1)!(R_{n-1}[\mathrm{e}^s]P_{n+1}(s)-R_n[\mathrm{e}^s]P_n(s))\\&=n!(n+1)!F_n(s)\\&<0,\end{aligned}$$

for $s\ne 0$ and $n\in\mathbb{N}$. Accordingly, the strict inequalities (3.1) and (3.11) are true for $s\ne 0$ and $n\in\mathbb{N}$.

When $s=0$, the strict inequalities (3.1) and (3.11) become

$$\frac{1}{\binom{n+2}{n+1}}\frac{1}{\binom{n}{n}}=\frac{1}{n+2}<\frac{1}{\binom{n+1}{n+1}}\frac{1}{\binom{n+1}{n}}=\frac{1}{n+1},$$

for $n\in\mathbb{N}_0$, which is obviously valid for $n\in\mathbb{N}_0$. The case $s=0$ and $n=0$ was also verified on page 14748 in this paper. In conclusion, the strict inequalities (3.1) and (3.11) are true for $s\in\mathbb{R}$ and $n\in\mathbb{N}_0$.

Since the normalized remainder $T_n[\mathrm{e}^s]$ is absolutely monotonic in $s\in\mathbb{R}$ for $n\in\mathbb{N}_0$, see [12, Theorem 2], all of its derivatives are positive, that is,

$$\frac{\mathrm{d}^m T_n[\mathrm{e}^s]}{\mathrm{d}s^m}=\sum_{\ell=0}^{\infty}\frac{1}{\binom{\ell+m+n+1}{n+1}}\frac{s^{\ell}}{\ell!}>0, \quad (3.16)$$

for $m, n \in \mathbb{N}_0$ and $s \in \mathbb{R}$; see [3, p. 98]. Considering the positivity in (3.16), we can write the inequality (3.1) as

$$\frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!}} < \frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n}{n}} \frac{s^\ell}{\ell!}},$$

for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$. This means that, for any given $s \in \mathbb{R}$, the sequence in (3.2) is decreasing in $n \in \mathbb{N}_0$. The proof of Theorem 1 is thus complete. □

4. Decreasing property

In this section, with the help of the inequality (3.1) in Theorem 1, we focus on verifying that the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ is decreasing in $s \in \mathbb{R}$ for $n \in \mathbb{N}_0$.

Theorem 2. *For $n \in \mathbb{N}_0$, the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ is decreasing in $s \in \mathbb{R}$.*

Proof. Making use of the Maclaurin series (2.1), we can write the normalized remainder $T_n[e^s]$ defined for $n \in \mathbb{N}_0$ by (2.2) as

$$T_n[e^s] = \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} > 0, \quad s \in \mathbb{R}.$$

The positivity of $T_n[e^s]$ for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$ was discussed in [12, Section 2]. Then, for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\frac{T_{n+1}[e^s]}{T_n[e^s]} = \frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+2}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!}}. \tag{4.1}$$

A direct differentiation on both sides of (4.1) yields

$$\begin{aligned}
\frac{d}{ds}\left(\frac{T_{n+1}[e^s]}{T_n[e^s]}\right) &= \frac{\sum_{\ell=1}^{\infty} \frac{1}{\binom{\ell+n+2}{n+2}} \frac{s^{\ell-1}}{(\ell-1)!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} - \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+2}} \frac{s^\ell}{\ell!} \sum_{\ell=1}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^{\ell-1}}{(\ell-1)!}}{\left[\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!}\right]^2} \\
&= \frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+3}{n+2}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} - \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+2}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}}{\left[\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!}\right]^2},
\end{aligned}$$

for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Accordingly, for proving the decreasing property of the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ in $s \in \mathbb{R}$, it suffices to show that the inequality (3.1) in Theorem 1 is true for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Because the inequality (3.1) in Theorem 1 has been proved to be true for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$ in Section 3 of this paper, we conclude that the decreasing property of the ratio $\frac{T_{n+1}[e^s]}{T_n[e^s]}$ for $n \in \mathbb{N}_0$ in $s \in \mathbb{R}$. The proof of Theorem 2 is thus complete. □

5. Several remarks

In this section, we list several remarks.

Remark 1. *A slight variant of the inequality* (3.1) *was announced as a problem at the site https://math.stackexchange.com/q/4956563 (accessed on 11 August 2024).*

For given $s \in \mathbb{R}$, as $n \to \infty$, what is the limit of the sequence in (3.2)*?*

The inequality (2.3) *can be reformulated as*

$$\frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!}} \le \frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+3}{n+1}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}} \le \cdots \le \frac{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+m+1}{n+1}} \frac{s^\ell}{\ell!}}{\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+m}{n+1}} \frac{s^\ell}{\ell!}}, \tag{5.1}$$

for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $s \in \mathbb{R}$. For given $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$, as $m \to \infty$, what is the limit of the sequence in the very right end of the inequality (5.1)*?*

These two questions on limits were also asked on the web site https://math.stackexchange.com/q/4956563 (accessed on 11 August 2024).

Remark 2. *By the definition in* (2.2)*, we acquire*

$$\frac{T_{n+1}[h(s)]}{T_n[h(s)]} = \frac{\frac{(n+2)!}{s^{n+2}}\left(\mathrm{e}^s - \sum_{\ell=0}^{n+1} \frac{s^\ell}{\ell!}\right)}{\frac{(n+1)!}{s^{n+1}}\left(\mathrm{e}^s - \sum_{\ell=0}^{n} \frac{s^\ell}{\ell!}\right)} = (n+2)\frac{\mathrm{e}^s - \sum_{\ell=0}^{n+1} \frac{s^\ell}{\ell!}}{s\left(\mathrm{e}^s - \sum_{\ell=0}^{n} \frac{s^\ell}{\ell!}\right)} \to \begin{cases} 0, & s \to \infty, \\ 1, & s \to 0, \\ \dfrac{n+2}{n+1}, & s \to -\infty, \end{cases}$$

for $n \in \mathbb{N}_0$. Making use of these limits and Theorem 2, we obtain the inequality

$$\frac{T_{n+1}[\mathrm{e}^s]}{T_n[\mathrm{e}^s]} < \frac{n+2}{n+1},$$

for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$. Equivalently, the inequality

$$\frac{R_{n+1}[\mathrm{e}^s]}{s^{n+2}} < \frac{1}{n+1} \frac{R_n[\mathrm{e}^s]}{s^{n+1}}$$

is true for $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$. Therefore, for $n \in \mathbb{N}_0$, the inequality

$$(n-s+1)\frac{R_n[\mathrm{e}^s]}{s^{n+1}} < \frac{1}{n!}$$

is valid in $s > 0$ and reverses in $s < 0$.

Remark 3. *For $n \in \mathbb{N}_0$ and $s \in \mathbb{R}$, let*

$$G_n(s) = \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n}} \frac{s^\ell}{\ell!} - \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n}{n}} \frac{s^\ell}{\ell!},$$

and

$$H_n(s) = \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+3}{n+1}} \frac{s^\ell}{\ell!} \sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+1}{n+1}} \frac{s^\ell}{\ell!} - \left[\sum_{\ell=0}^{\infty} \frac{1}{\binom{\ell+n+2}{n+1}} \frac{s^\ell}{\ell!}\right]^2.$$

These two functions $G_n(s)$ and $H_n(s)$ are slight modifications of those two functions in [12, Remark 7]. We guess that the functions $G_n(s)$ and $H_n(s)$ for $n \in \mathbb{N}_0$ should be absolutely monotonic in $s \in \mathbb{R}$, that is, $G_n^{(m)}(s) \ge 0$ and $H_n^{(m)}(s) \ge 0$ for $m, n \in \mathbb{N}_0$ and $s \in \mathbb{R}$. This guess is stronger than the inequalities (2.3) *and* (3.1)*. For detailed information on the notion of absolutely monotonic functions and its generalizations, please refer to [10, Chapter XIII], [19, Chapter IV], and the monograph [15].*

Remark 4. *It is known [4, Chapter XIII] that the quantity*

$$\Re_\mu\{f(x);y\} = \frac{1}{\Gamma(\mu)}\int_0^y f(x)(y-x)^{\mu-1}\,\mathrm{d}x$$

is called the Riemann-Liouville fractional integral of order μ*, where the classical Euler gamma function* $\Gamma(\mu)$ *is defined [16, Chapter 3] by*

$$\Gamma(\mu) = \lim_{n\to\infty}\frac{n!n^\mu}{\prod_{k=0}^{n}(\mu+k)}, \quad \mu\in\mathbb{C}\setminus\{0,-1,-2,\dots\},$$

whose reciprocal $\frac{1}{\Gamma(z)}$ *is an entire function on the complex plane* $\mathbb{C}$.

The integral representation (3.3) *in Lemma 1 can be reformulated in terms of the Riemann-Liouville fractional integral of order n as*

$$\sum_{\ell=0}^{\infty}\frac{\ell+1}{(\ell+n+1)!}s^\ell = \frac{1}{\Gamma(n)}\int_0^1 x\,\mathrm{e}^{sx}(1-x)^{n-1}\,\mathrm{d}x = \Re_n\{x\,\mathrm{e}^{sx};1\},$$

for $n\in\mathbb{N}$ *and* $s\in\mathbb{R}$.

A direct computation yields

$$\begin{aligned}
\Re_\mu\{x\,\mathrm{e}^{sx};y\} &= \frac{1}{\Gamma(\mu)}\int_0^y x\,\mathrm{e}^{sx}(y-x)^{\mu-1}\,\mathrm{d}x\\
&= \frac{y^{\mu+1}}{\Gamma(\mu)}\int_0^1 u\,\mathrm{e}^{syu}(1-u)^{\mu-1}\,\mathrm{d}u\\
&= \frac{y^{\mu+1}}{\Gamma(\mu)}\sum_{\ell=0}^{\infty}\biggl[\int_0^1 u^{\ell+1}(1-u)^{\mu-1}\,\mathrm{d}u\biggr]\frac{s^\ell y^\ell}{\ell!}\\
&= y^{\mu+1}\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+2)}{\Gamma(\ell+\mu+2)}\frac{s^\ell y^\ell}{\ell!}\\
&= y^{\mu+1}\sum_{\ell=0}^{\infty}\frac{\ell+1}{\Gamma(\ell+\mu+2)}y^\ell s^\ell,
\end{aligned}$$

for $\mu>0$ *and* $s,y\in\mathbb{R}$*. Making use of this expression, we may generalize the main results of this paper.*

6. Conclusions

In this paper, we demonstrated that the ratio $\frac{T_{n+1}[\mathrm{e}^s]}{T_n[\mathrm{e}^s]}$ between two normalized remainders $T_{n+1}[\mathrm{e}^s]$ and $T_n[\mathrm{e}^s]$ for $n\in\mathbb{N}_0$ is decreasing in $s\in\mathbb{R}$. This result extends [12, Theorem 1] from $(0,\infty)$ to $\mathbb{R}$.

The guess in Remark 3 is interesting in mathematics.

In [5, p. 19], several remainders are collected as follows.

> If a function $h(s)$ has derivatives of all orders throughout a neighborhood of a point η, then we may write the series
>
> $$h(\eta)+\frac{s-\eta}{1!}h'(\eta)+\frac{(s-\eta)^2}{2!}h''(\eta)+\frac{(s-\eta)^3}{3!}h'''(\eta)+\cdots,$$

which is known as the Taylor series of the function $h(s)$.

The Taylor series converges to the function $h(s)$ if the remainder

$$R_n(s) = h(s) - h(\eta) - \sum_{\ell=1}^{n} \frac{(s-\eta)^\ell}{\ell!} h^{(\ell)}(\eta)$$

approaches zero as $n \to \infty$. The following expressions are different forms for the remainder of a Taylor series:

$$R_n(s) = \frac{1}{n!} \int_\eta^s h^{(n+1)}(t)(s-t)^n \,\mathrm{d}t,$$

$$R_n(s) = \frac{(s-\eta)^{n+1}}{(n+1)!} h^{(n+1)}(\eta + \theta(s-\eta)), \qquad \text{(Lagrange)}, \tag{6.1}$$

$$R_n(s) = \frac{(s-\eta)^{n+1}}{n!} (1-\theta)^n h^{(n+1)}(\eta + \theta(s-\eta)), \qquad \text{(Cauchy)}, \tag{6.2}$$

and

$$R_n(s) = \frac{\varphi(s-\eta) - \varphi(0)}{\varphi'[(s-\eta)(1-\theta)]} \frac{(s-\eta)^n (1-\theta)^n}{n!} h^{(n+1)}(\eta + \theta(s-\eta)), \qquad \text{(Schlömilch)}, \tag{6.3}$$

for $0 < \theta < 1$, where $\varphi(s)$ is an arbitrary function satisfying the following two conditions:

(1) It and its derivative $\varphi'(s)$ are continuous in the interval $(0, s-\eta)$; and
(2) the derivative $\varphi'(s)$ does not change sign in that interval.

If we set $\varphi(s) = s^{p+1}$ for $0 < p \le n$, we obtain the following form for the remainder:

$$R_n(s) = \frac{(s-\eta)^{n+1}(1-\theta)^{n-p-1}}{(p+1)n!} h^{(n+1)}(\eta + \theta(s-\eta)), \qquad \text{(Rouché)}, \tag{6.4}$$

for $0 < \theta < 1$.

Like Lagrange's remainder (6.1), Cauchy's remainder (6.2), Schlömilch's remainder (6.3), and Rouché's remainder (6.4), Qi's normalized remainder $T_n[h(s)]$ in Definition 1 of the Maclaurin series (1.1) is worth being investigated deeply and extensively.

In the future, researchers can consider the following points.

(1) This paper would benefit from including applications to concrete real-life problems, demonstrating where the results apply and how the hypotheses are verified. Providing relevant examples would have made this paper much more interesting with not only the theoretic value.
(2) Are there other methods to verify the decreasing property in Theorem 2? Is there a simpler way to prove the decreasing property in Theorem 2?
(3) Can the inequalities established in this paper be extended to other series expansions?

More importantly, in [17, Remark 2] and [23, Remark 2], for example, Qi et al. found that the normalized remainder $T_n[\mathrm{e}^s]$ has something to do with the generating functions of the Bernoulli, Stirling, and Howard numbers and polynomials in combinatorial number theory.

Author contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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