



Research article

Generalized fractional Hermite-Hadamard-type inequalities for interval-valued s-convex functions

Guoshan Deng¹, Dafang Zhao^{1,*}, Sina Etemad^{2,3} and Jessada Tariboon^{4,*}

¹ Huangshi Key Laboratory of Metaverse and Virtual Simulation, School of Mathematics and Statistics, Hubei Normal University, Huangshi, Hubei 435002, China

² Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran

³ Mathematics in Applied Sciences and Engineering Research Group, Scientific Research Center, Al-Ayen University, Nasiriyah 64001, Iraq

⁴ Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

* **Correspondence:** Email: dafangzhao@163.com; jessada.t@sci.kmutnb.ac.th.

Abstract: Based on the generalized fractional integrals, we develop some Hermite-Hadamard-type inequalities for interval-valued s-convex functions. Further, we verify our results with graphical representation and concrete examples. Our research not only generalizes and extends the existing literature but also provides valuable insights for further research on interval-valued integral inequalities.

Keywords: fractional integral; interval-valued function; Hermite-Hadamard inequalities; s-convex function

Mathematics Subject Classification: 26E70, 34N05, 35A23

1. Introduction

The classical Hermite-Hadamard (H-H) inequality, serving as a litmus test for convexity, formally establishes that if the function $\mathcal{F} : [\mathfrak{J}_1, \mathfrak{J}_2] \rightarrow \mathbb{R}$ is a convex function satisfying the essential containment relationship between its midpoint value and integral mean. Specifically, the following inequalities are satisfied:

$$\mathcal{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \leq \frac{1}{\mathfrak{J}_2 - \mathfrak{J}_1} \int_{\mathfrak{J}_1}^{\mathfrak{J}_2} \mathcal{F}(\sigma) d\sigma \leq \frac{1}{2}(\mathcal{F}(\mathfrak{J}_1) + \mathcal{F}(\mathfrak{J}_2)).$$

Convexity inequalities have numerous applications in the study of different models from real-world applications [1, 2]. Because the H-H inequality is of great significance in convex analysis, it is widely applied in various fields such as integral inequalities, information theory, and optimization theory. In recent years, it has been extended and generalized through various forms of convexity, such as log-convexity [3, 4], harmonic convexity [5, 6], h-convexity [7–9], convexity in q -calculus [10–12] and especially s -convexity [13]. Since 1994, s -convexity has been a significant development and widespread application and various generalizations and results regarding H-H inequalities related to s -convex mappings have been established in [14–16].

From another perspective, interval analysis provides an effective numerical tool for solving uncertain and nonlinear problems. Since the publication of the first monograph in 1966 [17], it has evolved into a distinct branch of mathematics, with applications spanning data mining, machine learning, and various other fields. A key focus has been interval-valued (IV) function inequalities. Recently, based on interval calculus and generalized convexity, some authors like Ali et al. [18–20], Budak et al. [21, 22], Costa et al. [23–25], Du et al. [26, 27], Khan et al. [28, 29], Sarikaya et al. [30–32], and Zhao et al. [33–35] established the interval versions of the Chebyshev's inequality, Jensen's inequality, and H-H inequality. Furthermore, with the successive introduction of the bilateral Riemann-Liouville (R-L) fractional integral operators (left-hand and right-hand variants) for IV functions, the results related to inequalities for IV functions are more extensive and profound [36, 37]. Especially, in 2023, Budak et al. [38] introduced a novel generalized integral to demonstrate the generalized H-H-type inclusion of IV convex functions.

Motivated by the above-mentioned literature, we have established some novel interval forms of H-H inequalities by using generalized fractional integral operators and combining them with IV s -convex functions. The classical convexity theory is extended to the generalized s -convex framework, some interesting theorems are proved, and the exact inequality representation of the product of two s -convex functions is given. Our findings not only extend the main conclusions in [36, 38, 39], but also provide new insights for the study of IV inequalities.

The organization of this work is outlined below: Section 2 introduces essential background concepts, while Section 3 establishes a series of H-H-type inequalities for IV s -convex mappings using generalized fractional integral operators. Section 4 gives the conclusion.

2. Preliminaries

Let $\mathcal{A}$ denote a compact real interval, mathematically expressed as

$$[\mathcal{A}] = [\underline{\mathcal{A}}, \overline{\mathcal{A}}] = \{a \in \mathbb{R} | \underline{\mathcal{A}} \leq a \leq \overline{\mathcal{A}}\},$$

where $\underline{\mathcal{A}}, \overline{\mathcal{A}} \in \mathbb{R}$ satisfy $\underline{\mathcal{A}} \leq \overline{\mathcal{A}}$. If $\underline{\mathcal{A}} = \overline{\mathcal{A}}$, the interval becomes degenerate. The intervals discussed in this paper are all non-degenerate intervals. An interval $[\mathcal{A}]$ is termed positive when $\underline{\mathcal{A}} > 0$, and negative if $\overline{\mathcal{A}} < 0$. The sets $\mathbb{R}_I^-$, $\mathbb{R}_I^+$, and $\mathbb{R}_I$, respectively, denote all negative, positive, and arbitrary real-valued intervals. Additionally, we adopt the partial ordering “ $\supseteq$ ” defined by

$$[\underline{\mathcal{A}}, \overline{\mathcal{A}}] \supseteq [\underline{\mathcal{B}}, \overline{\mathcal{B}}] \iff \underline{\mathcal{A}} \leq \underline{\mathcal{B}} \quad \text{and} \quad \overline{\mathcal{B}} \leq \overline{\mathcal{A}}.$$

For $\eta \in \mathbb{R}$ and $\mathcal{A} \in \mathbb{R}_I$, then

$$\eta\mathcal{A} = \eta[\underline{\mathcal{A}}, \overline{\mathcal{A}}] = \begin{cases} [\eta\underline{\mathcal{A}}, \eta\overline{\mathcal{A}}], & \text{if } \eta \geq 0, \\ [\eta\overline{\mathcal{A}}, \eta\underline{\mathcal{A}}], & \text{if } \eta < 0. \end{cases}$$

For arbitrary $\mathcal{A}, \mathcal{B} \in \mathbb{R}_I$, the following four arithmetic operations are given:

$$\begin{aligned} \mathcal{A} + \mathcal{B} &= [\underline{\mathcal{A}}, \overline{\mathcal{A}}] + [\underline{\mathcal{B}}, \overline{\mathcal{B}}] = [\underline{\mathcal{A}} + \underline{\mathcal{B}}, \overline{\mathcal{A}} + \overline{\mathcal{B}}], \\ \mathcal{A} - \mathcal{B} &= [\underline{\mathcal{A}}, \overline{\mathcal{A}}] - [\underline{\mathcal{B}}, \overline{\mathcal{B}}] = [\underline{\mathcal{A}} - \overline{\mathcal{B}}, \overline{\mathcal{A}} - \underline{\mathcal{B}}], \\ \mathcal{A} \cdot \mathcal{B} &= [\min\{\underline{\mathcal{A}}\underline{\mathcal{B}}, \underline{\mathcal{A}}\overline{\mathcal{B}}, \overline{\mathcal{A}}\underline{\mathcal{B}}, \overline{\mathcal{A}}\overline{\mathcal{B}}\}, \max\{\underline{\mathcal{A}}\underline{\mathcal{B}}, \underline{\mathcal{A}}\overline{\mathcal{B}}, \overline{\mathcal{A}}\underline{\mathcal{B}}, \overline{\mathcal{A}}\overline{\mathcal{B}}\}], \\ \mathcal{B}/\mathcal{A} &= [\min\{\underline{\mathcal{B}}/\underline{\mathcal{A}}, \underline{\mathcal{B}}/\overline{\mathcal{A}}, \overline{\mathcal{B}}/\underline{\mathcal{A}}, \overline{\mathcal{B}}/\overline{\mathcal{A}}\}, \max\{\underline{\mathcal{B}}/\underline{\mathcal{A}}, \underline{\mathcal{B}}/\overline{\mathcal{A}}, \overline{\mathcal{B}}/\underline{\mathcal{A}}, \overline{\mathcal{B}}/\overline{\mathcal{A}}\}], \\ &\quad (0 \notin [\underline{\mathcal{A}}, \overline{\mathcal{A}}]). \end{aligned}$$

For additional information on interval arithmetic, refer to [40].

Definition 2.1. [13] A function $\mathcal{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}$ is classified as s -convex if it satisfies the convexity condition:

$$\mathcal{F}(\iota\hbar_1 + (1 - \iota)\hbar_2) \leq \iota^s \mathcal{F}(\hbar_1) + (1 - \iota)^s \mathcal{F}(\hbar_2), \quad (2.1)$$

for all $\hbar_1, \hbar_2 \in [\mathfrak{I}_1, \mathfrak{I}_2]$ and $\iota \in [0, 1]$ with $s \in (0, 1]$.

In [41], Breckner introduced the IV s -convex functions.

Definition 2.2. [41] $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$ is defined as an IV s -convex function if

$$\mathfrak{F}(\iota\hbar_1 + (1 - \iota)\hbar_2) \supseteq \iota^s \mathfrak{F}(\hbar_1) + (1 - \iota)^s \mathfrak{F}(\hbar_2), \quad (2.2)$$

for $\hbar_1, \hbar_2 \in [\mathfrak{I}_1, \mathfrak{I}_2]$ and $\iota \in [0, 1]$.

Consider $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$, we define $\int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \mathfrak{F}(\iota) d\iota$ by

$$\int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \mathfrak{F}(\iota) d\iota = \left[\int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \underline{\mathfrak{F}}(\iota) d\iota, \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \overline{\mathfrak{F}}(\iota) d\iota \right]$$

and say that the function $\mathfrak{F}$ is interval Lebesgue integrable on $[\mathfrak{I}_1, \mathfrak{I}_2]$ (or that $\mathfrak{F} \in IL_{[\mathfrak{I}_1, \mathfrak{I}_2]}$).

Definition 2.3. [42] Let $\mathcal{F} \in L[\mathfrak{I}_1, \mathfrak{I}_2]$. The bilateral R - L fractional integral operators, namely the left-sided $\mathcal{J}_{\mathfrak{I}_1+}^\alpha \mathcal{F}$ and the right-sided $\mathcal{J}_{\mathfrak{I}_2-}^\alpha \mathcal{F}$, are defined as

$$\mathcal{J}_{\mathfrak{I}_1+}^\alpha \mathcal{F}(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{I}_1}^\sigma (\sigma - \iota)^{\alpha-1} \mathcal{F}(\iota) d\iota,$$

and

$$\mathcal{J}_{\mathfrak{I}_2-}^\alpha \mathcal{F}(\sigma) = \frac{1}{\Gamma(\alpha)} \int_\sigma^{\mathfrak{I}_2} (\iota - \sigma)^{\alpha-1} \mathcal{F}(\iota) d\iota,$$

respectively, where $\alpha > 0$, $\mathfrak{I}_1 \geq 0$, $\mathcal{J}_{\mathfrak{I}_1+}^0 \mathcal{F}(\sigma) = \mathcal{J}_{\mathfrak{I}_2-}^0 \mathcal{F}(\sigma) = \mathcal{F}(\sigma)$ and $\Gamma(\alpha)$ is the Gamma function.

The H-H inequality in the form of R-L fractional integrals was proved by Sarikaya et al. in [32] as follows:

Theorem 2.4. Let $\mathcal{F} \in L[\mathfrak{I}_1, \mathfrak{I}_2]$ be a mapping from $[\mathfrak{I}_1, \mathfrak{I}_2]$ to $\mathbb{R}^+$. If $\mathcal{F}$ satisfies the convexity condition, then

$$\begin{aligned} \mathcal{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) &\leq \frac{\Gamma(\alpha + 1)}{2(\mathfrak{I}_2 - \mathfrak{I}_1)^\alpha} (\mathcal{J}_{\mathfrak{I}_1+}^\alpha \mathcal{F}(\mathfrak{I}_2) + \mathcal{J}_{\mathfrak{I}_2-}^\alpha \mathcal{F}(\mathfrak{I}_1)) \\ &\leq \frac{1}{2} (\mathcal{F}(\mathfrak{I}_1) + \mathcal{F}(\mathfrak{I}_2)), \end{aligned}$$

with $\alpha > 0$.

Definition 2.5. [36, 37] Let $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$. The IV R-L fractional integral operators, including both left-sided and right-sided variants, are defined as follows:

$$I_{\mathfrak{I}_1+}^\alpha \mathfrak{F}(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{I}_1}^\sigma (\sigma - \iota)^{\alpha-1} \mathfrak{F}(\iota) d\iota, \quad \sigma > \mathfrak{I}_1,$$

and

$$I_{\mathfrak{I}_2-}^\alpha \mathfrak{F}(\sigma) = \frac{1}{\Gamma(\alpha)} \int_\sigma^{\mathfrak{I}_2} (\iota - \sigma)^{\alpha-1} \mathfrak{F}(\iota) d\iota, \quad \sigma < \mathfrak{I}_2,$$

respectively. Here, $I_{\mathfrak{I}_1+}^\alpha \mathfrak{F}(\sigma) = [\mathcal{J}_{\mathfrak{I}_1+}^\alpha \mathfrak{F}(\sigma), \overline{\mathcal{J}}_{\mathfrak{I}_1+}^\alpha \mathfrak{F}(\sigma)]$, and $\alpha > 0$.

In [36], Budak et al. gave the fractional H-H inequality for IV convex functions as follows:

Theorem 2.6. Let $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$, and $\alpha > 0$. If $\mathfrak{F}$ is an IV convex function, then

$$\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2(\mathfrak{I}_2 - \mathfrak{I}_1)^\alpha} (I_{\mathfrak{I}_1+}^\alpha \mathfrak{F}(\mathfrak{I}_2) + I_{\mathfrak{I}_2-}^\alpha \mathfrak{F}(\mathfrak{I}_1)) \supseteq \frac{\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)}{2}. \quad (2.3)$$

Definition 2.7. [31] Let $\varphi : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}^+$ be a monotonically increasing mapping, and suppose $\mathcal{F}, \varphi \in L[\mathfrak{I}_1, \mathfrak{I}_2]$. The generalized R-L fractional integral operators $\mathcal{J}_{\mathfrak{I}_1+\varphi}^{\alpha,k} \mathcal{F}$ and $\mathcal{J}_{\mathfrak{I}_2-\varphi}^{\alpha,k} \mathcal{F}$ are defined by

$$\mathcal{J}_{\mathfrak{I}_1+\varphi}^{\alpha,k} (\mathcal{F})(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{I}_1}^\sigma (\sigma - \iota)^{\alpha-1} (\varphi(\sigma) - \varphi(\iota))^k \mathcal{F}(\iota) d\iota, \quad \sigma > \mathfrak{I}_1,$$

and

$$\mathcal{J}_{\mathfrak{I}_2-\varphi}^{\alpha,k} (\mathcal{F})(\sigma) = \frac{1}{\Gamma(\alpha)} \int_\sigma^{\mathfrak{I}_2} (\iota - \sigma)^{\alpha-1} (\varphi(\iota) - \varphi(\sigma))^k \mathcal{F}(\iota) d\iota, \quad \sigma < \mathfrak{I}_2,$$

respectively, where $k \in \mathbb{N} \cup \{0\}$, $\alpha > 0$, and $\mathfrak{I}_1 \geq 0$.

In 2023, Budak et al. introduced the generalized fractional integral for IV function as follows:

Definition 2.8. [38] Let $\varphi : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}^+$ be a monotonically increasing function, and $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$. The generalized R-L fractional integrals $I_{\mathfrak{I}_1+\varphi}^{\alpha,k} \mathfrak{F}$ and $I_{\mathfrak{I}_2-\varphi}^{\alpha,k} \mathfrak{F}$ of interval-valued functions are defined by

$$I_{\mathfrak{I}_1+\varphi}^{\alpha,k} (\mathfrak{F})(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{I}_1}^\sigma (\sigma - \iota)^{\alpha-1} (\varphi(\sigma) - \varphi(\iota))^k \mathfrak{F}(\iota) d\iota, \quad \sigma > \mathfrak{I}_1,$$

and

$$\mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(\mathfrak{F})(\sigma) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{\mathfrak{I}_2} (\iota - \sigma)^{\alpha-1} (\varphi(\iota) - \varphi(\sigma))^k \mathfrak{F}(\iota) d\iota, \quad \sigma < \mathfrak{I}_2,$$

respectively, where $\alpha > 0$, $\mathfrak{I}_1 \geq 0$ and $k \in \mathbb{N} \cup \{0\}$.

The families of all functions that are Riemann integrable and interval Riemann integrable over the closed interval $[\mathfrak{I}_1, \mathfrak{I}_2]$ are respectively represented by the notations $\mathcal{R}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$ and $\mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$.

Theorem 2.9. [38] Suppose $\mathfrak{F} \in \mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$, $\varphi : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}$ is a monotonically increasing function on $(\mathfrak{I}_1, \mathfrak{I}_2)$ with $\varphi \in L[\mathfrak{I}_1, \mathfrak{I}_2]$. Letting $\Phi(\iota) = \mathfrak{F}(\iota) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - \iota)$, then $\Phi \in \mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$, and we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right) (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1)) \\ & \supseteq \frac{1}{2} (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(\Phi)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(\Phi)(\mathfrak{I}_1)) \\ & \supseteq \frac{1}{2} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1)), \end{aligned}$$

where $\alpha > 0$ and $k \in \mathbb{N} \cup \{0\}$.

3. Main results

Let $\Phi(\varpi) = \mathfrak{F}(\varpi) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - \varpi)$ and $\Lambda(\varpi) = \mathcal{G}(\varpi) + \mathcal{G}(\mathfrak{I}_1 + \mathfrak{I}_2 - \varpi)$ for $\varpi \in [\mathfrak{I}_1, \mathfrak{I}_2]$. It is straightforward to demonstrate that if $\mathfrak{F}, \mathcal{G} \in \mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$, then $\Phi(\varpi), \Lambda(\varpi) \in \mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$.

Theorem 3.1. Let $\mathfrak{F} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_+^+$, $\mathfrak{F} \in \mathcal{IR}_{[\mathfrak{I}_1, \mathfrak{I}_2]}$, and $\varphi : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}^+$ be a monotonically increasing function. If $\mathfrak{F}$ is an IV s -convex function, then

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1)) \\ & \supseteq \frac{1}{2^s} (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(\Phi)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(\Phi)(\mathfrak{I}_1)) \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) (\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1)), \end{aligned} \tag{3.1}$$

where $\alpha > 0$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. By the assumption, we have

$$\mathfrak{F}\left(\frac{\hbar_1 + \hbar_2}{2}\right) \supseteq \frac{\mathfrak{F}(\hbar_1) + \mathfrak{F}(\hbar_2)}{2^s}. \tag{3.2}$$

Letting $\hbar_1 = \iota \mathfrak{I}_1 + (1 - \iota) \mathfrak{I}_2$, $\hbar_2 = (1 - \iota) \mathfrak{I}_1 + \iota \mathfrak{I}_2$ for $\iota \in [0, 1]$, we obtain

$$\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \supseteq \frac{1}{2^s} (\mathfrak{F}(\iota \mathfrak{I}_1 + (1 - \iota) \mathfrak{I}_2) + \mathfrak{F}((1 - \iota) \mathfrak{I}_1 + \iota \mathfrak{I}_2)). \tag{3.3}$$

Then, multiplying the two sides of (3.3) by $\iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k$ and integrating on $[0,1]$, we obtain

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k d\iota \\ & \supseteq \frac{1}{2^s} \left(\int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k \mathfrak{F}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2) d\iota \right. \\ & \quad \left. + \int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k \mathfrak{F}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) d\iota \right). \end{aligned}$$

Letting $y = \iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2$, we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy \\ & \supseteq \frac{1}{2^s} \left(\int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k \mathfrak{F}(y) dy \right. \\ & \quad \left. + \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y) dy \right). \end{aligned}$$

That is,

$$\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2) \supseteq \frac{1}{2^s} \mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_2). \quad (3.4)$$

By multiplying the two sides of (3.3) by $\iota^{\alpha-1}(\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k$ and integrating on $[0,1]$, we obtain

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \tau^{\alpha-1}(\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k d\iota \\ & \supseteq \frac{1}{2^s} \left(\int_0^1 \iota^{\alpha-1}(\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k \mathfrak{F}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2) d\iota \right. \\ & \quad \left. + \int_0^1 \iota^{\alpha-1}(\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k \mathfrak{F}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) d\iota \right). \end{aligned}$$

Letting $y = (1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2$, we have

$$\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1) \supseteq \frac{1}{2^s} \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_2). \quad (3.5)$$

Combining with conclusions (3.4) and (3.5), we obtain

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) (\mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1)) \\ & \supseteq \frac{1}{2^s} (\mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_1)). \end{aligned}$$

According to (2.2), we have

$$\begin{aligned} \mathfrak{F}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2) & \supseteq \iota^s \mathfrak{F}(\mathfrak{I}_1) + (1-\iota)^s \mathfrak{F}(\mathfrak{I}_2), \\ \mathfrak{F}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) & \supseteq \iota^s \mathfrak{F}(\mathfrak{I}_2) + (1-\iota)^s \mathfrak{F}(\mathfrak{I}_1). \end{aligned}$$

That is,

$$\mathfrak{F}(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2) + \mathfrak{F}((1 - \iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)). \quad (3.6)$$

Multiplying the two sides of (3.6) by $\iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2))^k$ and integrating on $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2))^k \mathfrak{F}(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2) d\iota \\ & + \int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2))^k \mathfrak{F}(\iota\mathfrak{I}_2 + (1 - \iota)\mathfrak{I}_1) d\iota \\ & \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_0^1 \iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2))^k d\iota. \end{aligned}$$

Letting $y = \iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2$, we have

$$\begin{aligned} & \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{\mathfrak{I}_2 - y}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k \mathfrak{F}(y) dy \\ & + \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{\mathfrak{I}_2 - y}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y) dy \\ & \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{\mathfrak{I}_2 - y}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy. \end{aligned}$$

That is,

$$\mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_2) \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2). \quad (3.7)$$

Similarly, multiplying the two sides of (3.6) by $\iota^{\alpha-1}(\varphi((1 - \iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k$ and integrating on $[0,1]$, let $y = \iota\mathfrak{I}_2 + (1 - \iota)\mathfrak{I}_1$, we have

$$\begin{aligned} & \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{y - \mathfrak{I}_1}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y) dy \\ & + \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{y - \mathfrak{I}_1}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k \mathfrak{F}(y) dy \\ & \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} \left(\frac{y - \mathfrak{I}_1}{\mathfrak{I}_2 - \mathfrak{I}_1} \right)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k dy. \end{aligned}$$

Then, we obtain

$$\mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_1) \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1). \quad (3.8)$$

By (3.7) and (3.8), we have

$$\mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(\Phi)(\mathfrak{I}_1) \supseteq (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) (\mathcal{I}_{\mathfrak{I}_1+, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2) + \mathcal{I}_{\mathfrak{I}_2-, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1)).$$

Hence, Theorem 3.1 is verified. $\square$

Corollary 3.2. If $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$ and $s = 1$, then

$$\mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) \supseteq \frac{\Gamma(\alpha + k + 1)}{2(\mathfrak{Y}_2 - \mathfrak{Y}_1)^{\alpha+k}} \left(\mathcal{I}_{\mathfrak{Y}_1+}^{\alpha+k} \mathfrak{F}(\mathfrak{Y}_2) + \mathcal{I}_{\mathfrak{Y}_2-}^{\alpha+k} \mathfrak{F}(\mathfrak{Y}_1) \right) \supseteq \frac{1}{2} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)).$$

Remark 3.3. If $s = 1$, Theorem 3.1 is reduced to Theorem 4.1 obtained by Budak et al. in [38].

Remark 3.4. If $k = 0$, then

$$\mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) \supseteq \frac{\Gamma(\alpha + 1)}{2^s(\mathfrak{Y}_2 - \mathfrak{Y}_1)^\alpha} \left(\mathcal{I}_{\mathfrak{Y}_1+}^\alpha \mathfrak{F}(\mathfrak{Y}_2) + \mathcal{I}_{\mathfrak{Y}_2-}^\alpha \mathfrak{F}(\mathfrak{Y}_1) \right) \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)).$$

Remark 3.5. If $s = 1$ and $k = 0$, then Theorem 3.1 simplifies to Theorem 3.2 as presented by Budak et al. in [36].

Example 3.6. Consider $\mathfrak{F} : [0, 2] \rightarrow \mathbb{R}_I^+$, where $\mathfrak{F}(x) = [2x^2 - 4, -8x^2 + 20x - 3]$, $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$, $\alpha = 2$, $s = 1$, and $k = 1$. Then we have

$$\begin{aligned} \Delta_1 &= \mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) \left(\mathcal{I}_{\mathfrak{Y}_1+,\varphi}^{\alpha,k}(1)(\mathfrak{Y}_2) + \mathcal{I}_{\mathfrak{Y}_2-,\varphi}^{\alpha,k}(1)(\mathfrak{Y}_1) \right) \\ &= \mathfrak{F}(1) \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) dx \\ &= \frac{16}{3} [-2, 9] = \left[-\frac{32}{3}, 48\right], \end{aligned}$$

$$\begin{aligned} \Delta_2 &= \frac{1}{2^s} \left(\mathcal{I}_{\mathfrak{Y}_1+,\varphi}^{\alpha,k}(\Phi)(\mathfrak{Y}_2) + \mathcal{I}_{\mathfrak{Y}_2-,\varphi}^{\alpha,k}(\Phi)(\mathfrak{Y}_1) \right) \\ &= \frac{1}{2} \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) (\mathfrak{F}(x) + \mathfrak{F}(2-x)) dx \\ &= \frac{1}{2} \left[-\frac{64}{5}, \frac{16}{3} + \frac{128}{5} \right] = \left[-\frac{32}{5}, \frac{464}{15} \right], \end{aligned}$$

and

$$\begin{aligned} \Delta_3 &= \frac{1}{2^s} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)) \left(\mathcal{I}_{\mathfrak{Y}_1+,\varphi}^{\alpha,k}(1)(\mathfrak{Y}_2) + \mathcal{I}_{\mathfrak{Y}_2-,\varphi}^{\alpha,k}(1)(\mathfrak{Y}_1) \right) \\ &= \frac{1}{2} (\mathfrak{F}(0) + \mathfrak{F}(2)) \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) dx \\ &= \frac{8}{3} ([-4, -3] + [4, 5]) = \left[0, \frac{16}{3} \right]. \end{aligned}$$

Then, we obtain

$$\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3.$$

The graphical representation (Figure 1) confirms the results.

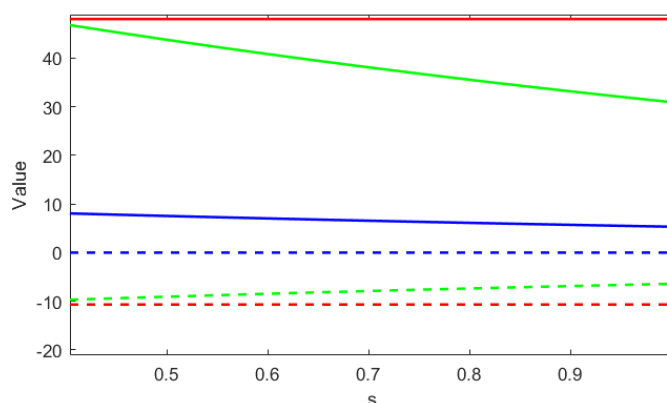


Figure 1. Illustration of Example 3.6: $\alpha = 2, k = 1$, and $s \in [0.4, 1]$. The red pattern represents Δ_1 , the blue pattern represents Δ_2 , and the green pattern represents Δ_3 .

Theorem 3.7. Assuming that the conditions of Theorem 3.1 are met, then

$$\begin{aligned}
 & \mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) \left(\mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) +, \varphi}^{\alpha, k}(1)(\mathfrak{Y}_2) + \mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) -, \varphi}^{\alpha, k}(1)(\mathfrak{Y}_1) \right) \\
 & \supseteq \frac{1}{2^s} \left(\mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) +, \varphi}^{\alpha, k}(\Phi)(\mathfrak{Y}_2) + \mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) -, \varphi}^{\alpha, k}(\Phi)(\mathfrak{Y}_1) \right) \\
 & \supseteq \frac{1}{2^s} \left(\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2) \right) \left(\mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) +, \varphi}^{\alpha, k}(1)(\mathfrak{Y}_2) + \mathcal{I}_{(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2)) -, \varphi}^{\alpha, k}(1)(\mathfrak{Y}_1) \right), \quad (3.9)
 \end{aligned}$$

where $\alpha > 0$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. First, consider $\hbar_1 = \frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2$ and $\hbar_2 = \frac{1}{2}\iota\mathfrak{Y}_2 + \frac{1}{2}(2-\iota)\mathfrak{Y}_1$ for $\iota \in [0, 1]$ in the inclusion (3.2). Then,

$$\begin{aligned}
 \mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) & \supseteq \frac{1}{2^s} \left(\mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) + \mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{Y}_2 + \frac{1}{2}(2-\iota)\mathfrak{Y}_1\right) \right) \\
 & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)). \quad (3.10)
 \end{aligned}$$

Then, multiplying the two sides of (3.10) by $\iota^{\alpha-1} \left(\varphi(\mathfrak{Y}_2) - \varphi\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) \right)^k$ and integrating on $[0, 1]$, we obtain

$$\begin{aligned}
 & \mathfrak{F}\left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2}\right) \int_0^1 \iota^{\alpha-1} \left(\varphi(\mathfrak{Y}_2) - \varphi\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) \right)^k d\iota \\
 & \supseteq \frac{1}{2^s} \int_0^1 \iota^{\alpha-1} \left(\varphi(\mathfrak{Y}_2) - \varphi\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) \right)^k \\
 & \quad \cdot \left(\mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) + \mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{Y}_2 + \frac{1}{2}(2-\iota)\mathfrak{Y}_1\right) \right) d\iota \\
 & \supseteq \frac{1}{2^s} \int_0^1 \iota^{\alpha-1} \left(\varphi(\mathfrak{Y}_2) - \varphi\left(\frac{1}{2}\iota\mathfrak{Y}_1 + \frac{1}{2}(2-\iota)\mathfrak{Y}_2\right) \right)^k (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)) d\iota.
 \end{aligned}$$

Letting $y = \frac{1}{2}\iota\mathfrak{I}_1 + \frac{1}{2}(2 - \iota)\mathfrak{I}_2$, we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right)\left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy \\ & \supseteq \frac{1}{2^s} \left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k (\mathfrak{F}(y) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y)) dy \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy, \end{aligned}$$

that is,

$$\begin{aligned} \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^+, \varphi}^{\alpha, k} (1)(\mathfrak{I}_2) & \supseteq \frac{1}{2^s} \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^+, \varphi}^{\alpha, k} (\Phi)(\mathfrak{I}_2) \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^+, \varphi}^{\alpha, k} (1)(\mathfrak{I}_2). \end{aligned} \quad (3.11)$$

Similarly, multiplying the two sides of (3.10) by $\iota^{\alpha-1} \left(\varphi\left(\frac{1}{2}(2 - \iota)\mathfrak{I}_1 + \frac{1}{2}\iota\mathfrak{I}_2\right) - \varphi(\mathfrak{I}_1) \right)^k$ and integrating on $[0, 1]$, we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \iota^{\alpha-1} \left(\varphi\left(\frac{1}{2}(2 - \iota)\mathfrak{I}_1 + \frac{1}{2}\iota\mathfrak{I}_2\right) - \varphi(\mathfrak{I}_1) \right)^k d\iota \\ & \supseteq \frac{1}{2^s} \int_0^1 \iota^{\alpha-1} \left(\varphi\left(\frac{1}{2}(2 - \iota)\mathfrak{I}_1 + \frac{1}{2}\iota\mathfrak{I}_2\right) - \varphi(\mathfrak{I}_1) \right)^k \\ & \quad \times \left(\mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{I}_1 + \frac{1}{2}(2 - \iota)\mathfrak{I}_2\right) + \mathfrak{F}\left(\frac{1}{2}\iota\mathfrak{I}_2 + \frac{1}{2}(2 - \iota)\mathfrak{I}_1\right) \right) d\iota \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_0^1 \iota^{\alpha-1} \left(\varphi\left(\frac{1}{2}(2 - \iota)\mathfrak{I}_1 + \frac{1}{2}\iota\mathfrak{I}_2\right) - \varphi(\mathfrak{I}_1) \right)^k d\iota. \end{aligned}$$

Letting $y = \frac{1}{2}(2 - \iota)\mathfrak{I}_1 + \frac{1}{2}\iota\mathfrak{I}_2$, we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right)\left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\mathfrak{I}_1}^{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}} (y - \mathfrak{I}_1)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k dy \\ & \supseteq \frac{1}{2^s} \left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\mathfrak{I}_1}^{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}} (y - \mathfrak{I}_1)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k (\mathfrak{F}(y) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y)) dy \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \left(\frac{2}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\mathfrak{I}_1}^{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}} (y - \mathfrak{I}_1)^{\alpha-1} (\varphi(y) - \varphi(\mathfrak{I}_1))^k dy, \end{aligned}$$

that is,

$$\begin{aligned} \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^-, \varphi}^{\alpha, k} (1)(\mathfrak{I}_1) & \supseteq \frac{1}{2^s} \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^-, \varphi}^{\alpha, k} (\Phi)(\mathfrak{I}_1) \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \mathcal{I}_{\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right)^-, \varphi}^{\alpha, k} (1)(\mathfrak{I}_1). \end{aligned} \quad (3.12)$$

By (3.11) and (3.12), we obtain the result. $\square$

Corollary 3.8. If $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$ and $s = 1$, then

$$\begin{aligned}\mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) &\supseteq \frac{2^{\alpha+k-1}\Gamma(\alpha+k+1)}{(\mathfrak{J}_2 - \mathfrak{J}_1)^{\alpha+k}} (I_{\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)+}^{\alpha+k} \mathfrak{F}(\mathfrak{J}_2) + I_{\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)-}^{\alpha+k} \mathfrak{F}(\mathfrak{J}_1)) \\ &\supseteq \frac{\mathfrak{F}(\mathfrak{J}_1) + \mathfrak{F}(\mathfrak{J}_2)}{2}.\end{aligned}$$

Remark 3.9. If $s = 1$, then Theorem 3.7 simplifies to Theorem 4.4 given by Budak et al. in [38].

Remark 3.10. If $k = 0$, then

$$\begin{aligned}\mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) &\supseteq \frac{2^\alpha \Gamma(\alpha+1)}{2^s (\mathfrak{J}_2 - \mathfrak{J}_1)^\alpha} (I_{\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)+}^{\alpha+k} \mathfrak{F}(\mathfrak{J}_2) + I_{\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)-}^{\alpha+k} \mathfrak{F}(\mathfrak{J}_1)) \\ &\supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{J}_1) + \mathfrak{F}(\mathfrak{J}_2)).\end{aligned}$$

Remark 3.11. If $s = 1$ and $k = 0$, then Theorem 3.7 which has been obtained by Zhao et al. in [39].

Example 3.12. Consider $\mathfrak{F} : [0, 1] \rightarrow \mathbb{R}_+^+$, where $\mathfrak{F}(x) = [x^2, 5 - (\frac{1}{2})^x]$, $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$, $\alpha = 2$, $s = 1$, and $k = 1$. Then we have

$$\begin{aligned}T_1 &= \mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \left(I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2))+, \varphi}^{\alpha,k} (1)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)-, \varphi}^{\alpha,k} (1)(\mathfrak{J}_1) \right) \\ &= \mathfrak{F}\left(\frac{1}{2}\right) \frac{1}{\Gamma(2)} \left(\int_{\frac{1}{2}}^1 (1-x)^2 dx + \int_0^{\frac{1}{2}} x^2 dx \right) \\ &= \frac{1}{12} \left[\frac{1}{4}, 5 - \left(\frac{1}{2}\right)^{\frac{1}{2}} \right] \approx [0.0208, 0.3750],\end{aligned}$$

$$\begin{aligned}T_2 &= \frac{1}{2^s} \left(I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2))+, \varphi}^{\alpha,k} (\Phi)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)-, \varphi}^{\alpha,k} (\Phi)(\mathfrak{J}_1) \right) \\ &= \frac{1}{2} \frac{1}{\Gamma(2)} \left(\int_{\frac{1}{2}}^1 (1-x)^2 (\mathfrak{F}(x) + \mathfrak{F}(1-x)) dx + \int_0^{\frac{1}{2}} x^2 (\mathfrak{F}(x) + \mathfrak{F}(1-x)) dx \right) \\ &\approx [0.0229, 0.3574],\end{aligned}$$

and

$$\begin{aligned}T_3 &= \frac{1}{2^s} (\mathfrak{F}(\mathfrak{J}_1) + \mathfrak{F}(\mathfrak{J}_2)) \left(I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2))+, \varphi}^{\alpha,k} (1)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1+\mathfrak{J}_2)-, \varphi}^{\alpha,k} (1)(\mathfrak{J}_1) \right) \\ &= \frac{1}{2} \frac{1}{\Gamma(2)} (\mathfrak{F}(0) + \mathfrak{F}(1)) \left(\int_{\frac{1}{2}}^1 (1-x)^2 dx + \int_0^{\frac{1}{2}} x^2 dx \right) \\ &= \frac{1}{24} \left([0, 4] + \left[1, \frac{9}{2} \right] \right) \approx [0.0417, 0.3542].\end{aligned}$$

Then, we obtain

$$T_1 \supseteq T_2 \supseteq T_3.$$

Theorem 3.13. Given that the assumptions of Theorem 3.1 hold, we can obtain

$$\mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \left(I_{\mathfrak{J}_1+, \varphi}^{\alpha,k} (1) \left(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2) \right) + I_{\mathfrak{J}_2-, \varphi}^{\alpha,k} (1) \left(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2) \right) \right)$$

$$\begin{aligned}
&\supseteq \frac{1}{2^s} \left(\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(\Phi) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(\Phi) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) \right) \\
&\supseteq \frac{1}{2^s} \left(\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2) \right) \left(\mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) + \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) \right),
\end{aligned} \tag{3.13}$$

for $\alpha > 0$ and $k \in \mathbb{N} \cup \{0\}$.

Proof. Considering $\hbar_1 = \frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2$ and $\hbar_1 = \frac{1}{2}(1 - \iota)\mathfrak{I}_1 + \frac{1}{2}(1 + \iota)\mathfrak{I}_2$ for $\iota \in [0, 1]$ in the inclusion (3.2). Then we obtain

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \\
&\supseteq \frac{1}{2^s} \left(\mathfrak{F}\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) + \mathfrak{F}\left(\frac{1}{2}(1 - \iota)\mathfrak{I}_1 + \frac{1}{2}(1 + \iota)\mathfrak{I}_2\right) \right) \\
&\supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)).
\end{aligned} \tag{3.14}$$

Then, multiplying two sides of (3.14) by

$$\iota^{\alpha-1} \left(\varphi\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right) - \varphi\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) \right)^k,$$

and integrating on $[0, 1]$, we obtain

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \iota^{\alpha-1} \left(\varphi\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) - \varphi\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) \right)^k d\iota \\
&\supseteq \frac{1}{2^s} \int_0^1 \tau^{\alpha-1} \left(\varphi\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) - \varphi\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) \right)^k \\
&\quad \cdot \left(\mathfrak{F}\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) + \mathfrak{F}\left(\frac{1}{2}(1 - \iota)\mathfrak{I}_1 + \frac{1}{2}(1 + \iota)\mathfrak{I}_2\right) \right) d\iota \\
&\supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_0^1 \iota^{\alpha-1} \left(\varphi\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) - \varphi\left(\frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2\right) \right)^k d\iota.
\end{aligned}$$

Letting $z = \frac{1}{2}(1 + \iota)\mathfrak{I}_1 + \frac{1}{2}(1 - \iota)\mathfrak{I}_2$, we have

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_{\mathfrak{I}_1}^{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}} \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} - z \right)^{\alpha-1} \left(\varphi\left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)\right) - \varphi(z) \right)^k dz \\
&\supseteq \frac{1}{2^s} \int_{\mathfrak{I}_1}^{\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}} \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} - z \right)^{\alpha-1} \left(\varphi\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) - \varphi(z) \right)^k (\mathfrak{F}(z) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - z)) dz \\
&\supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \int_{\mathfrak{I}_1}^{\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2)} \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} - z \right)^{\alpha-1} \left(\varphi\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) - \varphi(z) \right)^k dz,
\end{aligned}$$

that is,

$$\begin{aligned}
\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1) \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} \right) &\supseteq \frac{1}{2^s} \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(\Phi) \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} \right) \\
&\supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1) \left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2} \right).
\end{aligned} \tag{3.15}$$

Multiplying two sides of (3.14) by

$$\iota^{\alpha-1} \left(\varphi \left(\frac{1}{2}(1-\iota)\mathfrak{Y}_1 + \frac{1}{2}(1+\iota)\mathfrak{Y}_2 \right) - \varphi \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \right)^k,$$

and integrating on $[0,1]$, we obtain

$$\begin{aligned} & \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \int_0^1 \tau^{\alpha-1} \left(\varphi \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) - \varphi \left(\frac{1}{2}(1+\iota)\mathfrak{Y}_1 + \frac{1}{2}(1-\iota)\mathfrak{Y}_2 \right) \right)^k d\iota \\ & \supseteq \frac{1}{2^s} \int_0^1 \iota^{\alpha-1} \left(\varphi \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) - \varphi \left(\frac{1}{2}(1+\iota)\mathfrak{Y}_1 + \frac{1}{2}(1-\iota)\mathfrak{Y}_2 \right) \right)^k \\ & \quad \times \left(\mathfrak{F} \left(\frac{1}{2}(1+\iota)\mathfrak{Y}_1 + \frac{1}{2}(1-\iota)\mathfrak{Y}_2 \right) + \mathfrak{F} \left(\frac{1}{2}(1-\iota)\mathfrak{Y}_1 + \frac{1}{2}(1+\iota)\mathfrak{Y}_2 \right) \right) d\iota \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)) \int_0^1 \iota^{\alpha-1} \left(\varphi \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) - \varphi \left(\frac{1}{2}(1+\iota)\mathfrak{Y}_1 + \frac{1}{2}(1-\iota)\mathfrak{Y}_2 \right) \right)^k d\iota. \end{aligned}$$

Letting $z = \frac{1}{2}(1-\iota)\mathfrak{Y}_1 + \frac{1}{2}(1+\iota)\mathfrak{Y}_2$, we obtain

$$\begin{aligned} \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \mathcal{I}_{\mathfrak{Y}_2^-, \varphi}^{\alpha, k}(1) \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) & \supseteq \frac{1}{2^s} \mathcal{I}_{\mathfrak{Y}_2^-, \varphi}^{\alpha, k}(\Phi) \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \\ & \supseteq \frac{1}{2^s} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)) \mathcal{I}_{\mathfrak{Y}_2^-, \varphi}^{\alpha, k}(1) \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right). \end{aligned} \quad (3.16)$$

Adding the inclusion (3.15) to (3.16), we complete the proof. $\square$

Corollary 3.14. If $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$ and $s = 1$, then

$$\begin{aligned} \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) & \supseteq \frac{2^{\alpha+k-1} \Gamma(\alpha + k + 1)}{(\mathfrak{Y}_2 - \mathfrak{Y}_1)^{\alpha+k}} \left(\mathcal{I}_{\mathfrak{Y}_1^+}^{\alpha+k} \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) + \mathcal{I}_{\mathfrak{Y}_2^-}^{\alpha+k} \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \right) \\ & \supseteq \frac{1}{2} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)). \end{aligned}$$

Remark 3.15. If $s = 1$, Theorem 3.13 simplifies to Theorem 4.7 presented by Budak et al. in [38].

Remark 3.16. If $s = 1$ and $k = 0$, then

$$\begin{aligned} \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) & \supseteq \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(\mathfrak{Y}_2 - \mathfrak{Y}_1)^\alpha} \left(\mathcal{I}_{\mathfrak{Y}_1^+}^\alpha \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) + \mathcal{I}_{\mathfrak{Y}_2^-}^\alpha \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \right) \\ & \supseteq \frac{1}{2} (\mathfrak{F}(\mathfrak{Y}_1) + \mathfrak{F}(\mathfrak{Y}_2)), \end{aligned}$$

which has been obtained by Zhao et al. in [39].

Example 3.17. Consider $\mathfrak{F} : [0, 2] \rightarrow \mathbb{R}_I^+$, where $\mathfrak{F}(x) = [x^2, -x^2 + 8]$, $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$, $\alpha = 2$, $s = 1$, and $k = 1$. Then we have

$$\begin{aligned} D_1 & = \mathfrak{F} \left(\frac{\mathfrak{Y}_1 + \mathfrak{Y}_2}{2} \right) \left(\mathcal{I}_{\mathfrak{Y}_1^+, \varphi}^{\alpha, k}(1) \left(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2) \right) + \mathcal{I}_{\mathfrak{Y}_2^-, \varphi}^{\alpha, k}(1) \left(\frac{1}{2}(\mathfrak{Y}_1 + \mathfrak{Y}_2) \right) \right) \\ & = \mathfrak{F}(1) \frac{1}{\Gamma(2)} \left(\int_0^1 (1-x)^2 dx + \int_1^2 (x-1)^2 dx \right) \\ & = \frac{2}{3} [1, 7] = \left[\frac{2}{3}, \frac{14}{3} \right], \end{aligned}$$

$$\begin{aligned}
D_2 &= \frac{1}{2^s} \left(I_{\mathfrak{I}_1 +, \varphi}^{\alpha, k}(\Phi) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) + I_{\mathfrak{I}_2 -, \varphi}^{\alpha, k}(\Phi) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) \right) \\
&= \frac{1}{2} \frac{1}{\Gamma(2)} \int_0^2 (1-x)^2 (\mathfrak{F}(x) + \mathfrak{F}(2-x)) dx \\
&= \left[\frac{16}{15}, \frac{64}{15} \right],
\end{aligned}$$

and

$$\begin{aligned}
D_3 &= \frac{1}{2^s} (\mathfrak{F}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2)) \left(I_{\mathfrak{I}_1 +, \varphi}^{\alpha, k}(1) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) + I_{\mathfrak{I}_2 -, \varphi}^{\alpha, k}(1) \left(\frac{1}{2}(\mathfrak{I}_1 + \mathfrak{I}_2) \right) \right) \\
&= \frac{1}{2} \frac{1}{\Gamma(2)} (\mathfrak{F}(0) + \mathfrak{F}(2)) \int_0^2 (1-x)^2 dx \\
&= \frac{1}{3} ([0, 8] + [4, 4]) = \left[\frac{4}{3}, 4 \right].
\end{aligned}$$

Then, we obtain

$$D_1 \supseteq D_2 \supseteq D_3.$$

Theorem 3.18. Let $\mathfrak{F}, \mathcal{G} : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}_I^+$, $s_1, s_2 \in (0, 1]$, and $\varphi : [\mathfrak{I}_1, \mathfrak{I}_2] \rightarrow \mathbb{R}$ be a monotonically increasing function. If $\mathfrak{F}$ and $\mathcal{G}$ are both IV s -convex functions, then

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \left(I_{\mathfrak{I}_1 +, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2) + I_{\mathfrak{I}_2 -, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1) \right) \\
&\supseteq \frac{1}{2^{s_1+s_2}} \left(I_{\mathfrak{I}_1 +, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{I}_2) + I_{\mathfrak{I}_2 -, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{I}_1) \right) \\
&\supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) \left(I_{\mathfrak{I}_1 +, \varphi}^{\alpha, k}(1)(\mathfrak{I}_2) + I_{\mathfrak{I}_2 -, \varphi}^{\alpha, k}(1)(\mathfrak{I}_1) \right),
\end{aligned} \tag{3.17}$$

where

$$P(\mathfrak{I}_1, \mathfrak{I}_2) = \mathfrak{F}(\mathfrak{I}_1) \cdot \mathcal{G}(\mathfrak{I}_1) + \mathfrak{F}(\mathfrak{I}_2) \cdot \mathcal{G}(\mathfrak{I}_2),$$

and

$$Q(\mathfrak{I}_1, \mathfrak{I}_2) = \mathfrak{F}(\mathfrak{I}_1) \cdot \mathcal{G}(\mathfrak{I}_2) + \mathfrak{F}(\mathfrak{I}_2) \cdot \mathcal{G}(\mathfrak{I}_1).$$

Proof. By hypothesis, then

$$\mathfrak{F}\left(\frac{\hbar_1 + \hbar_2}{2}\right) \supseteq \frac{\mathfrak{F}(\hbar_1) + \mathfrak{F}(\hbar_2)}{2^{s_1}}, \mathcal{G}\left(\frac{\hbar_1 + \hbar_2}{2}\right) \supseteq \frac{\mathcal{G}(\hbar_1) + \mathcal{G}(\hbar_2)}{2^{s_2}}.$$

Considering $\hbar_1 = \iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2$ and $\hbar_2 = (1 - \iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2$, we obtain

$$\begin{aligned}
&\mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \\
&\supseteq \frac{1}{2^{s_1+s_2}} (\mathfrak{F}(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2) + \mathfrak{F}((1 - \iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) \\
&\quad \cdot (\mathcal{G}(\iota\mathfrak{I}_1 + (1 - \iota)\mathfrak{I}_2) + \mathcal{G}((1 - \iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) \\
&\supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)).
\end{aligned} \tag{3.18}$$

By multiplying two sides of (3.18) by $\iota^{\alpha-1}(\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k$ and integrating on $[0,1]$, we obtain

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \iota^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k d\iota \\ & \supseteq \frac{1}{2^{s_1+s_2}} \int_0^1 \iota^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k (\mathfrak{F}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2)) \\ & \quad + \mathfrak{F}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) \cdot (\mathcal{G}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2)) + \mathcal{G}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) d\tau \\ & \supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) \int_0^1 \iota^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2))^k d\iota. \end{aligned}$$

Letting $y = \iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2$, then

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \left(\frac{1}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy \\ & \supseteq \frac{1}{2^{s_1+s_2}} \left(\frac{1}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha \left(\int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k \right. \\ & \quad \cdot (\mathfrak{F}(y) + \mathfrak{F}(\mathfrak{I}_1 + \mathfrak{I}_2 - y)) \cdot (\mathcal{G}(y) + \mathcal{G}(\mathfrak{I}_1 + \mathfrak{I}_2 - y)) dy \\ & \supseteq \frac{1}{2^{s_1+s_2}} \left(\frac{1}{\mathfrak{I}_2 - \mathfrak{I}_1}\right)^\alpha (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) \\ & \quad \cdot \int_{\mathfrak{I}_1}^{\mathfrak{I}_2} (\mathfrak{I}_2 - y)^{\alpha-1} (\varphi(\mathfrak{I}_2) - \varphi(y))^k dy, \end{aligned}$$

that is,

$$\begin{aligned} \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2) & \supseteq \frac{1}{2^{s_1+s_2}} \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(\Phi\Lambda)(\mathfrak{I}_2) \\ & \supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) \mathcal{I}_{\mathfrak{I}_1+\varphi}^{\alpha,k}(1)(\mathfrak{I}_2). \end{aligned} \quad (3.19)$$

By multiplying two sides of (3.18) by $\iota^{\alpha-1}(\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k$ and integrating on $[0,1]$, we have

$$\begin{aligned} & \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \int_0^1 \iota^{\alpha-1} (\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k d\iota \\ & \supseteq \frac{1}{2^{s_1+s_2}} \int_0^1 \iota^{\alpha-1} (\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k \\ & \quad \cdot (\mathfrak{F}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2)) + \mathfrak{F}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) (\mathcal{G}(\iota\mathfrak{I}_1 + (1-\iota)\mathfrak{I}_2)) + \mathcal{G}((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2)) d\tau \\ & \supseteq \frac{1}{2^{s_1+s_2}} \int_0^1 \iota^{\alpha-1} (\varphi((1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2) - \varphi(\mathfrak{I}_1))^k (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) d\iota. \end{aligned}$$

Letting $y = (1-\iota)\mathfrak{I}_1 + \iota\mathfrak{I}_2$, that is,

$$\begin{aligned} \mathfrak{F}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{I}_1 + \mathfrak{I}_2}{2}\right) \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1) & \supseteq \frac{1}{2^{s_1+s_2}} \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(\Phi\Lambda)(\mathfrak{I}_1) \\ & \supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{I}_1, \mathfrak{I}_2) + Q(\mathfrak{I}_1, \mathfrak{I}_2)) \mathcal{I}_{\mathfrak{I}_2-\varphi}^{\alpha,k}(1)(\mathfrak{I}_1). \end{aligned} \quad (3.20)$$

Adding the inclusion (3.19) and (3.20), we obtained the desired result. $\square$

Example 3.19. Assume $\mathfrak{F}, \mathcal{G} : [0, 2] \rightarrow \mathbb{R}_+^+$, where $\mathfrak{F}(x) = [x^2, -x^2 + 10]$ and

$$\mathcal{G}(x) = \left[\frac{1}{2}x^2, -x^2 + 8 \right],$$

respectively. Let $\varphi(\iota) = \iota - \omega$ for $\omega \in \mathbb{R}$, $\alpha = 2, k = 1$, and $s = 1$. Then we have

$$\begin{aligned} \Theta_1 &= \mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}(1)(\mathfrak{J}_2) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}(1)(\mathfrak{J}_1) \right) \\ &= \mathfrak{F}(1) \cdot \mathcal{G}(1) \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) dx \\ &= \frac{16}{3} \left([1, 9] \cdot \left[\frac{1}{2}, 7 \right] \right) = \left[\frac{8}{3}, 336 \right], \end{aligned}$$

and

$$\begin{aligned} \Theta_2 &= \frac{1}{2^{s_1+s_2}} \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{J}_2) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{J}_1) \right) \\ &= \frac{1}{4} \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) (\mathfrak{F}(x) + \mathfrak{F}(2-x)) \cdot (\mathcal{G}(x) + \mathcal{G}(2-x)) dx \\ &= [5.4857, 303.2381], \end{aligned}$$

and

$$\begin{aligned} \Theta_3 &= \frac{1}{2^{s_1+s_2}} (P(\mathfrak{J}_1, \mathfrak{J}_2) + Q(\mathfrak{J}_1, \mathfrak{J}_2)) \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}(1)(\mathfrak{J}_2) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}(1)(\mathfrak{J}_1) \right) \\ &= \frac{1}{4} (\mathfrak{F}(0) + \mathfrak{F}(2)) \cdot (\mathcal{G}(0) + \mathcal{G}(2)) \frac{1}{\Gamma(2)} \int_0^2 ((2-x)^2 + x^2) dx \\ &= \frac{4}{3} ([8, 192]) = \left[\frac{32}{3}, 256 \right]. \end{aligned}$$

Then, we obtain

$$\Theta_1 \supseteq \Theta_2 \supseteq \Theta_3.$$

Analogously, we can derive the subsequent results.

Theorem 3.20. Assuming that the conditions of Theorem 3.18 hold, we consequently obtain

$$\begin{aligned} &\mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \left(I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) +, \varphi}^{\alpha, k}(1)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) -, \varphi}^{\alpha, k}(1)(\mathfrak{J}_1) \right) \\ &\supseteq \frac{1}{2^{s_1+s_2}} \left(I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) +, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) -, \varphi}^{\alpha, k}(\Phi\Lambda)(\mathfrak{J}_1) \right) \\ &\supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{J}_1, \mathfrak{J}_2) + Q(\mathfrak{J}_1, \mathfrak{J}_2)) \left(I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) +, \varphi}^{\alpha, k}(1)(\mathfrak{J}_2) + I_{(\frac{1}{2}(\mathfrak{J}_1 + \mathfrak{J}_2)) -, \varphi}^{\alpha, k}(1)(\mathfrak{J}_1) \right). \end{aligned} \quad (3.21)$$

Theorem 3.21. Assuming that the conditions of Theorem 3.18 hold, we obtain

$$\begin{aligned} &\mathfrak{F}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \cdot \mathcal{G}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \right) \\ &\supseteq \frac{1}{2^{s_1+s_2}} \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}(\Phi\Lambda)\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}(\Phi\Lambda)\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \right) \\ &\supseteq \frac{1}{2^{s_1+s_2}} (P(\mathfrak{J}_1, \mathfrak{J}_2) + Q(\mathfrak{J}_1, \mathfrak{J}_2)) \left(I_{\mathfrak{J}_1 +, \varphi}^{\alpha, k}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) + I_{\mathfrak{J}_2 -, \varphi}^{\alpha, k}\left(\frac{\mathfrak{J}_1 + \mathfrak{J}_2}{2}\right) \right). \end{aligned} \quad (3.22)$$

4. Conclusions

This study derives novel H-H-type inequalities through generalized fractional integrals applied to IV s-convex functions. These not only generalize but also refine the previously proposed inequalities established by Budak et al. and provide a foundation for further exploration of generalized convexity and IV estimation. The developed techniques offer new tools for uncertainty quantification in convex optimization problems with imprecise measurements. Future research directions include:

- (1) Develop new interval H-H inequalities based on more general convexity.
- (2) Extension to IV quasi-convex functions using variable-order fractional operators.
- (3) Applications in robust portfolio optimization with IV risk measures.

Author contributions

G. Deng and D. Zhao: Conceptualization, Methodology, Formal analysis; S. Etemad: Conceptualization, Software, Formal analysis, Investigation, Writing—original draft preparation; G. Deng: Writing—original draft preparation; D. Zhao: Validation, Investigation, Supervision, Project administration, Writing—review and editing; J. Tariboon: Validation, Formal analysis, Project administration, Writing—review and editing. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by the National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok, under Contract No. KMUTNB-FF-68-B-04, and the Foundation of Hubei Normal University (2024Z022).

Conflict of interest

The authors declare no conflict of interest.

References

1. T. X. Li, D. Acosta-Soba, A. Columbu, G. Viglialoro, Dissipative gradient nonlinearities prevent δ -formations in local and nonlocal attraction-repulsion chemotaxis models, *Stud. Appl. Math.*, **154** (2025), e70018. <https://doi.org/10.1111/sapm.70018>
2. T. X. Li, S. Frassu, G. Viglialoro, Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption, *Z. Angew. Math. Phys.*, **74** (2023), 109. <https://doi.org/10.1007/s00033-023-01976-0>
3. S. S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions, *Khayyam J. Math.*, **3** (2017), 98–115. <https://doi.org/10.22034/kjm.2017.47458>

4. C. P. Niculescu, The Hermite-Hadamard inequality for log-convex functions, *Nonlinear Anal.*, **75** (2012), 662–669. <https://doi.org/10.1016/j.na.2011.08.066>
5. I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, *Hacet. J. Math. Stat.*, **43** (2014), 935–942.
6. I. Iscan, S. H. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **238** (2014), 237–244. <https://doi.org/10.1016/j.amc.2014.04.020>
7. M. Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard-type inequalities for h -convex functions, *J. Math. Inequal.*, **2** (2008), 335–341. <https://doi.org/10.7153/jmi-02-30>
8. S. Varosanec, On h -convexity, *J. Math. Anal. Appl.*, **326** (2007), 303–311. <https://doi.org/10.1016/j.jmaa.2006.02.086>
9. D. F. Zhao, T. Q. An, G. J. Ye, W. Liu, New Jensen and Hermite-Hadamard type inequalities for h -convex interval-valued functions, *J. Inequal. Appl.*, **2018** (2018), 1–14. <https://doi.org/10.1186/s13660-018-1896-3>
10. S. I. Butt, M. N. Aftab, H. A. Nabwey, S. Etemad, Some Hermite-Hadamard and midpoint type inequalities in symmetric quantum calculus, *AIMS Math.*, **9** (2024), 5523–5549. <http://doi.org/10.3934/math.2024268>
11. T. Sitthiwiratttham, M. A. Ali, H. Budak, S. Etemad, S. Rezapour, A new version of (p, q) -Hermite-Hadamard's midpoint and trapezoidal inequalities via special operators in (p, q) -calculus, *Bound. Value Probl.*, **2022** (2022), 84. <https://doi.org/10.1186/s13661-022-01665-3>
12. S. Chasreechai, M. A. Ali, M. A. Ashraf, T. Sitthiwiratttham, S. Etemad, M. De la Sen, et al., On new estimates of q -Hermite-Hadamard inequalities with applications in quantum calculus, *Axioms*, **12** (2023), 1–14. <https://doi.org/10.3390/axioms12010049>
13. H. Hudzik, L. Maligranda, Some remarks on s -convex functions, *Aequationes Math.*, **48** (1994), 100–111. <https://doi.org/10.1007/BF01837981>
14. M. A. Khan, Y. M. Chu, T. U. Khan, J. Khan, Some new inequalities of Hermite-Hadamard type for s -convex functions with applications, *Open Math.*, **15** (2017), 1414–1430. <https://doi.org/10.1515/math-2017-0121>
15. P. Korus, An extension of the Hermite-Hadamard inequality for convex and s -convex functions, *Aequationes Math.*, **93** (2019), 527–534. <https://doi.org/10.1007/s00010-019-00642-z>
16. B. Meftah, A. Lakhdari, D. C. Bencettah, Some new Hermite-Hadamard type integral inequalities for twice differentiable s -convex functions, *Comput. Math. Model.*, **33** (2022), 330–353. <https://doi.org/10.1007/s10598-023-09576-3>
17. R. E. Moore, *Interval analysis*, Prentice-Hall, 1966.
18. M. A. Ali, H. Budak, M. Z. Sarikaya, E. Set, Quantum Ostrowski type inequalities for pre-invex functions, *Math. Slovaca*, **72** (2022), 1489–1500. <https://doi.org/10.1515/ms-2022-0101>
19. M. A. Ali, M. Z. Sarikaya, H. Budak, Z. Y. Zhang, Quantum Hermite-Hadamard type inequalities and related inequalities for subadditive functions, *Miskolc Math. Notes*, **24** (2023), 5–13. <http://doi.org/10.18514/MMN.2023.3769>

20. Z. Y. Zhang, M. A. Ali, H. Budak, M. Z. Sarikaya, On Hermite-Hadamard type inequalities for interval-valued multiplicative integrals, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.*, **69** (2020), 1428–1448. <https://doi.org/10.31801/cfsuasmas.754842>
21. H. Budak, H. Kara, M. A. Ali, S. Khan, Y. M. Chu, Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions, *Open Math.*, **19** (2021), 1081–1097. <https://doi.org/10.1515/math-2021-0067>
22. H. Budak, H. Kara, S. Erden, On Fejer type inclusions for products of interval-valued convex functions, *Filomat*, **35** (2021), 4937–4955. <https://doi.org/10.2298/FIL2114937B>
23. T. M. Costa, G. N. Silva, Y. Chalco-Cano, H. Roman-Flores, Gauss-type integral inequalities for interval and fuzzy-interval-valued functions, *Comput. Appl. Math.*, **38** (2019), 58. <https://doi.org/10.1007/s40314-019-0836-2>
24. T. M. Costa, Jensen's inequality type integral for fuzzy-interval-valued functions, *Fuzzy Sets Syst.*, **327** (2017), 31–47. <https://doi.org/10.1016/j.fss.2017.02.001>
25. T. M. Costa, H. Roman-Flores, Y. Chalco-Cano, Opial-type inequalities for interval-valued functions, *Fuzzy Sets Syst.*, **358** (2019), 48–63. <https://doi.org/10.1016/j.fss.2018.04.012>
26. T. S. Du, Y. Peng, Hermite-Hadamard type inequalities for multiplicative Riemann-Liouville fractional integrals, *J. Comput. Appl. Math.*, **440** (2024), 115582. <https://doi.org/10.1016/j.cam.2023.115582>
27. T. S. Du, S. H. Wu, S. J. Zhao, M. U. Awan, Riemann-Liouville fractional Hermite-Hadamard inequalities for h -preinvex functions, *J. Comput. Anal. Appl.*, **25** (2018), 364–384.
28. M. B. Khan, H. M. Srivastava, P. O. Mohammed, K. Nonlaopon, Y. S. Hamed, Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions, *AIMS Math.*, **7** (2022), 4338–4358. <https://doi.org/10.3934/math.2022241>
29. M. B. Khan, S. Treanta, H. Alrweili, T. Saeed, M. S. Soliman, Some new Riemann-Liouville fractional integral inequalities for interval-valued mappings, *AIMS Math.*, **7** (2022), 15659–15679. <https://doi.org/10.3934/math.2022857>
30. M. Z. Sarikaya, H. Budak, Generalized Hermite-Hadamard type integral inequalities for fractional integrals, *Filomat*, **30** (2016), 1315–1326. <https://doi.org/10.2298/FIL1605315S>
31. M. Z. Sarikaya, H. Budak, F. Usta, Some generalized integral inequalities via fractional integrals, *Acta Math. Univ. Comenian.*, **89** (2020), 27–38.
32. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, **57** (2013), 2403–2407. <https://doi.org/10.1016/j.mcm.2011.12.048>
33. D. F. Zhao, T. Q. An, G. J. Ye, W. Liu, Chebyshev type inequalities for interval-valued functions, *Fuzzy Sets Syst.*, **396** (2020), 82–101. <https://doi.org/10.1016/j.fss.2019.10.006>
34. D. F. Zhao, T. Q. An, G. J. Ye, D. F. M. Torres, On Hermite-Hadamard type inequalities for harmonical h -convex interval-valued functions, *Math. Inequal. Appl.*, **23** (2020), 95–105. <http://doi.org/10.7153/mia-2020-23-08>
35. D. F. Zhao, G. J. Ye, W. Liu, D. F. M. Torres, Some inequalities for interval-valued functions on time scales, *Soft Comput.*, **23** (2019), 6005–6015. <https://doi.org/10.1007/s00500-018-3538-6>

36. H. Budak, T. Tunc, M. Z. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Proc. Amer. Math. Soc.*, **148** (2020), 705–718. <https://doi.org/10.1090/proc/14741>
37. V. Lupulescu, Fractional calculus for interval-valued functions, *Fuzzy Sets Syst.*, **265** (2015), 63–85. <https://doi.org/10.1016/j.fss.2014.04.005>
38. H. Budak, H. Kara, F. Hezenci, Generalized Hermite-Hadamard inclusions for a generalized fractional integral, *Rocky Mountain J. Math.*, **53** (2023), 383–395. <https://doi.org/10.1216/rmj.2023.53.383>
39. D. F. Zhao, M. A. Ali, A. Kashuri, H. Budak, M. Z. Sarikaya, Hermite-Hadamard-type inequalities for the interval-valued approximately h -convex functions via generalized fractional integrals, *J. Inequal. Appl.*, **2020** (2020), 1–38. <https://doi.org/10.1186/s13660-020-02488-5>
40. R. E. Moore, *Methods and applications of interval analysis*, Philadelphia: SIAM, 1979.
41. W. W. Breckner, Continuity of generalized convex and generalized concave set-valued functions, *Rev. Anal. Numer. Theor. Approx.*, **22** (1993), 39–51.
42. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.



AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)