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A fast second-order block-centered finite difference method for the fractional Cattaneo equation

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Abstract: In this paper, a fast second-order block-centered finite difference method based on the $\mathcal{FL}2$-1_σ formula and a weighted approach was proposed for the time fractional Cattaneo equation. Using the special properties of the discrete coefficients and mathematical induction method, we derived the unconditional stability and convergence of the method rigorously. Second-order superconvergence in time and space for the velocity and pressure in discrete $L^\infty(L^2)$-norms was established on non-uniform rectangular grids. Numerical examples were provided to verify our theoretical results.

Keywords: fractional Cattaneo equation; Caputo derivative; block-centered finite difference; fast algorithm; $\mathcal{FL}2$-1_σ formula; superconvergence
Mathematics Subject Classification: 65N06, 65N15

1. Introduction

In this paper, we consider the following time fractional Cattaneo equation:

$$\frac{\partial p}{\partial t} + {}_0^C D_t^\alpha p - \nabla \cdot (a \nabla p) = f(x,y,t), \qquad (x,y,t) \in \Omega \times (0,T], \tag{1.1}$$

$$a \nabla p \cdot \boldsymbol{n} = 0, \qquad (x,y,t) \in \partial\Omega \times (0,T], \tag{1.2}$$

$$p(x,y,0) = \varphi(x,y), \quad \frac{\partial p(x,y,0)}{\partial t} = \psi(x,y), \qquad (x,y) \in \Omega, \tag{1.3}$$

where $\Omega = (x_L, x_R) \times (y_L, y_R)$, $\partial\Omega$ is the boundary, T is the final time, and $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. p is the pressure, $f(x,y,t)$, $\varphi(x,y)$, $\psi(x,y)$ are given known functions, and $a = \mathrm{diag}(a^x(x,y,t), a^y(x,y,t))$ is a positive definite matrix, i.e., there are two positive constants a_* and a^* such that

$$0 < a_* \le a^x \le a^*, \quad 0 < a_* \le a^y \le a^*.$$

The Caputo fractional derivative ${}_0^C D_t^\alpha p$ is defined by

$$ {}_0^C D_t^\alpha p(x,y,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 p(x,y,s)}{\partial s^2} (t-s)^{1-\alpha} \mathrm{d}s, \quad 1 < \alpha < 2. $$

Equations of this type describe the combined diffusion and wave-like behavior of heat conduction and have been widely used in various fields such as chemistry, biochemistry, thermoelasticity, and fluid mechanics [1–3]. More discussions of (1.1)–(1.3) are given in [4, 5].

In recent years, many works have been devoted to numerical methods for the fractional Cattaneo equation. For example, a high-order compact finite difference scheme for the generalized fractional Cattaneo equation was proposed in [6], where the Caputo fractional derivative was discretized by the classical $\mathcal{L}1$ formula. Zhao and Sun [7] constructed a compact Crank-Nicolson difference scheme for the one-dimensional fractional Cattaneo equation, where both the integer-order and fractional-order time derivatives were discretized by the Crank-Nicolson scheme. Ren and Gao [8] developed some compact alternating direction implicit difference schemes for the two-dimensional time fractional Cattaneo equation. Wei [9] presented a new finite difference method to approximate the Caputo fractional derivative and then applied it to introduce a fully discrete local discontinuous Galerkin method for the fractional Cattaneo equation. Nikan et al. [10] studied a local meshless method for the time fractional Cattaneo model.

The block-centered finite difference (BCFD) method is a powerful tool in computational fluid dynamics [11–13]. The key idea of this method is to approximate the pressure at the cell center, the x-component of velocity at the midpoint of the vertical edges of the cell, and the y-component of velocity at the midpoint of the horizontal edges of the cell. The advantage of the BCFD method is that it can guarantee mass conservation and approximate simultaneously the velocity and pressure with second-order accuracy on non-uniform rectangular grids. Recently, the BCFD method has been applied to solve various fractional partial differential equations, such as the time fractional diffusion equation [14–16], time fractional advection-dispersion equation [17], nonlinear fractional cable equation [18], and time fractional diffusion-wave equation [19]. In particular, the BCFD method combined with the Crank-Nicolson scheme for (1.1)–(1.3) was analyzed in [20].

To the best of our knowledge, most of the above methods can only achieve $(3-\alpha)$-order accuracy in time. Moreover, due to the nonlocal dependence of Caputo fractional derivatives, the above methods require the storage of the solutions at all previous time steps, which leads to large computation time and memory. In order to improve computational efficiency, Yan et al. employed the sum-of-exponentials (SOE) approximation to the kernel function and proposed a fast and second-order $\mathcal{FL}2\text{-}1_\sigma$ method for the time fractional diffusion equations in [21]. Based on the $\mathcal{FL}2\text{-}1_\sigma$ formula and a weighted approach, Lyu et al. [22] constructed a fast linearized finite difference scheme for the nonlinear time fractional wave equation. Their scheme can achieve second-order accuracy in time and greatly reduces the computational complexity. Subsequently, Bai et al. [23] presented a fast second-order finite-difference time-domain method for the time fractional Maxwell system. Although there are some results of fast BCFD methods for the fractional diffusion equations [24, 25], there are no studies on the fast second-order BCFD method for the fractional Cattaneo equation.

The purpose of this paper is to propose and analyze a fast second-order BCFD method for (1.1)–(1.3). For the spatial discretization, we adopt the BCFD method, and for the time discretization, we apply a proper linear weighted combination of the $\mathcal{FL}2\text{-}1_\sigma$ formula for the Caputo fractional derivative [22], and establish fitted time approximations for the integer-order terms. Using the special properties of the discrete coefficients and mathematical induction method, we derive the unconditional stability of the proposed method rigorously. Then, by establishing a proper relation between the velocity and the difference of the pressure, we show that the proposed method yields second-order superconvergence in both time and space for the velocity and pressure in discrete $L^\infty(L^2)$-norms on a non-uniform rectangular grid. Compared with the original BCFD method in [20], our new method is more accurate and efficient.

The structure of the paper is as follows. In Section 2, we first introduce some notation and basic results, and then develop a fast second-order BCFD method for (1.1)–(1.3), where a special approximation is used for the solution at the first time step. In Section 3, we apply the mathematical induction method to establish the stability of the method. The superconvergent error estimate is derived in Section 4. Finally, in Section 5, three numerical examples are presented to support our theoretical results.

Throughout the paper, we use C, with or without subscripts, to denote a generic positive constant, whose value may vary in each occurrence.

2. The fast second-order BCFD method

To illustrate our fast second-order BCFD method, we introduce the velocity $\boldsymbol{u} = (u^x, u^y) = -a\nabla p$ and rewrite (1.1) and (1.2) as

$$\frac{\partial p}{\partial t} + {}_0^C D_t^\alpha p + \nabla \cdot \boldsymbol{u} = f(x,y,t), \qquad (x,y,t) \in \Omega \times (0,T], \tag{2.1}$$

$$\boldsymbol{u} = -a\nabla p, \qquad (x,y,t) \in \Omega \times (0,T], \tag{2.2}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad (x,y,t) \in \partial\Omega \times (0,T]. \tag{2.3}$$

For the time discretization, a proper linear weighted combination of the $\mathcal{FL}2\text{-}1_\sigma$ formula is applied for the Caputo fractional derivative and fitted time approximations are established for the integer-order terms. Let N be a positive integer, $\tau = \frac{T}{N} < 1$ be the time step, $\sigma = \frac{3-\alpha}{2}$, $t_\sigma = \sigma\tau$, and $t_n = n\tau, 0 \le n \le N$. For a smooth function p on $[0,T]$, we define $p^\sigma = p(t_\sigma)$ and $p^n = p(t_n)$. Moreover, we set

$$\begin{aligned}
&d_t p^{n+\frac{1}{2}} = \frac{p^{n+1} - p^n}{\tau}, \quad \widetilde{p}^{n+\sigma} = \sigma p^{n+1} + (1-\sigma)p^n, \quad n \ge 0,\\
&\overline{p}^{n-\frac{1}{2}} = \frac{3-2\sigma}{2}\widetilde{p}^{n-1+\sigma} + \frac{2\sigma-1}{2}\widetilde{p}^{n-2+\sigma}, \quad n \ge 2,\\
&d_t\overline{p}^n = \frac{\overline{p}^{n+\frac{1}{2}} - \overline{p}^{n-\frac{1}{2}}}{\tau}, \quad d_{\hat{t}} p^n = \frac{p^{n+1} - p^{n-1}}{\tau}, \quad \widehat{p}^n = \frac{\overline{p}^{n+\frac{1}{2}} + \overline{p}^{n-\frac{1}{2}}}{2}, \quad n \ge 1,\\
&D_t p^{n+1-\sigma} = (2-2\sigma)d_t p^{n+\frac{1}{2}} + (2\sigma-1)d_{\hat{t}} p^n, \quad n \ge 1,\\
&\widetilde{d}_t p^{n+1} = \sigma D_t p^{n+1-\sigma} + (1-\sigma)D_t p^{n-\sigma}, \quad n \ge 1.
\end{aligned}$$

The SOE approximation proposed in [21,26] reads as:

Lemma 2.1. *Let $\beta = \alpha - 1$. Then for given tolerance error $\varepsilon \ll 1$, and final time T, there exist a positive integer N_e, positive quadrature nodes s_j, and corresponding positive weights ω_j such that*

$$\left|\frac{t^{-\beta}}{\Gamma(1-\beta)} - \sum_{j=1}^{N_e} \omega_j e^{-s_j t}\right| \le \varepsilon, \quad \forall t \in [\sigma\tau, T],$$

and the number of quadrature nodes needed is of the order

$$N_e = \mathcal{O}\left(\log\frac{1}{\varepsilon}\left(\log\log\frac{1}{\varepsilon} + \log\frac{T}{\sigma\tau}\right) + \log\frac{1}{\sigma\tau}\left(\log\log\frac{1}{\varepsilon} + \log\frac{1}{\sigma\tau}\right)\right).$$

Denote

$$\begin{aligned}
\lambda &= \frac{\tau^{1-\alpha}\sigma^{2-\alpha}}{\Gamma(3-\alpha)},\\
A_j &= \int_0^1 \left(\frac{3}{2} - s\right)e^{-s_j\tau(\sigma+1-s)}\mathrm{d}s,\\
B_j &= \int_0^1 \left(s - \frac{1}{2}\right)e^{-s_j\tau(\sigma+1-s)}\mathrm{d}s.
\end{aligned}$$

For $n = 0$, let

$$g_0^{(1)} = \frac{\lambda}{1-\sigma}.$$

For $n \geq 1$, let

$$g_i^{(n+1)} = \begin{cases}
\sum_{j=1}^{N_e} \omega_j e^{-(n-1)s_j\tau}[A_j - \frac{1}{2}B_j], & i = 0,\\
\sum_{j=1}^{N_e} \omega_j [e^{-(n-k-1)s_j\tau}A_j + e^{-(n-k)s_j\tau}B_j], & 1 \leq i \leq n-1,\\
\sum_{j=1}^{N_e} \omega_j B_j + \lambda, & i = n.
\end{cases}$$

Denote

$$\begin{aligned}
D_t p^{1-\sigma} &= 2(1-\sigma)d_t p^{\frac{1}{2}} + (2\sigma - 1)\frac{\partial p^0}{\partial t},\\
D_t p^{-\sigma} &= \tilde{b}_n(D_t p^{1-\sigma} - \frac{\partial p^0}{\partial t}) + \frac{\partial p^0}{\partial t},
\end{aligned}$$

where

$$b_n = \sum_{j=1}^{N_e} \omega_j e^{-(n-1)s_j\tau}B_j, \quad \tilde{b}_n = \frac{(3\sigma-1)b_n}{2(1-\sigma)g_0^{(n+1)}}.$$

Then, the weighted combination of the $\mathcal{FL}2\text{-}1_\sigma$ formula proposed in [22] for the Caputo derivative is:

$${}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, p^{n+1} = \begin{cases}
2(1-\sigma)g_0^{(1)}(d_t p^{\frac{1}{2}} - \frac{\partial p^0}{\partial t}), & n = 0,\\
\sum_{k=0}^{n} g_k^{(n+1)}(D_t p^{k+1-\sigma} - D_t p^{k-\sigma}), & 1 \leq n \leq N-1.
\end{cases}$$

According to [27–29], we have the following lemmas.

Lemma 2.2. *Assume that $p \in C^4[0,T]$. Then we have*

$$
\begin{aligned}
{}_0^C D_t^{\alpha}\, p(t_\sigma) &= {}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, p^1 + O(g_0^{(1)}\tau^2),\\
{}_0^C D_t^{\alpha}\, p(t_n) &= {}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, p^{n+1} + O(g_0^{(n+1)}\tau^2 + \varepsilon), \quad 1 \le n \le N-1.
\end{aligned}
$$

Lemma 2.3. *Assume that $p \in C^3[0,T]$. Then we have*

$$
\begin{aligned}
\frac{\partial p^{\sigma}}{\partial t} &= 2\sigma d_t p^{\frac{1}{2}} + (1-2\sigma)\frac{\partial p^0}{\partial t} + O(\tau^2),\\
\frac{\partial p^{n}}{\partial t} &= \widetilde{d}_t p^{n+1} + O(\tau^2), \quad 1 \le n \le N-1.
\end{aligned}
$$

Lemma 2.4. *The sequence $\{\tilde{b}_n\}(n \ge 1)$ satisfies*

$$
-1 < \tilde{b}_n < \frac{3\sigma - 1}{1-\sigma}.
$$

Lemma 2.5. *The sequences $\{g_0^{(i+1)}\}(0 \le i \le n)$ and $\{g_i^{(n+1)}\}(1 \le n \le N-1, 1 \le i \le n)$ satisfy*

$$
(a)\ g_1^{(n+1)} \le C g_0^{(n+1)}; \qquad (b)\ \tau \sum_{i=0}^{n} g_0^{(i+1)} < C \max_{1\le\gamma\le 2} t_n^{2-\gamma};
$$

$$
(c)\ \tau \sum_{i=0}^{n} \frac{1}{g_0^{(i+1)}} < C_\alpha \max_{1\le\gamma\le 2} t_{n+\sigma}^{\gamma}; \qquad (d)\ \tau \sum_{i=0}^{n} \frac{1}{g_i^{(n+1)}} < C t_{n+\sigma}^{2-\alpha}.
$$

Lemma 2.6. *Assume that $u \in C^2[0,T]$. Then we have*

$$
\begin{aligned}
u(t_\sigma) &= \widetilde{u}^{\sigma} + O(\tau^2),\\
u(t_n) &= \widehat{u}^{n} + O(\tau^2), \quad 1 \le n \le N-1.
\end{aligned}
$$

Lemma 2.7. *For any real sequence $\{F^n\}(1 \le n \le N-1)$, when τ and ε are sufficiently small, we have*

$$
\begin{aligned}
2\widetilde{d}_t p^{n+1}\left[{}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, p^{n+1} - F^n\right] \ \ge\ & \sum_{i=0}^{n} g_i^{(n+1)}(D_t p^{i+1-\sigma})^2 - \sum_{i=0}^{n-1} g_i^{(n)}(D_t p^{i+1-\sigma})^2\\
& -(g_1^{(n+1)} - g_1^{(n)})(D_t p^{1-\sigma})^2 - g_0^{(n+1)}\Big(D_t p^{-\sigma} + \frac{1}{g_0^{(n+1)}}F^n\Big)^2.
\end{aligned}
$$

Lemma 2.8. *For integer $n \ge 0$, let τ, H and $a_n, b_n, c_n, d_n, \gamma_n$ be non-negative numbers such that*

$$
a_M + \tau \sum_{n=0}^{M} b_n \le \tau \sum_{n=0}^{M} \gamma_n a_n + \tau \sum_{n=0}^{M} c_n + H, \quad M \ge 0,
$$

and $\tau\gamma_n < 1$. Then

$$
a_M + \tau \sum_{n=0}^{M} b_n \le \exp\Big(\tau \sum_{n=0}^{M} \frac{\gamma_n}{1-\tau\gamma_n}\Big)\Big(\tau \sum_{n=0}^{M} c_n + H\Big), \quad M \ge 0.
$$

For the spatial discretization, the BCFD method is considered. Let N_x and N_y be two positive integers. The domain Ω is partitioned by $\mathscr{G}_x \times \mathscr{G}_y$, where

$$\begin{aligned}
\mathscr{G}_x &: x_L = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = x_R, \\
\mathscr{G}_y &: y_L = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = y_R.
\end{aligned}$$

For $1 \le i \le N_x$ and $1 \le j \le N_y$, we use the following notations

$$\begin{aligned}
x_i &= \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}, \quad & y_j &= \frac{y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}}{2}, \\
h_i &= x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad & k_j &= y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, \\
h &= \max_{1\le i\le N_x} h_i, \quad & k &= \max_{1\le j\le N_y} k_j.
\end{aligned}$$

We assume that $\mathscr{G}_x \times \mathscr{G}_y$ is regular, that is, there exists a positive constant C such that

$$\min_{\substack{1\le i\le N_x \\ 1\le j\le N_y}} \{h_i, k_j\} \ge C \max_{\substack{1\le i\le N_x \\ 1\le j\le N_y}} \{h_i, k_j\}.$$

Define the grid function spaces:

$$\begin{aligned}
&\mathcal{P}_h = \{q_{i,j} | 1 \le i \le N_x, 1 \le j \le N_y\}, \\
&\mathcal{V}_h^x = \{v^x_{i+\frac{1}{2},j} | v^x_{\frac{1}{2},j} = v^x_{N_x+\frac{1}{2},j} = 0, 0 \le i \le N_x, 1 \le j \le N_y\}, \\
&\mathcal{V}_h^y = \{v^y_{i,j+\frac{1}{2}} | v^y_{i,\frac{1}{2}} = v^y_{i,N_y+\frac{1}{2}} = 0, 1 \le i \le N_x, 0 \le j \le N_y\}, \\
&\mathcal{V}_h = \mathcal{V}_h^x \times \mathcal{V}_h^y.
\end{aligned}$$

For a function $\omega(x, y)$, let $\omega_{l,m}$ denote $\omega(x_l, y_m)$, where l may take values $i, i+\frac{1}{2}$ for integer i, and m may take values $j, j+\frac{1}{2}$ for integer j. Then, we define the differential operators:

$$\begin{aligned}
[d_x\omega]_{i+\frac{1}{2},m} &= \frac{\omega_{i+1,m} - \omega_{i,m}}{h_{i+\frac{1}{2}}}, \qquad & [D_y\omega]_{l,j} &= \frac{\omega_{l,j+\frac{1}{2}} - \omega_{l,j-\frac{1}{2}}}{k_j}, \\
[D_x\omega]_{i,m} &= \frac{\omega_{i+\frac{1}{2},m} - \omega_{i-\frac{1}{2},m}}{h_i}, \qquad & [d_y\omega]_{l,j+\frac{1}{2}} &= \frac{\omega_{l,j+1} - \omega_{l,j}}{k_{j+\frac{1}{2}}}.
\end{aligned}$$

For functions ω and ψ, we define the following discrete L^2 inner products and norms:

$$\begin{aligned}
(\omega, \psi)_m &= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i k_j \omega_{i,j} \psi_{i,j}, \\
(\omega, \psi)_x &= \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} h_{i+\frac{1}{2}} k_j \omega_{i+\frac{1}{2},j} \psi_{i+\frac{1}{2},j}, \\
(\omega, \psi)_y &= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-1} h_i k_{j+\frac{1}{2}} \omega_{i,j+\frac{1}{2}} \psi_{i,j+\frac{1}{2}}, \\
\|\omega\|_F^2 &= (\omega, \omega)_F, \quad F = m, x, y.
\end{aligned}$$

For a vector-valued function $\boldsymbol{u} = (u^x, u^y)$, we define the discrete L^2 norm

$$\|\boldsymbol{u}\|_{TM}^2 = \|u^x\|_x^2 + \|u^y\|_y^2.$$

For $1 \le i \le N_x$ and $1 \le j \le N_y$, define

$$\delta_{i,j} = \frac{h_i^2}{8}\frac{\partial^2 p_{i,j}}{\partial x^2} + \frac{k_j^2}{8}\frac{\partial^2 p_{i,j}}{\partial y^2}.$$

Then according to [11, 20], we have the following lemmas.
Lemma 2.9. *If p is sufficiently smooth, then it holds that*

$$\begin{aligned}
&\frac{\partial \delta^0}{\partial t} = O(h^2 + k^2),\\
&D_t \delta^{1-\sigma} = O(h^2 + k^2),\\
&\widetilde{d}_t \delta_{i,j}^{n+1} = O(h^2 + k^2), \quad 1 \le n \le N-1,\\
&{}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, \delta^{n+1} = O(h^2 + k^2), \quad 0 \le n \le N-1.
\end{aligned}$$

Lemma 2.10. *If p and $\boldsymbol{u}$ are sufficiently smooth, then for $1 \le n \le N-1$, it holds that*

$$\begin{aligned}
\left[\frac{1}{a^x}\overline{u}^x\right]_{i+\frac{1}{2},j}^{n+\frac{1}{2}} &= -[d_x(\overline{p} - \overline{\delta})]_{i+\frac{1}{2},j}^{n+\frac{1}{2}} + \overline{\epsilon}_{i+\frac{1}{2},j}^{x}(p^{n+\frac{1}{2}}), \quad (x_{i+\frac{1}{2}}, y_j) \in \Omega,\\
\left[\frac{1}{a^y}\overline{u}^y\right]_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} &= -[d_y(\overline{p} - \overline{\delta})]_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} + \overline{\epsilon}_{i,j+\frac{1}{2}}^{y}(p^{n+\frac{1}{2}}), \quad (x_i, y_{j+\frac{1}{2}}) \in \Omega,
\end{aligned}$$

with the following approximate properties:

$$\begin{aligned}
\overline{\epsilon}_{i+\frac{1}{2},j}^{x}(p^{n+\frac{1}{2}}) &= O(\tau^2 + h^2 + k^2),\\
\overline{\epsilon}_{i,j+\frac{1}{2}}^{y}(p^{n+\frac{1}{2}}) &= O(\tau^2 + h^2 + k^2).
\end{aligned}$$

Lemma 2.11. *For $(v^x, v^y) \in \boldsymbol{\mathcal{V}}_h$ and $q \in P_h$, it holds that*

$$\begin{aligned}
(D_x v^x, q)_m &= -(v^x, d_x q)_x,\\
(D_y v^y, q)_m &= -(v^y, d_y q)_y.
\end{aligned}$$

In order to obtain high-order approximations at the first time step, we use the Taylor expansion to get

$$p^{\sigma} = \varphi + \sigma\tau\psi + O(\tau^2).$$

It follows from Lemmas 2.2 and 2.3 that

$${}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, p^1 + 2\sigma d_t p^{\frac{1}{2}} + (1-2\sigma)\psi - \nabla\cdot(a^{\sigma}\nabla(\varphi + \sigma\tau\psi)) = f^{\sigma} + O(g_0^{(1)}\tau^2),$$

which leads to

$$2(\sigma+\lambda) d_t p^{\frac{1}{2}} = (2\sigma + 2\lambda - 1)\psi + \nabla\cdot(a^{\sigma}\nabla(\varphi + \sigma\tau\psi)) + f^{\sigma} + O(g_0^{(1)}\tau^2).$$

Define

$$\varpi = \varphi + \frac{\tau}{2(\sigma+\lambda)}\Big\{(2\sigma+2\lambda-1)\psi + \nabla\cdot(a^{\sigma}\nabla(\varphi+\sigma\tau\psi)+f^{\sigma}\Big\}.$$

Then we have

$$\frac{p^1-\varpi}{\tau} = O(g_0^{(1)}\tau^2).$$

Therefore,

$$|\varpi - p^1| = O(\tau^3). \tag{2.4}$$

With the above preparations, our fast second-order BCFD method reads as follows: Find $P^n \in \mathcal{P}_h$ and $\boldsymbol{U}^n = (U^{x,n}, U^{y,n}) \in \mathcal{V}_h$ such that

$$[\widetilde{d_t P}]_{i,j}^{n+1} + [{}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha} P]_{i,j}^{n+1} + [D_x\widehat{U}^x]_{i,j}^n + [D_y\widehat{U}^y]_{i,j}^n = f_{i,j}^n, \qquad 1\le i\le N_x, 1\le j\le N_y, 1\le n\le N-1, \tag{2.5}$$

$$\overline{U}_{i+\frac{1}{2},j}^{x,n+\frac{1}{2}} = -a^{x,n+\frac{1}{2}}[d_x\overline{P}]_{i+\frac{1}{2},j}^{n+\frac{1}{2}}, \quad 1\le i\le N_x-1, 1\le j\le N_y, 0\le n\le N-1, \tag{2.6}$$

$$\overline{U}_{i,j+\frac{1}{2}}^{y,n+\frac{1}{2}} = -a^{y,n+\frac{1}{2}}[d_y\overline{P}]_{i,j+\frac{1}{2}}^{n}, \quad 1\le i\le N_x, 1\le j\le N_y-1, 0\le n\le N-1, \tag{2.7}$$

with

$$P_{i,j}^0 = \varphi_{i,j}, \quad 1\le i\le N_x, 1\le j\le N_y, \tag{2.8}$$

$$P_{i,j}^1 = \varpi_{i,j}, \quad 1\le i\le N_x, 1\le j\le N_y, \tag{2.9}$$

$$U_{i+\frac{1}{2},j}^{x,0} = u_{i+\frac{1}{2},j}^{x,0}, \quad 1\le i\le N_x-1, 1\le j\le N_y, \tag{2.10}$$

$$U_{i,j+\frac{1}{2}}^{y,0} = u_{i,j+\frac{1}{2}}^{y,0}, \quad 1\le i\le N_x, 1\le j\le N_y-1. \tag{2.11}$$

Here,

$$\begin{aligned}
u^{x,0} &= -\Big[a^x\frac{\partial\varphi}{\partial x}\Big]^0, \quad u^{y,0} = -\Big[a^y\frac{\partial\varphi}{\partial y}\Big]^0,\\
\Psi_{i,j} &= \psi_{i,j}, \quad 1\le i\le N_x, 1\le j\le N_y,\\
\overline{P}^{\frac{1}{2}} &= \frac{3-2\sigma}{2}\widetilde{P}^{\sigma} + \frac{2\sigma-1}{2}[\sigma P^0 + (1-\sigma)(P^1-2\tau\Psi)],\\
\overline{U}^{x,\frac{1}{2}} &= \frac{3-2\sigma}{2}\widetilde{U}^{x,\sigma} + \frac{2\sigma-1}{2}[\sigma U^{x,0} + (1-\sigma)(U^{x,1}-2\tau d_x\Psi)],\\
\overline{U}^{y,\frac{1}{2}} &= \frac{3-2\sigma}{2}\widetilde{U}^{y,\sigma} + \frac{2\sigma-1}{2}[\sigma U^{y,0} + (1-\sigma)(U^{y,1}-2\tau d_y\Psi)].
\end{aligned}$$

3. Stability of the method

In this section, we study the stability of the method (2.5)–(2.11). The existence and uniqueness of numerical solutions to (2.5)–(2.11) will follow from these stability results.

Taking $(V^x, V^y, Q_h) \in \mathcal{V}_h \times \mathcal{P}_h$, multiplying (2.5) by $h_i k_j Q_{ij}$, summing over $1 \le i \le N_x$, $1 \le j \le N_y$, and using Lemma 2.11, we obtain that for $1 \le n \le N-1$,

$$(\widetilde{d}_t P^{n+1}, Q)_m + ({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha} P^{n+1}, Q)_m - (\widehat{U}^{x,n}, d_x Q)_x - (\widehat{U}^{y,n}, d_y Q)_y = (f^n, Q)_m. \tag{3.1}$$

Similarly, for $0 \le n \le N-1$, it holds that

$$\left(\left(\frac{1}{a^x}\overline{U}^x\right)^{n+\frac{1}{2}}, V^x\right)_x = -(d_x \overline{P}^{n+\frac{1}{2}}, V^x)_x, \tag{3.2}$$

$$\left(\left(\frac{1}{a^y}\overline{U}^y\right)^{n+\frac{1}{2}}, V^y\right)_y = -(d_y \overline{P}^{n+\frac{1}{2}}, V^y)_y. \tag{3.3}$$

Theorem 3.1. *For sufficiently small τ and ε, the method (2.5)–(2.11) is unconditionally stable with the following estimate:*

$$\|P^n\|_m + \|\boldsymbol{U}^n\|_{TM} \le C\Big[\|f^{\sigma}\|_m + \sum_{i=1}^{4}(\|\varrho_i\|_m + \|d_x\varrho_i\|_x + \|d_y\varrho_i\|_y) + \Big(\tau\sum_{i=1}^{2}\|f^n\|_m^2\Big)^{\frac{1}{2}}\Big],$$

where

$$\varrho_1 = \varphi, \quad \varrho_2 = \psi, \quad \varrho_3 = \nabla\cdot(a\nabla\varphi), \quad \varrho_4 = \nabla\cdot(a\nabla\psi).$$

Proof. To prove the theorem, we only need to show that for $1 \le n \le N$,

$$S^n \le C\Big[\|f^0\|_m + \|f^1\|_m + \sum_{i=1}^{4}(\|\varrho_i\|_m + \|d_x\varrho_i\|_x + \|d_y\varrho_i\|_y) + \Big(\tau\sum_{i=0}^{2}\|f^n\|_m^2\Big)^{\frac{1}{2}}\Big], \tag{3.4}$$

where

$$S^n = \max\{(\|P^n\|_m + \|\widetilde{U}^{x,n-1+\sigma}\|_x + \|\widetilde{U}^{y,n-1+\sigma}\|_y, (2\sigma-1)\|\boldsymbol{U}^n\|_{TM}\}. \tag{3.5}$$

Now we prove (3.4) by the mathematical induction method. From (2.6)–(2.11), we easily know that (3.4) holds for $n = 1$. Assume that (3.4) holds for $1 \le n \le N-1$. Then it is sufficient to show that (3.4) also holds for $n = N$.

A simple computation yields

$$\widetilde{d}_t P^{n+1} = d_t \overline{P}^n, \quad 1 \le n \le N-1.$$

Thus, choosing $Q = \widetilde{d}_t P^{n+1}$ in (3.1) gives

$$\|\widetilde{d}_t P^{n+1}\|_m^2 + ({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha} P^{n+1}, \widetilde{d}_t P^{n+1})_m - (\widehat{U}^{x,n}, d_x(d_t\overline{P}^n))_x - (\widehat{U}^{y,n}, d_y(d_t\overline{P}^n))_y = (f^n, \widetilde{d}_t P^{n+1})_m. \tag{3.6}$$

Applying Lemma 2.7 yields

$$2({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha} P^{n+1}, \widetilde{d}_t P^{n+1})_m \ \ge \ \sum_{i=0}^{n} g_i^{(n+1)}\|D_t P^{i+1-\sigma}\|_m^2 - \sum_{i=0}^{n-1} g_i^{(n)}\|D_t P^{i+1-\sigma}\|_m^2$$

$$- (g_1^{(n+1)} - g_0^{(n)})\|D_tP^{1-\sigma}\|_m^2 - g_0^{(n+1)}\|D_tP^{-\sigma}\|_m^2. \tag{3.7}$$

Due to (3.2), we can get

$$\left(d_t\Big(\frac{1}{a^x}\overline{U}^x\Big)^n, V^x\right)_x = -(d_x(d_t\overline{P}^n), V^x)_x.$$

It follows then that

$$\begin{aligned}
-(\widehat{U}^{x,n}, d_x(d_t\overline{P}^n))_x &= \left(d_t\Big(\frac{1}{a^x}\overline{U}^x\Big)^n, \widehat{U}^{x,n}\right)_x \\
&= \frac{1}{2\tau}\Big(\frac{1}{a^{x,n+\frac{1}{2}}}\overline{U}^{x,n+\frac{1}{2}} - \frac{1}{a^{x,n-\frac{1}{2}}}\overline{U}^{x,n-\frac{1}{2}}, \overline{U}^{x,n+\frac{1}{2}} + \overline{U}^{x,n-\frac{1}{2}}\Big)_x \\
&= \frac{1}{2}d_t\left\|\frac{1}{\sqrt{a^{x,n}}}\overline{U}^{x,n}\right\|_x^2 + \frac{1}{2}\Big(d_t\Big(\frac{1}{a^{x,n}}\Big)\overline{U}^{x,n+\frac{1}{2}}, \overline{U}^{x,n-\frac{1}{2}}\Big)_x \\
&\geq \frac{1}{2}d_t\left\|\frac{1}{\sqrt{a^{x,n}}}\overline{U}^{x,n}\right\|_x^2 - C(\|\overline{U}^{x,n+\frac{1}{2}}\|_x^2 + \|\overline{U}^{x,n-\frac{1}{2}}\|_x^2).
\end{aligned} \tag{3.8}$$

Similarly, using (3.3), we can easily obtain

$$-(\widehat{U}^{y,n}, d_y(d_t\overline{P}^n))_y \geq \frac{1}{2}d_t\left(\left\|\frac{1}{\sqrt{a^y}}\overline{U}^y\right\|_y^2\right)^n - C(\|\overline{U}^{y,n+\frac{1}{2}}\|_y^2 + \|\overline{U}^{y,n-\frac{1}{2}}\|_y^2). \tag{3.9}$$

Using the Cauchy-Schwarz inequality, we have

$$(f^n, \widetilde{d}_tP^{n+1})_m \leq \frac{1}{2}(\|\widetilde{d}_tP^{n+1}\|_m^2 + \|f^n\|_m^2). \tag{3.10}$$

Substituting (3.7)–(3.10) into (3.6), we conclude that

$$\begin{aligned}
&\|\widetilde{d}_tP^{n+1}\|_m^2 + \sum_{i=0}^{n} g_i^{(n+1)}\|D_tP^{i+1-\sigma}\|_m^2 - \sum_{i=0}^{n-1} g_i^{(n)}\|D_tP^{i+1-\sigma}\|_m^2 \\
&\quad + d_t\left(\left\|\frac{1}{\sqrt{a^x}}\overline{U}^x\right\|_x^2\right)^n + d_t\left(\left\|\frac{1}{\sqrt{a^y}}\overline{U}^y\right\|_y^2\right)^n \\
&\leq \left|g_1^{(n+1)} - g_0^{(n)}\right|\|D_tP^{1-\sigma}\|_m^2 + g_0^{(n+1)}\|D_tP^{-\sigma}\|_m^2 + \|f^n\|_m^2 \\
&\quad + C(\|\overline{U}^{x,n+\frac{1}{2}}\|_x^2 + \|\overline{U}^{x,n-\frac{1}{2}}\|_x^2 + \|\overline{U}^{y,n-\frac{1}{2}}\|_y^2 + \|\overline{U}^{y,n+\frac{1}{2}}\|_y^2).
\end{aligned} \tag{3.11}$$

Summing up (3.11) for n from 1 to $N-1$ and multiplying by τ, we get

$$\begin{aligned}
&\tau\sum_{n=1}^{N-1}\|\widetilde{d}_tP^{n+1}\|_m^2 + \tau\sum_{i=0}^{N-1} g_i^{(N)}\|D_tP^{i+1-\sigma}\|_m^2 + \frac{1}{a^*}(\|\overline{U}^{x,N-\frac{1}{2}}\|_x^2 + \|\overline{U}^{y,N-\frac{1}{2}}\|_y^2) \\
&\leq \frac{1}{a_*}(\|\overline{U}^{x,\frac{1}{2}}\|_x^2 + \|\overline{U}^{y,\frac{1}{2}}\|_y^2) + \tau g_0^{(1)}\|D_tP^{1-\sigma}\|_m^2 + \tau\sum_{n=1}^{N-1} g_0^{(n+1)}\|D_tP^{-\sigma}\|_m^2 \\
&\quad + \tau\sum_{n=1}^{N-1}\left|g_1^{(n+1)} - g_0^{(n)}\right|\|D_tP^{1-\sigma}\|_m^2 + \tau\sum_{n=1}^{N-1}\|f^n\|_m^2 + \tau\sum_{n=1}^{N-1}(\|\overline{U}^{x,n-\frac{1}{2}}\|_x^2 + \|\overline{U}^{y,n-\frac{1}{2}}\|_y^2).
\end{aligned} \tag{3.12}$$

Applying Lemma 2.8 to (3.12) gives

$$\tau\sum_{n=1}^{N-1}\|\widetilde{d}_tP^{n+1}\|_m^2+\tau\sum_{i=0}^{N-1}g_i^{(N)}\|D_tP^{i+1-\sigma}\|_m^2+\frac{1}{a^*}(\|\overline{U}^{x,N-\frac{1}{2}}\|_x^2+\|\overline{U}^{y,N-\frac{1}{2}}\|_y^2)$$
$$\le C\Big(\|\overline{U}^{x,\frac{1}{2}}\|_x^2+\|\overline{U}^{y,\frac{1}{2}}\|_y^2+\tau g_0^{(1)}\|D_tP^{1-\sigma}\|_m^2+\tau\sum_{n=1}^{N-1}g_0^{(n+1)}\|D_tP^{-\sigma}\|_m^2 \quad (3.13)$$
$$+\tau\sum_{n=1}^{N-1}\left|g_1^{(n+1)}-g_0^{(n)}\right|\|D_tP^{1-\sigma}\|_m^2+\tau\sum_{n=1}^{N-1}\|f^n\|_m^2\Big).$$

By Lemma 2.5, we have

$$\begin{aligned}\Big\|\sum_{i=0}^{N-1}\tau D_tP^{i+1-\sigma}\Big\|_m^2 &\le \Big(\tau\sum_{i=0}^{N-1}\frac{1}{g_i^{(N)}}\Big)\tau\sum_{i=0}^{N-1}g_i^{(N)}\|D_tP^{i+1-\sigma}\|_m^2\\ &\le C\Big(\tau\sum_{i=0}^{N-1}g_i^{(N)}\|D_tP^{i+1-\sigma}\|_m^2\Big).\end{aligned} \quad (3.14)$$

A direct computation shows that

$$\begin{aligned}&\Big\|(\frac{3}{2}-\sigma)P^0+(\sigma-\frac{1}{2})P^1-(2\sigma-1)\psi\Big\|_m^2+\Big\|\sum_{i=0}^{N-1}\tau D_tP^{i+1-\sigma}\Big\|_m^2\\ &\ge\frac{1}{2}\Big\|(\frac{3}{2}-\sigma)P^0+(\sigma-\frac{1}{2})P^1-(2\sigma-1)p_1+\sum_{i=0}^{N-1}\tau D_tP^{i+1-\sigma}\Big\|_m^2\\ &=\frac{1}{2}\Big\|(\frac{3}{2}-\sigma)P^{M+1}+(\sigma-\frac{1}{2})P^{N-1}\Big\|_m^2.\end{aligned} \quad (3.15)$$

From [22], we know that

$$\tau\sum_{n=1}^{N-1}\left|g_0^{(n+1)}\right|\le C, \quad (3.16)$$

and

$$\tau\sum_{n=1}^{N-1}\left|g_1^{(n+1)}-g_0^{(n)}\right|\le C. \quad (3.17)$$

Using Lemma 2.4, (2.8), and (2.9), we obtain

$$\begin{aligned}\|D_tP^{1-\sigma}\|_m^2 &= \|2(1-\sigma)d_tP^{\frac{1}{2}}+(2\sigma-1)\psi\|_m^2\\ &\le 2(\|d_tP^{\frac{1}{2}}\|_m^2+\|\psi\|_m^2)\\ &\le C(\|\psi\|_m^2+\|\varrho_3\|_m^2+\|\varrho_4\|_m^2+\|f^\sigma\|_m^2),\end{aligned} \quad (3.18)$$

and

$$\|D_tP^{-\sigma}\|_m^2 = \|\tilde{b}_nD_tp^{1-\sigma}+(1-\tilde{b}_n)\psi\|_m^2$$

$$\leq C(\|D_t p^{1-\sigma}\|_m^2 + \|\psi\|_m^2) \tag{3.19}$$
$$\leq C(\|\psi\|_m^2 + \|\varrho_3\|_m^2 + \|\varrho_4\|_m^2 + \|f^\sigma\|_m^2).$$

Combining estimates (3.13)–(3.19) yields

$$\begin{aligned}&\Big(\tau\sum_{n=1}^{N-1}\|\widetilde{d}_tP^{n+1}\|_m^2\Big)^{\frac{1}{2}} + \Big\|(\frac{3}{2}-\sigma)P^N + (\sigma-\frac{1}{2})P^{N-1}\Big\|_m + \|\overline{U}^{x,N-\frac{1}{2}}\|_x + \|\overline{U}^{y,N-\frac{1}{2}}\|_y\\ &\leq C\Big[\|f^\sigma\|_m + \sum_{i=1}^{4}(\|\varrho_i\|_m + \|d_x\varrho_i\|_x + \|d_y\varrho_i\|_y) + \Big(\tau\sum_{i=0}^{2}\|f^n\|_m^2\Big)^{\frac{1}{2}}\Big].\end{aligned} \tag{3.20}$$

On the other hand, using the triangle inequality, we have

$$\begin{aligned}&\Big\|(\frac{3}{2}-\sigma)P^N + (\sigma-\frac{1}{2})P^{N-1}\Big\|_m + \|\overline{U}^{x,N-\frac{1}{2}}\|_x + \|\overline{U}^{y,N-\frac{1}{2}}\|_y\\ &\geq (\frac{3}{2}-\sigma)[\|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y]\\ &\quad - (\sigma-\frac{1}{2})[\|P^{N-1}\|_m + \|\widetilde{U}^{x,N-2+\sigma}\|_x + \|\widetilde{U}^{y,N-2+\sigma}\|_y].\end{aligned} \tag{3.21}$$

Substituting (3.21) into (3.20), we get

$$\begin{aligned}&(\frac{3}{2}-\sigma)[\|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y]\\ &\quad - (\sigma-\frac{1}{2})[\|P^{N-1}\|_m + \|\widetilde{U}^{x,N-2+\sigma}\|_x + \|\widetilde{U}^{y,N-2+\sigma}\|_y]\\ &\leq C\Big[\|f^\sigma\|_m + \sum_{i=1}^{4}(\|\varrho_i\|_m + \|d_x\varrho_i\|_x + \|d_y\varrho_i\|_y) + \Big(\tau\sum_{i=0}^{2}\|f^n\|_m^2\Big)^{\frac{1}{2}}\Big].\end{aligned} \tag{3.22}$$

Now, we discuss the following two cases:

Case (I): $\|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y \leq \|P^{N-1}\|_m + \|\widetilde{U}^{x,N-2+\sigma}\|_x + \|\widetilde{U}^{y,N-2+\sigma}\|_y$.

(i) If $\|\boldsymbol{U}^N\|_{TM} \leq \|\boldsymbol{U}^{N-1}\|_{TM}$, then $S^N \leq S^{N-1}$, and (3.4) follows directly.

(ii) If $\|\boldsymbol{U}^N\|_{TM} \geq \|\boldsymbol{U}^{N-1}\|_{TM}$, then

$$\begin{aligned}\|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y &\geq \sigma\|\boldsymbol{U}^N\|_{TM} - (1-\sigma)\|\boldsymbol{U}^{N-1}\|_{TM}\\ &\geq (2\sigma-1)\|\boldsymbol{U}^N\|_{TM}.\end{aligned}$$

From (3.5), we get

$$S^N = \|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y \leq S^{N-1},$$

and thus (3.4) follows directly.

Case (II): $\|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y \geq \|P^{N-1}\|_m + \|\widetilde{U}^{x,N-2+\sigma}\|_x + \|\widetilde{U}^{y,N-2+\sigma}\|_y$.

In this situation, we have

$$(\frac{3}{2}-\sigma)[\|P^N\|_m + \|\widetilde{U}^{x,N-1+\sigma}\|_x + \|\widetilde{U}^{y,N-1+\sigma}\|_y]$$

$$
\begin{aligned}
&-(\sigma-\frac{1}{2})[\|P^{N-1}\|_m+\|\widetilde{U}^{x,N-2+\sigma}\|_x+\|\widetilde{U}^{y,N-2+\sigma}\|_y]\\
&\geq(2-2\sigma)[\|P^{N}\|_m+\|\widetilde{U}^{x,N-1+\sigma}\|_x+\|\widetilde{U}^{y,N-1+\sigma}\|_y].
\end{aligned}
$$

From (3.22), it follows that

$$
\begin{aligned}
&\|P^{N}\|_m+\|\widetilde{U}^{x,N-1+\sigma}\|_x+\|\widetilde{U}^{y,N-1+\sigma}\|_y\\
&\leq C\Big[\|f^{\sigma}\|_m+\sum_{i=1}^{4}(\|\varrho_i\|_m+\|d_x\varrho_i\|_x+\|d_y\varrho_i\|_y)+\Big(\tau\sum_{i=0}^{2}\|f^n\|_m^2\Big)^{\frac{1}{2}}\Big].
\end{aligned}
\tag{3.23}
$$

(i) If $\|\boldsymbol{U}^N\|_{TM}\leq\|\boldsymbol{U}^{N-1}\|_{TM}$, then we have

Subcase (1): $S^N=\|P^N\|_m+\|\widetilde{U}^{x,N-1+\sigma}\|_x+\|\widetilde{U}^{y,N-1+\sigma}\|_y$. Then, (3.4) easily follows from (3.23).

Subcase (2): $S^N=(2\sigma-1)\|\boldsymbol{U}^N\|_{TM}\leq(2\sigma-1)\|\boldsymbol{U}^{N-1}\|_{TM}\leq S^{N-1}$, which immediately leads to (3.4).

(ii) If $\|\boldsymbol{U}^N\|_{TM}\geq\|\boldsymbol{U}^{N-1}\|_{TM}$, then

$$
\|\widetilde{U}^{x,N-1+\sigma}\|_x+\|\widetilde{U}^{y,N-1+\sigma}\|_y\geq(2\sigma-1)\|\boldsymbol{U}^N\|_{TM}.
$$

From (3.5), we know that

$$
S^N=\|P^N\|_m+\|\widetilde{U}^{x,N-1+\sigma}\|_x+\|\widetilde{U}^{y,N-1+\sigma}\|_y.
$$

Using (3.23), (3.4) is clarified, and so the proof is finished. □

4. Superconvergence of the method

In this section, we investigate the superconvergence of the method (2.5)–(2.11).

Let us introduce the error functions:

$$
\begin{aligned}
&\xi_{i,j}^n=(P-p-\delta)_{i,j}^n, && 1\leq i\leq N_x, 1\leq j\leq N_y,\\
&\eta_{i+\frac{1}{2},j}^{x,n}=(U^x-u^x)_{i+\frac{1}{2},j}^n, && 0\leq i\leq N_x, 1\leq j\leq N_y,\\
&\eta_{i,j+\frac{1}{2}}^{y,n}=(U^x-u^x)_{i,j+\frac{1}{2}}^n, && 1\leq i\leq N_x, 0\leq j\leq N_y.
\end{aligned}
$$

By (2.4),(2.8)–(2.11), we can easily obtain

$$
\|\xi^0\|_m=\|\eta^{x,0}\|_x=\|\eta^{y,0}\|_y=0,\quad \overline{\xi}^{\frac{1}{2}}=(-2\sigma^2+3\sigma-\frac{1}{2})\xi^1=O(\tau^3),\tag{4.1}
$$

$$
\overline{\eta}^{x,\frac{1}{2}}=(-2\sigma^2+3\sigma-\frac{1}{2})\eta^{x,1}+O(\tau^2+h^2),\quad \overline{\eta}^{y,\frac{1}{2}}=(-2\sigma^2+3\sigma-\frac{1}{2})\eta^{y,1}+O(\tau^2+k^2).\tag{4.2}
$$

Lemma 4.1. *Assume that the analytical solution p is sufficiently smooth. Then, it holds that*

$$
\|\eta^{x,1}\|_x+\|\eta^{y,1}\|_y\leq C(\tau^2+h^2+k^2).
$$

Proof. Using the Taylor expansion, we have

$$
\Big[\frac{1}{a^x}\overline{u}^x\Big]_{i+\frac{1}{2},j}^{\frac{1}{2}}=-[d_x(\overline{p}-\overline{\delta})]_{i+\frac{1}{2},j}^{\frac{1}{2}}+O(\tau^2+h^2+k^2),\qquad (x_{i+\frac{1}{2}},y_j)\in\Omega,
$$

$$\left[\frac{1}{a^y}\overline{u}^y\right]^{\frac{1}{2}}_{i,j+\frac{1}{2}} = -[d_y(\overline{p}-\overline{\delta})]^{\frac{1}{2}}_{i,j+\frac{1}{2}} + O(\tau^2+h^2+k^2), \qquad (x_i, y_{j+\frac{1}{2}}) \in \Omega,$$

which, in combination with (2.6) and (2.7), leads to

$$\begin{aligned}
\left[\frac{1}{a^x}\overline{\eta}^x\right]^{\frac{1}{2}}_{i+\frac{1}{2},j} &= -[d_x(\overline{\xi})]^{\frac{1}{2}}_{i+\frac{1}{2},j} + O(\tau^2+h^2+k^2), \qquad (x_{i+\frac{1}{2}}, y_j) \in \Omega,\\
\left[\frac{1}{a^y}\overline{\eta}^y\right]^{\frac{1}{2}}_{i,j+\frac{1}{2}} &= -[d_y(\overline{\xi})]^{\frac{1}{2}}_{i,j+\frac{1}{2}} + O(\tau^2+h^2+k^2), \qquad (x_i, y_{j+\frac{1}{2}}) \in \Omega.
\end{aligned}$$

From (4.1) and (4.2), we then get

$$\begin{aligned}
\left[\frac{1}{a^{x,\frac{1}{2}}}\eta^{x,1}\right]_{i+\frac{1}{2},j} &= -[d_x(\xi^1)]_{i+\frac{1}{2},j} + O(\tau^2+h^2+k^2), \qquad (x_{i+\frac{1}{2}}, y_j) \in \Omega,\\
\left[\frac{1}{a^{y,\frac{1}{2}}}\eta^{y,1}\right]_{i,j+\frac{1}{2}} &= -[d_y(\xi^1)]_{i,j+\frac{1}{2}} + O(\tau^2+h^2+k^2), \qquad (x_i, y_{j+\frac{1}{2}}) \in \Omega,
\end{aligned}$$

and consequently,

$$\begin{aligned}
\|\eta^{x,1}\|_x &\leq C\|d_x(p^1-\varpi)\|_x + C(\tau^2+h^2+k^2)\\
&\leq C(\tau^2+h^2+k^2).
\end{aligned} \tag{4.3}$$

Similarly,

$$\|\eta^{y,1}\|_y \leq C(\tau^2+h^2+k^2). \tag{4.4}$$

We conclude the proof by combining (4.3) and (4.4). □

Using Lemmas 2.2, 2.3, 2.6, 2.9, and (3.1), we easily obtain that for $1 \leq n \leq N-1$,

$$(\widetilde{d_t}\xi^{n+1}, Q)_m + ({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\xi^{n+1}, Q)_m - (\widehat{\eta}^{x,n}, d_xQ)_x - (\widehat{\eta}^{y,n}, d_yQ)_y = (R^n, Q)_m, \tag{4.5}$$

where

$$D_t\xi^{1-\sigma} = (2-2\sigma)\tfrac{\xi^1}{\tau}, \qquad D_t\xi^{-\sigma} = \tilde{b}_n D_t\xi^{1-\sigma}, \tag{4.6}$$

and

$$\begin{aligned}
R^n_{i,j} &= \frac{\partial p^n_{i,j}}{\partial t} - \widetilde{d_t}(p-\delta)^{n+1}_{i,j} + [{}^{C}_{0}D^{\alpha}_t\, p]^n_{i,j} - [{}^{\mathcal{F}}\widehat{\mathcal{D}}^{\alpha}_t(p-\delta)]^{n+1}_{i,j}\\
&\quad + \frac{\partial u^{x,n}_{i,j}}{\partial x} - [D_x\widehat{u}^x]^n_{i,j} + \frac{\partial u^{y,n}_{i,j}}{\partial y} - [D_y\widehat{u}^y]^n_{i,j}\\
&= O(g_0^{(n+1)}\tau^2 + h^2 + k^2 + \varepsilon).
\end{aligned} \tag{4.7}$$

Using (3.2), (3.3), and Lemmas 2.10 and 4.1, we conclude that for $0 \leq n \leq N$,

$$\left(\left(\frac{1}{a^x}\overline{\eta}^x\right)^{n+\frac{1}{2}}, V^x\right)_x = -(d_x\overline{\xi}^{n+\frac{1}{2}}, V^x)_x - (\overline{\epsilon}^{x,n+\frac{1}{2}}(p), V^x)_x, \tag{4.8}$$

$$\left(\left(\frac{1}{a^y}\overline{\eta}^y\right)^{n+\frac{1}{2}}, V^y\right)_y = -(d_y\overline{\xi}^{n+\frac{1}{2}}, V^y)_y - (\overline{\epsilon}^{y,n+\frac{1}{2}}(p), V^y)_y, \tag{4.9}$$

where

$$\overline{\epsilon}^{x,n+\frac{1}{2}}(p) = O(\tau^2 + h^2 + k^2), \quad \overline{\epsilon}^{y,n+\frac{1}{2}}(p) = O(\tau^2 + h^2 + k^2).$$

Theorem 4.1. *Assume that the analytical solution p is sufficiently smooth. Then for sufficiently small τ and ε, it holds that*

$$\|\xi^n\|_m + \|\eta^{x,n}\|_x + \|\eta^{y,n}\|_y \le C(\tau^2 + h^2 + k^2 + \varepsilon), \quad 0 \le n \le N.$$

Proof. The proof is similar to that of Theorem 3.1. We only need to apply the mathematical induction method to show

$$S^n \le C(\tau^2 + h^2 + k^2 + \varepsilon), \quad 1 \le n \le N, \tag{4.10}$$

where

$$S^n = \max\{(\|\xi^n\|_m + \|\widehat{\eta}^{x,n-1+\sigma}\|_x + \|\widehat{\eta}^{y,n-1+\sigma}\|_y, (2\sigma - 1)(\|\eta^{x,n}\|_x + \|\eta^{y,n}\|_y)\}. \tag{4.11}$$

By (4.1) and Lemma 4.1, we know that (4.10) holds for $n = 1$. Assume that (4.10) holds for $1 \le n \le N - 1$. Then it suffices to prove that (4.10) also holds for $n = N$.

Taking $Q = \widetilde{d}_t\xi^{n+1} = d_t\overline{\xi}^n$ in (4.5) leads to

$$\|\widetilde{d}_t\xi^{n+1}\|_m^2 + ({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, \xi^{n+1} - R^n, \widetilde{d}_t\xi^{n+1})_m - (\widehat{\eta}^{x,n}, d_x(d_t\overline{\xi}^n))_x - (\widehat{\eta}^{y,n}, d_y(d_t\overline{\xi}^n))_y = 0. \tag{4.12}$$

Applying Lemma 2.7 yields

$$\begin{aligned} 2({}^{\mathcal{F}}\widehat{\mathcal{D}}_t^{\alpha}\, \xi^{n+1} - R^n, \widetilde{d}_t\xi^{n+1})_m \ \ge\ & \sum_{i=0}^{n} g_i^{(n+1)}\|D_t\xi^{i+1-\sigma}\|_m^2 - \sum_{i=0}^{n-1} g_i^{(n)}\|D_t\xi^{i+1-\sigma}\|_m^2 \\ & - (g_1^{(n+1)} - g_0^{(n)})\|D_t\xi^{1-\sigma}\|_m^2 \\ & - g_0^{(n+1)}\left\|D_t\xi^{-\sigma} + \frac{1}{g_0^{(n+1)}}R^n\right\|_m^2. \end{aligned} \tag{4.13}$$

By (4.8), we have

$$\left(d_t\left(\frac{1}{a^x}\overline{\eta}^x\right)^n, V^x\right)_x = -(d_x(d_t\overline{\xi}^n), V^x)_x - (d_t(\overline{\epsilon}^{x,n}(p)), V^x)_x,$$

and therefore

$$\begin{aligned} -(\widehat{\eta}^{x,n}, d_x(d_t\overline{\xi}^n))_x &= \left(d_t\left(\frac{1}{a^x}\overline{\eta}^x\right)^n, \widehat{\eta}^{x,n}\right)_x + (d_t(\overline{\epsilon}^x(p^n)), \widehat{\eta}^{x,n})_x \\ &= \frac{1}{2}d_t\left\|\frac{1}{\sqrt{a^{x,n}}}\overline{\eta}^{x,n}\right\|_x^2 + (d_t(\overline{\epsilon}^{x,n}(p)), \widehat{\eta}^{x,n})_x \end{aligned}$$

$$+\frac{1}{2}\Big(d_t\Big(\frac{1}{a^{x,n}}\Big)\overline{\eta}^{x,n+\frac{1}{2}},\overline{\eta}^{x,n-\frac{1}{2}}\Big)_x \tag{4.14}$$
$$\geq \frac{1}{2}d_t\Big\|\frac{1}{\sqrt{a^{x,n}}}\overline{\eta}^{x,n}\Big\|_x^2 - C(\|d_t(\overline{\epsilon}^{x,n}(p))\|_x^2 + \|\overline{\eta}^{x,n+\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{x,n-\frac{1}{2}}\|_x^2).$$

Similarly, using (4.9), we can obtain

$$-(\widehat{\eta}^{y,n}, d_y(d_t\overline{\xi}^n))_y \geq \frac{1}{2}d_t\Big\|\frac{1}{\sqrt{a^{y,n}}}\overline{\eta}^{y,n}\Big\|_y^2 - C(\|d_t(\overline{\epsilon}^{y,n}(p))\|_y^2 + \|\overline{\eta}^{y,n+\frac{1}{2}}\|_y^2 + \|\overline{\eta}^{y,n-\frac{1}{2}}\|_y^2). \tag{4.15}$$

Substituting (4.13)–(4.15) into (4.12) gives

$$\begin{aligned}
&2\|\widetilde{d}_t\xi^{n+1}\|_m^2 + \sum_{i=0}^{n} g_i^{(n+1)}\|D_t\xi^{i+1-\sigma}\|_m^2 - \sum_{i=0}^{n-1} g_i^{(n)}\|D_t\xi^{i+1-\sigma}\|_m^2\\
&+ d_t\Big\|\frac{1}{\sqrt{a^{x,n}}}\overline{\eta}^{x,n}\Big\|_x^2 + d_t\Big\|\frac{1}{\sqrt{a^{y,n}}}\overline{\eta}^{y,n}\Big\|_y^2\\
&\leq \Big|g_1^{(n+1)} - g_0^{(n)}\Big|\|D_t\xi^{1-\sigma}\|_m^2 + g_0^{(n+1)}\Big\|D_t\xi^{-\sigma} + \frac{1}{g_0^{(n+1)}}R^n\Big\|_m^2\\
&+ C(\|d_t(\overline{\epsilon}^{x,n}(p))\|_x^2 + \|d_t(\overline{\epsilon}^{y,n}(p))\|_y^2 + \|\overline{\eta}^{x,n+\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{x,n-\frac{1}{2}}\|_x^2).
\end{aligned} \tag{4.16}$$

Summing up (4.16) for n from 1 to $N-1$ and multiplying by τ, we conclude that

$$\begin{aligned}
&\frac{1}{a^*}(\|\overline{\eta}^{x,N-\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{y,N-\frac{1}{2}}\|_y^2) + 2\tau\sum_{n=1}^{N-1}\|\widetilde{d}_t\xi^{n+1}\|_m^2 + \tau\sum_{i=0}^{N-1} g_i^{(N)}\|D_t\xi^{i+1-\sigma}\|_m^2\\
&\leq \frac{1}{a_*}(\|\overline{\eta}^{x,\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{y,\frac{1}{2}}\|_y^2) + \tau\sum_{n=1}^{N-1}\frac{1}{g_0^{(n+1)}}\|R^n\|_m^2 + \tau\sum_{n=1}^{N-1} g_0^{(n+1)}\|D_t\xi^{-\sigma}\|_m^2\\
&+ \tau g_0^{(1)}\|D_t\xi^{1-\sigma}\|_m^2 + \tau\sum_{n=1}^{N-1}\Big|g_1^{(n+1)} - g_0^{(n)}\Big|\|D_t\xi^{1-\sigma}\|_m^2\\
&+ C\tau\sum_{n=1}^{N-1}(\|d_t(\overline{\epsilon}^{x,n}(p))\|_x^2 + \|d_t(\overline{\epsilon}^{y,n}(p))\|_y^2)\\
&+ C\tau\sum_{n=0}^{N-1}(\|\overline{\eta}^{x,n+\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{x,n-\frac{1}{2}}\|_x^2).
\end{aligned}$$

Using Lemma 2.8, one gets

$$\|\overline{\eta}^{x,N-\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{y,N-\frac{1}{2}}\|_y^2 + \tau\sum_{n=1}^{N-1}\|\widetilde{d}_t\xi^{n+1}\|_m^2 + \tau\sum_{i=0}^{N-1} g_i^{(N)}\|D_t\xi^{i+1-\sigma}\|_m^2 \leq C\sum_{i=0}^{5} T_i, \tag{4.17}$$

where

$$T_1 = \|\overline{\eta}^{x,\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{y,\frac{1}{2}}\|_y^2,$$

$$T_2 = \tau \sum_{n=1}^{N-1} \frac{1}{g_0^{(n+1)}} \|R^n\|_m^2,$$

$$T_3 = \tau \sum_{n=1}^{N-1} g_0^{(n+1)} \|D_t \xi^{-\sigma}\|_m^2,$$

$$T_4 = \tau g_0^{(1)} \|D_t \xi^{1-\sigma}\|_m^2 + \tau \sum_{n=1}^{N-1} \left| g_1^{(n+1)} - g_0^{(n)} \right| \|D_t \xi^{1-\sigma}\|_m^2,$$

$$T_5 = \tau \sum_{n=1}^{N-1} (\|d_t(\overline{\epsilon}^{x,n}(p))\|_x^2 + \|d_t(\overline{\epsilon}^{y,n}(p))\|_y^2).$$

Using (4.2) and Lemma 4.1, we have

$$T_1 \le C(\tau^2 + h^2 + k^2)^2. \tag{4.18}$$

Using (4.7) and Lemma 2.5, we obtain

$$\begin{aligned} T_2 &\le C\tau \sum_{n=1}^{N-1} \frac{1}{g_0^{(n+1)}} (g_0^{(n+1)} \tau^2 + h^2 + k^2 + \varepsilon)^2 \\ &\le C\tau \sum_{n=1}^{N-1} g_0^{(n+1)} \tau^4 + C\tau \sum_{n=1}^{N-1} \frac{1}{g_0^{(n+1)}} (h^2 + k^2 + \varepsilon)^2 \\ &\le C(\tau^2 + h^2 + k^2 + \varepsilon)^2. \end{aligned} \tag{4.19}$$

Using (3.16), (3.17), (4.1), and (4.6), we get

$$T_3 \le C \|D_t \xi^{-\sigma}\|_m^2 \le \frac{C}{\tau^2} \|\xi^1\|_m^2 \le C\tau^4, \tag{4.20}$$

$$T_4 \le C \|D_t \xi^{1-\sigma}\|_m^2 \le \frac{C}{\tau^2} \|\xi^1\|_m^2 \le C\tau^4. \tag{4.21}$$

By Lemma 2.10, we have

$$T_5 = C\tau \sum_{n=1}^{N-1} (\|\overline{\epsilon}^{x,n}(d_t(p))\|_x^2 + \|\overline{\epsilon}^{y,n}(d_t(p))\|_y^2) \le C(\tau^2 + h^2 + k^2)^2. \tag{4.22}$$

Substituting (4.18)–(4.22) into (4.17), we obtain

$$\|\overline{\eta}^{x,N-\frac{1}{2}}\|_x^2 + \|\overline{\eta}^{y,N-\frac{1}{2}}\|_y^2 + \tau \sum_{n=1}^{N-1} (\|\widetilde{d_t} \xi^{n+1}\|_m^2 + g_i^{(N)} \|D_t \xi^{i+1-\sigma}\|_m^2) \le C(\tau^2 + h^2 + k^2 + \varepsilon)^2.$$

Similar to the proof of Theorem 3.1, we can conclude that

$$\begin{aligned} &(\frac{3}{2} - \sigma)[\|\xi^N\|_m + \|\widetilde{\eta}^{x,N-\sigma}\|_x + \|\widetilde{\eta}^{y,N-\sigma}\|_y] \\ &\qquad - (\sigma - \frac{1}{2})[\|\xi^{N-1}\|_m + \|\widetilde{\eta}^{x,N-2+\sigma}\|_x + \|\widetilde{\eta}^{y,N-2+\sigma}\|_y] \end{aligned} \tag{4.23}$$

$$\leq C(\tau^2 + h^2 + k^2 + \varepsilon)^2.$$

Again, we discuss the following two cases:
Case (I): $\|\xi^N\|_m + \|\widetilde{\eta}^{x,N-1+\sigma}\|_x + \|\widetilde{\eta}^{y,N-1+\sigma}\|_y \leq \|\xi^{N-1}\|_m + \|\widetilde{\eta}^{x,N-2+\sigma}\|_x + \|\widetilde{\eta}^{y,N-2+\sigma}\|_y$.
In this case, we can obtain $S^N \leq S^{N-1}$, and (4.10) follows directly.
Case (II): $\|\xi^N\|_m + \|\widetilde{\eta}^{x,N-1+\sigma}\|_x + \|\widetilde{\eta}^{y,N-1+\sigma}\|_y \geq \|\xi^{N-1}\|_m + \|\widetilde{\eta}^{x,N-2+\sigma}\|_x + \|\widetilde{\eta}^{y,N-2+\sigma}\|_y$.
We use (4.23) to obtain

$$\|\xi^N\|_m + \|\widetilde{\eta}^{x,N-1+\sigma}\|_x + \|\widetilde{\eta}^{y,N-1+\sigma}\|_y \leq C(\tau^2 + h^2 + k^2 + \varepsilon)^2. \tag{4.24}$$

Following similar arguments as in Theorem 3.1, we can get

$$S^N \leq S^{N-1} \quad \text{or} \quad S^N = \|\xi^N\|_m + \|\widetilde{\eta}^{x,N-1+\sigma}\|_x + \|\widetilde{\eta}^{y,N-1+\sigma}\|_y. \tag{4.25}$$

By combining (4.24) and (4.25), we obtain (4.10), which completes the proof. □

Theorem 4.2. *Assume that the analytical solution p is sufficiently smooth. Then for sufficiently small τ and ε, it holds that*

$$\|(p - P)^n\|_m + \|(\boldsymbol{u} - \boldsymbol{U})^n\|_{TM} \leq C(\tau^2 + h^2 + k^2 + \varepsilon), \quad 0 \leq n \leq N.$$

Proof. The proof follows from the triangle inequality, the estimation of δ, and Theorem 4.1. □

Remark 4.1. Theorem 4.2 shows that our method has the second-order temporal accuracy, regardless of the order of the Caputo fractional derivative. It is noteworthy that instead of presenting error estimates for the velocity in discrete $L^2(L^2)$-norm [20], we provide rigorous error estimates for the velocity in discrete $L^\infty(L^2)$-norm. Moreover, the application of the SOE approach reduces storage and computational cost. Thus, our method is more accurate and efficient than the original BCFD method in [20].

5. Numerical experiments

In this section, we provide four numerical examples to verify our theoretical results. The first three examples are chosen to demonstrate the accuracy and efficiency of the proposed method for smooth solutions, and the fourth example is chosen to demonstrate the performance of the proposed method for solutions with sharp boundary layers. These examples are taken from [20,30] with slight modifications. In all experiments, we choose $\Omega = (0, 1) \times (0, 1), T = 1, a^x = a^y = 1, \varepsilon = 10^{-9}$, and use the Fast-BCFD method and CN-BCFD method to denote our fast second-order BCFD method (2.5)–(2.11) and the Crank-Nicolson BCFD method proposed in [20].

Example 5.1. The initial condition and the right side are chosen according to the exact solution

$$p(x, y, t) = 100t^{5+\alpha}x^2(1 - x)^2y^2(1 - y)^2.$$

Example 5.2. The initial condition and the right side are chosen according to the exact solution

$$p(x, y, t) = (1 + t^{4.1})x^2(1 - x)^2y^2(1 - y)^2.$$

Example 5.3. The initial condition and the right side are chosen according to the exact solution

$$p(x, y, t) = t^{3+\alpha}\cos(\pi x)\cos(\pi y).$$

Example 5.4. The initial condition and the right side are chosen according to the exact solution

$$p(x, y, t) = t^{3+\alpha}\tanh(20x)\tanh(20y).$$

To compare the accuracy of the Fast-BCFD method and the CN-BCFD method, we take $N_x = N_y = N$ and compute the solutions on three consecutive meshes with $N = 40, 80, 160$. The errors and convergent rates for Examples 5.1–5.3 with different α are shown in Tables 1–3, respectively. These numerical results show that the convergence rates of the velocity and pressure in discrete $L^\infty(L^2)$-norms are all around 2 for the Fast-BCFD method, and are all around $3-\alpha$ for the CN-BCFD method. Clearly, the Fast-BCFD method is more accurate than the CN-BCFD method when α tends to 2.

Table 1. Errors and convergence rates of both methods for Example 5.1.

Method	α	N	$\|\|p^n - P^n\|\|_m$	Order	$\|\|\boldsymbol{u}^n - \boldsymbol{U}^n\|\|_{TM}$	Order
	1.1	40	4.5547E-4	–	3.8636E-3	–
		80	1.1413E-4	2.00	9.6888E-4	2.00
		160	2.8565E-5	2.00	2.4252E-4	2.00
	1.3	40	5.8065E-4	–	4.5369E-3	–
		80	1.4581E-4	1.99	1.1405E-3	1.99
		160	3.6518E-5	2.00	2.8578E-4	2.00
	1.5	40	6.5259E-4	–	4.9172E-3	–
Fast-BCFD		80	1.6379E-4	1.99	1.2368E-3	1.99
		160	4.0972E-5	2.00	3.0989E-4	2.00
	1.7	40	6.5791E-4	–	4.9401E-3	–
		80	1.6473E-4	2.00	1.2410E-3	1.99
		160	4.1100E-5	2.00	3.1059E-4	2.00
	1.9	40	5.9131E-4	–	4.5537E-3	–
		80	1.4773E-4	2.00	1.1412E-3	2.00
		160	3.6809E-5	2.00	2.8520E-4	2.00
	1.1	40	2.5531E-4	–	1.6247E-3	–
		80	6.6141E-5	1.95	4.0348E-4	2.01
		160	1.7156E-5	1.95	1.0008E-4	2.01
	1.3	40	4.2173E-4	–	1.3922E-3	–
		80	1.2976E-4	1.70	3.1577E-4	2.14
		160	4.0069E-5	1.70	6.9280E-5	2.19
	1.5	40	1.0260E-3	–	6.1583E-4	–
CN-BCFD		80	3.7730E-4	1.44	2.5303E-4	1.28
		160	1.3722E-4	1.46	1.3157E-4	0.94
	1.7	40	2.6163E-3	–	3.9839E-3	–
		80	1.1005E-3	1.25	1.9826E-3	1.01
		160	4.5691E-4	1.27	8.9774E-4	1.14
	1.9	40	6.2850E-3	–	1.5363E-2	–
		80	2.9822E-3	1.08	7.7734E-3	0.98
		160	1.4050E-3	1.09	3.7784E-3	1.04

Table 2. Errors and convergence rates of both methods for Example 5.2.

Method	α	N	$\|\|p^n - P^n\|\|_m$	Order	$\|\|\boldsymbol{u}^n - \boldsymbol{U}^n\|\|_{TM}$	Order
Fast-BCFD	1.1	40	2.1773E-5	–	1.3400E-5	–
		80	5.5446E-6	1.97	3.3530E-6	2.00
		160	1.3988E-6	1.99	8.3881E-7	2.00
	1.3	40	2.0073E-5	–	1.5197E-5	–
		80	5.1249E-6	1.97	3.8112E-6	2.00
		160	1.2946E-6	1.99	9.5415E-7	2.00
	1.5	40	1.8491E-5	–	1.5953E-5	–
		80	4.7351E-6	1.97	4.0012E-6	2.00
		160	1.1980E-6	1.98	1.0017E-6	2.00
	1.7	40	1.6968E-5	–	1.5891E-5	–
		80	4.3597E-6	1.96	3.9693E-6	2.00
		160	1.1051E-6	1.98	9.9180E-7	2.00
	1.9	40	1.5427E-5	–	1.6415E-5	–
		80	3.9770E-6	1.96	4.0669E-6	2.01
		160	1.0097E-6	1.98	1.0124E-6	2.01
CN-BCFD	1.1	40	2.4095E-5	–	5.1024E-6	–
		80	6.0357E-6	2.00	1.2736E-6	2.00
		160	1.5120E-6	2.00	3.1744E-7	2.00
	1.3	40	2.3674E-5	–	4.7435E-6	–
		80	6.0253E-6	1.97	1.1553E-6	2.04
		160	1.5392E-6	1.97	2.8038E-7	2.04
	1.5	40	2.5165E-5	–	4.2276E-6	–
		80	6.8076E-6	1.89	1.1653E-6	1.86
		160	1.8851E-6	1.85	3.7917E-7	1.62
	1.7	40	3.1189E-5	–	8.6029E-6	–
		80	9.7784E-6	1.67	3.8927E-6	1.14
		160	3.2519E-6	1.59	1.7141E-6	1.18
	1.9	40	4.7383E-5	–	2.7019E-5	–
		80	1.8426E-5	1.36	1.3965E-5	0.95
		160	7.6872E-6	1.26	6.8800E-6	1.02

Table 3. Errors and convergence rates of both methods for Example 5.3.

Method	α	N	$\|\|p^n - P^n\|\|_m$	Order	$\|\|\boldsymbol{u}^n - \boldsymbol{U}^n\|\|_{TM}$	Order
Fast-BCFD	1.1	40	8.3329E-4	–	4.2722E-3	–
		80	2.0857E-4	2.00	1.0693E-3	2.00
		160	5.2175E-5	2.00	2.6749E-4	2.00
	1.3	40	1.0588E-3	–	5.2738E-3	–
		80	2.6540E-4	2.00	1.3218E-3	2.00
		160	6.6430E-5	2.00	3.3082E-4	2.00
	1.5	40	1.2119 E-3	–	5.9539E-3	–
		80	3.0384E-4	2.00	1.4926E-3	2.00
		160	7.6042E-5	2.00	3.7352E-4	2.00
	1.7	40	1.2623E-3	–	6.1778E-3	–
		80	3.1613E-4	2.00	1.5472E-3	2.00
		160	7.9031E-5	2.00	3.8680E-4	2.00
	1.9	40	1.1776E-3	–	5.8014E-3	–
		80	2.9437E-4	2.00	1.4505E-3	2.00
		160	7.3489E-5	2.00	3.6218E-4	2.00
CN-BCFD	1.1	40	2.3487E-4	–	1.6142E-3	–
		80	5.7677E-5	2.03	3.9897E-4	2.02
		160	1.4143E-5	2.03	9.8518E-4	2.02
	1.3	40	1.3782E-4	–	1.1831E-3	–
		80	2.1693E-5	2.67	2.3911E-4	2.31
		160	1.5014E-6	3.85	4.2355E-4	2.50
	1.5	40	3.6029E-4	–	1.0294E-3	–
		80	1.7237E-4	1.06	6.2302E-4	0.72
		160	7.2154E-5	1.26	2.8488E-4	1.13
	1.7	40	2.1646E-3	–	9.0437E-3	–
		80	9.6238E-4	1.17	4.1327E-3	1.13
		160	4.1150E-4	1.23	1.7925E-3	1.21
	1.9	40	7.8199E-3	–	3.4163E-2	–
		80	3.7992E-3	1.04	1.6736E-2	1.03
		160	1.8100E-3	1.07	8.0056E-3	1.06

To compare the computational efficiency of the Fast-BCFD method and CN-BCFD method, we compare the CPU time in seconds of both methods. Since the results for different α are very similar, we only present the results for $\alpha = 1.5$ for brevity. From Table 4, one can check that for small time steps, the Fast-BCFD method is more efficient than the CN-BCFD method.

Table 4. CPU time in seconds with $\alpha = 1.5$ and $N_x = N_y = 10$.

	N	Fast-BCFD	CN-BCFD
Example 5.1	4000	16.07	25.58
	8000	31.85	83.22
	16000	63.91	316.00
	32000	129.42	1197.31
Example 5.2	4000	15.05	23.48
	8000	29.94	82.55
	16000	59.82	308.80
	32000	119.39	1174.04
Example 5.3	4000	18.54	26.51
	8000	36.95	88.77
	16000	73.47	320.66
	32000	147.52	1204.43

For Example 5.4, the exact solution exhibits a sharp layer at $x = 0$ and $y = 0$. In order to obtain better accuracy, we adopt the following mesh function:

$$x_{i+\frac{1}{2}} = \left(\frac{i}{N_x}\right)^2, \quad 0 \le i \le N_x,$$
$$y_{j+\frac{1}{2}} = \left(\frac{j}{N_y}\right)^2, \quad 0 \le j \le N_y.$$

Numerical results for this example with $\alpha = 1.9$ are summarized in Table 5. Obviously, it can be seen that numerical results using a non-uniform grid are also in agreement with our theoretical analysis. Figures 1–3 depict the graphs of the exact solutions and the numerical solutions computed by the Fast-BCFD method at time $T = 1$ on the non-uniform 40×40 mesh when $\alpha = 1.9$. For a better comparison, Figure 4 shows the computed pressure profiles along $y = x$, u^x-velocity profiles along $y = 0.5$, and u^y-velocity profiles along $x = 0.5$. One can observe that our numerical solutions are in good agreement with the exact solutions.

Table 5. Errors and convergence rates of the Fast-BCFD method for Example 5.4.

N	$\|p^n - P^n\|_m$	Order	$\|\boldsymbol{u}^n - \boldsymbol{U}^n\|_{TM}$	Order
20	3.6921E-2	–	8.6083E-2	–
40	9.2329E-3	2.00	2.1518E-2	2.00
80	2.3087E-3	2.00	5.3842E-3	2.00
160	5.7695E-4	2.00	1.3470E-3	2.00

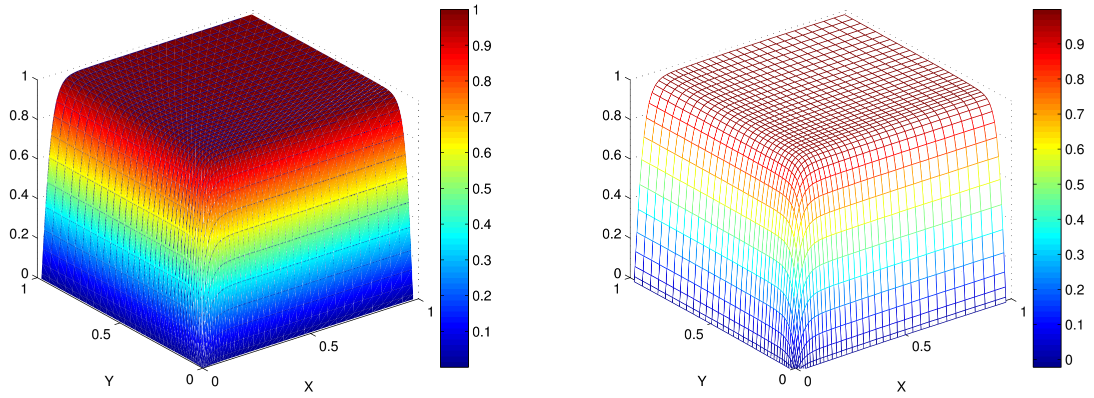

Figure 1. p^n and P^n at $T = 1$ for Example 5.4.

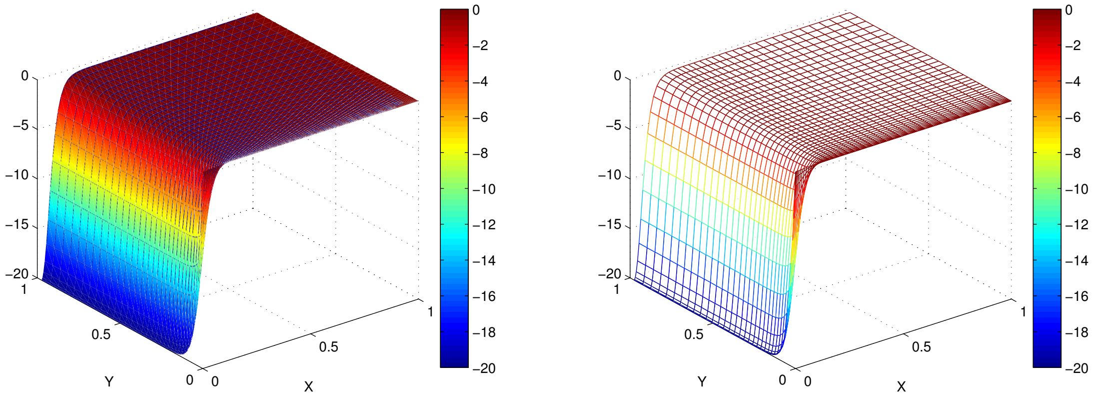

Figure 2. $u^{x,n}$ and $U^{x,n}$ at $T = 1$ for Example 5.4.

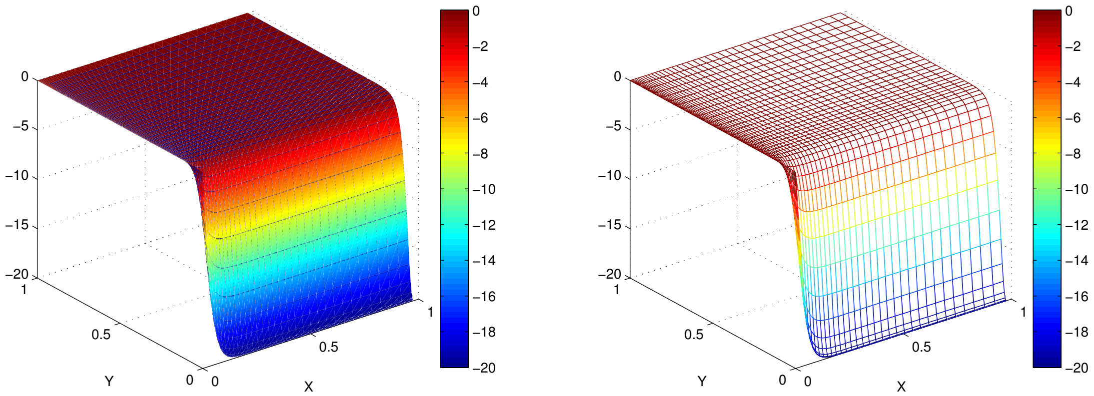

Figure 3. $u^{y,n}$ and $U^{y,n}$ at $T = 1$ for Example 5.4.

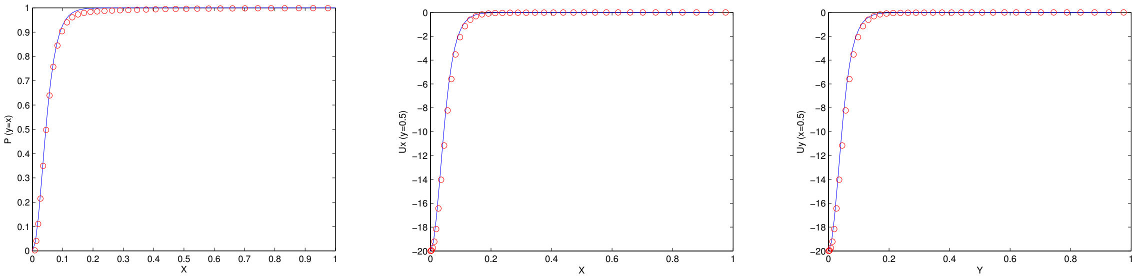

Figure 4. The computed pressure profiles along $y = x$, u^x-velocity profiles along $y = 0.5$, and u^y-velocity profiles along $x = 0.5$ of the Fast-BCFD method at $T = 1$ for Example 5.4.

6. Conclusions

In this paper, a fast second-order BCFD method based on the $\mathcal{FL}2\text{-}1_\sigma$ formula and a weighted approach was presented for the two-dimensional time fractional Cattaneo equation. The corresponding stability and superconvergence analysis on non-uniform rectangular grids were obtained. The accuracy and efficiency of the proposed method were shown by four numerical examples.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

References

1. A. Compte, R. Metzler, The generalized Cattaneo equation for the description of anomalous transport processes, *J. Phys. A: Math. Gen.*, **30** (1997), 7277–7289. https://doi.org/10.1088/0305-4470/30/21/006
2. Y. Z. Povstenko, Fractional Cattaneo-type equations and generalized thermoelasticity, *J. Therm. Stresses*, **34** (2011), 97–114. https://doi.org/10.1080/01495739.2010.511931
3. H. T. Qi, X. Y. Jiang, Solutions of the space-time fractional Cattaneo diffusion equation, *Phys. A*, **390** (2011), 1876–1883. https://doi.org/10.1016/j.physa.2011.02.010
4. Z. N. Xue, G. Q. Cao, Y. J. Yu, J. L. Liu, Coupled thermoelastic fracture analysis of a cracked fiber reinforced composite hollow cylinder by fractional Cattaneo-Vernotte models, *Theor. Appl. Fract. Mec.*, **121** (2022), 103538. https://doi.org/10.1016/j.tafmec.2022.103538

5. Y. J. Yu, Z. C. Deng, Fractional order theory of Cattaneo-type thermoelasticity using new fractional derivatives, *Appl. Math. Model.*, **87** (2020), 731–751. https://doi.org/10.1016/j.apm.2020.06.023

6. S. Vong, H. Pang, X. Q. Jin, A high-order difference scheme for the generalized Cattaneo equation, *East Asian J. Appl. Math.*, **2** (2012), 170–184. https://doi.org/10.4208/eajam.110312.240412a

7. X. Zhao, Z. Z. Sun, Compact Crank-Nicolson schemes for a class of fractional Cattaneo equation in inhomogeneous medium, *J. Sci. Comput.*, **62** (2014), 747–771. https://doi.org/10.1007/s10915-014-9874-5

8. J. C. Ren, G. H. Gao, Efficient and stable numerical methods for the two-dimensional fractional Cattaneo equation, *Numer. Algor.*, **69** (2015), 795–818. https://doi.org/10.1007/s11075-014-9926-9

9. L. L. Wei, Analysis of a new finite difference/local discontinuous Galerkin method for the fractional Cattaneo equation, *Numer. Algor.*, **77** (2018), 675–690. https://doi.org/10.1007/ s11075-017-0334-9

10. O. Nikan, Z. Avazzadeh, J. A. T. Machado, Numerical approach for modeling fractional heat conduction in porous medium with the generalized Cattaneo model, *Appl. Math. Model.*, **100** (2021), 107–124. https://doi.org/10.1016/j.apm.2021.07.025

11. A. Weiser, M. F. Wheeler, On convergence of block-centered finite-differences for elliptic problems, *SIAM J. Numer. Anal.*, **25** (1988), 351–375. https://doi.org/10.1137/0725025

12. H. Rui, H. Pan, A block-centered finite difference method for the Darcy-Forchheimer model, *SIAM J. Numer. Anal.*, **5** (2012), 2612–2631. https://doi.org/10.1137/110858239

13. W. Liu, J. T. Cui, J. Xin, A block-centered finite difference method for an unsteady asymptotic coupled model in fractured media aquifer system, *J. Comput. Appl. Math.*, **337** (2018), 319–340. https://doi.org/10.1016/j.cam.2017.12.035

14. Y. L. Jing, C. Li, Block-centered finite difference method for a tempered subdiffusion model with time-dependent coefficients, *Comput. Math. Appl.*, **145** (2023) 202–223. https://doi.org/10.1016/j.camwa.2023.06.014

15. S. Y. Zhai, X. L. Feng, A block-centered finite-difference method for the time-fractional diffusion equation on nonuniform grids, *Numer. Heat Tr. B-Fund.*, **69** (2016), 217–233. https://doi.org/10.1080/10407790.2015.1097101

16. Z. G. Liu, X. L. Li, A parallel CGS block-centered finite difference method for a nonlinear time-fractional parabolic equation, *Comput. Meth. Appl. Mech. Eng.*, **308** (2016), 330–348. https://doi.org/10.1016/j.cma.2016.05.028

17. X. L. Li, H. X. Rui, A high-order fully conservative block-centered finite difference method for the time-fractional advection-dispersion equation, *Appl. Numer. Math.*, **124** (2018), 89–109. https://doi.org/ 10.1016/j.apnum.2017.10.004

18. X. L. Li, H. X. Rui, Stability and convergence based on the finite difference method for the nonlinear fractional cable equation on non-uniform staggered grids, *Appl. Numer. Math.*, **152** (2020), 403–421. https://doi.org/10.1016/j.apnum.2019.11.013

19. X. L. Li, H. X. Rui, A block-centered finite difference method for the distributed-order time-fractional diffusion-wave equation, *Appl. Numer. Math.*, **131** (2018), 123–139. https://doi.org/10.1016/j.apnum.2018.04.013

20. X. L. Li, H. X. Rui, Z. G. Liu, A block-centered finite difference method for fractional Cattaneo equation, *Numer. Meth. Part. D. E.*, **34** (2017), 296–316. https://doi.org/10.1002/num.22198

21. Y. G. Yan, Z. Z. Sun, J. W. Zhang, Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations: a second-order scheme, *Commun. Comput. Phys.*, **22** (2017), 1028–1048. https://doi.org/10.4208/cicp.OA-2017-0019

22. P. Lyu, Y. X. Liang, Z. B. Wang, A fast linearized finite difference method for the nonlinear multi-term time-fractional wave equation, *Appl. Numer. Math.*, **151** (2020), 448–471. https://doi.org/10.1016/j.apnum.2019.11.012

23. X. X. Bai, J. Huang, H. X. Rui, S. Wang, Numerical simulation for 2D/3D time fractional Maxwell's system based on a fast second-order FDTD algorithm, *J. Comput. Appl. Math.*, **416** (2022), 114590. https://doi.org/10.1016/j.cam.2022.114590

24. L. Ma, H. F. Fu, B. Y. Zhang, S. S. Xie, A fast compact block-centered finite difference method on graded meshes for time-fractional reaction-diffusion equations and its robust analysis, *Numer. Math. Theory Me.* , **17** (2024), 429–462. https://doi.org/10.4208/nmtma.OA-2023-0108

25. X. Zhao, X. L. Li, Z. Y. Li, Fast and efficient finite difference method for the distributed-order diffusion equation based on the staggered grids, *Appl. Numer. Math.*, **174** (2022), 34–45. https://doi.org/10.1016/j.apnum.2022.01.006

26. S. D. Jiang, J. W. Zhang, Q. Zhang, Z. M. Zhang, Fast evaluation of the Caputo fractional derivative and its applications to fractional diffusion equations, *Commun. Comput. Phys.*, **21** (2017), 650–678. https://doi.org/10.4208/cicp.OA-2016-0136

27. Z. Guan, J. G. Wang, Y. Liu, Y. F. Nie, Unconditionally optimal convergence of a linearized Galerkin FEM for the nonlinear time-fractional mobile/immobile transport equation, *Appl. Numer. Math.*, **172** (2022), 133–156. https://doi.org/10.1016/j.apnum.2021.10.004

28. P. Lyu, S. Vong, A linearized second-order scheme for nonlinear time fractional Klein-Gordon type equations, *Numer. Algor.*, **78** (2018), 485–511. https://doi.org/10.1007/s11075-017-0385-y

29. J. G. Heywood, R. Rannacher, Finite-element approximation of the nonstationary Navier-Stokes problem part IV: Error analysis for second-order time discretization, *SIAM J. Numer. Anal.*, **27** (1990), 353–384. https://doi.org/10.1137/0727022

30. L. J. Nong, Q. Yi, A. Chen, A fast second-order ADI finite difference scheme for the two-dimensional time-fractional Cattaneo equation with spatially variable coefficients, *Fractal Fract.*, **8** (2024), 453. https://doi.org/10.3390/fractalfract8080453

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