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Research article

An analysis on the boundary control for nonlinear integro-differential evolution systems with impulsive effects and time delays via fixed point theorems

Kamalendra Kumar[1], Rohit Patel[2], Mohammad Sajid[3,*] and Rakesh Kumar[4]

[1] Department of Basic Science, Shri Ram Murti Smarak College of Engineering and Technology, Bareilly, India
[2] Department of Mathematics, Government P.G. College Bisalpur, Pilibhit, India
[3] Department of Mechanical Engineering, College of Engineering, Qasim University, Saudi Arabia
[4] Department of Mathematics, Hindu College, Moradabad (Affiliated to MJP Rohilkhand University, Bareilly), India

* **Correspondence:** Email: msajd@qu.edu.sa.

Abstract: This study investigated the boundary controllability of nonlinear impulsive integro-differential evolution systems (NIIESs) with time-varying delays within Banach spaces. Two classes of NIIESs were considered, and sufficient conditions for their controllability were established using fixed point theorems and semigroup theory. For the first class, Schaefer's fixed point theorem was employed in combination with compact semigroup theory, whereas for the second class, Schauder's fixed point theorem was utilized. The research defined essential hypotheses and mathematical structures to ensure the robustness and applicability of the results. Illustrative examples were provided to confirm the applicability and effectiveness of the developed theoretical framework. This work significantly contributes to the study of partial functional integro-differential equations in nonlinear systems, particularly systems influenced by impulsive effects and time delays, addressing gaps in the existing literature.

Keywords: boundary controllability; impulsive systems; nonlinear systems; integro-differential equations; fixed point theory; time-delay systems
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1. Introduction

Physical and biological systems, such as blood flow, heartbeats, bursting rhythms in medicine, and population dynamics, often exhibit abrupt transitions. Similarly, in economic models, price fluctuations and market behavior may lead to sudden changes in the system's state. These phenomena are typically difficult to model using only traditional continuous differential equations because they experience sudden jumps or impulsive effects that traditional equations cannot represent. Thus, these systems require more advanced models like impulsive differential equations or evolution systems with impulses. Readers may refer to the works in [1–4].

In the context of control theory, the goal is to design systems that can influence and regulate the behavior of another system to meet a desired objective. This can range from controlling a simple temperature regulation system to more complex control problems such as the regulation of space exploration systems. Nonlinear systems, which are common in real-world scenarios, present unique challenges due to their complex dynamics. The study of controllability, the ability to steer the system's state, from any initial condition to a desired final state, is crucial in understanding and designing control systems, especially for infinite-dimensional systems.

Control theory, an integral part of engineering and applied mathematics, focuses on the principles and methodologies required to design and analyze control systems. "System control" refers to the process of modifying or directing a system's behavior to achieve specific desired outcomes. This often requires designing devices that interact with the target system to regulate its performance. Control theory has found applications ranging from simple mechanisms, such as temperature controllers, to highly sophisticated technologies like space exploration systems [5].

Real-world systems are often inherently nonlinear, underscoring the importance of investigating nonlinear systems. Controllability, a crucial concept in control theory, has been widely studied for both finite and infinite-dimensional systems. The seminal work of Kalman [6] introduced controllability in linear systems. Nonlinear integro-differential systems, often arising as abstract formulations of partial integro-differential equations, are used to model complex phenomena such as viscoelastic behavior. Among the mathematical tools available for analyzing controllability in nonlinear systems, fixed point theory stands out as particularly effective. Nonlinear systems are more representative of real-world phenomena because most systems in nature, economics, and engineering are nonlinear in nature. Controllability in nonlinear integro-differential systems has been a topic of extensive research. These systems, which model phenomena like viscoelastic behavior or population dynamics, involve both differential and integral components. Fixed point theory has been effectively used to study the controllability of such systems, with key contributions from researchers like Diallo et al. [7] and Chalishajar et al. [8]. Additionally, various researchers have derived sufficient conditions for controllability in impulsive nonlinear systems [9–11].

The concept of approximate controllability, which deals with the ability to drive the system close to a desired state within a tolerance, has also been explored in nonlinear systems. Researchers have used analytic semigroups and fractional power operators to investigate this concept. These methodologies provide an understanding of the system's dynamics under different conditions, including the neutral integro-differential systems that arise in various contexts. Several researchers [12, 13] explored sufficient conditions for approximate controllability in linear and nonlinear integro-differential systems. For example, Huang and Fu [14] analyzed the approximate controllability of semilinear neutral integro-differential systems using analytic semigroups, fractional power operators, and resolvent theory. Similarly, Lian et al. [15] applied C-semigroups to study fractional linear evolution systems, while Gunasekar et al. [16] investigated Volterra-Fredholm integro-differential equations with neutral and non-integer orders.

The problem of boundary controllability is a very important issue in various applications in which control acts on the interior of the domain or the boundary of the domain sequentially, see, for instance, [17–19]. Lasiecka and Triggiani [20, 21] introduced abstract boundary control models to handle partial differential equations. A comprehensive regularity theory for parabolic systems was initially developed by Lasiecka [22] and after that it was elaborated by the authors [23, 24], while Fattorini [25] developed semigroup techniques for modeling boundary input problems in first- and second-order partial differential equations. Balakrishnan [26] later improved this approach, introducing mild solutions for boundary control models by utilizing semigroup theory. Barbu [27] made significant contributions to boundary-distributed control systems, and further developments on sufficient conditions for boundary control problems were discussed in [28–31].

More recent studies have extended boundary controllability to complex scenarios. For example, Sutrima et al. [32] analyzed the well-posedness and approximate controllability of mixed boundary control problems for non-autonomous systems using C_0-quasi semigroups. The boundary controllability for a cascade system coupling a bilaplacian operator of heat equation are investigated in [33]. Wang and Tian [34] investigated the boundary controllability of time-fractional nonlinear Korteweg-de Vries equations using semigroup theory. Kumar et al. [35] and Radhakrishnan et al. [36] employed fixed point theorems to study boundary controllability in neutral systems. Furthermore, several authors [37, 38] explored stochastic equations and regional controllability for systems such as coupled hyperbolic equations, parabolic systems, and time-fractional diffusion processes. Li et al. [39] and Hu et al. [40] explored impulsive differential systems, focusing on the existence and convergence of solutions. They applied advanced methods like monotone iterative techniques, quasilinearization, and refined boundary conditions to establish rigorous analytical results.

This study is motivated by the growing need to address the boundary controllability of nonlinear impulsive integro-differential evolution systems (NIIESs), a class of systems often overlooked in existing literature [17–19, 41–43]. By leveraging fixed point theorems (Schaefer and Schauder) and compact semigroup theory, we investigate boundary control for two distinct classes of NIIESs with time-varying delays (i.e., time-varying control systems).

The main contributions of this manuscript are:

1) Developing novel and significant results on the boundary controllability of nonlinear impulsive integro-differential evolution systems (NIIESs) with time-varying delays. Unlike prior works that mainly address standard impulsive or delay-free systems, this study incorporates both impulsive effects and time-varying delays into the boundary control framework. By applying fixed point theorems in combination with semigroup theory, our results extend and generalize existing controllability conditions found in the literature, addressing gaps not previously covered.
2) Employing Schaefer's fixed point theorem for the first class of NIIESs and Schauder's fixed point theorem for the second class, combined with semigroup theory.
3) Introducing hypotheses $(H_1) - (H_9)$ for analyzing the first class and $(H_{10}) - (H_{12})$ for the second class, providing a framework for deriving controllability results.
4) Enhancing existing findings by addressing systems not previously studied in the literature.

The paper is structured into the following sections: Section 2 introduces preliminary concepts and assumptions essential for understanding the mathematical framework used in the study. Sections 3 and

4 provide the derivation of sufficient conditions for the two classes of boundary control systems under consideration. Section 5 offers illustrative examples to demonstrate the practical application of the theoretical results. Section 6 concludes the paper and outlines directions for future research, including potential extensions to more complex models or systems.

This paper addresses an important gap in the existing literature by providing new results for the boundary controllability of nonlinear impulsive integro-differential evolution systems (NIIESs) with time-varying delays. The use of fixed point theory and semigroup theory offers a robust mathematical framework for analyzing these complex systems, and the theoretical results are supported by illustrative examples. The study opens the door for further research in this area, particularly in the context of impulsive systems and their control.

2. Preliminaries

Throughout the paper, Y and U are assumed to be two Banach spaces along with the norms $\|.\|$ and $|.|$ consecutively. Also, μ is a linear, closed, and densely defined operator together with $dom(\mu) \subseteq Y$ and $R(\mu) \subseteq Y$. Moreover, α is termed as a linear operator along with $dom(\alpha) \subseteq Y$ and range $R(\alpha) \subseteq X$, which is a Banach space with norm $\|.\|_X$.

First, we examine the boundary control nonlinear system of the type

$$\begin{cases} x'(\zeta) = \mu x(\zeta) + f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big), \\ \qquad\qquad \zeta \in J = [0, a], \ \zeta \neq \zeta_k, \\ \alpha x(\zeta) = B_1 u(\zeta), \\ x(0) = x_0, \\ \Delta x|_{\zeta=\zeta_k} = I_k(x(\zeta_k^-)), k = 1, 2, ..., m, \end{cases} \tag{2.1}$$

in which $B_1 : U \to X$ is a linear continuous operator and $[0, a]$ is a fixed interval. Also, the control function $u(.)$ is in $L^1(J, U)$, which is a Banach space of admissible control functions. The nonlinear operators $f : J \times Y^{l+1} \to Y; e : \Theta \times Y \to Y$ are continuous functions and $\Theta = \{(\zeta, s) : 0 \leq s \leq \zeta \leq a\}$. Moreover, $\xi_i : [0, a] \to [0, a], i = 1, 2, ..., l+1$, are continuous functions such that $\xi_i(\zeta) \leq \zeta, i = 1, 2, ..., l+1$. Also $\Delta x|_{\zeta=\zeta_k} = x(\zeta_k^+) - x(\zeta_k^-)$, where the characters $x(\zeta_k^+)$ and $x(\zeta_k^-)$ describe the right and left limits of $x(\zeta)$ at $\zeta = \zeta_k$, respectively, for $0 = \zeta_0 < \zeta_1 < \zeta_2 < ... < \zeta_k < \zeta_{k+1} = a$.

Throughout the paper, it is supposed that A from Y into Y is the linear operator expressed as:

$$dom(A) = \{x \in dom(\mu); \alpha x = 0\}, Ax = \mu x \text{ for } x \in dom(A).$$

We state the characterization of a compact semigroup with regard to the resolvent operators $R(\lambda : A)$ of its generator A as follows.

Lemma 2.1. *[44]. Let A denote the infinitesimal generator of a uniformly continuous semigroup $T(t)$. If the resolvent $R(\lambda, A)$ associated with A is compact for every $\lambda \in \rho(A)$ (the resolvent set of A), then the semigroup $T(t)$ is compact.*

Let A represent the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(\zeta), \zeta \geq 0\}$ defined on the Banach space Y. Suppose there exists a constant $M > 0$ such that $\|T(\zeta)\| \leq M$ for all $\zeta \geq 0$, ensuring the semigroup is uniformly bounded.

When introducing impulsive conditions into the framework of the assumed equations (2.1), it becomes necessary to extend the notation and framework. For clarity, let us define $J_0 = [0, \zeta_1]$ and $J_k = (\zeta_k, \zeta_{k+1}]$, where $k = 1, 2, \dots, m$. Furthermore, let $PC([0, a], Y)$ denote the set of piecewise continuous functions x mapping $[0, a]$ into Y such that:

1) $x(\zeta)$ is continuous for $\zeta \neq \zeta_k$.
2) $x(\zeta)$ is left-continuous at the points $\zeta = \zeta_k$.
3) The right-hand limit $x(\zeta_k^+)$ exists for $k = 1, 2, \dots, m$.

This formulation accommodates the impulsive nature of the system by allowing discontinuities at specified points while maintaining the required regularity elsewhere. In [45], $PC(J, Y)$ is specified as a Banach space with norm

$$||x||_{PC} = \sup_{\zeta \in J} ||x(\zeta)||.$$

Barbu [27] presented the following assumption and its implication in his work on boundary controllability which is also necessary in our work.

(A_1) It is essential to the operator α that it satisfies the condition, which is stated as: $dom(\mu) \subset dom(\alpha)$, and the restriction of α to $dom(\alpha)$ is called continuous relative to the graph norm of $dom(\mu)$.

(A_2) Let $B : U \to Y$ be a linear continuous operator such that $\mu B \in L(U, Y)$ and $\alpha(Bu) = B_1 u$ for all $u \in U$. Additionally, $Bu(t)$ is continuously differentiable, and $||Bu|| \leq C||B_1 u||$ for all $u \in U$, where $C > 0$ is a constant.

(A_3) For every $0 < \zeta \leq a$ and $u \in U$, we have $T(\zeta)Bu \in \mathrm{dom}(A)$, indicating that the operator A acts on $T(\zeta)Bu$ for these values of ζ and u. Additionally, we introduce a positive function $\gamma \in L^1(0, a)$ such that the operator norm $||AT(\zeta)B||_{L(U,Y)}$ is bounded above by $\gamma(\zeta)$ almost everywhere on the interval $(0, a)$. This ensures that the action of A on $T(\zeta)B$ is controlled in terms of the function γ, which belongs to the integrable space $L^1(0, a)$.

Suppose the solution of (2.1) is expressed by $x(\zeta)$. Now, by defining a function $z(\zeta) = x(\zeta) - Bu(\zeta)$ and, from assumption (A_2), we notice that $z(\zeta) \in dom(A)$. Thus, in terms of A and B, equation (2.1) is noted as

$$\begin{cases} \frac{d}{d\zeta}x(\zeta) = Az(\zeta) + \mu Bu(\zeta) + f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \\ \qquad\qquad \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big), \ \zeta \in [0, a], \zeta \neq \zeta_k, \\ x(\zeta) = z(\zeta) + Bu(\zeta), \\ x(0) = x_0, \\ \Delta x|_{\zeta = \zeta_k} = I_k(x(\zeta_k^-)), k = 1, 2, ..., m. \end{cases} \tag{2.2}$$

Also, z can be interpreted as a mild solution to the Cauchy problem specified below if u is continuously differentiable on $[0, a]$.

$$\begin{cases} \frac{dz(\zeta)}{d\zeta} &= Az(\zeta) + \mu Bu(\zeta) - B\frac{d}{d\zeta}u(\zeta) \\ &+ f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big), \\ z(0) &= x_0 - Bu(0), \\ \Delta z|_{\zeta=\zeta_k} &= I_k(x(\zeta_k^-)), k = 1, 2, ..., m. \end{cases} \tag{2.3}$$

Since $z \in \text{dom}(A)$, we apply the variation of constants formula to the linear inhomogeneous system. By integrating this equation over $[0, \zeta]$, and applying the strongly continuous semigroup $T(\zeta)$ generated by A, we obtain the expression:

$$\begin{aligned} z(\zeta) = T(\zeta)(x_0 - Bu(0)) + \int_0^\zeta T(\zeta - s)\Big[\mu Bu(s) - B\frac{d}{ds}u(s) \\ + f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big), \Big]ds \\ + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-)). \end{aligned}$$

Recalling $x(\zeta) = z(\zeta)+Bu(\zeta)$, which expresses the mild solution $x(\zeta)$ in terms of the initial condition, the semigroup $T(\zeta)$, the control input u, the nonlinear function f, and the impulsive terms I_k. The contribution of Az is embedded in the integral representation via the semigroup framework, and is no longer explicitly present after applying the Duhamel principle. So, the solution of boundary control nonlinear system (2.1) is presented by

$$\begin{aligned} x(\zeta) = T(\zeta)(x_0 - Bu(0)) + Bu(\zeta) + \int_0^\zeta T(\zeta - s)\Big\{\mu Bu(s) - B\frac{d}{ds}u(s)\Big\}ds \\ + \int_0^\zeta T(\zeta - s)f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\ + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-)). \end{aligned} \tag{2.4}$$

Definition 2.2. A continuous function $x(\cdot) : PC[0, a] \to Y$ is considered a mild solution of the boundary control nonlinear system (2.1) if the following conditions are met:

1) The initial condition is satisfied, i.e., $x(0) = x_0$;

2) For each $\zeta \in [0, a]$, the impulsive condition holds at $\zeta = \zeta_k$, specifically $\Delta x|_{\zeta=\zeta_k} = I_k(x(\zeta_k^-))$, where $k = 1, 2, \dots, m$;

3) The function $x(\cdot)$ is continuous on each interval J_k (where $k = 0, 1, 2, \dots, m$).

These conditions ensure that $x(\cdot)$ adheres to the prescribed initial and boundary conditions, with continuity across the specified intervals and the appropriate impulsive jumps at the designated times ζ_k and the following condition is fulfilled:

$$\begin{cases} x(\zeta) &= T(\zeta)x_0 + \int_0^\zeta [T(\zeta - s)\mu - AT(\zeta - s)]Bu(s)ds \\ &+ \int_0^\zeta T(\zeta - s)f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\ &+ \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-)), t \in J. \end{cases} \tag{2.5}$$

As a consequence of the above, Eq (2.5) is said to be a mild solution of system (2.1).

Definition 2.3. The NIIESs (2.1) is defined to be controllable on the interval J if, for every pair of initial and target states $x_0, x_1 \in Y$, there exists a control function $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of system (2.1) transitions from the initial state to the desired target state, satisfying the condition $x(a) = x_1$. This implies that the system can be driven to any final state within Y using an appropriate control input u.

Now, we require Schaefer's fixed point theorem [46] as mentioned below.

Schaefer's Theorem: Let Z be a normed linear space. Let $G : Z \to Z$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set. Let $\Phi = \{x \in Z : x = \varpi G x \text{ for some } 0 < \varpi < 1\}$.

Then, either $\Phi(G)$ is unbounded or G has a fixed point.

Further, the following hypotheses are considered:

(H_1) The linear operator $N : L^2(J, U) \to Y$ explained by
$Nu = \int_0^{\zeta} [T(a-s)\mu - AT(a-s)]Bu(s)ds$
stimulates an invertible operator $\tilde{N}^{-1}$ described on $L^2(J, U)/kerN$ and suppose $M_1 > 0$ in such a way that $\|\tilde{N}^{-1}\| \leq M_1$.

(H_2) For every $0 \leq \zeta \leq a$, the function $f(\zeta, ., ..., .) : Y^{l+1} \to Y$ is continuous and to each $x_1, x_2, ..., x_{l+1} \in Y$, the function $f(., x_1, ..., x_{l+1}) : J \to Y$ is strongly measurable.

(H_3) For each $r > 0$, there exists $K_r \in L^1(0, a)$ in such a manner that
$\sup_{\|x\| \leq r} \|f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^{\zeta} e(\zeta, s, x(\xi_{l+1}(s)))ds\Big)\| \leq K_r(\zeta)$.

(H_4) There exists a continuous function $p : J \to [0, \infty)$ such that
$\|f(\zeta, x_1, ..., x_{l+1})\| \leq p(\zeta)\Omega_0(\|x_1\| + \|x_2\| + ... + \|x_{l+1}),$
where $\Omega_0 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(H_5) For each $(\zeta, s) \in \Theta$, the function $e(\zeta, s, .) : Y \to Y$ is continuous and $\forall x \in Y$, the function $e(., ., x) : \Theta \to Y$ is strongly measurable.

(H_6) An integrable function $m : J \to [0, \infty)$ exists in such a manner that
$\|e(\zeta, s, x)\| \leq m(\zeta)\Omega(\|x\|),\ 0 \leq s \leq \zeta \leq a,\ x \in Y.$

(H_7) There exist constants $M_2,\ M_3 > 0$ in such a manner that $\|\mu B\|_{L(U,Y)} \leq M_2$ and $\int_0^a \gamma(\zeta)d\zeta \leq M_3$.

(H_8) The function $I_k : Y \to Y$ is continuous and there exists a constant M'_k in such a way that
$\|I_k(x) - I_k(y)\| \leq M'_k,\ k = 1, 2, 3, ..., m$, for each $x, y \in Y$.
Moreover, suppose that $\beta = \sum_{k=1}^{m} M'_k$.

(H_9) $\int_0^a m^*(s)ds < \int_c^{\infty} \frac{ds}{\Omega_0(s)+\Omega(s)}$,
where $c = M\|x_0\| + (MM_2a + M_3)Q$,

$m^*(\zeta) = \max\{lMp(\zeta), m(\zeta)\}$,

$$Q = M_1\Big[\|x_1\| + M\|x_0\| + M\int_0^a p(s)\Omega_0(\|x(\xi_1(s))\| + ... + \|x(\xi_l(s))\| + \int_0^s m(w)\Omega(\|x(\xi_{l+1}(w))\|)dw)ds + M\beta\Big].$$

3. Controllability of the impulsive integro-differential system via Schaefer's fixed point theorem

Theorem 3.1. *The boundary control NIIESs* (2.1) *is controllable on J if the assumptions* $(A_1)-(A_3)$ *along with the hypotheses* $(H_1)-(H_9)$ *are satisfied.*

Proof. By referring H_1 for any arbitrary function $x(.)$, we define the control formally as

$$u(\zeta)=\tilde{N}^{-1}\Big[x_1-T(a)x_0-\int_0^a T(a-s)f\Big(s,x(\xi_1(s)),...,x(\xi_l(s)),\\ \int_0^s e(s,w,x(\xi_{l+1}(w)))dw\Big)ds-\sum_{0<\zeta_k<a}T(a-\zeta_k)I_k(x(\zeta_k^-))\Big](\zeta).$$

Let the Banach space $Z=PC(J,Y)$ with norm $\|x\|=sup\{|x(\zeta)|:\zeta\in J\}$. First, we derive a priori bounds for the equation as stated below,

$$x(\zeta)=\varpi T(\zeta)x_0+\varpi\int_0^\zeta[T(\zeta-\eta)\mu-AT(\zeta-\eta)]\times\\ B\tilde{N}^{-1}\Big[x_1-T(a)x_0-\int_0^a T(a-s)f\Big(s,x(\xi_1(s)),...,x(\xi_l(s)),\\ \int_0^s e(s,w,x(\xi_{l+1}(w)))dw\Big)ds-\sum_{0<\zeta_k<a}T(a-\zeta_k)I_k(x(\zeta_k^-))\Big]d\eta\\ +\varpi\int_0^\zeta T(\zeta-s)f\Big(s,x(\xi_1(s)),...,x(\xi_l(s)),\int_0^s e(s,w,x(\xi_{l+1}(w)))dw\Big)ds\\ +\sum_{0<\zeta_k<\zeta}T(\zeta-\zeta_k)I_k(x(\zeta_k^-)).$$

We have

$$\|x(\zeta)\|\le M\|x_0\|+\int_0^\zeta(MM_2+\gamma(\eta))M_1\Big[\|x_1\|+M\|x_0\|+M\int_0^a p(s)\Omega_0\times\\ \Big(\|x(\xi_1(s))\|+...+\|x(\xi_l(s))\|+\int_0^s m(w)\Omega\|x(\xi_{l+1}(w))\|dw\Big)ds\\ +M\sum_{k=1}^m M_k'\Big]d\eta+M\int_0^\zeta p(s)\Omega_0\Big(\|x(\xi_1(s))\|+...+\|x(\xi_l(s))\|\\ +\int_0^s m(w)\Omega\|x(\xi_{l+1}(w))\|dw\Big)ds+M\sum_{k=1}^m M_k'\\ \le M\|x_0\|+(MM_2a+M_3)M_1\Big[\|x_1\|+M\|x_0\|\\ +M\int_0^a p(s)\Omega_0\Big(\|x(\xi_1(s))\|+...+\|x(\xi_l(s))\|+\int_0^s m(w)\Omega\|x(\xi_{l+1}(w))\|dw\Big)ds+M\beta\Big]\\ +M\int_0^\zeta p(s)\Omega_0\Big(\|x(\xi_1(s))\|+...+\|x(\xi_l(s))\|+\int_0^s m(w)\Omega\|x(\xi_{l+1}(w))\|dw\Big)ds+M\beta.$$

Let

$$Q = M_1\Big[\|x_1\| + M\|x_0\| + M\int_0^a p(s)\Omega_0\Big(\|x(\xi_1(s))\| + ... + \|x(\xi_l(s))\|$$

$$+ \int_0^s m(w)\Omega\Big(\|x(\xi_{l+1}(w))\|dw\Big)ds + M\beta\Big],$$

and then

$$x(\zeta) \leq \ M\|x_0\| + (MM_2a + M_3)Q + M\int_0^\zeta p(s)\Omega_0\Big(\|x(\xi_1(s))\| + ...$$

$$+ \|x(\xi_l(s))\| + \int_0^s m(w)\Omega\|x(\xi_{l+1}(w))\|dw\Big)ds + M\beta.$$

We denote the right-hand side (R.H.S.) of the above mentioned inequality as $v(\zeta)$. Consequently, we get

$$c = v(0) = M\|x_0\| + (MM_2a + M_3)Q, \|x(\zeta)\| \leq v(\zeta)$$

and

$$v'(\zeta) \leq Mp(\zeta)\Omega_0\Big(\|x(\xi_1(\zeta))\| + ... + \|x(\xi_l(\zeta))\| + \int_0^\zeta m(s)\Omega\|x(\xi_{l+1}(s))\|ds\Big).$$

This infers

$$\begin{aligned} v'(\zeta) &\leq Mp(\zeta)\Omega_0\Big(v(\xi_1(\zeta)) + ... + v(\xi_l(\zeta)) + \int_0^\zeta m(s)\Omega v(\xi_{l+1}(s))ds\Big) \\ &\leq Mp(\zeta)\Omega_0\Big(v(\zeta) + ... + v(\zeta) + \int_0^\zeta m(s)\Omega v(s)ds\Big) \\ &\leq Mp(\zeta)\Omega_0\Big(lv(\zeta) + \int_0^\zeta m(s)\Omega v(s)ds\Big), \end{aligned}$$

since v is explicitly increasing and $\xi_i(\zeta) \leq \zeta, i = 1, 2, 3, ..., l + 1$.

Let $\omega(\zeta) = lv(\zeta) + \int_0^\zeta m(s)\Omega(v(s))ds$.

Then, $\omega(0) = lv(0) = c$, $v(\zeta) \leq \omega(\zeta)$, and

$$\omega'(\zeta) = lv'(\zeta) + m(\zeta)\Omega(v(\zeta)) = lMp(\zeta)\Omega_0\omega(\zeta) + m(\zeta)\Omega(v(\zeta))$$

$$= lMp(\zeta)\Omega_0\omega(\zeta) + m(\zeta)\Omega(\omega(\zeta)) \leq m^*(\zeta)[\Omega_0(\omega(\zeta)) + \Omega(\omega(\zeta))].$$

This implies that

$$\int_{\omega(0)}^{\omega(\zeta)} \frac{ds}{\Omega_0(s) + \Omega(s)} \leq \int_0^a m^*(s)ds \leq \int_c^\infty \frac{ds}{\Omega_0(s) + \Omega(s)}, \zeta \in J.$$

The above inequality implies that there is a constant $\omega(\zeta) \leq K^*, \zeta \in J$, and hence, $\|x(\zeta)\| \leq K^*, \zeta \in J$, in which K^* depends on the functions m^*, Ω_0, Ω as well as on constant a.

Further, we define the operator $G : Z \to Z$ as follows:

$$
\begin{aligned}
(Gx)(\zeta) = T(\zeta)x_0 + \int_0^{\zeta} [T(\zeta-\eta)\mu - AT(\zeta-\eta)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
- \int_0^a T(a-s)f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\
- \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big]d\eta + \int_0^{\zeta} T(\zeta-s)f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\
\int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds + \sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(x(\zeta_k^-)),\ \zeta \in J.
\end{aligned}
$$

Now, we demonstrate that the operator G is a completely continuous operator. For this purpose, we assume that $B_r = x \in Z : \|x\| \leq r$ for some $r \geq 1$. Obviously B_r is a nonempty set that is a bounded, closed, and convex set in $PC([0, a], Y)$. First, we demonstrate that G maps B_r into an equicontinuous family. Consider $x \in B_r$ and $\zeta_1, \zeta_2 \in J$. Then, if $0 < \zeta_1 < \zeta_2 \leq a$, we have

$$
\begin{aligned}
\|(Gx)(\zeta_1) - (Gx)(\zeta_2)\| \leq \|T(\zeta_1) - T(\zeta_2)\|\|x_0\| + \Big\|\int_0^{\zeta_1} [T(\zeta_1-\eta) \\
- T(\zeta_2-\eta)]\mu B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 - \int_0^a T(a-s)f\Big(s, x(\xi_1(s)), ..., \\
x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds - \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big]d\eta\Big\| \\
+ \Big\|\int_{\zeta_1}^{\zeta_2} T(\zeta_2-\eta)\mu B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 - \int_0^a T(a-s)f\Big(s, x(\xi_1(s)), ... \\
x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds - \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big]d\eta\Big\| \\
+ \Big\|\int_0^{\zeta_1} A[T(\zeta_1-\eta) - T(\zeta_2-\eta)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 - \int_0^a T(a-s) \\
f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\
- \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big]d\eta\Big\| + \Big\|\int_{\zeta_1}^{\zeta_2} AT(\zeta_2-\eta)B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
- \int_0^a T(a-s)f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\
- \sum_{0<\zeta_k<a} T(a-t_k)I_k(x(\zeta_k^-))\Big]d\eta\Big\| + \Big\|\int_0^{\zeta_1} [T(\zeta_1-s) - T(\zeta_2-s)]
\end{aligned}
$$

$$
\begin{aligned}
&\times f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\
&+ \sum_{0<\zeta_k<\zeta_1} [T(\zeta_1 - \zeta_k) - T(\zeta_2 - \zeta_k)]I_k(x(\zeta_k^-))\Bigg\| \\
&+ \Bigg\| \int_{\zeta_1}^{\zeta_2} T(\zeta_2 - s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\
&\int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds + \sum_{\zeta_1<\zeta_k<\zeta_2} T(\zeta_2 - \zeta_k)I_k(x(\zeta_k^-))\Bigg\| \\
\leq{}& \|T(\zeta_1) - T(\zeta_2)\|\|x_0\| + \int_0^{\zeta_1} \Big\|T(\zeta_1 - \eta) - T(\zeta_2 - \eta)\Big\| M_1 M_2 \Big[\|x_1\| \\
&+ M\|x_0\| + M\int_0^a K_r(s)ds + M\beta\Big]d\eta + \int_{\zeta_1}^{\zeta_2} \|T(\zeta_2 - \eta)\| M_1 M_2 \\
&\Big[\|x_1\| + M\|x_0\| + M\int_0^a K_r(s)ds + M\beta\Big]d\eta + \int_0^{\zeta_1} \Big\|A[T(\zeta_1 - \eta) \\
&- T(\zeta_2 - \eta)]B\Big\| M_1 \Big[\|x_1\| + M\|x_0\| + M\int_0^a K_r(s)ds + M\beta\Big]d\eta \\
&+ \int_{\zeta_1}^{\zeta_2} \|AT(\zeta_2 - \eta)B\| M_1 \Big[\|x_1\| + M\|x_0\| + M\int_0^a K_r(s)ds + M\beta\Big]d\eta \\
&+ \int_0^{\zeta_1} \Big\|T(\zeta_1 - s) - T(\zeta_2 - s)\Big\| K_r(s)ds + \Bigg\| \sum_{0<\zeta_k<\zeta_1} [T(\zeta_1 - \zeta_k) \\
&- T(\zeta_2 - \zeta_k)]I_k(x(\zeta_k^-))\Bigg\| + \int_{\zeta_1}^{\zeta_2} \|T(\zeta_2 - s)\| K_r(s)ds + M \sum_{\zeta_1<\zeta_k<\zeta_2} M_k'.
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side (R.H.S.) tends to zero. This implies that the compactness of $T(\zeta)$ for $\zeta > 0$ leads to continuity in the uniform operator topology.

Therefore, the operator G maps B_r into an equicontinuous family of functions. Additionally, the family GB_r is uniformly bounded, which is straightforward to verify. Moreover, we demonstrate that $\overline{GB_r}$ is compact. Since GB_r is equicontinuous, we can apply the Arzel-Ascoli theorem, which guarantees that $\{(Gx)(\zeta) : x \in B_r\}$ is precompact in Y. Fix $0 < \zeta \leq a$, and let ε be a real number such that $0 < \varepsilon < \zeta$. For all $x \in B_r$, we obtain the following:

$$
\begin{aligned}
(G_\varepsilon x)(\zeta) ={}& T(\zeta)x_0 + \int_0^{\zeta-\varepsilon} T(\zeta - \eta)\mu B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a - s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds \\
&- \sum_{0<\zeta_k<a} T(a - \zeta_k)I_k(x(\zeta_k^-))\Big]d\eta - \int_0^{\zeta-\varepsilon} AT(\zeta - \eta)B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a - s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{0<\zeta_k<a} T(a-\zeta_k) I_k(x(\zeta_k^-))\Big] d\eta + \int_0^{\zeta-\varepsilon} T(\zeta - s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\
&\int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds + \sum_{0<\zeta_k<\zeta-\varepsilon} T(\zeta-\zeta_k) I_k(x(\zeta_k^-)) \\
&\leq T(\zeta)x_0 + T(\varepsilon) \int_0^{\zeta-\varepsilon} T(\zeta-\eta-\varepsilon)\mu B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a-s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds \\
&- \sum_{0<\zeta_k<a} T(a-\zeta_k) I_k(x(\zeta_k^-))\Big] d\eta - T(\varepsilon) \int_0^{\zeta-\varepsilon} AT(\zeta-\eta-\varepsilon) B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a-s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds \\
&- \sum_{0<\zeta_k<a} T(a-\zeta_k) I_k(x(\zeta_k^-))\Big] d\eta + T(\varepsilon) \int_0^{\zeta-\varepsilon} T(\zeta-s-\varepsilon) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\
&\int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds + T(\varepsilon) \sum_{0<\zeta_k<\zeta-\varepsilon} T(\zeta-\varepsilon-\zeta_k) I_k(x(\zeta_k^-)).
\end{aligned}
$$

Thus $T(\zeta)$ is compact, and the set $\{G_\varepsilon x(\zeta) : x \in B_r\}$ is precompact in Y to each $\varepsilon, 0 < \varepsilon < \zeta$. Also, $\forall\ x \in B_r$, we have

$$
\begin{aligned}
\|(Gx)(\zeta) - (G_\varepsilon x)(\zeta)\| \leq& \int_{\zeta-\varepsilon}^{\zeta} \Big\| T(\zeta-\eta)\mu B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a-s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds \\
&- \sum_{0<\zeta_k<a} T(a-\zeta_k) I_k(x(\zeta_k^-))\Big\| d\eta + \int_{\zeta-\varepsilon}^{\zeta} \Big\| AT(\zeta-\eta) B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 \\
&- \int_0^a T(a-s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big) ds \\
&- \sum_{0<\zeta_k<a} T(a-\zeta_k) I_k(x(\zeta_k^-))\Big\| d\eta + \int_{\zeta-\varepsilon}^{\zeta} \Big\| T(\zeta-s) f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\
&\int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big\| ds + \sum_{\zeta-\varepsilon<\zeta_k<\zeta} \Big\| T(\zeta-\zeta_k) I_k(x(\zeta_k^-))\Big\| \\
\leq& \int_{\zeta-\varepsilon}^{\zeta} \|T(\zeta-\eta)\| M_1 M_2 \Big[\|x_1\| + M\|x_0\| + \int_0^a \|T(a-s)\| K_r(s) ds + M\beta\Big] d\eta \\
&+ \int_{\zeta-\varepsilon}^{\zeta} \|AT(\zeta-\eta)B\| M_1 \Big[\|x_1\| + M\|x_0\| + \int_0^a \|T(a-s)\| K_r(s) ds + M\beta\Big] d\eta \\
&+ \int_{\zeta-\varepsilon}^{\zeta} \|T(\zeta-s)\| K_r(s) ds + \sum_{\zeta-\varepsilon<\zeta_k<\zeta} \|T(\zeta-\zeta_k) I_k(x(\zeta_k^-))\|.
\end{aligned}
$$

Since there are precompact sets arbitrarily close to the set $\{(Gx)(\zeta) : x \in B_r\}$, the set is precompact in Y. It remains to demonstrate that $G : Z \to Z$ is continuous. Suppose $\{x_n\}_0^\infty$ with $x_n \to x$ in Z. Then, there exists an integer r in such a manner that $\|x_n(\zeta)\| \le r$, for each n and $\zeta \in J$, and therefore $x_n \in B_r$. From (H_2) and (H_5), we have

$$f\Big(\zeta, x_n(\xi_1(\zeta)), ..., x_n(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x_n(\xi_{l+1}(s)))ds\Big) \to f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big),$$

for every $\zeta \in J$, and hence

$$\Big\|f\Big(\zeta, x_n(\xi_1(\zeta)), ..., x_n(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x_n(\xi_{l+1}(s)))ds\Big) - f\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e(\zeta, s, x(\xi_{l+1}(s)))ds\Big)\Big\| \le 2K_r(\zeta).$$

Employing dominated convergence theorem, we procure

$$\begin{aligned}
\|Gx_n - Gx\| = \sup_{\zeta \in J}\Big\| \int_0^\zeta & [- T(\zeta - \eta)\mu + AT(\zeta - \eta)]B\tilde{N}^{-1} \\
& \Big[\int_0^a T(a - s)\Big\{f\Big(s, x_n(\xi_1(s)), ..., x_n(\xi_l(s)), \int_0^s e(s, w, x_n(\xi_{l+1}(w)))dw\Big) \\
& - f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)\Big\}ds \\
& + \sum_{0<\zeta_k<a} T(a - \zeta_k)\Big[I_k(x_n(\zeta_k^-)) - I_k(x(\zeta_k^-))\Big]d\eta + \int_0^\zeta T(\zeta - s) \\
& \Big[f\Big(s, x_n(\xi_1(s)), ..., x_n(\xi_l(s)), \int_0^s e(s, w, x_n(\xi_{l+1}(w)))dw\Big) \\
& - f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)\Big]ds \\
& + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)\Big[I_k(x_n(\zeta_k^-)) - I_k(x(\zeta_k^-))\Big]\Big\| \\
& \le \sup_{\zeta \in J} \int_0^\zeta [\|T(\zeta - \eta)\|\|\mu B\| + \|AT(\zeta - \eta)B\|]\|\tilde{N}^{-1}\|\Big[\int_0^a \|T(a - s)\| \\
& \Big\|f\Big(s, x_n(\xi_1(s)), ..., x_n(\xi_l(s)), \int_0^s e(s, w, x_n(\xi_{l+1}(w)))dw\Big) \\
& - f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e(s, w, x(\xi_{l+1}(w)))dw\Big)\Big\|ds \\
& + \sum_{0<\zeta_k<a} \|T(a - \zeta_k)\|\Big\|I_k(x_n(\zeta_k^-)) - I_k(x(\zeta_k^-))\Big\|\Big]d\eta + \int_0^\zeta \|T(\zeta - s)\|
\end{aligned}$$

$$
\begin{aligned}
&\Big\|f\Big(s, x_n(\xi_1(s)), ..., x_n(\xi_l(s)), \int_0^s e\big(s, w, x_n(\xi_{l+1}(w))\big)dw\Big)\\
&- f\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e\big(s, w, x(\xi_{l+1}(w))\big)dw\Big)\Big\|ds\\
&+ \sum_{0<\zeta_k<\zeta} \|T(\zeta-\zeta_k)\|\Big\|I_k(x_n(\zeta_k^-)) - I_k(x(\zeta_k^-))\Big\| \to 0,\ as\ n \to\ \infty.
\end{aligned}
$$

Thus, G is continuous and, hence, it is completely continuous.

At last, as we have showed in the first step, the set $\Phi(G) = \{x \in Z : x = \bar{\omega}Gx, \bar{\omega} \in (0, 1)\}$ is bounded. Therefore, according to Schaefer's theorem, the operator G has a fixed point in Z. This implies that any fixed point of G is a mild solution of (2.1) on J satisfying $(Gx)(\zeta) = x(\zeta)$. □

4. Controllability of a nonlinear integro-differential system via Schauder's fixed point theorem

Assume we have the boundary control system of the following type:

$$
\begin{cases}
x'(\zeta) = \mu x(\zeta) + f_1\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e_1(\zeta, s, x(\xi_{l+1}(s))ds\Big)\\
\qquad\quad +f_2\Big(\zeta, x(\iota_1(\zeta)), ..., x(\iota_p(\zeta)), \int_0^\zeta e_2(\zeta, s, x(\iota_{p+1}(s))ds\Big),\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \zeta \in J = [0, a], \zeta \neq \zeta_k,\\
\alpha x(\zeta) = B_1 u(\zeta),\\
x(0) = x_0,\\
\Delta x|_{\zeta=\zeta_k} = I_k(x(\zeta_k^-)), k = 1, 2, ..., m,
\end{cases}
\tag{4.1}
$$

where the nonlinear operators f_1, f_2 are continuous functions to be specified later and $\Theta = \{(\zeta, s) : 0 \leq s \leq \zeta \leq a\}$. Moreover, $\xi_i : J \to J, i = 1, 2, ..., l+1$, and $\iota_j : J \to J, j = 1, 2, ..., p+1$, are continuous functions in such a way that $\xi_i(\zeta) \leq \zeta, i = 1, 2, ..., l+1$, and $\iota_j(\zeta) \leq \zeta, j = 1, 2, ..., p+1$. Applying the same argument as in Section 2, the mild solution of system (4.1) is expressed as follows:

$$
\begin{aligned}
x(\zeta) = T(\zeta)x_0 + \int_0^\zeta [T(\zeta-s)\mu - AT(\zeta-s)]Bu(s)ds + \int_0^\zeta T(\zeta-s)\Big[f_1\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)),\\
\int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)),\\
\int_0^s e_2(s, w, x(\iota_{p+1}(w))dw\Big)\Big]ds + \sum_{0<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(x(\zeta_k^-)), t \in J.
\end{aligned}
$$

Here, assumptions (A_1) and (A_2) and hypothesis (H_1) are considered as we assumed in Section 2. Consider the following additional hypotheses as follows:

(H_{10}) For all $\zeta \in (0, a]$ and $u \in U, T(\zeta)Bu \in dom(A)$. In addition, there exists $C_1 > 0$ such that $\|AT(\zeta)\| \leq C_1$. Also, we take constant $C_2 > 0$ such that $\|B\| \leq C_2$.

(H_{11}) The nonlinear operators $f_1 : J \times Y^{l+1} \to Y, f_2 : J \times Y^{p+1} \to Y$ are continuous and there exist contants $C_3, C_4 > 0$ in such a manner that

$$
\Big\|f_1\Big(\zeta, x(\xi_1(\zeta)), ..., x(\xi_l(\zeta)), \int_0^\zeta e_1(\zeta, s, x(\xi_{l+1}(s))ds\Big)\Big\| \leq C_3,
$$

$$\left\|f_2\Big(\zeta, x(\iota_1(\zeta)), ..., x(\iota_p(\zeta)), \int_0^{\zeta} e_2(\zeta, s, x(\iota_{p+1}(s))ds\Big)\right\| \le C_4,$$

for $\zeta \in J$ and $(\zeta, s) \in \Theta$.

(H_{12}) The function $I_k : Y \to Y$ is continuous and there exists a constant d_k in such a way that $\|I_k(x)\| \le d_k, k = 1, 2, ..., m, \forall\ x \in Y$.

Theorem 4.1. *The boundary control NIIESs* (4.1) *is controllable on* J *if the assumptions* $(A_1), (A_2)$ *along with hypotheses* (H_1) *and* $(H_{10}) - (H_{12})$ *are satisfied.*

Proof. By referring (H_1) for any arbitrary function $x(.)$, we define the control formally as

$$\begin{aligned} u(\zeta) = \tilde{N}^{-1}\Bigg[& x_1 - T(a)x_0 - \int_0^a T(a-s)\Big\{f_1\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \\ & \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)), \\ & \int_0^s e_2(s, w, x(\iota_{p+1}(w))dw\Big)\Big\}ds - \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Bigg](\zeta),\ \zeta \in J. \end{aligned}$$

Applying the above control, we will demonstrate that the operator P expressed by

$$\begin{aligned} (Px)(\zeta) = T(\zeta)x_0 + & \int_0^{\zeta} [T(\zeta - s)\mu - AT(\zeta - s)]Bu(s)ds + \int_0^{\zeta} T(\zeta - s) \\ & \Big\{f_1\Big(s, x(\xi_1(s)), ..., x(\xi_l(s)), \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), \\ & ..., x(\iota_p(s)), \int_0^s e_2(s, w, x(\iota_{p+1}(w))dw\Big)\Big\}ds + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-)), \zeta \in J, \end{aligned}$$

has a fixed point $x(.)$, by employing Schauder's fixed point theorem. We observe that this fixed point is then a solution of (4.1). Explicitly, $(Px)(a) = x_1$. This implies that the control function u steers the NIIESs from the initial state x_0 to x_1 in time a.

Assume that $V = PC(J, Y)$ and $V_0 = \{x \in V : \|x(\zeta)\| \le \rho, \text{for } \zeta \in J\}$, where ρ is the positive constant that is specified as follows:

$$\begin{aligned} \rho = M\|x_0\| + a(M\|\mu\| + C_1)M_1C_2\Big\{ & \|x_1\| + M\|x_0\| + Ma(C_3 + C_4) \\ & + M\sum_{k=1}^{m} d_k\Big\} + aM(C_3 + C_4) + M\sum_{k=1}^{m} d_k. \end{aligned} \tag{4.2}$$

Then, V_0 is a subset of the closed, bounded, and convex set V. We define a mapping $P : V \to V_0$ by

$$\begin{aligned} (Px)(\zeta) = T(\zeta)x_0 + \int_0^{\zeta} & [T(\zeta - s)\mu - AT(\zeta - s)]B\tilde{N}^{-1}\Bigg[x_1 - T(a)x_0 \\ & - \int_0^a T(a - v)\Big\{f_1\Big(v, x(\xi_1(v)), ..., x(\xi_l(v)), \int_0^v e_1(v, w, x(\xi_{l+1}(w))dw\Big) \end{aligned}$$

$$+ f_2\Big(v, x(\iota_1(v)), ..., x(\iota_p(v)), \int_0^v e_2(v, w, x(\iota_{p+1}(w))dw)\Big\}dv$$

$$- \sum_{0<\zeta_k<a} T(a - \zeta_k)I_k(x(\zeta_k^-))\Big](s)ds + \int_0^\zeta T(\zeta - s)\Big\{f_1\Big(s, x(\xi_1(s)),$$

$$..., x(\xi_l(s)), \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)),$$

$$\int_0^s e_2(s, w, x(\iota_{p+1}(w))dw)\Big\}ds + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-)).$$

Consider

$$\|(Px)(\zeta)\| \leq \|T(\zeta)x_0\| + \Big\| \int_0^\zeta [T(\zeta - s)\mu - AT(\zeta - s)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0$$

$$- \int_0^a T(a - v)\Big\{f_1\Big(v, x(\xi_1(v)), ..., x(\xi_l(v)), \int_0^v e_1(v, w, x(\xi_{l+1}(w))dw\Big)$$

$$+ f_2\Big(v, x(\iota_1(v)), ..., x(\iota_p(v)), \int_0^v e_2(v, w, x(\iota_{p+1}(w))dw)\Big\}dv$$

$$- \sum_{0<\zeta_k<a} T(a - \zeta_k)I_k(x(\zeta_k^-))\Big](s)ds\Big\| + \Big\| \int_0^\zeta T(\zeta - s)\Big\{f_1\Big(s, x(\xi_1(s)),$$

$$..., x(\xi_l(s)), \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)),$$

$$\int_0^s e_2(s, w, x(\iota_{p+1}(w))dw)\Big\}ds + \sum_{0<\zeta_k<\zeta} T(\zeta - \zeta_k)I_k(x(\zeta_k^-))\Big\|$$

$$\leq M\|x_0\| + \int_0^\zeta (M\|\mu\| + C_1)M_1C_2\Big\{\|x_1\| + M\|x_0\| + (MC_3 + MC_4)a + M\sum_{k=1}^m d_k\Big\}ds$$

$$+ \int_0^\zeta M(C_3 + C_4)ds + M\sum_{k=1}^m d_k$$

$$\leq M\|x_0\| + a(M\|\mu\| + C_1)M_1C_2\Big\{\|x_1\| + M\|x_0\| + (MC_3 + MC_4)a$$

$$+ M\sum_{k=1}^m d_k\Big\} + aM(C_3 + C_4) + M\sum_{k=1}^m d_k.$$

Since f_1 and f_2 are continuous, and $\|(Px)(\zeta)\| \leq \rho$, it follows that P is continuous and maps V_0 into itself. Moreover, P maps V_0 to precompact subsets of V_0. To verify this, we first need to show that for each fixed $\zeta \in J$, the set $V_0(\zeta) = \{(Px)(\zeta) : x \in V_0\}$ is precompact in Y. This is immediately clear for $\zeta = 0$, since $V_0(0) = x_0$. For a fixed $\zeta > 0$, and for any $0 < \varepsilon < \zeta$, we proceed to show that

$$(P_\varepsilon x)(\zeta) = T(\zeta)x_0 + \int_0^{\zeta-\varepsilon} [T(\zeta - s)\mu - AT(\zeta - s)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0$$

$$- \int_0^a T(a - v)\Big\{f_1\Big(v, x(\xi_1(v)), ..., x(\xi_l(v)), \int_0^v e_1(v, w, x(\xi_{l+1}(w))dw\Big)$$

$$+ f_2\Big(v, x(\iota_1(v)), ..., x(\iota_p(v)), \int_0^v e_2(v, w, x(\iota_{p+1}(w))dw)\Big)\Big\}dv$$
$$- \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big](s)ds + \int_0^{\zeta-\varepsilon} T(\zeta-s)\Big\{f_1\Big(s, x(\xi_1(s)),$$
$$..., x(\xi_l(s)), \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)),$$
$$\int_0^s e_2(s, w, x(\iota_{p+1}(w))dw\Big)\Big\}ds + \sum_{0<\zeta_k<\zeta-\varepsilon} T(\zeta-\zeta_k)I_k(x(\zeta_k^-)).$$

Hence $T(\zeta)$ is compact for every $\zeta > 0$, and the set $V_\varepsilon(\zeta) = \{(P_\varepsilon x)(\zeta) : x \in V_0\}$ is precompact in Y to each $\varepsilon, 0 < \varepsilon < \zeta$. Furthermore, for $x \in V_0$, we obtain

$$\|(Px)(\zeta) - (P_\varepsilon x)(\zeta)\| \leq \Big\| \int_{\zeta-\varepsilon}^{\zeta} [T(\zeta-s)\mu - AT(\zeta-s)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0$$
$$- \int_0^a T(a-v)\Big\{f_1\Big(v, x(\xi_1(v)), ..., x(\xi_l(v)), \int_0^v e_1(v, w, x(\xi_{l+1}(w))dw\Big)$$
$$+ f_2\Big(v, x(\iota_1(v)), ..., x(\iota_p(v)), \int_0^v e_2(v, w, x(\iota_{p+1}(w))dw)\Big)\Big\}dv$$
$$- \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big](s)ds\Big\| + \Big\| \int_{\zeta-\varepsilon}^{\zeta} T(\zeta-s)\Big\{f_1\Big(s, x(\xi_1(s)),$$
$$..., x(\xi_l(s)), \int_0^s e_1(s, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(s, x(\iota_1(s)), ..., x(\iota_p(s)),$$
$$\int_0^s e_2(s, w, x(\iota_{p+1}(w))dw\Big)\Big\}ds + \sum_{\zeta-\varepsilon<\zeta_k<\zeta} T(\zeta-\zeta_k)I_k(x(\zeta_k^-))\Big\|$$
$$\leq \varepsilon(M\|\mu\| + C_1)M_1C_2\Big\{\|x_1\| + M\|x_0\| + (MC_3 + MC_4)a + M\sum_{k=1}^{m} d_k\Big\}$$
$$+ \varepsilon(MC_3 + MC_4) + M \sum_{\zeta-\varepsilon<\zeta_k<\zeta}^{m} d_k$$

and this means that $V_0(\zeta)$ is totally bounded, which means that it is precompact in Y. We have to demonstrate that $P(V_0) = \{Px : x \in V_0\}$ is an equicontinuous family of functions. For this purpose, we set $\zeta_2 > \zeta_1 > 0$ and then we obtain

$$\|(Px)(\zeta_1) - (Px)(\zeta_2)\| \leq \|(T(\zeta_1) - T(\zeta_2))x_0\|$$
$$+ \Big\| \int_0^{\zeta_1} [\{T(\zeta_1-s)\mu - AT(\zeta_1-s)\} - \{T(\zeta_2-s)\mu - AT(\zeta_2-s)\}]$$
$$B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 - \int_0^a T(a-v)\Big\{f_1\Big(v, x(\xi_1(v)), ..., x(\xi_l(v)),$$
$$\int_0^v e_1(v, w, x(\xi_{l+1}(w))dw\Big) + f_2\Big(v, x(\iota_1(v)), ..., x(\iota_p(v)),$$

$$\int_0^v e_2(v,w,x(\iota_{p+1}(w))dw\Big)\Big\}dv - \sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big](s)ds\Big\|$$

$$+\Big\|\int_{\zeta_1}^{\zeta_2}[T(\zeta_2-s)\mu - AT(\zeta_2-s)]B\tilde{N}^{-1}\Big[x_1 - T(a)x_0 - \int_0^a T(a-v)\times$$

$$\Big\{f_1\Big(v, x(\xi_1(v)),...,, x(\xi_l(v)), \int_0^v e_1(v,w,x(\xi_{l+1}(w))dw\Big)$$

$$+ f_2\Big(v, x(\iota_1(v)),..., x(\iota_p(v)), \int_0^v e_2(v,w,x(\iota_{p+1}(w))dw\Big)\Big\}dv$$

$$-\sum_{0<\zeta_k<a} T(a-\zeta_k)I_k(x(\zeta_k^-))\Big](s)ds\Big\| + \Big\|\int_0^{\zeta_1}[T(\zeta_1-s)-T(\zeta_2-s)]\times$$

$$\Big\{f_1\Big(s, x(\xi_1(s)),...,, x(\xi_l(s)), \int_0^s e_1(s,w,x(\xi_{l+1}(w))dw\Big)$$

$$+ f_2\Big(s, x(\iota_1(s)),..., x(\iota_p(s)), \int_0^s e_2(s,w,x(\iota_{p+1}(w))dw\Big)\Big\}ds\Big\|$$

$$+\Big\|\int_{\zeta_1}^{\zeta_2} T(\zeta_2-s)\Big\{f_1\Big(s, x(\xi_1(s)),...,, x(\xi_l(s)), \int_0^s e_1(s,w,x(\xi_{l+1}(w))dw\Big)$$

$$+ f_2\Big(s, x(\iota_1(s)),..., x(\iota_p(s)), \int_0^s e_2(s,w,x(\iota_{p+1}(w))dw\Big)\Big\}ds\Big\|$$

$$+\sum_{0<\zeta_k<\zeta_1}\|T(\zeta_1-\zeta_k)-T(\zeta_2-\zeta_k)\|\|I_k(x(\zeta_k^-))\|$$

$$+\sum_{\zeta_1<\zeta_k<\zeta_2}\|T(\zeta_2-\zeta_k)\|\|I_k(x(\zeta_k^-))\|.$$

By using the conditions (A_1), (A_2) and the hypotheses $(H_1), (H_{10}) - (H_{12})$, we get

$$\begin{aligned}\|(Px)(\zeta_1)-(Px)(\zeta_2)\| \le\ & \|(T(\zeta_1)-T(\zeta_2))x_0\| \\ & + \int_0^{\zeta_1}\|\{T(\zeta_1-s)\mu - AT(\zeta_1-s)\} - \{T(\zeta_2-s)\mu \\ & - AT(\zeta_2-s)\}\|M_1C_2\Big[\|x_1\| + M\|x_0\| + (MC_3+MC_4)a + M\sum_{k=1}^m d_k\Big]ds \\ & + \Big[(M\|\mu\|+C_1)M_1C_2\Big\{\|x_1\|+M\|x_0\|+(MC_3+MC_4)a + M\sum_{k=1}^m d_k\Big\}\Big] \\ & (\zeta_2-\zeta_1) + \int_0^{\zeta_1}\|T(\zeta_1-s)-T(\zeta_2-s)\|(C_3+C_4)ds + M(C_3+C_4)(\zeta_2-\zeta_1) \\ & + \sum_{0<\zeta_k<\zeta_1}\|T(\zeta_1-\zeta_k)-T(\zeta_2-\zeta_k)\|d_k + M\sum_{\zeta_1<\zeta_k<\zeta_2} d_k. \end{aligned} \tag{4.3}$$

The compactness of $T(\zeta)$ for $\zeta > 0$ implies that $T(\zeta)$ is continuous in the uniform operator topology for $\zeta > 0$. As a result, the right-hand side of (4.3) tends to zero as $\zeta_2 \to \zeta_1$. Therefore, $P(V_0)$ forms an equicontinuous family of functions. Additionally, $P(V_0)$ is bounded in V, and by applying the

Arzel-Ascoli theorem, we conclude that $P(V_0)$ is precompact. Consequently, by Schauder's fixed point theorem in V_0, any fixed point of P is a mild solution to (4.1), satisfying $(Px)(\zeta) = x(\zeta) \in Y$. This shows that system (4.1) is controllable on J. $\square$

5. Examples

To illustrate the theoretical results, we take nonlinear partial functional integro-differential equations of the following form:

Example 5.1. Let us take the following partial functional integro-differential equations as the boundary control nonlinear system:

$$\begin{cases} \frac{\partial z(\zeta,y)}{\partial \zeta} = \frac{\partial^2 z(\zeta,y)}{\partial y^2} + \frac{1}{(1+\zeta)(1+\zeta^2)}\Big(z(\sin\zeta, y) + \sin z(\zeta, y)\int_0^\zeta e^{-z(\sin s,y)}ds\Big), y \in \Omega, \zeta \in J = [0,a], \zeta \neq \zeta_k; \\ z(\zeta,0) = u(\zeta,0), on \sum = (0,a)\times\Gamma, \zeta \in [0,a], \zeta \neq \zeta_k; \\ z(\zeta_k^+) - z(\zeta_k^-) = I_k(z(\zeta_k^-)),\ k = 1,2,3,...,m; \\ z(\zeta,y) = 0, z(0,y) = z_0(y), y \in \Omega, \end{cases} \tag{5.1}$$

where $z_0 \in L^2(\Omega)$ and $u \in L^2(\sum)$. Consider that $\Omega \subset \mathbb{R}^n$, which is bounded and open, and let Γ be a sufficiently smooth boundary of Ω. Problem (5.1) can be formulated in an abstract form of boundary control system (2.1) if we set $Y = L^2(\Omega), X = H^{-\frac{1}{2}}(\Gamma), U = L^2(\Gamma), B_1 = I$, the identity operator, and $\mu z = \frac{\partial^2 z}{\partial y^2}$ with $dom(\mu) = \Big\{z \in L^2(\Omega) : \frac{\partial^2 z}{\partial y^2} \in L^2(\Omega)\Big\}$. The trace operator α is well-defined and expressed as $\alpha z = z|_\Gamma$ and for every $z \in dom(\mu)$, $\alpha \in H^{-\frac{1}{2}}(\Gamma)$.

Express the operator $A : dom(A) \subset Y \to Y$ that is specified as $Az = \Delta z$ with $dom(A) = H_0^1(\Omega) \cap H^2(\Omega)$ where $H^k(\Omega), H^\alpha(\Omega)$ are applicable Sobolev spaces on Ω and Γ (see [29, 35, 47]).

We introduce the following functions:

$$\int_0^\zeta e(\zeta, s, z(\xi(s)))(y)ds = \frac{\sin z(\zeta,y)}{(1+\zeta)(1+\zeta^2)}\int_0^\zeta e^{-z(\sin s,y)}ds,$$

$$f\Big(\zeta, z(\xi(\zeta)), \int_0^\zeta e(\zeta, s, z(\xi(s)))ds\Big)(y) = \frac{1}{(1+\zeta)(1+\zeta^2)} \times \Big[z(\sin\zeta, y) + \sin z(\zeta,y)\int_0^\zeta e^{-z(\sin s,y)}ds\Big].$$

Further, A can be expressed as

$$Az = \sum_{n=1}^{\infty}(-n^2)(z, z_n)z_n, z \in dom(A),$$

where $z_n(y) = \sqrt{2}\sin ny, n = 1, 2, ...,$ is an orthogonal set of eigenvectors of A. In view of ([25] (see Assumptions 2, 3 and Remark 1.4) and [47]), A generates strongly continuous semigroup $T(\zeta) : \zeta \geq 0$ in Y and $T(\zeta)$ is compact in such a manner that $|T(\zeta)| \leq e^{-\zeta}$ for all $\zeta > 0$. Hence $(A1)$ is satisfied.

Again, we find

$$\left|\frac{1}{(1+\zeta)(1+\zeta^2)}\Big[z(\sin\zeta, y) + \sin z(\zeta,y)\int_0^\zeta e^{-z(\sin s,y)}ds\Big]\right| \leq \frac{1}{(1+\zeta^2)}|z|.$$

To verify assumptions $(A2)$ and $(A3)$, we select the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ to be specified as $Bu = v_u$, where $v_u \in L^2(\Omega)$ is the unique solution to the Dirichlet boundary value problem,

$$\begin{cases} \Delta v_u = 0 \; in \; \Omega, \\ v_u = u \; in \; \Gamma. \end{cases}$$

In other words

$$\int_\Omega v_u \Delta\Psi dx = \int_\Gamma u \frac{\partial\Psi}{\partial\eta} dx, \qquad \forall\Psi \in H_0^1(\Omega) \cup H^2(\Omega) \tag{5.2}$$

where $\frac{\partial\Psi}{\partial\eta}$ denotes the outward normal derivative of Ψ. This outward normal is well-defined as an element of $H^{\frac{1}{2}}(\Gamma)$. From (5.2), it follows that

$$\|v_u\|_{L^2(\Omega)} \leq C_5 \|u\|_{H^{\frac{-1}{2}}(\Gamma)}, \qquad \forall u \in H^{\frac{-1}{2}}(\Gamma),$$

and

$$\|v_u\|_{H^1(\Omega)} \leq C_6 \|u\|_{H^{\frac{1}{2}}(\Gamma)}, \qquad \forall u \in H^{\frac{1}{2}}(\Gamma).$$

From the above estimates, it follows by an interpolation argument [47] that

$$\|AT(\zeta)B\|_{L(L^2(\Omega),L^2(\Omega))} \leq C_7 \zeta^{\frac{-3}{4}}, \quad \forall\zeta \geq 0,$$

with $v(\zeta) = C_7\zeta^{\frac{-3}{4}}$ where C_5, C_6, C_7 are positive and independent of u.

Thus, all the assumptions $(A1) - (A3)$ are satisfied.

Verification of hypotheses:

In the context of Example 5.1, we verify the hypotheses required for the controllability results.

$(H1)$–$(H9)$: The nonlinear terms and integral kernels in the equation are smooth and bounded, fulfilling the measurability and continuity conditions in (H2), (H4)–(H6). Growth bounds of the nonlinearity and impulses ensure the integrability in (H3) and the contractive conditions in (H8). The compactness of the semigroup $T(\zeta)$, along with the boundedness of the operators, verifies (H1) and (H7). Finally, the function $m^*(\zeta)$ defined from the nonlinearity satisfies the integral inequality in (H9), ensuring the applicability of the fixed point theorem. Thus, Theorem 3.1 can be applied for (5.1) and so the system (5.1) is controllable on $[0, a]$.

Example 5.2. Assume the following partial functional integro-differential equations as the boundary control nonlinear system:

$$\begin{cases} \frac{\partial z(\zeta,y)}{\partial\zeta} = \frac{\partial^2 z(\zeta,y)}{\partial y^2} + c_1(\zeta) z(\sin\zeta, y) + c_2(\zeta)\sin z(\zeta, y) + \frac{1}{1+\zeta^2}\int_0^\zeta c_3(s) z(\sin s, y) ds \\ \qquad + \tilde{c}_1(\zeta) z(\sin\zeta, y) + \tilde{c}_2(\zeta)\sin z(\zeta, y) \\ \qquad + \frac{1}{1+\zeta^2}\int_0^\zeta \tilde{c}_3(s) z(\sin s, y) ds, y \in \Omega, \zeta \in J = [0, a], \zeta \neq \zeta_k; \\ z(\zeta, 0) = u(\zeta, 0), on \sum = (0, a) \times \Gamma, \zeta \in [0, a], \zeta \neq \zeta_k; \\ z(\zeta, y) = 0, z(0, y) = z_0(y), y \in \Omega; \\ \Delta z(\zeta_k, y) = z(\zeta_k^+, y) - z(\zeta_k^-, y) = I_k(z(\zeta_k^-)), \end{cases} \tag{5.3}$$

where $z_0(y) \in L^2(\Omega)$ and $u \in L^2(\sum)$. Problem (5.3) can be formulated in abstract form of boundary control system (4.1) by taking a suitable choice of spaces and operators as we described in the previous example.

We assume the following condition: The functions $c_i(.)$ and $\tilde{c}_i, i = 1,2,3$, are continuous on $[0,1]$; $\delta_i = \sup_{0\le s\le 1} |c_i(s)| < 1, i = 1,2,3$; and $\tilde{\delta}_i = \sup_{0\le s\le 1} |\tilde{c}_i(s)| < 1, i = 1,2,3$.

We introduce the following functions:

$$
\begin{aligned}
f_1\Big(\zeta, z(\xi(\zeta)), \int_0^\zeta e_1(\zeta, s, z(\xi(s)))ds\Big)(y) &= c_1(\zeta)z(\sin\zeta, y) \\
&\quad + c_2(\zeta)\sin z(\zeta, y) + \frac{1}{1+\zeta^2}\int_0^\zeta c_3(s)z(\sin s, y)ds; \\
f_2\Big(\zeta, z(\iota(\zeta)), \int_0^\zeta e_2(\zeta, s, z(\iota(s)))ds\Big)(y) &= \tilde{c}_1(\zeta)z(\sin\zeta, y) \\
&\quad + \tilde{c}_2(\zeta)\sin z(\zeta, y) + \frac{1}{1+\zeta^2}\int_0^\zeta \tilde{c}_3(s)z(\sin s, y)ds;
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\zeta e_1(\zeta, s, z(\xi(s)))(y)ds &= \frac{1}{1+\zeta^2}\int_0^\zeta c_3(s)z(\sin s, y)ds; \\
\int_0^\zeta e_2(\zeta, s, z(\iota(s)))(y)ds &= \frac{1}{1+\zeta^2}\int_0^\zeta \tilde{c}_3(s)z(\sin s, y)ds.
\end{aligned}
$$

The remainder is the same as is given in Example 5.1. So, we omitted it here. Thus, Theorem 4.1 can be employed for (5.3) and so system (5.3) is controllable on $[0, a]$.

6. Discussion and conclusions

This research comprehensively investigates the boundary controllability of nonlinear impulsive integro-differential evolution systems (NIIESs) with time-varying delays in Banach spaces. By employing advanced mathematical frameworks, including fixed point theorems and semigroup theory, the study successfully establishes sufficient conditions for the controllability of two distinct classes of systems.

The study's primary contributions include introducing rigorous hypotheses tailored to impulsive systems, deriving novel controllability results, and validating theoretical constructs through illustrative examples. These advancements address significant gaps in existing literature and provide a robust foundation for analyzing complex real-world systems with impulsive conditions and nonlinearity.

The findings have profound implications for a variety of fields, including engineering, biological modeling, and economics, where controlling systems with sudden state changes is crucial. For instance, applications in population dynamics, medical rhythms, and viscoelastic systems can directly benefit from the methodologies developed here.

Future research directions could focus on extending these results to stochastic systems, fractional-order models, and multidimensional systems. Additionally, exploring the numerical implementation of the theoretical findings can bridge the gap between mathematical theory and practical applications, enabling further innovation in system control.

Author contributions

Kamalendra Kumar: Conceptualization, Investigation, Methodology, Writing-original draft; Rohit Patel: Methodology,Writing-original draft; Mohammad Sajid: Funding acquisition, Writing-review and editing; Rakesh Kumar: Investigation, Writing-original draft.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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