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Research article

On deformable fractional order implicit differential equations involving orthogonal super metric spaces

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Abstract: In this manuscript, we use the notion of orthogonality in a super metric space and prove some fixed point theorems that focus on orthogonal contractions and orthogonal $\mathcal{F}$-contractions, which are types of mappings exhibiting particular contraction properties while satisfying orthogonality. To illustrate and support our theoretical results, we provide concrete examples that demonstrate the application of these findings. Furthermore, we show how our results can be utilized in practical scenarios by presenting a solution to a deformable implicit differential equation, highlighting the relevance of our work in both theoretical and applied contexts.

Keywords: orthogonal F-contraction; orthogonal set; super metric space
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1. Introduction

In 1906, the concept of a metric space (MS) was proposed by Fréchet [1] using a distance function, which is known as a metric. Since then, the notion has been generalized and extended by eminent researchers in various ways like semi metric [2], partial metric [3], etc. On the other hand, other generalizations were obtained by modifying the triangle inequality, such as in a b-MS [4]. Recently, one such generalization was given by Karapinar and Khojasteh [5], known as a super metric space (SMS). It is defined as given in the next definition.

Definition 1.1. *Consider* $\mathfrak{U} \neq \emptyset$ *and a mapping* $\sigma^* : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$. *Let for any* $a, b \in \mathfrak{U}$, *we have:*

(1) $\sigma^*(a, b) \geq 0$ *and* $\sigma^*(a, b) = 0$ *if and only if* $a = b$;
(2) $\sigma^*(a, b) = \sigma^*(b, a)$;
(3) For distinct sequences $\{a_n\}, \{b_n\} \in \mathfrak{U}$ *which satisfy* $\lim_{n\to\infty} \sigma^*(a_n, b_n) = 0$, *there exists* $s \geq 1$ *such that*

$$\lim_{n\to\infty} \sup \sigma^*(a_n, c) \leq s \lim_{n\to\infty} \sup \sigma^*(b_n, c), \text{ for all } c \in \mathfrak{U}.$$

Then, $(\mathfrak{U}, \sigma^*)$ *is known as a SMS.*

The study of contractions in supermetric spaces has seen significant recent developments. In 2022, Karapinar and Fulga [6] pioneered the examination of rational-form contractions within supermetric space framework. Building on this foundation, Abodayeh et al. [7] advanced the field in 2024 by investigating Ćirić-type generalized $\mathcal{F}$-contractions, almost $\mathcal{F}$-contractions, and their hybrid variants in super metric spaces. Most recently, Shah et al. [8] have contributed to this growing body of research by analyzing the characteristics of generalized (ψ, ϕ)-type contraction mappings featuring rational-type expressions in super-metric space settings.

In 2017, Zulfeqarr et al. [9] introduced an innovative concept called the deformable fractional derivative by modifying the traditional limit approach used in standard derivatives. This concept is termed 'deformable' because of its unique capability to smoothly transition a function to its derivative. These deformable derivatives can be viewed as fractional-order derivatives. The authors of [10] further explored the properties of this new concept and applied their findings to examine the following Cauchy problem with a non-local condition:

$$\begin{cases} D_0^\alpha(t) = \mathfrak{T}(t, a(t)), t \in (0, k] \\ a(0) = g(a(t)) + a_0. \end{cases}$$

Here, g is a continuous function on $\mathbb{R}$, and D_0^α represents a deformable derivative of order $\alpha \in (0, 1)$. In [11], Mebrat and Guerekata utilized Weissinger's and Krasnoselskij's fixed point theorems to investigate the existence of solutions for the following problem:

$$\begin{cases} D_0^\alpha(t) = \phi(a(t)) + \psi(t, a(t)) + \int_0^t \gamma(t, s, a(s))ds, t \in (0, k] \\ a(0) = a_0, \end{cases}$$

where $\psi : P \times \mathbb{R} \to \mathbb{R}$ and $\gamma : P \times P \times \mathbb{R} \to \mathbb{R}$ are continuous functions. In 2023, Krim et al. [12] demonstrated the existence and uniqueness of solutions for a specific class of deformable fractional differential equations. Shortly afterward, Sreedharan et al. [13] derived sufficient conditions to guarantee the existence of solutions for a perturbed fractional neutral integro-differential system formulated within the deformable derivative framework in a Banach space.

In 1922, Banach [14] formulated the Banach contraction principle, a landmark concept in the realm of fixed point theory. This principle asserts that any Banach contraction self-mapping $\mathfrak{T}$ acting on a complete MS $(\mathfrak{U}, \sigma^*)$ possesses a unique fixed point. Due to its wide applicability, it has various generalizations, extensions, and applications. In 2012, Wardowski [15] proved a generalization of the Banach contraction principle. Let $\mathcal{F} : (0, \infty) \to \mathbb{R}$ fulfill the given requirements:
$(P1)$ The function $\mathcal{F}$ is increasing;
$(P2)$ $\mathcal{F}(t_n) \to -\infty$ as $n \to \infty$ if and only if the sequence $\{t_n\} \in (0, \infty)$ converges to zero;
$(P3)$ $\lim_{t \to 0^+} t^k \mathcal{F}(t) = 0$ for some $k \in (0, 1)$.
The set of all the functions satisfying $(P1)$, $(P2)$, and $(P3)$ is denoted by $\mathfrak{F}$. Some examples of such functions are as follows:

Example 1.2. *Define mappings* $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, *and* $\mathcal{F}_4$ *from* $(0, \infty)$ *into* $\mathbb{R}$ *as* $\mathcal{F}_1(t) = e^t, \mathcal{F}_2(t) = -\frac{1}{t}, \mathcal{F}_3(t) = \log t, \mathcal{F}_4(t) = -\frac{1}{\sqrt{t}}$. *Then*
(i) $\mathcal{F}_1$ *fulfills conditions* $(P1)$ *and* $(P3)$*;*
(ii) $\mathcal{F}_2$ *and* $\mathcal{F}_4$ *fulfill conditions* $(P1)$ *and* $(P2)$*;*
(iii) $\mathcal{F}_3$ *and* $\mathcal{F}_5$ *fulfill all the three properties.*

If a function $\mathcal{F}$ fulfills the property $(P1)$, then following two properties hold:
(i) $\mathcal{F}$ is continuous almost everywhere.
(ii) For all $s \in (0, \infty)$, left- and right-hand limits exist, i.e.,

$$\lim_{t \to s^+} \mathcal{F}(t) = \mathcal{F}(s^+) \text{ and } \lim_{t \to s^-} \mathcal{F}(t) = \mathcal{F}(s^-).$$

Using this, Wardowski [15] proposed the idea of an $\mathcal{F}$-contraction. In the past few years, $\mathcal{F}$-contraction has been generalized and extended by eminent mathematicians (see, for instance, [16–19]). Also, Gordji et al. [20] proposed the idea of an orthogonal set and proved some fixed point results in Banach spaces. Gordji et al. [21] proved the existence and uniqueness of fixed points for mappings defined on ϵ-connected orthogonal metric spaces. Later, Gungor et al. [22] extended these results by modifying distance functions to derive new fixed point theorems in orthogonal metric spaces. Further contributions include Yang et al. [23], who introduced orthogonal (Q, ψ)-contractions, and Sawangsup et al. [24], who investigated orthogonal Q-contraction mappings. In a subsequent work, Sawangsup et al. [25] developed the concept of orthogonal F-contraction mappings, broadening the scope of fixed point theory in orthogonal spaces. For further results using orthogonality, refer to [26, 27].

This paper makes significant contributions to fixed point theory by introducing innovative concepts in the context of supermetric spaces with orthogonality. New classes of orthogonal contractions and orthogonal $\mathcal{F}$-contractions are specifically designed for supermetric spaces, extending traditional fixed-point results to this more general setting. Also, some comprehensive existence and uniqueness theorems for these contraction mappings in the setting of orthogonal supermetric spaces are provided. To support our results, some examples have been provided along with an important application to deformable implicit differential equations.

2. Preliminaries

In this section, we give some basic preliminaries and already known important results.

Definition 2.1. *Consider $\mathcal{F} \in \mathfrak{F}$ and $\mathfrak{T}$ to be a self-mapping on a MS $(\mathfrak{U}, \sigma^*)$. Let there is a positive number $\xi > 0$ for which $\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) > 0$ implies*

$$\xi + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) \leq \mathcal{F}(\sigma^*(a, b)), \tag{2.1}$$

for all $a, b \in \mathfrak{U}$. Then the mapping $\mathfrak{T}$ is known as an $\mathcal{F}$-contraction. This notion was introduced by Wardowski [15].

Theorem 2.2. *[28] Let $(\mathfrak{U}, \sigma^*)$ be a complete metric space, and let $\mathfrak{T}\colon \mathfrak{U} \to \mathfrak{U}$ be a self-mapping. Suppose there exist a function $\mathcal{F} : (0, \infty) \to \mathbb{R}$ satisfying property (P1) and a constant $\xi > 0$ such that for all $a, b \in \mathfrak{U}$ with $\sigma^*(\mathfrak{T}a, \mathfrak{T}b) > 0$, the following inequality holds:*

$$\xi + \mathcal{F}\left(\sigma^*(\mathfrak{T}a, \mathfrak{T}b)\right) \leq \mathcal{F}\left(\sigma^*(a, b)\right). \tag{2.2}$$

Then $\mathfrak{T}$ admits a unique fixed point in $\mathfrak{U}$.

Definition 2.3. *[20] Let $\mathfrak{U}$ be a non-empty set and $\perp \subseteq \mathfrak{U} \times \mathfrak{U}$ be a binary relation. $(\mathfrak{U}, \perp)$ is called an orthogonal set if $\perp$ satisfies the following condition:*

$$\exists\, a_0 \in \mathfrak{U} : (\forall a \in \mathfrak{U}, a \perp a_0) \text{ or } (\forall a \in \mathfrak{U}, a_0 \perp a).$$

Also, this a_0 is called an orthogonal element.

Definition 2.4. *[20] Let $(\mathfrak{U}, \perp)$ be an orthogonal set.*
(i) The sequence $\{a_n\}$ is named an orthogonal sequence (O-sequence) if we have

$$(\forall\, n, a_n \perp a_{n+1}) \text{ or } (\forall\, n, a_{n+1} \perp a_n).$$

(ii) The mapping $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ is referred to as an orthogonal preserving (O-preserving) mapping if we have

$$a \perp b \iff \mathfrak{T}(a) \perp \mathfrak{T}(b).$$

Lemma 2.5. *Consider $\{a_n\}$ to be a sequence in a complete SMS $(\mathfrak{U}, \sigma^*)$. If $\lim_{n\to\infty} \sigma^*(a_n, a_{n+1}) = 0$, then the sequence $\{a_n\}$ is Cauchy and converges in $\mathfrak{U}$.*

Definition 2.6. *[9] Let $a(t)$ be a real-valued function on $[0, k]$. The deformable derivative of a of order α at $t \in (0, k)$ is defined as*

$$D_0^\alpha(a(t)) = \lim_{\epsilon\to 0} \frac{(1 + \epsilon\beta)a(t + \epsilon\alpha) - a(t)}{\epsilon},$$

where $\alpha + \beta = 1$, and $\alpha \in (0, 1]$. a is differentiable at t, if the limit exists.

If $\alpha = 1$ and $\beta = 0$, then the deformable derivative becomes a usual derivative.

Definition 2.7. *[9] Let $a(t)$ be a continuous function on $[0, k]$. The deformable integral of a of order α is defined as*

$$I_0^\alpha(a(t)) = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha} t} \int_0^t e^{\frac{\beta}{\alpha} s} a(s) ds,$$

where $\alpha + \beta = 1$, and $\alpha \in (0, 1]$.

If $\alpha = 1$ and $\beta = 0$, then the deformable integral becomes a usual Riemann integral.

Theorem 2.8. *[9] Let $\alpha, \alpha_1, \alpha_2 \in (0, 1]$ such that $\alpha + \beta = 1$ and $\alpha_i + \beta_i = 1$, for $i = 1, 2$. Then*

$$\begin{aligned}
&(a)\ \ D^{\alpha}(sa + tb) = sD^{\alpha}(a) + tD^{\alpha}(b)\\
&(b)\ \ D^{\alpha_1}D^{\alpha_2} = D^{\alpha_2}D^{\alpha_1}\\
&(c)\ \ D^{\alpha}(r) = \beta r \text{ for some constant } r\\
&(d)\ \ D^{\alpha}(fg) = (D^{\alpha}f)g + \alpha f Dg\\
&(e)\ \ I^{\alpha}(sa + tb) = sI^{\alpha}(a) + tI^{\alpha}(b)\\
&(f)\ \ I_a^{\alpha_1}I_a^{\alpha_2} = I_a^{\alpha_2}I_a^{\alpha_1}.
\end{aligned}$$

Lemma 2.9. *[9] Let $a(t)$ be a continuous function on $[0, k]$ and $I_0^{\alpha}(a)$ be α-differentiable on $(0, k)$. Then*

$$D_0^{\alpha}(I_0^{\alpha}(a))(t) = a(t) \ \text{ and } \ I_0^{\alpha}(D_0^{\alpha}(a))(t) = a(t).$$

Lemma 2.10. *[11] Let $\phi \in L^1(P)$ and $0 < \alpha < 1$; then the initial value problem*

$$\begin{cases} D_0^{\alpha}(a(t)) = \phi(t, a(t)),\ t \in P = (0, k],\ k > 0\\ a(0) = g(a(t)) + a_0, \end{cases}$$

is analogous to the following integral equation

$$a(t) = (a_0 - g(a))e^{\frac{\beta}{\alpha}t} + e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\phi(s)ds.$$

3. Main results

We start with the following definition.

Definition 3.1. *Let $(\mathfrak{U}, \perp)$ be an orthogonal set. Then $(\mathfrak{U}, \sigma^*, \perp)$ is referred to as an O-SMS if σ^* is a super metric on $(\mathfrak{U}, \perp)$.*

Example 3.2. *Consider $\mathfrak{U} = \mathbb{R}$ and define the binary relation $\perp$ as: $a \perp b$ if and only if $ab = a \vee b$, where $\vee$ means or. Then letting $a_0 = 0$ or 1, we see that $(\mathfrak{U}, \perp)$ is an orthogonal set. Also, $\sigma^* : \mathfrak{U}\times\mathfrak{U} \to [0, \infty)$ is given as*

$$\begin{aligned}
&\sigma^*(a, b) = (a - b)^2 \text{ for } a, b \in \mathbb{R} - \{1\},\\
&\sigma^*(1, b) = \sigma^*(b, 1) = (1 - b^3)^2 \text{ for } b \in \mathbb{R}.
\end{aligned}$$

Clearly, σ^* satisfies conditions (1) and (2) of Definition 1.1. Now, for condition (3), let $b \neq 1$, and $\{a_n\}, \{b_n\}$ be two sequences in $\mathfrak{U}$ such that $\limsup_{n\to\infty} \sigma^*(a_n, b_n) = 0$. This yields that $\lim_{n\to\infty} a_n = v = \lim_{n\to\infty} b_n$. So, we attain

$$\begin{aligned}
\limsup_{n\to\infty} \sigma^*(a_n, c) &= (v - c)^2\\
&\leq (v - c)^2
\end{aligned}$$

$$\leq \limsup_{n\to\infty} \sigma^*(b_n, c),$$

for all $c \in \mathfrak{U}$. Let $b = 1$, then taking both the sequences $\{a_n\}, \{b_n\}$ identical, we attain the desired result. Thus, $(\mathfrak{U}, \perp, \sigma^*)$ is an O-SMS.

Definition 3.3. *Let $(\mathfrak{U}, \sigma^*, \perp)$ be a an O-SMS. Then,*
(i) $\mathfrak{U}$ is referred to as orthogonally complete (O-complete) if every Cauchy O-sequence converges in $\mathfrak{U}$.
(ii) $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ is called orthogonally continuous (O-continuous or $\perp$-continuous) at $a \in \mathfrak{U}$ if for an O-sequence $\{a_n\}$ in $\mathfrak{U}$ with $a_n \to a$, we have $\mathfrak{T}(a_n) \to \mathfrak{T}(a)$. Also, $\mathfrak{T}$ is called $\perp$-continuous on $\mathfrak{U}$ if $\mathfrak{T}$ is $\perp$-continuous at each $a \in \mathfrak{U}$.

Definition 3.4. *Let $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ be a self-mapping on the O-SMS $(\mathfrak{U}, \sigma^*, \perp)$. Then $\mathfrak{T}$ is called an O-contraction if there is $0 \leq \theta < 1$ such that*

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\, \sigma^*(a, b), \tag{3.1}$$

for all $a, b \in \mathfrak{U}$.

Theorem 3.5. *Let $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ be an O-preserving O-contraction on an O-complete O-SMS $(\mathfrak{U}, \sigma^*, \perp)$. Then $\mathfrak{T}$ admits a unique fixed point in $\mathfrak{U}$.*

Proof. Since $(\mathfrak{U}, \perp)$ is an orthogonal set, there is $a_0 \in \mathfrak{U}$ such that $(\forall x \in \mathfrak{U}, a_0 \perp a)$ or $(\forall x \in \mathfrak{U}, a \perp a_0)$. Let $a_1 = \mathfrak{T}(a_0)$; then $a_0 \perp a_1$. Since $\mathfrak{T}$ is O-preserving, $a_0 \perp a_1$ implies $\mathfrak{T}(a_0) \perp \mathfrak{T}(a_1)$. Now, let $a_2 = \mathfrak{T}(a_1)$; then $a_1 \perp a_2$. Continuing in a similar fashion, we attain an O-sequence $\{a_n\}$.

Letting $a = a_n$ and $b = a_{n+1}$, then from Eq (3.1), we attain

$$\begin{aligned}
\sigma^*(a_{n+1}, a_{n+2}) &= \sigma^*(\mathfrak{T}(a_n), \mathfrak{T}(a_{n+1})) \\
&\leq \theta\, \sigma^*(a_n, a_{n+1}) \\
&\leq \ldots \\
&\leq \ldots \\
&\leq \theta^n \sigma^*(a_0, a_{n+1}).
\end{aligned}$$

Letting $n \to \infty$, we attain $d(a_{n+1}, a_{n+2}) \to 0$. From Lemma 2.5, we attain that the O-sequence $\{a_n\}$ is Cauchy. Since $(\mathfrak{U}, \perp)$ is O-complete, we obtain that $\{a_n\}$ converges in $\mathfrak{U}$ to some a^*. Since $\mathfrak{T}$ is continuous, we attain that a^* is a fixed point of $\mathfrak{T}$.

Uniqueness: Let a^* and b^* be two distinct fixed points of $\mathfrak{T}$; then from Eq (3.1), we attain

$$\begin{aligned}
\sigma^*(a^*, b^*) &= \sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \\
&\leq \theta\, \sigma^*(a^*, b^*),
\end{aligned}$$

which results in a contradiction as $0 \leq \theta < 1$. □

Example 3.6. *Consider $\mathfrak{U} = [1, 2]$ and define a binary relation $\perp$ as $a \perp b$ if and only if $ab \geq (a \vee b)$, where $\vee$ means or. Letting $a_0 = 1$, we see that $(\mathfrak{U}, \perp)$ is an orthogonal set. Also, $\sigma^* : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ is given by*

$$\sigma^*(a, b) = (a - b)^2, \ \ \textit{for } a, b \in (1, 2],$$

$$\sigma^*(1,b) = \sigma^*(b,1) = (1-b^2)^2.$$

Define a self-mapping $\mathfrak{T}$ *as*

$$\mathfrak{T}(a) = \begin{cases} 1 & \text{if } a \in [1,2) \\ \frac{3}{2} & \text{if } a = 2. \end{cases}$$

Now, we show that $\mathfrak{T}$ *is an orthogonal contraction.*
Case(i) If $a = 1$ *and* $b \in [1,2)$*, then* $\mathfrak{T}(a) = 1$ *and* $\mathfrak{T}(b) = 1$*. For any* $0 \leq \theta < 1$*, we have*

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\, \sigma^*(a,b).$$

Case(ii) If $a = 1, b = 2$*, then* $\mathfrak{T}(a) = 1, \mathfrak{T}(b) = \frac{3}{2}$*. For any* $\frac{25}{144} \leq \theta < 1$*, we have*

$$\begin{aligned} \sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) &= \left(1 - \frac{9}{2}\right)^2 \\ &= \frac{25}{16} \\ &\leq \theta \times 9 \\ &= \theta(1-(2)^2)^2 \\ &= \theta\, \sigma^*(a,b). \end{aligned}$$

Case(iii) If $a =\in [1,2), b \in [1,2)$*, then* $\mathfrak{T}(a) = 1, \mathfrak{T}(b) = 1$*. For any* $0 \leq \theta < 1$*, we have*

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\, \sigma^*(a,b).$$

Case(iv) If $a = 2, b = 2$*, then* $\mathfrak{T}(a) = \frac{3}{2}, \mathfrak{T}(b) = \frac{3}{2}$*. For any* $0 \leq \theta < 1$*, we have*

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\, \sigma^*(a,b).$$

Thus, it is clear that $\mathfrak{T}$ *is an orthogonal contraction and* 1 *is its unique fixed point.*

Definition 3.7. *Let* $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ *be a self-mapping on the O-SMS* $(\mathfrak{U}, \sigma^*, \perp)$*. Then* $\mathfrak{T}$ *is called an orthogonal* $\mathcal{F}$*-contraction, if for* $\mathcal{F} \in \mathfrak{F}$ *there is some* $\tau > 0$ *such that*

$$\tau + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) \leq \mathcal{F}(\sigma^*(a,b)), \tag{3.2}$$

for all $a, b \in \mathfrak{U}$*.*

Theorem 3.8. *Let* $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ *be an O-preserving mapping on an O-complete O-SMS* $(\mathfrak{U}, \sigma^*, \perp)$*. Also,* $\mathcal{F}$ *satisfies property* $(P1)$ *such that*

$$\tau + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) \leq \mathcal{F}(\sigma^*(a,b)),$$

for all $a, b \in \mathfrak{U}$*. Then* $\mathfrak{T}$ *admits a unique fixed point in* $\mathfrak{U}$*.*

Proof. Since $(\mathfrak{U}, \perp)$ is an orthogonal set, there is $a_0 \in \mathfrak{U}$ such that $(\forall x \in \mathfrak{U}, a_0 \perp a)$ or $(\forall x \in \mathfrak{U}, a \perp a_0)$. Let $a_1 = f(a_0)$; then $a_0 \perp a_1$. Since $\mathfrak{T}$ is O-preserving, $a_0 \perp a_1$ implies $\mathfrak{T}(a_0) \perp \mathfrak{T}(a_1)$. Let $a_2 = \mathfrak{T}(a_1)$; then $a_1 \perp a_2$. Continuing in a similar fashion, we attain an O-sequence $\{a_n\}$.

Letting $a = a_n, b = a_{n+1}$, then from Eq (3.2), we have

$$\begin{aligned}\tau + \mathcal{F}(\sigma^*(a_{n+1}, a_{n+2})) &= \tau + \mathcal{F}(\sigma^*(f(a_n), f(a_{n+1}))) \\ &\leq \mathcal{F}(\sigma^*(a_n, a_{n+1})).\end{aligned}$$

From property (P1) of $\mathcal{F}$, we have

$$\sigma^*(a_{n+1}, a_{n+2}) < \sigma^*(a_n, a_{n+1}),$$

for all $n \geq 1$. This shows that the sequence $\{\sigma^*_n = \sigma^*(a_n, a_{n+1})\}$ is decreasing and bounded below by 0. Hence, it converges to some $c^* \geq 0$. Suppose $c^* > 0$ and let $n \to \infty$, we attain

$$\tau + \mathcal{F}(c^*) \leq \mathcal{F}(c^*),$$

which is a contradiction. Hence, $c^* = 0$, i.e., $\lim_{n\to\infty} \sigma^*(a_n, a_{n+1}) \to 0$. From Lemma 2.5, we attain that the O-sequence $\{a_n\}$ is Cauchy. As $(\mathfrak{U}, \perp)$ is O-complete, so $\{a_n\}$ converges in $\mathfrak{U}$ to, say, a^*. Using the continuity of $\mathfrak{T}$, we attain that a^* is a fixed point of $\mathfrak{T}$.

Uniqueness: Let a^* and b^* be two distinct fixed points of $\mathfrak{T}$; then from Eq (3.1), we attain

$$\begin{aligned}\sigma^*(a^*, b^*) &= \sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \\ &\leq \theta\, \sigma^*(a^*, b^*),\end{aligned}$$

which results in a contradiction as $0 \leq \theta < 1$. □

Example 3.9. *Let $\mathfrak{U} = [0, 1]$ be endowed with a binary operation $\perp$ such that $a \perp b$ if and only if $ab = a \vee b$. A mapping $\sigma^* : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ is defined as*

$$\sigma^*(a, b) = \begin{cases} ab & \text{if } a \neq b \\ 0 & \text{if } a = b. \end{cases}$$

Next, define a self-mapping $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ such that

$$\mathfrak{T}(a) = \begin{cases} \frac{a}{2} & \text{if } a \in [0, 1) \\ \frac{1}{4} & \text{if } a = 1. \end{cases}$$

Now, we show that $\mathfrak{T}$ is an orthogonal contraction.
Case(i) If $a = 0, b \in [0, 1)$, then $\mathfrak{T}(a) = 0, \mathfrak{T}(b) = \frac{b}{2}$. For any $0 \leq \theta < 1$, we attain

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\sigma^*(a, b).$$

Case(ii) If $a = 0, b = 1$, then $\mathfrak{T}(a) = 0, \mathfrak{T}(b) = \frac{1}{4}$. For any $0 \leq \theta < 1$, we attain

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\sigma^*(a, b).$$

Case(iii) If $a = 1, b \in [0,1)$, *then* $\mathfrak{T}(a) = \frac{1}{4}, \mathfrak{T}(b) = \frac{b}{2}$. *For any* $\frac{1}{4} \leq \theta < 1$, *we attain*

$$\begin{aligned}\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) &= \frac{b}{8}\\ &\leq \theta\frac{b}{2}\\ &\leq \theta\sigma^*(a,b).\end{aligned}$$

Case(iv) If $a = 1, b = 1$, *then* $\mathfrak{T}(a) = \frac{1}{4}, \mathfrak{T}(b) = \frac{1}{4}$. *For any* $0 \leq \theta < 1$, *we attain*

$$\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b)) \leq \theta\sigma^*(a,b).$$

Thus, it is clear that $\mathfrak{T}$ *is an orthogonal* $\mathcal{F}$*-contraction and* 0 *is the unique fixed point of* $\mathfrak{T}$.

Corollary 3.10. *Let* $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ *be an O-preserving orthogonal* $\mathcal{F}$*-contraction on an O-complete O-SMS* $(\mathfrak{U}, \sigma^*, \perp)$. *Then* $\mathfrak{T}$ *admits a unique fixed point in* $\mathfrak{U}$.

Proof. As $\mathcal{F} \in \mathfrak{F}$, it satisfies $(P1)$ also. Hence the result holds. □

4. Application

Define a deformable fractional order implicit differential equation as follows:

$$\begin{cases} D_0^\alpha(a(t)) = \psi(t, a(t), D_0^\alpha(a(t))), \ t \in P = (0,k] \\ a(0) = a_0 \geq 1. \end{cases} \tag{4.1}$$

Here, D_0^α is the deformable fractional derivative and $0 < \alpha < 1$.

Let $\mathfrak{U} = \{u \in C(P, \mathbb{R}) : u(t) > 0\}$ and define $\sigma^* : \mathfrak{U} \times \mathfrak{U} \to [0, \infty)$ such that

$$\sigma^*(a(t), b(t)) = \sup_{t \in P} |a(t) - b(t)|^2.$$

Also, consider a binary relation $\perp$ such that $a(t) \perp b(t)$ if and only if $a(t)b(t) \geq (a(t) \vee b(t))$. Then $(\mathfrak{U}, \sigma^*, \perp)$ is an O-SMS. From Lemma 2.10, the problem (4.1) is equivalent to

$$a(t) = a_0 e^{\frac{\beta}{\alpha}t} + e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\phi(s)ds,$$

where $\phi(t) \in C(P)$ with $\phi(t) = \psi(t, a(t), D_0^\alpha(t))$.

Take into account the following hypotheses:

(H_{01}) There are γ, γ_1 and $\gamma_2 : C(P) \times C(P) \to [0, \infty)$ such that for $a, b \in C(P)$ and $t \in P$,

$$|\psi(t, a(t), \phi(t)) - \psi'(t, b(t), \phi'(t)| \leq \gamma_1(a,b)||a - b||_\infty^2 + \gamma_2(a,b)||\phi(t) - \phi'(t)||_\infty^2,$$

with

$$\gamma_1(a,b)||a - b||_\infty^2 + \gamma_2(a,b)||\phi(t) - \phi'(t)||_\infty^2 \leq \gamma(a,b)||a - b||_\infty^2 \text{ where } ||a(t)||_\infty = \sup\{|a(t)| : t \in P\}.$$

(H_{02}) For an increasing function $\mathcal{F} \in \mathfrak{F}$, we have

$$\tau + \mathcal{F}\left(e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\gamma(a,b)||a - b||_\infty ds\right) \leq \mathcal{F}(||a - b||_\infty^2).$$

Theorem 4.1. *Suppose that the hypotheses* (H_{01}) *and* (H_{02}) *hold. Consequently, the initial value problem (4.1) possesses a unique solution.*

Proof. Define a self-mapping $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ such that

$$\mathfrak{T}(a(t)) = a_0 e^{\frac{\beta}{\alpha}t} + e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\phi(s)ds.$$

To prove the existence of solutions, it is enough to deduce that $\mathfrak{T}$ is an orthogonal $\mathcal{F}$-contraction. In this process, first we prove that $\mathfrak{T}$ is O-preserving.

Since $a_0 \geq 1$ and $e^{\frac{\beta}{\alpha}t} \geq 1$, one writes $\mathfrak{T}(a(t)) \geq 1$. Recall that $a(t) \perp b(t)$, so we have $a(t)b(t) \geq a(t)$. We deduce that $\mathfrak{T}(a(t))\mathfrak{T}(b(t)) \geq \mathfrak{T}(a(t))$. Hence, $\mathfrak{T}$ is O-preserving.

Now, we show that the SMS $(\mathfrak{U}, \sigma^*, \perp)$ is O-complete. Let $\{a_n\}$ be a Cauchy O-sequence in $\mathfrak{U}$. In this regard, one has

$$a_n(t)a_{n+1}(t) \geq a_n(t).$$

Since $a_n(t) > 0$ for all $n \geq 1$, there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k}(t) \geq 1$ for all $k \geq 1$, which converges to $a(t) \geq 1$. The sequence $\{a_n\}$ is Cauchy, so it will converge to $a(t) \geq 1 > 0$. Hence, the SMS $(\mathfrak{U}, \sigma^*, \perp)$ is O-complete. Now, from the contraction condition, we attain

$$\tau + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) = \tau + \mathcal{F}\left(\sup_{t\in P}\left(e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|\psi(s, a(s), \phi(s)) - \psi'(s, b(s), \phi'(s))|ds\right)^2\right).$$

From property $(P1)$ and hypothesis (H_{01}), we obtain

$$\begin{aligned}\tau + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) &\leq \tau + \mathcal{F}\left(\sup_{t\in P} e^{\frac{-\beta}{2\alpha}t}\int_0^t (e^{\frac{\beta}{\alpha}s}\gamma_1(a,b)\|a-b\|_\infty^2 + \gamma_2(a,b)\|\phi(t) - \phi'(t)\|_\infty^2)ds\right)\\ &\leq \tau + \mathcal{F}\left(\sup_{t\in P} e^{\frac{-\beta}{2\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\gamma(a,b)\|a-b\|_\infty^2 ds\right).\end{aligned}$$

Taking $0 < \tau < 1$ and using hypothesis (H_{02}), we attain

$$\leq \tau + \mathcal{F}\left(\sup_{t\in P} e^{\frac{-\beta}{2\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\gamma(a,b)\|a-b\|_\infty^2 ds\right) \leq \mathcal{F}\left(\|a-b\|_\infty^2\right),$$

which implies that

$$\begin{aligned}\tau + \mathcal{F}(\sigma^*(\mathfrak{T}(a), \mathfrak{T}(b))) &\leq \mathcal{F}\left(\|a-b\|_\infty^2\right)\\ &= \mathcal{F}(\sigma^*(a,b)).\end{aligned}$$

Hence, it is clear that $\mathfrak{T}$ satisfies all the requirements of Theorem 3.8. Therefore, a unique solution to problem 4.1 exists. □

5. Conclusions

In this manuscript, we have investigated the concept of orthogonality within super metric spaces, leading to the establishment of several fixed point theorems pertaining to orthogonal contractions and orthogonal $\mathcal{F}$-contractions. These mappings exhibit unique contraction properties while adhering to orthogonality, making them significant in various mathematical contexts. To support our theoretical contributions, we provided concrete examples that not only illustrate the application of our findings but also clarify their practical implications. Moreover, we demonstrated how these results can be applied in real-world scenarios by presenting a solution to a deformable implicit differential equation.

Author contributions

A.G.: Conceptualization, Formal analysis, Data curation, Writing original draft, Writing–review and editing; S.R.: Data curation, Formal analysis, Investigation; H.A.: Investigation, Writing–review and editing; S.M.: Investigation, Writing–review and editing, Funding. All authors have read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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