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Research article

On a Diophantine equation with four prime variables

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Abstract: Let $[\alpha]$ denote the integral part of the real number α, and let N be a sufficiently large integer. In this paper, we proved that for $1 < c < \frac{38}{29}$, almost all $n \in (N, 2N]$ can be represented as $[p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] = n$, where p_1, p_2, p_3, p_4 are prime numbers.

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1. Introduction

The Diophantine equation is a classical problem in number theory. Let $[\alpha]$ denote the integral part of the real number α, and let N be a sufficiently large integer. In 1933, Segal [1, 2] firstly studied additive problems with non-integer degrees and proved that for $c > 1$ being not an integer, there exists $k(c) > 0$ such that the Diophantine equation

$$[x_1^c] + [x_2^c] + \cdots + [x_k^c] = N \tag{1.1}$$

is solvable for $k > k(c)$. Later, Deshouillers [3] and Arkhilov and Zhitkov [4] improved the Segal's bound for $k(c)$. Laporta [5] demonstrated in 1999 that the equation

$$[p_1^c] + [p_2^c] = n \tag{1.2}$$

is solvable in primes p_1, p_2 provided that $1 < c < \frac{17}{16}$ and N is sufficiently large. Recently, the range of c in (1.2) was enlarged to $1 < c < \frac{14}{11}$ by Zhu [6].

In 1995, Laporta and Tolev [7] considered the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = n \tag{1.3}$$

with prime variables p_1, p_2, p_3. Denote the weighted number of solutions of the Eq (1.3) by

$$\mathcal{R}(n) = \sum_{[p_1^c]+[p_2^c]+[p_3^c]=n} (\log p_1)(\log p_2)(\log p_3), \tag{1.4}$$

where $N/2 < n \le N$ and N is a sufficiently large integer. They established the following asymptotic formula

$$\mathcal{R}(n) = \frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} n^{\frac{3}{c}-1} + O\left(N^{\frac{3}{c}-1}\exp\left(-\log^{\frac{1}{3}-\delta} N\right)\right)$$

for any $0 < \delta < \frac{1}{3}$ and $1 < c < \frac{17}{16}$. Afterwards, the range of c was enlarged to $1 < c < \frac{12}{11}$ by Kumchev and Nedeva [8], to $1 < c < \frac{258}{235}$ by Zhai and Cao [9], to $1 < c < \frac{137}{119}$ by Cai [10], to $1 < c < \frac{3113}{2703}$ by Li and Zhang [11], and to $1 < c < \frac{3581}{3106}$ by Baker [12].

In this paper, we shall investigate the solvability of the following Diophantine equation

$$n = [p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] \tag{1.5}$$

in prime variables p_1, p_2, p_3, p_4. Denote the weighted number of solutions of the above equation by

$$R(n) = \sum_{[p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]=n} (\log p_1)(\log p_2)(\log p_3)(\log p_4) \tag{1.6}$$

and establish the following theorem.

Theorem 1. Let N be a sufficiently large integer. Then for $1 < c < \frac{38}{29}$ and $n \in (\frac{N}{2}, N]$ but

$$O(N\exp(-\log^{\frac{1}{5}} N))$$

exceptions, we have

$$R(n) = \frac{\Gamma^4\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{4}{c}\right)} n^{\frac{4}{c}-1} + O\left(N^{\frac{4}{c}-1}\exp\left(-\log^{\frac{1}{4}} N\right)\right), \tag{1.7}$$

where the implied constant in the O-term depends only on c.

Notation. Throughout the paper, we assume that $1 < c < \frac{38}{29}$. The symbol N always denotes a sufficiently large integer. Let $\varepsilon \in \left(0, 10^{-10}(\frac{38}{29} - c)\right)$. Let p, with or without subscripts, be reserved for a prime number. We denote the fractional part of x by $\{x\}$ and the distance from x to the nearest integer by $\|x\|$. Let

$$P = N^{\frac{1}{c}}, \quad \tau = P^{1-c-\varepsilon}, \quad e(x) = e^{2\pi i x}, \quad S(\alpha) = \sum_{p\le P}(\log p)e(\alpha[p^c]),$$
$$S(\alpha, X) = \sum_{X<p\le 2X}(\log p)e(\alpha[p^c]), \quad T(\alpha, X) = \sum_{X<n\le 2X} e([n^c]\alpha).$$

2. Auxiliary lemmas

To prove Theorem 1, we need the following lemmas.

Lemma 2.1. [13, Lemma 5] Suppose that z_n is a sequence of complex numbers, then we have

$$\left|\sum_{N\le n\le 2N} z_n\right|^2 \le \left(1+\frac{N}{Q}\right)\sum_{q=0}^{Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\left(\sum_{N\le n\le 2N-q}\overline{z_n}z_{n+q}\right),$$

where $\mathrm{Re}(t)$ and $\bar{t}$ denote the real part and the conjugate of the complex number t, respectively.

Lemma 2.2. [14, (3.3.4)] Suppose that $|x|>0$ and $c>1$. Then for any exponent pair (κ,λ), $M\le a<b\le 2M$, we have

$$\sum_{a\le n\le b} e(xn^c) \ll (|x|M^c)^{\kappa}M^{\lambda-\kappa}+\frac{M^{1-c}}{|x|}.$$

Lemma 2.3. [15, Lemma 12] Suppose that t is not an integer and $H\ge 3$. Then for any $\alpha\in(0,1)$, we have

$$e(-\alpha\{t\}) = \sum_{|h|\le H} c_h(\alpha)e(ht) + O\left(\min\left(1,\frac{1}{H\|t\|}\right)\right),$$

where

$$c_h(\alpha)=\frac{1-e(-\alpha)}{2\pi i(h+\alpha)}.$$

Lemma 2.4. [13, Lemma 3] Suppose that $3<U<V<Z<X$, and $\{Z\}=\frac{1}{2}$, $X\ge 64Z^2U$, $Z\ge 4U^2$, $V^3\ge 32X$. We further assume that $F(n)$ is a complex valued function such that $|F(n)|\le 1$. Then the sum

$$\sum_{X\le n\le 2X}\Lambda(n)F(n)$$

may be decomposed into $O(\log^{10}X)$ sums, each of which either of type I:

$$\sum_{M\le m\le 2M} a(m)\sum_{N\le n\le 2N} F(mn)$$

with $N>Z$, where $a(m)\ll m^{\varepsilon}$ and $X\ll MN\ll X$, or of type II:

$$\sum_{M\le m\le 2M} a(m)\sum_{N\le n\le 2N} b(n)F(mn)$$

with $U\ll M\ll V$, where $a(m)\ll m^{\varepsilon}, b(n)\ll n^{\varepsilon}$ and $X\ll MN\ll X$.

Lemma 2.5. Let $P^{\frac{8}{11}}\ll X\ll P$, $H=X^{\frac{1}{58}}$ and $c_h(\alpha)$ denote complex numbers such that $c_h(\alpha)\ll(1+|h|)^{-1}$. Then uniformly with respect to $\alpha\in(\tau,1-\tau)$, we have

$$S_I=\sum_{|h|\sim H} c_h(\alpha)\sum_{M\le m\le 2M} a(m)\sum_{N\le n\le 2N} e\left((h+\alpha)(mn)^c\right)\ll X^{\frac{57}{58}+2\varepsilon} \tag{2.1}$$

for any $a(m)\ll m^{\varepsilon}$, where $X\ll MN\ll X$ and $M\ll X^{\frac{183}{290}}$.

Proof. We have

$$S_I \ll X^{\varepsilon} \max_{|\lambda|\in(\tau,H+1)} \sum_{M\le m\le 2M} \left| \sum_{N\le n\le 2N} e(\lambda(mn)^c) \right|. \tag{2.2}$$

For the inner sum over n in (2.2), we have

$$\begin{aligned} S_I &\ll X^{\varepsilon} \max_{|\lambda|\in(\tau,H+1)} \sum_{M\le m\le 2M} \left((|\lambda| X^c)^{\frac{1}{30}} N^{\frac{25}{30}} + \frac{N}{|\lambda| X^c} \right) \\ &\ll X^{\frac{57}{58}+2\varepsilon}, \end{aligned}$$

where Lemma 2.2 with the exponential pair $(\kappa,\lambda) = (\frac{1}{30}, \frac{26}{30})$ is used. □

Lemma 2.6. Let $P^{\frac{8}{11}} \ll X \ll P$, $H = X^{\frac{1}{58}}$, and let $c_h(\alpha)$ denote complex numbers such that $c_h(\alpha) \ll (1+|h|)^{-1}$. Then uniformly with respect to $\alpha \in (\tau, 1-\tau)$, we have

$$S_{II} = \sum_{|h|\sim H} c_h(\alpha) \sum_{M\le m\le 2M} a(m) \sum_{N\le n\le 2N} b(n) e\left((h+\alpha)(mn)^c\right) \ll X^{\frac{57}{58}+2\varepsilon} \tag{2.3}$$

for any $a(m) \ll m^{\varepsilon}$, $b(n) \ll n^{\varepsilon}$, $X \ll MN \ll X$ and $X^{\frac{1}{29}} \ll M \ll X^{\frac{83}{116}}$.

Proof. Taking $Q = X^{\frac{1}{29}}$, then $Q = o(N)$. According to Cauchy's inequality and Lemma 2.1, we get

$$|S_{II}| \ll X^{2\varepsilon} \sum_{|h|\le H} |c_h(\alpha)| \left(\frac{X^2}{Q} + \frac{X}{Q} \sum_{q\le Q} \sum_{M\le m\le 2M} \left| \sum_{N\le n\le 2N} e(f_h(m,n,q)) \right| \right)^{\frac{1}{2}}, \tag{2.4}$$

where $f_h(m,n,q) = (h+\alpha) n^c ((m+q)^c - m^c)$. Thus, it is sufficient to estimate the sum

$$S' := \sum_{N\le n\le 2N} e(f_h(m,n,q)).$$

By Lemma 2.2 with the exponential pair $(\kappa,\lambda) = (\frac{1}{6}, \frac{2}{3})$, we have

$$S' \ll \left((qHX^c M^{-1})^{\frac{1}{6}} N^{\frac{1}{2}} + \frac{X}{q\tau X^c} \right).$$

Putting the above estimate into (2.4), we can obtain

$$\begin{aligned} |S_{II}| &\ll X^{2\varepsilon} \sum_{|h|\le H} |c_h(\alpha)| \left(\frac{X^2}{Q} + \frac{X}{Q} \sum_{q\le Q} \sum_{M\le m\le 2M} \left((qHX^c M^{-1})^{\frac{1}{6}} N^{\frac{1}{2}} + \frac{X}{q\tau X^c} \right) \right)^{\frac{1}{2}} \\ &\ll X^{\frac{57}{58}+2\varepsilon}. \end{aligned}$$

Thus we complete the proof of Lemma 2.6. □

Lemma 2.7. [16, Theorem 2] Suppose $K > 1, \gamma > 0, c > 1, c \notin \mathbb{Z}$. Let $\mathfrak{A}(K; c, \gamma)$ denote the number of solutions of the inequality

$$|n_1^c + n_2^c - n_3^c - n_4^c| < \gamma,\ K < n_1, n_2, n_3, n_4 \le 2K,$$

then we have

$$\mathfrak{A}(K; c, \gamma) \ll (\gamma K^{4-c} + K^2)K^{\varepsilon}.$$

Lemma 2.8. For $1 < c < 3$ $(c \neq 2)$, we have

$$\int_0^1 |S(\alpha)|^4 d\alpha \ll (P^{4-c} + P^2)P^{\varepsilon}.$$

Proof. By a splitting argument, it is sufficient to show that

$$\int_0^1 \left| S\left(\alpha, \frac{P}{2}\right)\right|^4 d\alpha \ll (P^{4-c} + P^2)P^{\varepsilon}.$$

We have

$$\begin{aligned} & \int_0^1 \left| S\left(\alpha, \frac{P}{2}\right)\right|^4 d\alpha \\ = & \sum_{\frac{P}{2} < p_1, p_2, p_3, p_4 \le P} (\log p_1) \cdots (\log p_4) \int_0^1 e\left(([p_1^c] + [p_2^c] - [p_3^c] - [p_4^c])\alpha\right) d\alpha \\ = & \sum_{\substack{\frac{P}{2} < p_1, p_2, p_3, p_4 \le P \\ [p_1^c] + [p_2^c] = [p_3^c] + [p_4^c]}} (\log p_1) \cdots (\log p_4) \ll (\log P)^4 \sum_{\substack{\frac{P}{2} < n_1, n_2, n_3, n_4 \le P \\ [n_1^c] + [n_2^c] = [n_3^c] + [n_4^c]}} 1. \end{aligned}$$

If $[n_1^c] + [n_2^c] = [n_3^c] + [n_4^c]$, we can obtain

$$|n_1^c + n_2^c - n_3^c - n_4^c| = |\{n_1^c\} + \{n_2^c\} - \{n_3^c\} - \{n_4^c\}| \le 2.$$

From Lemma 2.7, we have

$$\int_0^1 |S(\alpha)|^4 d\alpha \ll (\log P)^4 \cdot \mathfrak{A}\left(\frac{P}{2}; c, 2\right) \ll (P^{4-c} + P^2)P^{\varepsilon},$$

which completes the proof of Lemma 2.8. □

3. The estimation of $S(\alpha)$

Lemma 3.1. For $\tau \le \alpha \le 1 - \tau$, we have

$$S(\alpha) \ll P^{\frac{57}{58} + 2\varepsilon}.$$

Proof. Throughout the proof of this lemma, we write $H = X^{\frac{1}{58}}$ for convenience. We need only to show that the estimation

$$\sum_{X < n \le 2X} \Lambda(n) e(\alpha[n^c]) \ll X^{\frac{57}{58} + 2\varepsilon} \tag{3.1}$$

holds for $P^{\frac{8}{11}} \le X \le P$ and $\tau \le \alpha \le 1-\tau$. By Lemma 2.3, we can obtain

$$\begin{aligned}\sum_{X<n\le 2X}\Lambda(n)e(\alpha[n^c]) &= \sum_{|h|\le H}c_h(\alpha)\sum_{X<n\le 2X}\Lambda(n)e((h+\alpha)n^c)\\ &\quad +O\left((\log X)\sum_{X<n\le 2X}\min\left(1,\frac{1}{H\|n^c\|}\right)\right).\end{aligned} \tag{3.2}$$

By the expansion

$$\min\left(1,\frac{1}{H\|n^c\|}\right)=\sum_{h=-\infty}^{\infty}a_he(hn^c) \tag{3.3}$$

with

$$|a_h|\le \min\left(\frac{\log 2H}{H},\frac{1}{|h|},\frac{H}{h^2}\right), \tag{3.4}$$

we get

$$\begin{aligned}\sum_{X<n\le 2X}\min\left(1,\frac{1}{H\|n^c\|}\right) &\le \sum_{h=-\infty}^{\infty}a_h\left|\sum_{X<n\le 2X}e(hn^c)\right|\\ &\ll \frac{X\log 2H}{H}+\sum_{1\le h\le H}\frac{1}{h}\left((hX^c)^{\frac{1}{2}}+\frac{X}{hX^c}\right)\\ &\quad +\sum_{h>H}\frac{H}{h^2}\left((hX^c)^{\frac{1}{2}}+\frac{X}{hX^c}\right)\\ &\ll X^{\frac{57}{58}}\log X,\end{aligned} \tag{3.5}$$

where we estimate the sum over n by Lemma 2.2 with the exponent pair $(\kappa,\lambda)=\left(\frac{1}{2},\frac{1}{2}\right)$.

Taking $U=X^{\frac{1}{29}}$, $V=X^{\frac{1}{3}}$, $Z=[X^{\frac{14}{29}}]+\frac{1}{2}$. By Lemma 2.4 with $F(n)=e((h+\alpha)n^c)$, we get that the sum

$$\sum_{|h|\le H}c_h(\alpha)\sum_{X<n\le 2X}\Lambda(n)e((h+\alpha)n^c)$$

can be represented as $O(\log^{10}X)$ sums, either of type I

$$S_I'=\sum_{|h|\le H}c_h(\alpha)\sum_{M<m\le 2M}a(m)\sum_{N<n\le 2N}e((h+\alpha)(mn)^c),\ N>Z,$$

or type II

$$S_{II}'=\sum_{|h|\le H}c_h(\alpha)\sum_{M<m\le 2M}a(m)\sum_{N<n\le 2N}b(n)e((h+\alpha)(mn)^c),\ U<M<V.$$

By Lemma 2.5, we get

$$S_I'\ll X^{\frac{57}{58}+2\varepsilon}. \tag{3.6}$$

By Lemma 2.6, we get

$$S_{II}'\ll X^{\frac{57}{58}+2\varepsilon}. \tag{3.7}$$

From (3.6) and (3.7), we can obtain

$$\sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} \Lambda(n) e((h+\alpha)n^c) \ll X^{\frac{57}{58}+2\varepsilon}. \tag{3.8}$$

From (3.2), (3.5), and (3.8), we complete the proof of Lemma 3.1. □

Lemma 3.2. For $\alpha \in (0,1)$, we have

$$T(\alpha, X) \ll X^{\frac{4+c}{7}} \log X + \frac{1}{\alpha X^{c-1}}.$$

Proof. Let $H' = X^{\frac{3-c}{7}}$. By Lemma 2.3, we obtain

$$T(\alpha, X) = \sum_{|h|\le H'} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) + O\left((\log X) \sum_{X<n\le 2X} \min\left(1, \frac{1}{H' \parallel n^c \parallel}\right)\right).$$

From (3.3) and (3.4), we have

$$\begin{aligned}
\sum_{X<n\le 2X} \min\left(1, \frac{1}{H' \parallel n^c \parallel}\right) &\le \sum_{h=-\infty}^{\infty} |a_h| \left| \sum_{X<n\le 2X} e(hn^c) \right| \\
&\ll \frac{X \log 2H'}{H'} + \sum_{1\le h\le H'} \frac{1}{h} \left((hX^c)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{X}{hX^c} \right) \\
&\quad + \sum_{h\ge H'} \frac{H'}{h^2} \left((hX^c)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{X}{hX^c} \right) \\
&\ll X^{\frac{c+4}{7}} \log X,
\end{aligned} \tag{3.9}$$

where we used Lemma 2.2 with the exponent pair $(\kappa, \lambda) = (\frac{1}{6}, \frac{2}{3})$. Similarly, we have

$$\begin{aligned}
&\sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) \\
=\ & c_0(\alpha) \sum_{X<n\le 2X} e(\alpha n^c) + \sum_{1\le |h|\le H'} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) \\
\ll\ & X^{\frac{c+4}{7}} \log X + \frac{X}{\alpha X^c}.
\end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we complete the proof of Lemma 3.2. □

Lemma 3.3. Let $P^{\frac{8}{11}} \ll X \ll P$, we have

$$\int_{\tau}^{1-\tau} |S(\alpha)|^4 d\alpha \ll P^{\frac{431-87c}{116}+\varepsilon} + P^{\frac{2734-377c}{812}+\varepsilon} + P^{\frac{36+2c}{14}+\varepsilon}. \tag{3.11}$$

Proof. Let $\Theta = (\tau, 1-\tau)$ and $K_l(\alpha) = \overline{S(\alpha)}|S(\alpha)|^l$ ($l = 1$ or 2). Then we have

$$\left| \int_{\Theta} S(\alpha) K_l(\alpha) d\alpha \right|$$

$$\ll \quad (\log N) \max_{P^{\frac{8}{11}} \le X \le P} \left| \int_{\Theta} S(\alpha, X) K_l(\alpha) d\alpha \right| + P^{\frac{8}{11}} (\log P) \int_{\Theta} |K_l(\alpha)| \, d\alpha. \tag{3.12}$$

In addition, we get

$$\begin{aligned}
\left| \int_{\Theta} S(\alpha, X) K_l(\alpha) d\alpha \right| &= \left| \sum_{X \le p \le 2X} \log p \int_{\Theta} e(\alpha [p^c]) K_l(\alpha) d\alpha \right| \\
&\le \sum_{X \le p \le 2X} \log p \left| \int_{\Theta} e(\alpha [p^c]) K_l(\alpha) d\alpha \right| \\
&\le (\log X) \sum_{X \le n \le 2X} \left| \int_{\Theta} e(\alpha [n^c]) K_l(\alpha) d\alpha \right|. \qquad (3.13)
\end{aligned}$$

By (3.13) and Cauchy's inequality, we have

$$\begin{aligned}
&\left| \int_{\Theta} S(\alpha, X) K_l(\alpha) d\alpha \right|^2 \le X \left(\log^2 X \right) \sum_{X \le n \le 2X} \left| \int_{\Theta} e(\alpha [n^c]) K_l(\alpha) d\alpha \right|^2 \\
\le \quad & X \left(\log^2 X \right) \int_{\Theta} \overline{K_l(\beta)} d\beta \int_{\Theta} K_l(\alpha) T(\alpha - \beta, X) d\alpha \\
\le \quad & X \left(\log^2 X \right) \int_{\Theta} |K_l(\beta)| \, d\beta \int_{\Theta} |K_l(\alpha)| |T(\alpha - \beta, X)| \, d\alpha. \qquad (3.14)
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_{\Theta} |K_l(\alpha) T(\alpha - \beta, X)| \, d\alpha \\
\ll \quad & \int_{\substack{\Theta \\ |\alpha - \beta| \le X^{-c}}} |K_l(\alpha) T(\alpha - \beta, X)| \, d\alpha + \int_{\substack{\Theta \\ |\alpha - \beta| > X^{-c}}} |K_l(\alpha) T(\alpha - \beta, X)| \, d\alpha. \qquad (3.15)
\end{aligned}$$

For the first integral in (3.15), we use the trivial bound $T(\alpha, X) \le X$ and get

$$\begin{aligned}
&\int_{\substack{\Theta \\ |\alpha - \beta| \le X^{-c}}} |K_l(\alpha) T(\alpha - \beta, X)| \, d\alpha \\
\ll \quad & X \max_{\alpha \in \Theta} |K_l(\alpha)| \int_{|\alpha - \beta| \le X^{-c}} 1 d\alpha \ll X^{1-c} \max_{\alpha \in \Theta} |K_l(\alpha)|. \qquad (3.16)
\end{aligned}$$

For the second integral in (3.15), we use Lemma 3.2 and get

$$\begin{aligned}
&\int_{\substack{\Theta \\ |\alpha - \beta| > X^{-c}}} |K_l(\alpha)| |T(\alpha - \beta, X)| \, d\alpha \\
\ll \quad & \int_{\substack{\Theta \\ |\alpha - \beta| > X^{-c}}} |K_l(\alpha)| \left(X^{\frac{4+c}{7}} \log X + \frac{X^{1-c}}{|\alpha - \beta|} \right) d\alpha \\
\ll \quad & X^{\frac{4+c}{7}} \log X \int_{\Theta} |K_l(\alpha)| \, d\alpha + X^{1-c} \max_{\alpha \in \Theta} |K_l(\alpha)| \int_{X-c < |\alpha - \beta| \le 2} \frac{1}{|\alpha - \beta|} d\alpha \\
\ll \quad & X^{\frac{4+c}{7}} (\log X) \int_{\Theta} |K_l(\alpha)| \, d\alpha + X^{1-c} \max_{\alpha \in \Theta} |K_l(\alpha)| \log X. \qquad (3.17)
\end{aligned}$$

From (3.12) and (3.14)–(3.17), we obtain

$$\left|\int_{\Theta} S(\alpha)K_l(\alpha)d\alpha\right|^2 \ll X^{\frac{11+c}{7}+\varepsilon}\left(\int_{\Theta}|K_l(\alpha)|\,d\alpha\right)^2 + P^{\frac{16}{11}+\varepsilon}\left(\int_{\Theta}|K_l(\alpha)|\,d\alpha\right)^2 + X^{2-c+\varepsilon}\max_{\alpha\in\Theta}|K_l(\alpha)|\int_{\Theta}|K_l(\alpha)|\,d\alpha. \tag{3.18}$$

By applying Lemma 3.1 and the bound

$$\int_{\tau}^{1-\tau}|S^2(\alpha)|d\alpha \ll \int_0^1 |S^2(\alpha)|d\alpha \ll P\log^2 P,$$

we can deduce from (3.18) with $l = 1$ that

$$\begin{aligned}
\int_{\Theta}|S(\alpha)|^3 d\alpha &= \int_{\Theta} S(\alpha)K_1(\alpha)d\alpha \\
&\ll X^{1-\frac{c}{2}+\varepsilon}\max_{\alpha\in\Theta}|S(\alpha)|\left(\int_0^1 |S(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\
&\quad + \left(X^{\frac{11+c}{14}+\varepsilon} + P^{\frac{8}{11}+\varepsilon}\right)\left(\int_0^1 |S(\alpha)|^2 d\alpha\right) \\
&\ll X^{1-\frac{c}{2}}P^{\frac{57}{58}+\frac{1}{2}+\varepsilon} + X^{\frac{11+c}{14}}P^{1+\varepsilon} + P^{\frac{19}{11}+\varepsilon} \\
&\ll P^{\frac{144-29c}{58}+\varepsilon} + P^{\frac{25+c}{14}+\varepsilon}.
\end{aligned} \tag{3.19}$$

Then it follows from (3.18) with $r = 2$ and (3.19) that

$$\begin{aligned}
\int_{\Theta}|S(\alpha)|^4 d\alpha &= \int_{\Theta} S(\alpha)K_2(\alpha)d\alpha \\
&\ll X^{1-\frac{c}{2}+\varepsilon}\max_{\alpha\in\Theta}|S(\alpha)|^{\frac{3}{2}}\left(\int_{\Theta}|S(\alpha)|^3 d\alpha\right)^{\frac{1}{2}} + \left(X^{\frac{11+c}{14}+\varepsilon} + P^{\frac{8}{11}+\varepsilon}\right)\left(\int_{\Theta}|S(\alpha)|^3 d\alpha\right) \\
&\ll P^{\frac{287}{116}-\frac{c}{2}+\varepsilon}\left(P^{\frac{144-29c}{58}} + P^{\frac{25+c}{14}}\right)^{\frac{1}{2}} + P^{\frac{11+c}{14}+\varepsilon}\left(P^{\frac{144-29c}{58}} + P^{\frac{25+c}{14}}\right) \\
&\ll P^{\frac{431-87c}{116}+\varepsilon} + P^{\frac{1327-174c}{406}+\varepsilon} + P^{\frac{36+2c}{14}+\varepsilon},
\end{aligned}$$

which completes the proof of Lemma 3.3. □

4. Proof of the theorem

By the definition of $R(n)$, we have

$$\begin{aligned}
R(n) &= \int_{-\tau}^{1-\tau} S^4(\alpha)e(-\alpha n)d\alpha \\
&= \int_{-\tau}^{\tau} S^4(\alpha)e(-\alpha n)d\alpha + \int_{\tau}^{1-\tau} S^4(\alpha)e(-\alpha n)d\alpha \\
&= R_1(N) + R_2(N).
\end{aligned} \tag{4.1}$$

4.1. Evaluation of $R_1(N)$

In this subsection, we shall prove the following equation

$$R_1(n) = \frac{\Gamma^4(1+\frac{1}{c})}{\Gamma(\frac{4}{c})} n^{\frac{4}{c}-1} + O\left(N^{\frac{4}{c}-1} \exp(-(\log n)^{\frac{1}{4}})\right). \tag{4.2}$$

Define

$$G(\alpha) = \sum_{m \le N} \frac{1}{c} m^{\frac{1}{c}-1} e(m\alpha),$$
$$B_1(n) = \int_{-\tau}^{\tau} G^4(\alpha) e(-n\alpha) d\alpha,$$
$$B(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} G^4(\alpha) e(-n\alpha) d\alpha.$$

Then

$$R_1(n) = (R_1(n) - B_1(n)) + (B_1(n) - B(n)) + B(n). \tag{4.3}$$

As is shown in Theorem 2.3 of Vaughan [17], we can obtain

$$B(n) = \frac{\Gamma^4(1+\frac{1}{c})}{\Gamma(\frac{4}{c})} P^{4-c} + O(P^{3-c}). \tag{4.4}$$

From Lemma 2.8 of Vaughan [17], for $\nu > 0$, we have

$$B_1(n) - B(n) \ll \int_{\tau}^{\frac{1}{2}} |G(\alpha)|^4 d\alpha \ll \int_{\tau}^{\frac{1}{2}} \alpha^{-\frac{4}{c}} d\alpha \ll \tau^{1-\frac{4}{c}} \ll P^{4-c-\nu}. \tag{4.5}$$

Next we estimate $|R_1(n) - B_1(n)|$. Let $W_1(N)$ denote the set of integers n in the interval $(\frac{N}{2}, N]$ such that

$$|R_1(n) - B_1(n)| = \left| \int_{-\tau}^{\tau} (S^4(\alpha) - G^4(\alpha)) e(-n\alpha) d\alpha \right| \ge \frac{n^{\frac{4}{c}-1}}{\log n}. \tag{4.6}$$

We take $W_1 = |W_1(N)|$, and choose the complex number $\varphi_1(n)$ satisfying $|\varphi_1(n)| = 1$ and

$$\varphi_1(n) \int_{-\tau}^{\tau} (S^4(\alpha) - G^4(\alpha)) e(-n\alpha) d\alpha = \left| \int_{-\tau}^{\tau} (S^4(\alpha) - G^4(\alpha)) e(-n\alpha) d\alpha \right|. \tag{4.7}$$

Thus, for $n \in W_1(N)$, by (4.6) and (4.7), we can obtain

$$\frac{N^{\frac{4}{c}-1} W_1}{\log N} \ll \int_{-\tau}^{\tau} (S^4(\alpha) - G^4(\alpha)) L(\alpha) d\alpha, \tag{4.8}$$

where $L(\alpha) = \sum_{n \in W_1(N)} \varphi_1(n) e(-\alpha n)$. By Lemma 2.8 of Vaughan [17], we have

$$G(\alpha) \ll \min(N^{\frac{1}{c}}, |\alpha|^{-\frac{1}{c}}).$$

Thus,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}}|G(\alpha)|^4d\alpha &\ll \int_0^{\frac{1}{2}}\min(N^{\frac{1}{c}},|\alpha|^{-\frac{1}{c}})^4d\alpha\\ &\ll \int_0^{\frac{1}{N}}N^{\frac{4}{c}}d\alpha+\int_{\frac{1}{N}}^{\frac{1}{2}}\alpha^{-\frac{4}{c}}d\alpha\\ &\ll N^{\frac{4}{c}-1}\ll P^{4-c}. \end{aligned}\tag{4.9}$$

For $|\alpha|\le\tau$, we have

$$S(\alpha)=\sum_{p\le P}(\log p)e(p^c\alpha)+O(\tau P)=S^*(\alpha)+O(\tau P).\tag{4.10}$$

Now we consider the upper bound of $|S(\alpha)-G(\alpha)|$ under the condition $|\alpha|\le\tau$. By (4.10), we have

$$\begin{aligned}S(\alpha) &= \sum_{n\le P}\Lambda(n)e(n^c\alpha)+O(P^{\frac{1}{2}})+O(\tau P)\\ &= \sum_{n\le P}\Lambda(n)e(n^c\alpha)+O(P^{1-\varepsilon}).\end{aligned}\tag{4.11}$$

For $|\alpha|\le\tau$ and $u\ge 2$, by Lemma 1.2 of Ivić [18], we have

$$\sum_{1<m\le u}e(m\alpha)=\int_1^u e(t\alpha)dt+O(1).$$

According to partial summation and the above identity, we can obtain

$$\begin{aligned}\sum_{n\le P}\Lambda(n)e(n^c\alpha) &= \int_1^P e(t^c\alpha)dt+O(P\exp(-(\log P)^{\frac{1}{3}}))\\ &= \int_1^N \frac{1}{c}u^{\frac{1}{c}-1}e(u\alpha)du+O(P\exp(-(\log P)^{\frac{1}{3}}))\\ &= \sum_{m\le N}\frac{1}{c}m^{\frac{1}{c}-1}e(m\alpha)+O(P\exp(-(\log P)^{\frac{1}{3}}))\\ &= G(\alpha)+O(P\exp(-(\log P)^{\frac{1}{3}})).\end{aligned}\tag{4.12}$$

By (4.11) and (4.12), we have

$$\sup_{|\alpha|\le\tau}|S(\alpha)-G(\alpha)|\ll P\exp(-(\log P)^{\frac{1}{3}}).\tag{4.13}$$

We use Hölder's inequality, Lemma 2.8, (4.9), (4.13), and the obvious bound $\int_0^1|L(\alpha)|^4d\alpha\ll W_1^3$ and get

$$\int_{-\tau}^{\tau}(S^4(\alpha)-G^4(\alpha))L(\alpha)d\alpha$$

$$
\begin{aligned}
&\ll \int_{-\tau}^{\tau}|S(\alpha)-G(\alpha)|(|S(\alpha)|^3+|G(\alpha)|^3)|L(\alpha)|d\alpha \\
&\ll \sup_{|\alpha|\le\tau}|S(\alpha)-G(\alpha)|\left(\int_{-\tau}^{\tau}|S(\alpha)|^4+|G(\alpha)|^4 d\alpha\right)^{\frac{3}{4}}\left(\int_{-\tau}^{1-\tau}|L(\alpha)|^4 d\alpha\right)^{\frac{1}{4}} \\
&\ll N^{\frac{4}{c}-\frac{3}{4}}\exp(-\log^{\frac{1}{4}}N)W_1^{\frac{3}{4}}. \qquad (4.14)
\end{aligned}
$$

From (4.8) and (4.14), we obtain

$$\frac{N^{\frac{4}{c}-1}W_1}{\log N} \ll N^{\frac{4}{c}-\frac{3}{4}}\exp(-\log^{\frac{1}{4}}N)W_1^{\frac{3}{4}},$$

which yields that

$$W_1 \ll N\exp(-\log^{\frac{1}{5}}N). \qquad (4.15)$$

4.2. Evaluation of $R_2(N)$

In this subsection, let $W_2(N)$ denote the set of integers n in the interval $(\frac{N}{2}, N]$ such that

$$|R_2(n)| = \left|\int_{\tau}^{1-\tau}S^4(\alpha)e(-n\alpha)d\alpha\right| \gg \frac{n^{\frac{4}{c}-1}}{\log n}. \qquad (4.16)$$

By Bessel's inequality and taking $W_2 = |W_2(N)|$, we have

$$W_2\left(\frac{N^{\frac{4}{c}-1}}{\log N}\right)^2 \ll \sum_{n\in W_2(N)}\left|\int_{\tau}^{1-\tau}S^4(\alpha)e(-n\alpha)d\alpha\right|^2 \ll \int_{\tau}^{1-\tau}|S^8(\alpha)|d\alpha. \qquad (4.17)$$

Since $1 < c < \frac{38}{29}$ and $\varepsilon \in (0, 10^{-10}(\frac{38}{29}-c))$, we can deduce from (4.17) and Lemma 3.3 that

$$
\begin{aligned}
W_2 &\ll N^{2-\frac{8}{c}+\varepsilon}P^{\frac{57}{58}\times 4+\varepsilon}\left(P^{\frac{431-87c}{116}+\varepsilon}+P^{\frac{1327-174c}{406}+\varepsilon}+P^{\frac{36+2c}{14}+\varepsilon}\right) \\
&\ll P^{c-\varepsilon} \ll N^{1-\varepsilon}. \qquad (4.18)
\end{aligned}
$$

Let $\mathcal{W}(N)$ denote the number of integers n in the interval $(\frac{N}{2}, N]$ such that

$$\left|R(n)-\frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{4}{c}\right)}n^{\frac{4}{c}-1}\right| \ge \frac{n^{\frac{4}{c}-1}}{\log n}. \qquad (4.19)$$

Then, by (4.1)–(4.4), (4.15), and (4.18), we have

$$\left|R(n)-\frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)}n^{\frac{4}{c}-1}\right| \le |R_1(n)-B_1(n)|+|R_2(n)|+O\left(n^{\frac{4}{c}-1-\varepsilon}\right).$$

From the above formula and (4.19), we can get

$$\mathcal{W}(N) \le W_1+W_2 \ll N\exp(-\log^{\frac{1}{5}}N).$$

Thus we complete the proof of Theorem 1.

5. Conclusions

In this paper, we proved that almost all $n \in (N, 2N]$ can be represented as $n = [p_1^c]+[p_2^c]+[p_3^c]+[p_4^c]$, where p_1, p_2, p_3, p_4 are prime numbers and $[x]$ denotes the integer part of x. Our method also yields an asymptotic formula for the number of representations of these n.

Author contributions

J. Huang: Conceptualization, formal analysis, investigation, resources, writing—original draft, writing—review and editing; W. G. Zhai: Conceptualization, investigation, writing—original draft, writing—review and editing; D. Y. Zhang: Data curation, funding acquisition, methodology, project administration, supervision, validation, visualization, writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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