Mathematics
https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(6): 14183–14195.
DOI: 10.3934/math.2025639
Received: 09 April 2025
Revised: 23 May 2025
Accepted: 05 June 2025
Published: 19 June 2025

Research article

Sharp bounds for multilinear Hardy operators on central Morrey spaces with power weights

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Abstract: In this paper, we studied the precise norm of the multilinear Hardy operators P^m and Q^m on central Morrey spaces with power weights. Furthermore, the precise norm of the multilinear Hardy operator Q^m on Lebesgue spaces with power weights was also obtained.

Keywords: sharp bound; multilinear Hardy operator; Morrey space with power weight
Mathematics Subject Classification: 42B25, 40A30

1. Introduction

Let f be a locally integrable function on $\mathbb{R}^+ = (0, \infty)$, and the classical Hardy operator H and its adjoint H^* are defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(y)dy, \quad H^*f(x) = \int_x^\infty \frac{f(y)}{y}dy.$$

Hardy [9, 10] proved that

$$\|Hf\|_{L^p(\mathbb{R}^+)} \le p'\|f\|_{L^p(\mathbb{R}^+)}, \quad \|H^*f\|_{L^p(\mathbb{R}^+)} \le p\|f\|_{L^p(\mathbb{R}^+)},$$

where $1 < p < \infty$, $1/p + 1/p' = 1$, and the constants p' and p are best possible, i.e.,

$$\|H\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = p', \quad \|H^*\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = p.$$

Faris [6] introduced the following n-dimensional Hardy operator:

$$Pf(x) = \frac{1}{\Omega_n|x|^n}\int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

for nonnegative functions on $\mathbb{R}^n$, where $\Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ is the volume of the unit ball in $\mathbb{R}^n$ and $\Gamma(1+\frac{n}{2}) = \int_0^\infty t^{\frac{n}{2}} e^{-t} dt$. Christ and Grafakos [3] obtained that for $1 < p < \infty$,

$$\|P\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = p'.$$

Zhao, Fu, and Lu [20] obtained the sharp bound for the weak-type (p, p) inequality,

$$\|P\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} = 1, \quad 1 \le p \le \infty.$$

We use the following notation: $d\vec{y} = dy_1 \cdots dy_m$, $B(0, R)$ denotes a ball of radius R centered at the origin, for $1 \le i \le m$, $y_i = (y_{1i}, \cdots, y_{ni})$ denotes elements of $\mathbb{R}^n$, and the Euclidean norm of each y_i is $|y_i| = (\sum_{j=1}^{n} |y_{ji}|^2)^{1/2}$ and of the m-tuple $(y_1, \cdots, y_m)$ is $|(y_1, \cdots, y_m)| = (\sum_{i=1}^{m} |y_i|^2)^{1/2}$.

The Hardy operator was extended to the multilinear setting by Fu, Grafakos, Lu, and Zhao [8]. Let $m \in \mathbb{N}$, $f_1, \cdots, f_m$ be locally integrable functions on $\mathbb{R}^n$, and the multilinear Hardy operator P^m is defined by

$$P^m(f_1, \cdots, f_m)(x) = \frac{1}{\Omega_{mn}|x|^{mn}} \int_{|(y_1,\cdots,y_m)|<|x|} f_1(y_1) \cdots f_m(y_m) d\vec{y}, \tag{1.2}$$

where $x \in \mathbb{R}^n \backslash \{0\}$. They obtained the sharp bounds for the multilinear Hardy operator P^m mapping from product Lebesgue spaces to Lebesgue spaces (both equipped with power weights) and from product central Morrey spaces to central Morrey spaces. After that, many researchers studied sharp estimates for the multilinear Hardy operators and their variants on Lebesgue spaces and Morrey-type spaces (see e.g., [13, 14, 16] and the references therein). Wei and Yan [18] gave the sharp bounds for the multilinear Hardy operators P^m on mixed radial-angular central Morrey spaces. Readers are referred to [7, 11, 17] for more details. Two other variants of multilinear Hardy operators were introduced and studied by Bényi and Oh [2].

The adjoint operator of the n-dimensional Hardy operator P is

$$Qf(x) = \frac{1}{\Omega_n} \int_{|y|\ge|x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n, \tag{1.3}$$

defined for locally integrable functions on $\mathbb{R}^n$. Usually, we refer to both P and Q as n-dimensional Hardy operators. Using the fact that

$$\|P\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = p', \quad 1 < p < \infty,$$

and by duality, we obtain $\|Q\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le p$. Consider $f(x) = |x|^{-\frac{n}{p}+\varepsilon} \chi_{B(0,1)}(x)$, and it is easy to get that

$$\|Q\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = p, \quad 1 < p < \infty.$$

Duoandikoetxea, Martín-Reyes, and Ombrosi [4] gave weighted inequalities for the n-dimensional Hardy operators P and Q.

Let $m \in \mathbb{N}$, $f_1, \cdots, f_m$ be locally integrable functions on $\mathbb{R}^n$, and we define the multilinear Hardy operator Q^m as

$$Q^m(f_1, \cdots, f_m)(x) = \frac{1}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|\ge|x|} \frac{f_1(y_1) \cdots f_m(y_m)}{|(y_1, \cdots, y_m)|^{mn}} d\vec{y}, \tag{1.4}$$

and obviously, $Q^1 = Q$.

Let $1 \leq p < \infty$, $-1/p \leq \lambda < 0$, $\alpha \in \mathbb{R}$, and f be a measurable function on $\mathbb{R}^n$, and we define the central (also known as local) Morrey spaces with power weights $\dot{B}^{p,\lambda}(|x|^\alpha dx)$ by

$$\dot{B}^{p,\lambda}(|x|^\alpha dx) = \{f \in L^p_{loc}(|x|^\alpha dx) : \|f\|_{\dot{B}^{p,\lambda}(|x|^\alpha dx)} < \infty\},$$

where

$$\|f\|_{\dot{B}^{p,\lambda}(|x|^\alpha dx)} = \sup_{R>0}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|f(x)|^p|x|^\alpha dx\Big)^{\frac{1}{p}}.$$

Yee and Ho [19] obtained the boundedness of the Hardy operators on weighted local Morrey spaces. When $\alpha = 0$, $\dot{B}^{p,\lambda}(|x|^\alpha dx)$ is the central homogeneous Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ introduced by Alvarez, Guzmán-Partida, and Lakey [1]. The classical Morrey space was introduced by Morrey [12], and the Morrey spaces with general weights were introduced in [15]. The case $\lambda = -1/p$ corresponds to the Lebesgue spaces with power weights.

Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $1 \leq p < \infty$, $\alpha_i < pn(1 - 1/p_i)$, $i = 1, \cdots, m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, and $\alpha = \sum_{j=1}^m \alpha_j$. Fu, Grafakos, Lu, and Zhao [8] obtained the multilinear Hardy operator P^m maps $L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx) \times \cdots \times L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)$ to $L^p(|x|^\alpha dx)$ with norm equal to the constant

$$\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(pmn - n - \alpha)}\frac{\prod\limits_{i=1}^m \Gamma(\frac{n}{2}(1 - \frac{1}{p_i} - \frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m - \frac{1}{p} - \frac{\alpha}{pn}))},$$

where $\omega_n = n\Omega_n$. Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $-1/p_i < \lambda_i < 0$, $1 \leq p < \infty$, $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, $p\lambda = p_i\lambda_i$, $i = 1, \cdots, m$, and $\lambda = \sum_{j=1}^m \lambda_j$. They also obtained that P^m maps $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\dot{B}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ with norm

$$\|P^m\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\dot{B}^{p_m,\lambda_m}(\mathbb{R}^n)\to\dot{B}^{p,\lambda}(\mathbb{R}^n)} = \frac{m\omega_n^m}{2^{m-1}\omega_{mn}(m+\lambda)}\frac{\prod_{i=1}^m \Gamma(\frac{n}{2}(1+\lambda_i))}{\Gamma(\frac{n}{2}(m+\lambda))}.$$

In this paper, we establish the sharp bounds for the multilinear Hardy operators P^m and Q^m mapping from product central Morrey spaces to central Morrey spaces (both equipped with power weights) in Sections 2 and 3, respectively. We also obtain the precise norm of the multilinear Hardy operator Q^m on Lebesgue spaces with power weights.

We recall the definitions of the beta function $B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt$ and the gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, where z and w are complex numbers with positive real parts. These functions satisfy the identity $B(z, w)\Gamma(z + w) = \Gamma(z)\Gamma(w)$.

2. The sharp bounds for P^m on central Morrey spaces with power weights

In this section, we obtain sharp bounds for the multilinear Hardy operator P^m on the central Morrey spaces with power weights. Our results include, as special cases, the sharp bounds for the central Morrey spaces.

Theorem 2.1. Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $-1/p_i < \lambda_i < 0$, $1 \leq p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\alpha_i < pn\lambda_i + pn$, $i = 1, \cdots, m$, $\alpha = \sum_{j=1}^m \alpha_j$, and $\lambda = \sum_{j=1}^m \lambda_j$. Then the multilinear Hardy operator P^m defined in (1.2) maps the product of central Morrey spaces with power weights $\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)$ to $\dot{B}^{p,\lambda}(|x|^{\alpha}dx)$ and

$$\|P^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)}\leq C_1,$$

where

$$C_1=\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(pmn+pn\lambda-\alpha)}\frac{\prod_{i=1}^m\Gamma(\frac{n}{2}(1+\lambda_i-\frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m+\lambda-\frac{\alpha}{pn}))}.$$

If for $i=1,\cdots,m$, $p\lambda=p_i\lambda_i$, then

$$\|P^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)}=C_1.$$

Remark 2.2. For $1<p<\infty$ and $-1/p<\lambda<0$, the operator norm from $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ to itself of the n-dimensional Hardy operator P defined in (1.1) was evaluated in [8]. It was found to be independent of n.

Remark 2.3. Assume that $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$ and $\lambda=\sum_{j=1}^m\lambda_j$. Then for all $i=1,\cdots,m$, the inequality $p\lambda\leq p_i\lambda_i$ holds if and only if $p\lambda=p_i\lambda_i$.

Proof of Theorem 2.1. As in the proof of [8], the operator P^m and its restriction to radial functions have the same operator norm on the spaces $\dot{B}^{p,\lambda}(|x|^{\alpha}dx)$. Taking radial functions $f_i\in\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_i p_i}{p}}dx)$, $i=1,\cdots,m$, by the Minkowski integral inequality and the Hölder inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|P^m(f_1,\cdots,f_m)(x)|^p|x|^{\alpha}dx\Big)^{\frac{1}{p}}\\
&=\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\Big|\frac{1}{\Omega_{mn}|x|^{mn}}\int_{|(y_1,\cdots,y_m)|<|x|}f_1(y_1)\cdots f_m(y_m)d\vec{y}\Big|^p|x|^{\alpha}dx\Big)^{\frac{1}{p}}\\
&=\frac{1}{\Omega_{mn}}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\Big|\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^m f_i(|x|y_i)d\vec{y}\Big|^p|x|^{\alpha}dx\Big)^{\frac{1}{p}}\\
&\leq\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\Big|\prod_{i=1}^m f_i(x|y_i|)\Big|^p|x|^{\alpha}dx\Big)^{\frac{1}{p}}d\vec{y}\\
&\leq\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^m\Big(\frac{1}{|B(0,R)|^{1+\lambda_i p_i}}\int_{B(0,R)}|f_i(x|y_i|)|^{p_i}|x|^{\frac{\alpha_i p_i}{p}}dx\Big)^{\frac{1}{p_i}}d\vec{y}\\
&=\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^m\Big(\frac{1}{|B(0,R)|^{1+\lambda_i p_i}}\int_{B(0,|y_i|R)}|f_i(t)|^{p_i}|t|^{\frac{\alpha_i p_i}{p}}dt\frac{1}{|y_i|^{n+\frac{\alpha_i p_i}{p}}}\Big)^{\frac{1}{p_i}}d\vec{y}\\
&\leq\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}d\vec{y}\prod_{i=1}^m\|f_i\|_{\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_i p_i}{p}}dx)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}d\vec{y}\\
&=\frac{mn\omega_n^m}{\omega_{mn}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\int_0^1 r^{mn+n\lambda-\frac{\alpha}{p}-1}
\end{aligned}$$

$$\cdot\prod_{i=1}^{m-1}(\cos\theta_i)^{n+n\lambda_i-\frac{\alpha_i}{p}-1}(\sin\theta_i)^{m-i-1+\sum\limits_{j=i+1}^{m}(n+n\lambda_j-\frac{\alpha_j}{p}-1)}drd\theta_1\cdots d\theta_{m-1}.$$

By an elementary calculation, we obtain that

$$\begin{aligned}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2z-1}(\sin\theta)^{2w-1}d\theta &= \int_0^1(1-t^2)^{z-1}t^{2w-1}dt\ \ (t=\sin\theta)\\ &= \frac{1}{2}\int_0^1(1-x)^{z-1}x^{w-1}dx\ \ (x=t^2)\\ &= \frac{1}{2}B(w,z). \end{aligned}\tag{2.1}$$

Therefore, we have

$$\begin{aligned}&\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^{m}|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}d\vec{y}\\ &=\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(pmn+pn\lambda-\alpha)}B\Big(\frac{\sum\limits_{i=2}^{m}(n+n\lambda_i-\frac{\alpha_i}{p})}{2},\frac{n+n\lambda_1-\frac{\alpha_1}{p}}{2}\Big)\\ &\quad\cdot B\Big(\frac{\sum\limits_{i=3}^{m}(n+n\lambda_i-\frac{\alpha_i}{p})}{2},\frac{n+n\lambda_2-\frac{\alpha_2}{p}}{2}\Big)\cdots B\Big(\frac{n+n\lambda_m-\frac{\alpha_m}{p}}{2},\frac{n+n\lambda_{m-1}-\frac{\alpha_{m-1}}{p}}{2}\Big)\\ &=C_1.\end{aligned}$$

Thus

$$\|P^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_mp_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)}\le C_1.$$

If for $i=1,\cdots,m$, $p\lambda=p_i\lambda_i$, let $\widetilde{f_i}(x)=|x|^{n\lambda_i-\frac{\alpha_i}{p}}$, $x\in\mathbb{R}^n$, and then

$$\begin{aligned}\|\widetilde{f_i}\|_{\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_ip_i}{p}}dx)} &= \sup_{R>0}\Big(\frac{1}{|B(0,R)|^{1+\lambda_ip_i}}\int_{B(0,R)}|\widetilde{f_i}(x)|^{p_i}|x|^{\frac{\alpha_ip_i}{p}}dx\Big)^{\frac{1}{p_i}}\\ &= \frac{n^{\lambda_i}}{\omega_n^{\lambda_i}(1+p_i\lambda_i)^{\frac{1}{p_i}}}.\end{aligned}$$

We have

$$\begin{aligned}P^m(\widetilde{f_1},\cdots,\widetilde{f_m})(x) &= \frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^{m}f_i(|x|y_i)d\vec{y}\\ &= \frac{1}{\Omega_{mn}}|x|^{n\lambda-\frac{\alpha}{p}}\int_{|(y_1,\cdots,y_m)|<1}\prod_{i=1}^{m}|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}d\vec{y}\\ &= C_1\prod_{i=1}^{m}\widetilde{f_i}(x),\end{aligned}$$

and

$$
\begin{aligned}
&\|P^m(\widetilde{f_1},\cdots,\widetilde{f_m})\|_{\dot{B}^{p,\lambda}(|x|^\alpha dx)}\\
&=\sup_{R>0}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|P^m(\widetilde{f_1},\cdots,\widetilde{f_m})(x)|^p|x|^\alpha dx\Big)^{\frac{1}{p}}\\
&=C_1\sup_{R>0}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\prod_{i=1}^m|\widetilde{f_i}(x)|^p|x|^\alpha dx\Big)^{\frac{1}{p}}\\
&=C_1\sup_{R>0}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|x|^{pn\lambda}dx\Big)^{\frac{1}{p}}\\
&=C_1\frac{n^\lambda}{\omega_n^\lambda(1+p\lambda)^{\frac{1}{p}}}\\
&=C_1\prod_{i=1}^m\frac{n^{\lambda_i}}{\omega_n^{\lambda_i}(1+p_i\lambda_i)^{\frac{1}{p_i}}}\\
&=C_1\prod_{i=1}^m\|\widetilde{f_i}\|_{\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_i p_i}{p}}dx)}.
\end{aligned}
$$

This ends the proof.

When $\alpha_i = 0$ for all $i = 1,\cdots,m$, the result above was proved in [8].

3. The sharp bounds for Q^m on central Morrey spaces with power weights

In this section, we obtain sharp bounds for the multilinear Hardy operator Q^m on the central Morrey spaces with power weights. Additionally, the sharp bounds for central Morrey spaces are derived.

Theorem 3.1. Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $-1/p_i < \lambda_i < 0$, $1 \le p < \infty$, $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$, $\alpha_i < pn\lambda_i + pn$, $i = 1,\cdots,m$, $\alpha = \sum_{j=1}^m \alpha_j$, $\lambda = \sum_{j=1}^m \lambda_j$, and $pn\lambda < \alpha$. Then the multilinear Hardy operator Q^m defined in (1.4) maps $\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)$ to $\dot{B}^{p,\lambda}(|x|^\alpha dx)$ and

$$
\|Q^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^\alpha dx)} \le C_2,
$$

where

$$
C_2 = \frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(\alpha-pn\lambda)}\frac{\prod_{i=1}^m\Gamma(\frac{n}{2}(1+\lambda_i-\frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m+\lambda-\frac{\alpha}{pn}))}.
$$

If $p\lambda = p_i\lambda_i,\ i = 1,\cdots,m$, then

$$
\|Q^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1 p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_m p_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^\alpha dx)} = C_2.
$$

Remark 3.2. For $1 < p < \infty$ and $-1/p < \lambda < 0$, the operator norm from $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ to itself of the n-dimensional Hardy operator Q defined in (1.3) is independent of n.

Proof of Theorem 3.1. As before, we note that the operator Q^m and its restriction to radial functions have the same operator norm in $\dot{B}^{p,\lambda}(|x|^\alpha dx)$, and taking radial functions $f_i \in \dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_i p_i}{p}}dx)$, $i = 1,\cdots m$, then

$$
Q^m(f_1,\cdots,f_m)(x) = \frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{f_1(|x|y_1)\cdots f_m(|x|y_m)}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}.
$$

By the Minkowski integral inequality and the Hölder inequality, we have

$$
\begin{aligned}
&\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|Q^m(f_1,\cdots,f_m)(x)|^p|x|^\alpha dx\Big)^{\frac{1}{p}}\\
&=\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\Big|\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{f_1(x|y_1|)\cdots f_m(x|y_m|)}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}\Big|^p|x|^\alpha dx\Big)^{\frac{1}{p}}\\
&\le\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\Big(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\Big|\frac{f_1(x|y_1|)\cdots f_m(x|y_m|)}{|(y_1,\cdots,y_m)|^{mn}}\Big|^p|x|^\alpha dx\Big)^{\frac{1}{p}}d\vec{y}\\
&\le\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{1}{|(y_1,\cdots,y_m)|^{mn}}\prod_{i=1}^m\Big(\frac{1}{|B(0,R)|^{1+\lambda_ip_i}}\int_{B(0,R)}|f_i(x|y_i|)|^{p_i}|x|^{\frac{\alpha_ip_i}{p}}dx\Big)^{\frac{1}{p_i}}d\vec{y}\\
&=\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}}{|(y_1,\cdots,y_m)|^{mn}}\prod_{i=1}^m\Big(\frac{1}{|B(0,|y_i|R)|^{1+\lambda_ip_i}}\int_{B(0,|y_i|R)}|f_i(t)|^{p_i}|t|^{\frac{\alpha_ip_i}{p}}dt\Big)^{\frac{1}{p_i}}d\vec{y}\\
&\le\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}\prod_{i=1}^m\|f_i\|_{\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_ip_i}{p}}dx)},
\end{aligned}
$$

and by Eq (2.1), we obtain that

$$
\begin{aligned}
&\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}\\
&=\frac{mn\omega_n^m}{\omega_{mn}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\int_1^\infty r^{n\lambda-\frac{\alpha}{p}-1}\\
&\cdot\prod_{i=1}^{m-1}(\cos\theta_i)^{n+n\lambda_i-\frac{\alpha_i}{p}-1}(\sin\theta_i)^{m-i-1+\sum\limits_{j=i+1}^m(n+n\lambda_j-\frac{\alpha_j}{p}-1)}drd\theta_1\cdots d\theta_{m-1}\\
&=\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(\alpha-pn\lambda)}B\Big(\frac{\sum\limits_{i=2}^m(n+n\lambda_i-\frac{\alpha_i}{p})}{2},\frac{n+n\lambda_1-\frac{\alpha_1}{p}}{2}\Big)\\
&\cdot B\Big(\frac{\sum\limits_{i=3}^m(n+n\lambda_i-\frac{\alpha_i}{p})}{2},\frac{n+n\lambda_2-\frac{\alpha_2}{p}}{2}\Big)\cdots B\Big(\frac{n+n\lambda_m-\frac{\alpha_m}{p}}{2},\frac{n+n\lambda_{m-1}-\frac{\alpha_{m-1}}{p}}{2}\Big)\\
&=C_2.
\end{aligned}
$$

Thus

$$\|Q^m\|_{\dot{B}^{p_1,\lambda_1}(|x|^{\frac{\alpha_1p_1}{p}}dx)\times\cdots\times\dot{B}^{p_m,\lambda_m}(|x|^{\frac{\alpha_mp_m}{p}}dx)\to\dot{B}^{p,\lambda}(|x|^\alpha dx)}\le C_2.$$

If for $i=1,\cdots,m$, $p\lambda=p_i\lambda_i$, as in the proof of Theorem 3.1, let $\widetilde{f_i}(x)=|x|^{n\lambda_i-\frac{\alpha_i}{p}}$, $x\in\mathbb{R}^n$, and we have

$$
\begin{aligned}
Q^m(\widetilde{f_1},\cdots,\widetilde{f_m})(x)&=\frac{1}{\Omega_{mn}}\int_{|(y_1,\cdots,y_m)|>1}\frac{\widetilde{f_1}(x|y_1|)\cdots\widetilde{f_m}(x|y_m|)}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}\\
&=\frac{1}{\Omega_{mn}}|x|^{n\lambda-\frac{\alpha}{p}}\int_{|(y_1,\cdots,y_m)|>1}\frac{\prod_{i=1}^m|y_i|^{n\lambda_i-\frac{\alpha_i}{p}}}{|(y_1,\cdots,y_m)|^{mn}}d\vec{y}
\end{aligned}
$$

$$= C_2 \prod_{i=1}^{m} \widetilde{f_i}(x),$$

and

$$\begin{aligned}
&\|Q^m(\widetilde{f_1}, \cdots, \widetilde{f_m})\|_{\dot{B}^{p,\lambda}(|x|^\alpha dx)} \\
&= \sup_{R>0} \Big(\frac{1}{|B(0,R)|^{1+\lambda p}} \int_{B(0,R)} |Q^m(\widetilde{f_1}, \cdots, \widetilde{f_m})(x)|^p |x|^\alpha dx \Big)^{\frac{1}{p}} \\
&= C_2 \sup_{R>0} \Big(\frac{1}{|B(0,R)|^{1+\lambda p}} \int_{B(0,R)} \prod_{i=1}^{m} |\widetilde{f_i}(x)|^p |x|^\alpha dx \Big)^{\frac{1}{p}} \\
&= C_2 \prod_{i=1}^{m} \frac{n^{\lambda_i}}{\omega_n^{\lambda_i} (1 + p_i \lambda_i)^{\frac{1}{p_i}}} \\
&= C_2 \prod_{i=1}^{m} \|\widetilde{f_i}\|_{\dot{B}^{p_i,\lambda_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)}.
\end{aligned}$$

This ends the proof.

Let $\alpha_i = 0$, $i = 1, \cdots, m$, and we have the following result.

Corollary 3.4. Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $-1/p_i < \lambda_i < 0$, $1 \le p < \infty$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$, $p\lambda = p_i \lambda_i$, $i = 1, \cdots, m$, and $\lambda = \sum_{j=1}^{m} \lambda_j$. Then the multilinear Hardy operator Q^m maps $\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ with norm

$$\|Q^m\|_{\dot{B}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \dot{B}^{p_m,\lambda_m}(\mathbb{R}^n) \to \dot{B}^{p,\lambda}(\mathbb{R}^n)} = -\frac{m\omega_n^m}{2^{m-1}\omega_{mn}\lambda} \frac{\prod_{i=1}^{m} \Gamma(\frac{n}{2}(1+\lambda_i))}{\Gamma(\frac{n}{2}(m+\lambda))}.$$

4. The sharp bounds for Q^m on Lebesgue spaces with power weights

In this section, we establish sharp bounds for the multilinear Hardy operator Q^m on Lebesgue spaces with power weights.

Theorem 4.1. Let $m \in \mathbb{N}$, $1 < p_i < \infty$, $1 \le p < \infty$, $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$, $\alpha_i < pn(1 - 1/p_i)$, $i = 1, \cdots, m$, $\alpha = \sum_{j=1}^{m} \alpha_j$, and $n + \alpha > 0$. Then the multilinear Hardy operator Q^m maps the product of weighted Lebesgue spaces $L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}} dx) \times \cdots \times L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}} dx)$ to $L^p(|x|^\alpha dx)$ with norm equal to the constant

$$\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(n+\alpha)} \frac{\prod_{i=1}^{m} \Gamma(\frac{n}{2}(1 - \frac{1}{p_i} - \frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m - \frac{1}{p} - \frac{\alpha}{pn}))}.$$

Proof. As before, we observe that the operator Q^m and its restriction to radial functions have the same operator norm on $L^p(|x|^\alpha dx)$. Let $f_i \in L^{p_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)$, $i = 1, \cdots, m$, be radial functions, and by Minkowski's integral inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|Q^m(f_1, \cdots, f_m)\|_{L^p(|x|^\alpha dx)} \\
&= \frac{1}{\Omega_{mn}} \Big(\int_{\mathbb{R}^n} \Big| \int_{|(y_1,\cdots,y_m)|>1} \frac{f_1(|x|y_1) \cdots f_m(|x|y_m)}{|(y_1, \cdots, y_m)|^{mn}} d\vec{y} \Big|^p |x|^\alpha dx \Big)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|>1} \Big(\int_{\mathbb{R}^n} \Big| \frac{f_1(|x|y_1)\cdots f_m(|x|y_m)}{|(y_1,\cdots,y_m)|^{mn}} \Big|^p |x|^\alpha dx \Big)^{\frac{1}{p}} d\vec{y} \\
&= \frac{1}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|>1} \Big(\int_{\mathbb{R}^n} \Big| \frac{f_1(x|y_1|)\cdots f_m(x|y_m|)}{|(y_1,\cdots,y_m)|^{mn}} \Big|^p |x|^\alpha dx \Big)^{\frac{1}{p}} d\vec{y} \\
&= \frac{1}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|>1} \frac{1}{|(y_1,\cdots,y_m)|^{mn}} \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^n} |f_i(x|y_i|)|^{p_i} |x|^{\frac{\alpha_i p_i}{p}} dx \Big)^{\frac{1}{p_i}} d\vec{y} \\
&\leq \frac{1}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|>1} \frac{1}{|(y_1,\cdots,y_m)|^{mn} \prod\limits_{i=1}^{m} |y_i|^{\frac{n}{p_i}+\frac{\alpha_i}{p}}} d\vec{y} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)} \\
&= C_3 \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)},
\end{aligned}
$$

and by Eq (2.1), we obtain that

$$
\begin{aligned}
C_3 = &\frac{mn\omega_n^m}{\omega_{mn}} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_1^{\infty} r^{-\frac{n}{p}-\frac{\alpha}{p}-1} \\
&\cdot \prod_{i=1}^{m-1} (\cos\theta_i)^{n-\frac{n}{p_i}-\frac{\alpha_i}{p}-1} (\sin\theta_i)^{m-i-1+\sum\limits_{j=i+1}^{m}(n-\frac{n}{p_j}-\frac{\alpha_j}{p}-1)} dr d\theta_1 \cdots d\theta_{m-1} \\
= &\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(n+\alpha)} B\Big(\frac{\sum\limits_{i=2}^{m}(n-\frac{n}{p_i}-\frac{\alpha_i}{p})}{2}, \frac{n-\frac{n}{p_1}-\frac{\alpha_1}{p}}{2} \Big) \\
&\cdot B\Big(\frac{\sum\limits_{i=3}^{m}(n-\frac{n}{p_i}-\frac{\alpha_i}{p})}{2}, \frac{n-\frac{n}{p_2}-\frac{\alpha_2}{p}}{2} \Big) \cdots B\Big(\frac{n-\frac{n}{p_m}-\frac{\alpha_m}{p}}{2}, \frac{n-\frac{n}{p_{m-1}}-\frac{\alpha_{m-1}}{p}}{2} \Big) \\
= &\frac{pmn\omega_n^m}{2^{m-1}\omega_{mn}(n+\alpha)} \frac{\prod\limits_{i=1}^{m} \Gamma(\frac{n}{2}(1-\frac{1}{p_i}-\frac{\alpha_i}{pn}))}{\Gamma(\frac{n}{2}(m-\frac{1}{p}-\frac{\alpha}{pn}))}.
\end{aligned}
$$

To show that C_3 is the best possible constant, we should obtain that

$$
\|Q^m\|_{L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}} dx)\times\cdots\times L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}} dx)\to L^p(|x|^\alpha dx)} \geq C_3.
$$

For a sufficiently small ε, $0<\varepsilon<\min\{1, \frac{n+\alpha}{p_m}, \frac{1}{\sqrt{m}}\}$, and we define

$$
f_i^\varepsilon(x) = \begin{cases} |x|^{-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i}}, & |x| \leq \frac{1}{\sqrt{m}}, \\ 0, & |x| > \frac{1}{\sqrt{m}}, \end{cases}
$$

where $i=1,\cdots,m$. We have that

$$
\|f_1^\varepsilon\|^{p_1}_{L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}} dx)} = \cdots = \|f_m^\varepsilon\|^{p_m}_{L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}} dx)} = \frac{\omega_n}{p_m\varepsilon}\Big(\frac{1}{\sqrt{m}}\Big)^{p_m\varepsilon}.
$$

$Q^m(f_1^\varepsilon, \cdots, f_m^\varepsilon)(x) = 0$ when $|x| \geq 1$, and that

$$Q^m(f_1^\varepsilon, \cdots, f_m^\varepsilon)(x) = \frac{|x|^{-\frac{n}{p}-\frac{\alpha}{p}+\frac{p_m\varepsilon}{p}}}{\Omega_{mn}} \int_{|(y_1,\cdots,y_m)|>1;|y_1|<\frac{\sqrt{m}}{m|x|};\cdots;|y_m|<\frac{\sqrt{m}}{m|x|}} \frac{\prod_{i=1}^m |y_i|^{-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i}}}{|(y_1,\cdots,y_m)|^{mn}} d\vec{y}$$

when $|x| < 1$.

By Eq (2.1), we have

$$\begin{aligned}
&\|Q^m(f_1^\varepsilon, \cdots, f_m^\varepsilon)\|_{L^p(|x|^\alpha dx)} \\
&= \frac{1}{\Omega_{mn}}\Big(\int_{|x|<1} \Big||x|^{-\frac{n}{p}-\frac{\alpha}{p}+\frac{p_m\varepsilon}{p}} \int_{|(y_1,\cdots,y_m)|>1;|y_1|,\cdots,|y_m|<\frac{\sqrt{m}}{m|x|}} \frac{\prod_{i=1}^m |y_i|^{-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i}}}{|(y_1,\cdots,y_m)|^{mn}} d\vec{y}\Big|^p |x|^\alpha dx\Big)^{\frac{1}{p}} \\
&\geq \frac{1}{\Omega_{mn}}\Big(\int_{|x|<\varepsilon} \Big||x|^{-\frac{n}{p}-\frac{\alpha}{p}+\frac{p_m\varepsilon}{p}} \int_{|(y_1,\cdots,y_m)|>1;|y_1|,\cdots,|y_m|<\frac{\sqrt{m}}{m\varepsilon}} \frac{\prod_{i=1}^m |y_i|^{-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i}}}{|(y_1,\cdots,y_m)|^{mn}} d\vec{y}\Big|^p |x|^\alpha dx\Big)^{\frac{1}{p}} \\
&\geq \Big(\frac{\omega_n \varepsilon^{p_m\varepsilon}}{p_m\varepsilon}\Big)^{\frac{1}{p}} \frac{mn\omega_n^m}{\omega_{mn}} \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_1^{\frac{\sqrt{m}}{m\varepsilon}} r^{-\frac{n}{p}-\frac{\alpha}{p}+\frac{p_m\varepsilon}{p}-1} \\
&\quad \cdot \prod_{i=1}^{m-1} (\cos\theta_i)^{n-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i}-1} (\sin\theta_i)^{m-i-1+\sum_{j=i+1}^m (n-\frac{n}{p_j}-\frac{\alpha_j}{p}+\frac{p_m\varepsilon}{p_j}-1)} dr d\theta_1 \cdots d\theta_{m-1} \\
&= \frac{mn\omega_n^m \sqrt{m}^{\frac{p_m\varepsilon}{p}} \varepsilon^{\frac{p_m\varepsilon}{p}} (1-(\sqrt{m}\varepsilon)^{\frac{n}{p}+\frac{\alpha}{p}-\frac{p_m\varepsilon}{p}})}{2^{m-1}\omega_{mn}(\frac{n}{p}+\frac{\alpha}{p}-\frac{p_m\varepsilon}{p})} B\Big(\frac{\sum_{i=2}^m (n-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i})}{2}, \frac{n-\frac{n}{p_1}-\frac{\alpha_1}{p}+\frac{p_m\varepsilon}{p_1}}{2}\Big) \\
&\quad \cdot B\Big(\frac{\sum_{i=3}^m (n-\frac{n}{p_i}-\frac{\alpha_i}{p}+\frac{p_m\varepsilon}{p_i})}{2}, \frac{n-\frac{n}{p_2}-\frac{\alpha_2}{p}+\frac{p_m\varepsilon}{p_2}}{2}\Big) \cdots \\
&\quad \cdot B\Big(\frac{n-\frac{n}{p_m}-\frac{\alpha_m}{p}+\varepsilon}{2}, \frac{n-\frac{n}{p_{m-1}}-\frac{\alpha_{m-1}}{p}+\frac{p_m\varepsilon}{p_{m-1}}}{2}\Big) \prod_{i=1}^m \|f_i^\varepsilon\|_{L^{p_i}(|x|^{\frac{\alpha_i p_i}{p}} dx)}.
\end{aligned}$$

Finally, let $\varepsilon \to 0$, and we get

$$\|Q^m\|_{L^{p_1}(|x|^{\frac{\alpha_1 p_1}{p}} dx)\times\cdots\times L^{p_m}(|x|^{\frac{\alpha_m p_m}{p}} dx)\to L^p(|x|^\alpha dx)} \geq C_3.$$

This ends the proof.

Duoandikoetxea, Martín-Reyes, and Ombrosi [4] introduced the n-dimensional maximal operator

$$Nf(x) = \sup_{r>|x|} \frac{1}{\Omega_n r^n} \int_{|y|<r} |f(y)| dy$$

for locally integrable functions on $\mathbb{R}^n$. The operator N plays a crucial role in proving the boundedness of both the Hardy operator P and the Calderón operator S (defined as $S = P+Q$) on weighted Lebesgue spaces. In [5], the operator N was further employed to build weights that yield the boundedness of the fractional hardy operator. Let f be the nonnegative integrable on $\mathbb{R}^n$, and we obtain

$$Pf(x) \leq Nf(x) \leq Pf(x) + Qf(x).$$

For $1 < p < \infty$, $-\frac{1}{p} < \lambda < 0$, and $\alpha \in \mathbb{R}$, by Minkowski's integral inequality, we have

$$\begin{aligned}||P||_{\dot{B}^{p,\lambda}(|x|^{\alpha}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)} &\leq ||N||_{\dot{B}^{p,\lambda}(|x|^{\alpha}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)}\\ &\leq ||P||_{\dot{B}^{p,\lambda}(|x|^{\alpha}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)} + ||Q||_{\dot{B}^{p,\lambda}(|x|^{\alpha}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)},\end{aligned}$$

and

$$\begin{aligned}||P||_{L^{p}(|x|^{\alpha}dx)\to L^{p}(|x|^{\alpha}dx)} &\leq ||N||_{L^{p}(|x|^{\alpha}dx)\to L^{p}(|x|^{\alpha}dx)}\\ &\leq ||P||_{L^{p}(|x|^{\alpha}dx)\to L^{p}(|x|^{\alpha}dx)} + ||Q||_{L^{p}(|x|^{\alpha}dx)\to L^{p}(|x|^{\alpha}dx)}.\end{aligned}$$

An immediate consequence of Theorems 2.1 and 3.1 is that for all $1 < p < \infty$, $-\frac{1}{p} < \lambda < 0$, $pn\lambda < \alpha < pn\lambda + pn$,

$$\frac{pn}{pn + pn\lambda - \alpha} \leq ||N||_{\dot{B}^{p,\lambda}(|x|^{\alpha}dx)\to\dot{B}^{p,\lambda}(|x|^{\alpha}dx)} \leq \frac{pn}{pn + pn\lambda - \alpha} + \frac{pn}{\alpha - pn\lambda}.$$

By Theorems 4.1 and 1 in [8], we obtain that for $1 < p < \infty$ and $-n < \alpha < pn - n$,

$$\frac{pn}{pn - n - \alpha} \leq ||N||_{L^{p}(|x|^{\alpha}dx)\to L^{p}(|x|^{\alpha}dx)} \leq \frac{pn}{pn - n - \alpha} + \frac{pn}{n + \alpha}.$$

5. Conclusions

We establish the precise norm of the multilinear Hardy operators P^m and Q^m on central Morrey spaces with power weights. Following the method developed in the proof of [8], we also obtain the exact operator norms of the multilinear Hardy operator Q^m on Lebesgue spaces with power weights. This approach may be adapted to study the multilinear Hardy operators P^m and Q^m on Herz spaces and Herz-Morrey spaces.

Use of Generative-AI tools declaration

The authors declare they have not used Artiffcial Intelligence (AI) tools in the creation of this article.

Author contributions

Meichuan Lv: Conceptualization, writing-original draft, methodology, writing-review and editing; Wenming Li: Conceptualization, methodology, writing-review, supervision, language editing and funding acquisition. All the authors have read and agreed to the published version of the manuscript.

Acknowledgments

The work was supported by the Natural Science Foundation of Hebei Province (No. A2021205013) and the Innovation Fund of the School of Mathematical Sciences, Hebei Normal University (No. ycxzzbs202502).

Conflict of interest

The authors state that there are no conflicts of interest in this paper.

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