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Research article

A method for solving the Cauchy problem for Duffng type integro-differential equation

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Abstract: The paper explored the applicability of the Dzhumabaev parametrization method to Cauchy problems with nonlinear integrodifferential operator equation arising in applied mathematics. The proposed method reformulated the problem under consideration as an equivalent parametric multipoint problem. Subsequently, linearization of the problem and stepwise solution of a sequence of linear approximations was used to solve the parametric problem. Particular emphasis was placed on solving the nonlinear special Cauchy problem for Fredholm integrodifferential equations. A novel strategy was employed to determine solutions of the original problem through this approach. The effectiveness of the proposed approach was demonstrated by numerical examples.

Keywords: Duffing type integro-differential equation, Dzhumabaev parametrization method; algorithm; iterative process; initial guess solution; special Cauchy problem
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1. Introduction

In applied sciences, the Duffing equation is a well-established nonlinear model used to address several significant applied phenomena, including periodic orbit extraction, nonuniform behavior due to infinite domains, nonlinear mechanical oscillators, and disease prediction. For the latest developments in this area, see [1–6] and related references.

Integrodifferential equations (IDEs) are frequently encountered in scientific applications where aftereffects or time delays are significant factors. They are particularly relevant for modeling hereditary systems. In biology, these equations are essential for analyzing the spread of infectious diseases through dispersal and reaction-diffusion models that estimate invasion speeds in ecological systems. For further theoretical insights, refer to [7].

In [8], the authors studied a boundary value problem (BVP) for the Duffing type IDE with separated boundary conditions and applied the generalized quasi-linearization algorithm [9] to construct monotone sequences of lower and upper solutions. In a related study [10], the generalized quasi-linearization technique was employed to obtain analytical approximations for the solutions of the Duffing type IDE with multipoint conditions. Both works demonstrated that the resulting sequences of approximate solutions converge uniformly and quadratically to the unique solution of the respective problems.

In [11], a novel iterative algorithm was introduced to solve the Duffing type IDE within the reproducing kernel space [12, 13]. The desired solution was expressed as a series and it was demonstrated that the $n-$term approximation $u_n(x)$ converges to the exact solution $u(x)$.

In [14], an improved variational iteration method [15, 16] was introduced to solve the Duffing type IDE. This modification has been shown to improve computational efficiency by eliminating redundant steps in identifying the unknown parameters of the initial solution, in contrast to the standard variational iteration method.

The nonlinear BVPs for Fredholm IDEs on a finite interval were investigated in [17, 18]. Using the Dzhumabaev parametrization method, these equations were transformed into parametric special Cauchy problems for IDEs. Conditions were established to ensure the existence of a unique solution to these problems.

A numerical approach for solving parameter-dependent problems for a system of Fredholm IDEs was presented in [19]. In [20], a numerical approach based on spline approximations was proposed to solve linear BVPs for IDEs.

This study considers the initial value problem (IVP) for Duffing type IDE

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \beta x^3 + \gamma x^5 = \eta cos(\omega t) + \int_0^T \varphi_1(t)\psi_1(\tau)x(\tau)d\tau \tag{1.1}$$

$$+ \int_0^T \varphi_2(t)\psi_2(\tau)\frac{\partial x(\tau)}{\partial\tau}d\tau + p(t), t \in (0, T),$$

$$x(0) = a, \quad \frac{dx(0)}{dt} = 0, \tag{1.2}$$

where x is displacement; α, β, γ, η, ω are arbitrary constants; and $\varphi_k(t)$, $\psi_k(\tau)$, $k = 1, 2$, and $p(t)$ are continuous on $[0, T]$.

Denote by $C([0, T], R)$ the space of continuous functions $x : [0, T] \to R$ with the norm $\|x\|_1 = \max_{t\in[0,T]} \|x(t)\|$. A function $x(t) \in C([0, T], R)$ that is continuously differentiable on $(0, T)$ is said to be a solution to problem (1.1), (1.2) if it satisfies Eq (1.1) along with initial condition (1.2).

By a solution to problem (1.1), (1.2) we mean a continuously differentiable on the $(0, T)$ function $x(t) \in C([0, T], R)$ that satisfies Eq (1.1) and initial condition (1.2).

The integral term significantly influences the properties of Eq (1.1). Even when the differential part of this equation is linear it may not have a solution unless additional conditions are imposed [21].

This paper aims to propose a constructive approach to solving IVP (1.1), (1.2). To achieve this, we employ the parametrization method along with the results of [21].

2. Application of the Dzhumabaev parametrization method to the IVP

Let us rewrite problems (1.1) and (1.2) in the form

$$\frac{dy}{dt} = A(t)y + \int_0^T \varphi(t)\psi(\tau)y(\tau)d\tau + f(t,y) + f_0(t), \quad t \in (0,T), \quad y \in R^2, \tag{2.1}$$

$$y(0) = d, \quad d \in R^2, \tag{2.2}$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} 0 & 0 \\ \varphi_1(t) & \varphi_2(t) \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} \psi_1(t) & 0 \\ 0 & \psi_2(t) \end{pmatrix}, \quad f(t,y) = \begin{pmatrix} 0 \\ -\beta y_1^3 - \gamma y_1^5, \end{pmatrix},$$

$$f_0(t) = \begin{pmatrix} 0 \\ p(t) \end{pmatrix}, \quad d = \begin{pmatrix} a \\ 0 \end{pmatrix}.$$

The interval $[0,T]$ is divided into N subintervals using the partition Δ_N: $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$.

Let Δ_N be a regular partition [21, p.83].

The function spaces given below are introduced: $C([0,T],\Delta_N,R^{2N})$ is the space of function systems $y[t] = (y_1(t),\dots,y_N(t))$, where functions $y_r : [t_{r-1},t_r) \to R^2$, $r = \overline{1,N}$, are continuous and have finite left-sided limits $\lim_{t\to t_r-0} y_r(t)$, with the norm $\left\|y[\cdot]\right\|_2 = \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \|y_r(t)\|$;

$PC([0,T],\Delta_N,R^2)$ is the space of piecewise continuous functions $y : [0,T] \to R^2$ with possible discontinuity points t_s, $s = \overline{1,N-1}$, with the norm $\|y\|_3 = \sup_{t\in[0,T]} \|y(t)\|$;

$\widetilde{C}([0,T],\Delta_N,R^{2N})$ is the space of function systems $v[t] = (v_1(t),\dots,v_N(t))$, where functions $v_r : [t_{r-1},t_r] \to R^2$, $r = \overline{1,N}$, are continuous, with the norm $\left\|v[\cdot]\right\|_4 = \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \|v_r(t)\|$.

Suppose that $y^{(0)}(t)$ is a unique solution to a linear IVP corresponding to problems (2.1) and (2.2). We identify the vector $\lambda^{(0)} \in R^{2N}$ and the function system $u^{(0)}[t] \in C([0,T],\Delta_N,R^{2N})$ as follows:

$$\lambda_r^{(0)} = y^{(0)}(t_{r-1}), \quad r = \overline{1,N}, \tag{2.3}$$

$$u_r^{(0)}(t) = y^{(0)}(t) - \lambda_r^{(0)}, \quad t \in [t_{r-1},t_r), \quad r = \overline{1,N}. \tag{2.4}$$

Given positive numbers $\rho, \rho_\lambda, \rho_u$, we compose the following balls:

$$S(\lambda^{(0)},\rho_\lambda) = \left\{\lambda = (\lambda_1,\dots,\lambda_N) \in R^{2N} : \max_{r=\overline{1,N}} \left\|\lambda_r - \lambda_r^{(0)}\right\| < \rho_\lambda\right\},$$

$$S(u^{(0)}[t],\rho_u) = \left\{u[t] \in C([0,T],\Delta_N,R^{2N}) : \left\|u[\cdot] - u^{(0)}[\cdot]\right\|_2 < \rho_u\right\},$$

$$S(y^{(0)}(t),\rho) = \left\{y(t) \in PC([0,T],\Delta_N,R^2) : \|y - y^{(0)}\|_3 < \rho\right\},$$

and introduce the sets:

$$G^0(\rho) = \left\{(t,y) : t \in [0,T], \|y - y^{(0)}(t)\| < \rho\right\},$$

$$G_r^0(\rho) = \Big\{(t, y) : t \in [t_{r-1}, t_r), \|y - y^{(0)}(t)\| < \rho\Big\}, \quad r = \overline{1, N}.$$

Suppose a function $y(t)$ satisfies (2.1), (2.2), and $(t, y(t)) \in G^0(\rho)$. Then, the functions $y_r(t)$, $r = \overline{1, N}$, defined by the equalities $y_r(t) = y(t)$, $t \in [t_{r-1}, t_r)$, satisfy the multipoint BVP

$$\frac{dy_r}{dt} = A(t)y_r + \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau)y_j(\tau)d\tau + f(t, y_r) + f_0(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}, \tag{2.5}$$

$$y_1(0) = d, \tag{2.6}$$

$$\lim_{t\to t_p-0} y_p(t) = y_{p+1}(t_p), \quad p = \overline{1, N-1}, \tag{2.7}$$

and $(t, y_r(t)) \in G_r^0(\rho)$, $r = \overline{1, N}$. By a solution to (2.5)–(2.7), we mean a function system $y^*[t] = (y_1^*(t), \dots, y_N^*(t)) \in C([0, T], \Delta_N, R^{2N})$, where the functions $y_r^*(t)$, $r = \overline{1, N}$, are continuously differentiable on their domains and satisfy the system of Eqs (2.5)–(2.7).

Introducing the parameters $\lambda_r = y_r(t_{r-1})$ and setting $u_r(t) = y_r(t) - \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, yields the BVP with parameters

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau)[u_j(\tau) + \lambda_j]d\tau \tag{2.8}$$

$$+ f(t, u_r + \lambda_r) + f_0(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N},$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{2.9}$$

$$\lambda_1 = d, \tag{2.10}$$

$$\lambda_p + \lim_{t\to t_p-0} u_p(t) = u_{p+1}(t_p), \quad p = \overline{1, N-1}. \tag{2.11}$$

A pair $(\lambda^*, u^*[t])$ with elements $\lambda^* \in R^{2N}$, $u^*[t] \in C([0, T], \Delta_(N), R^{2N})$ is said to be a solution to problems (2.8)–(2.11) if the functions $u_r^*(t)$ are continuously differentiable on $[t_{r-1}, t_r)$, $r = \overline{1, N}$ and satisfy Eqs (2.8)–(2.11) with $\lambda_r = \lambda_r^*$, $r = \overline{1, N}$.

Clearly, $u^*[t] = u[t, \lambda^*]$. Moreover, if the pair $(\lambda^*, u^*[t])$ is a solution to problems (2.8)–(2.11), then the function $y^*(t)$, defined by the equalities $y^*(t) = \lambda_r^* + u_r^*(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, and $y^*(T) = \lambda_N^* + \lim_{t\to T-0} u_N^*(t)$, is a solution to the original problems (2.1) and (2.2).

Problems (2.8) and (2.9) form the special Cauchy problem for the system of nonlinear IDEs with parameters on subintervals.

3. Solvability of the special Cauchy problem and an iterative method for finding its solution

In order to solve original problems (2.5)–(2.7), we employ the limit values of the solution to problems (2.8) and (2.9). Consequently, it is appropriate to study the following special Cauchy problem on the closed subintervals:

$$\frac{dv_r}{dt} = A(t)(v_r + \lambda_r) + \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau)[v_j(\tau) + \lambda_j]d\tau \tag{3.1}$$

$$+ f(t, v_r + \lambda_r) + f_0(t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N},$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{3.2}$$

It is obvious that if, for some $\lambda = \tilde{\lambda}$, the function systems $u[t, \tilde{\lambda}]$ and $v[t, \tilde{\lambda}]$ are solutions to problems (2.8), (2.9) and (3.1), (3.2), respectively, then the following relations hold:

$$u_r(t, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N) = v_r(t, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N), \quad t \in [t_{r-1}, t_r), \tag{3.3}$$

$$\lim_{t \to t_r - 0} u_r(t, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N) = v_r(t_r, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_N), \quad r = \overline{1, N}. \tag{3.4}$$

The special Cauchy problem for a system of linear Fredholm IDEs with parameters has been investigated in [21, p.81]. As observed in this study, the special Cauchy problem for linear Fredholm IDEs does not always have a solution.Therefore, we analyze the solvability of problems (3.1) and (3.2) and develop an algorithm for finding its solution.

An initial guess solution $v^{(0)}[t] = (v_1^{(0)}(t), \ldots, v_N^{(0)}(t))$ to problems (3.1) and (3.2) is defined by setting $v_r^{(0)}(t) = u_r^{(0)}(t)$, $t \in [t_{r-1}, t_r)$, and $v_r^{(0)}(t_r) = \lim_{t \to t_r - 0} u_r^{(0)}(t), \quad r = \overline{1, N}$. Assume that

$$S(v^{(0)}[t], \rho_u) = \left\{ v[t] \in \widetilde{C}([0, T], \Delta_N, R^{2N}) : \left\| v[\cdot] - v^{(0)}[\cdot] \right\|_4 < \rho_u \right\} \textit{ and } \rho_u + \rho_\lambda \le \rho.$$

Condition A. *The function $f(t, x)$ has uniformly continuous partial derivative $f'_x(t, x)$ in $G^0(\rho)$.*

A powerful technique for solving nonlinear BVPs involves transforming them into nonlinear operator equations and using iterative processes to determine their solutions (see [22–24]).

Define the space $X = \left\{ v[t] = (v_1(t), \ldots, v_N(t)) \in \widetilde{C}([0, T], \Delta_N, R^{2N}) : v_r(t_{r-1}) = 0, \, r = \overline{1, N} \right\}$, and let $Y = \widetilde{C}([0, T], \Delta_N, R^{2N})$.

Then, introduce the linear operator $H : X \to Y$ as follows:

$$Hv[t] = w^{(1)}[t],$$

where

$$w^{(1)}[t] = (w_1^{(1)}(t), \ldots, w_N^{(1)}(t)),$$

$$w_r^{(1)}(t) = \dot{v}_r(t) - A(t)v_r(t) - \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau) v_j(\tau) d\tau - f_0(t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N},$$

and the nonlinear operator

$$F(v[t], \hat{\lambda}) = (w_1^{(2)}(t, \hat{\lambda}), \ldots, w_N^{(2)}(t, \hat{\lambda})),$$

$$w_r^{(2)}(t, \hat{\lambda}) = -A(t)\hat{\lambda}_r - f(t, v_r(t) + \hat{\lambda}_r) - \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau) d\tau \hat{\lambda}_j, \quad t \in [t_{r-1}, t_r], \; r = \overline{1, N}.$$

The domain of H is $D(H) = \left\{ v[t] = (v_1(t), \ldots, v_N(t)) \in X, \text{ where } v_r(t) \text{ is continuously differentiable on } [t_{r-1}, t_r], \, r = \overline{1, N} \right\}$. It is straightforward to verify that H is a closed unbounded linear operator.

We formulate problems (3.1) and (3.2) as the operator equation

$$Hv[t] + F(v[t], \hat{\lambda}) = 0 \tag{3.5}$$

and apply the results from [25, 26].

Condition A guarantees the existence of the uniformly continuous Frechet derivative $F'_v(v[t], \hat{\lambda})$ in $S(v^{(0)}[t], \rho_u)$ [24, p.646], which can be written in the form:

$$F'_v(v[t], \hat{\lambda}) = diag\Big\{ -\frac{\partial f(t, v_1(t) + \hat{\lambda}_1)}{\partial y}, \dots, -\frac{\partial f(t, v_N(t) + \hat{\lambda}_N)}{\partial y} \Big\}.$$

The closed linear operator $H + F'_v(v[t], \hat{\lambda}) : X \to Y$ has a bounded inverse, if the linear operator equation

$$\Big(H + F'_v(v[t], \hat{\lambda})\Big)\vartheta = g[t], \;\; g[t] = (g_1(t), \dots, g_N(t)) \in Y, \tag{3.6}$$

is uniquely solvable. Equation (3.6) is equivalent to the parametric special Cauchy problem for the system of linear IDEs

$$\frac{d\vartheta_r}{dt} = (A(t) + f'_x(t, v_r(t) + \hat{\lambda}_r))\vartheta_r + \varphi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \psi(\tau)\vartheta_j(\tau)d\tau + g_r(t), \;\; t \in [t_{r-1}, t_r], \tag{3.7}$$

$$\vartheta_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{3.8}$$

Problems (3.7) and (3.8) form a linearized special Cauchy problem for nonlinear problems (3.1) and (3.2). Criteria ensuring the existence and uniqueness of a solution to the parametric special Cauchy problem for a system of linear IDEs on subintervals has been obtained in [21].

Problems (3.7) and (3.8) are called well-posed if for any $g[t] \in Y$ it has a unique solution $\vartheta[t] \in \widetilde{C}([0, T], \Delta_N, R^{2N})$ and the inequality $\big\|\vartheta[\cdot]\big\|_4 \le \chi\big\|g[\cdot]\big\|_4$ holds, where χ is a constant independent of $g[t]$.

The constant χ is referred to as the well-posedness constant of problems (3.7) and (3.8). Let $L(Y, X)$ be the space of linear bounded operators $\Lambda : Y \to X$ with the induced norm. It is clear that if problems (3.7) and (3.8) are well-posed with constant χ, then

$$\Big\|\Big[H + F'_v(v[t], \hat{\lambda})\Big]^{-1}\Big\|_{L(Y,X)} \le \chi.$$

Fix $\hat{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$, $\hat{v}^{(0)}[t] \in S(v^{(0)}[t], \rho_u) \cap D(H)$, and $\hat{\rho}_u > 0$.

Theorem 1. *Assume the following conditions hold:*

(i) $F'_v(v[t], \hat{\lambda})$ is uniformly continuous in $S(\hat{v}^{(0)}[t], \hat{\rho}_u)$;

(ii) the operator $H + F'_v(v[t], \hat{\lambda}) : X \to Y$ has a bounded inverse and

$$\Big\|\Big[H + F'_v(v[t], \hat{\lambda})\Big]^{-1}\Big\|_{L(Y,X)} \le \hat{\chi} \quad \textit{for all} \quad v[t] \in S(\hat{v}^{(0)}[t], \hat{\rho}_u), \quad \hat{\chi} \quad \textit{is a constant};$$

(iii) $\hat{\chi} \cdot \Big\|H\hat{v}^{(0)}[\cdot] + F(\hat{v}^{(0)}[\cdot], \hat{\lambda})\Big\|_4 < \hat{\rho}_u.$

Then, there exist numbers $\alpha_k \geq 1$, $k = 0, 1, 2, \ldots$, such that the sequence $\{\hat{v}^{(k)}[t]\}$, generated by the iterative process

$$\hat{v}^{(k+1)}[t] = \hat{v}^{(k)}[t] - \frac{1}{\alpha_k}\Big[H + F'_v(\hat{v}^{(k)}[t], \hat{\lambda})\Big]^{-1}\Big[H\hat{v}^{(k)}[t] + F(\hat{v}^{(k)}[t], \hat{\lambda})\Big], \quad k = 0, 1, 2, \ldots, \tag{3.9}$$

converges to $v[t, \hat{\lambda}]$, an isolated solution to Eq (3.5) in $S(\hat{v}^{(0)}[t], \hat{\rho}_u)$, and the following estimate holds:

$$\Big\|v[\cdot, \hat{\lambda}] - \hat{v}^{(0)}[\cdot]\Big\|_4 \leq \chi \Big\|H\hat{v}^{(0)}[\cdot] + F(\hat{v}^{(0)}[\cdot], \hat{\lambda})\Big\|_4. \tag{3.10}$$

Theorem 1, along with the interrelation between the special Cauchy problems (3.1) and (3.2) and the operator Eq (3.5), implies the following result.

Theorem 2. *Let Condition A be fulfilled, the special Cauchy problems* (3.1) *and* (3.2) *be well-posed with constant $\hat{\chi}$ for all $v[t] \in S(\hat{v}^{(0)}[t], \hat{\rho}_u)$, and the following inequality be valid:*

$$\hat{\chi} \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \Bigg\|\dot{\hat{v}}_r^{(0)}(t) - A(t)[\hat{v}_r^{(0)}(t) + \hat{\lambda}_r] - \varphi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi(\tau)[\hat{v}_j^{(0)}(\tau) + \hat{\lambda}_j]d\tau$$
$$- f(t, \hat{v}_r^{(0)}(t) + \hat{\lambda}_r) - f_0(t)\Bigg\| < \hat{\rho}_u.$$

Then, there exist numbers $\alpha_k \geq 1$, such that the sequence $\{\hat{v}^{(k)}[t]\}$, generated by the iterative process

$$\hat{v}^{(k+1)}[t] = \hat{v}^{(k)}[t] + \Delta v^{(k)}[t, \hat{\lambda}], \quad k = 0, 1, 2, \ldots, \tag{3.11}$$

where $\Delta v^{(k)}[t, \hat{\lambda}]$ is the solution to the parametric special Cauchy problem for the system of linear IDEs

$$\frac{d\Delta v_r}{dt} = \Big(A(t) + f'_x(t, v_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_r)\Big)\Delta v_r + \varphi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi(\tau)\Delta v_j(\tau)d\tau \tag{3.12}$$
$$- \frac{1}{\alpha_k}\Big\{\dot{v}_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) - A(t)(v_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_r) - f_0(t)$$
$$- \varphi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi(\tau)\Big[v_j^{(k)}(\tau, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_j\Big]d\tau - f(t, v_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_r)\Big\}, \quad t \in [t_{r-1}, t_r],$$

$$\Delta v_r(t_{r-1}, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) = 0, \quad r = \overline{1, N}, \tag{3.13}$$

converges to $v[t, \hat{\lambda}]$, an isolated solution to problems (3.1) and (3.2) in $S(\hat{v}^{(0)}[t], \hat{\rho}_u)$, and

$$\Big\|v[\cdot, \hat{\lambda}] - \hat{v}^{(0)}[\cdot]\Big\|_4 \leq \hat{\chi} \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \Bigg\|\dot{\hat{v}}_r^{(0)}(t) - A(t)[\hat{v}_r^{(0)}(t) + \hat{\lambda}_r] \tag{3.14}$$
$$- f(t, \hat{v}_r^{(0)}(t) + \hat{\lambda}_r) - \varphi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi(\tau)\Big[\hat{v}_j^{(0)}(\tau) + \hat{\lambda}_j\Big]d\tau - f_0(t)\Bigg\|.$$

For given $\varepsilon > 0$, we use the estimate $\left\|\Delta v^{(k)}[\cdot, \hat{\lambda}]\right\|_4 \leq \varepsilon$ as a termination criterion of iterative process (3.11) for finding approximate solution to problems (3.1) and (3.2).

Suppose that a function system $v[t, \lambda] = (v_1(t, \lambda), \dots, v_N(t, \lambda))$ is a solution to the parametric special Cauchy problem for the system of nonlinear IDEs (3.1), (3.2), i.e.,

$$\frac{dv_r(t, \lambda_1, \dots, \lambda_N)}{dt} = A(t)[v_r(t, \lambda_1, \dots, \lambda_N) + \lambda_r] + \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau)[v_j(\tau, \lambda_1, \dots, \lambda_N) + \lambda_j] d\tau$$

$$+ f(t, v_r(t, \lambda_1, \dots, \lambda_N) + \lambda_r) + f_0(t), \quad t \in [t_{r-1}, t_r], \tag{3.15}$$

$$v_r(t_{r-1}, \lambda_1, \dots, \lambda_N) = 0, \quad r = \overline{1, N}. \tag{3.16}$$

Furthermore, by constructing an algorithm to solve the IVP (2.5), (2.6), we employ the partial derivatives of the functions $v_r(t, \lambda)$ with respect to parameters λ_i, $r, i = \overline{1, N}$. Therefore, below we compose a problem which allows us to find these partial derivatives.

Under conditions of Theorem 2, applying Peano's theorem [27, p.95], we can easily show the existence of

$$\frac{\partial v_r(t, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i}, \quad r, i = \overline{1, N},$$

for all $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$. Differentiating (3.15) and (3.16) with respect to λ_i, $i = \overline{1, N}$ yields

$$\frac{d}{dt}\left(\frac{\partial v_r(t, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i}\right) = (A(t) + f'_x(t, v_r(t, \lambda_1, \dots, \lambda_N) + \lambda_r))\left[\frac{\partial v_r(t, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i} + \sigma_{ri}\right]$$

$$+ \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau) \frac{\partial v_j(\tau, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i} d\tau + \sum_{k=1}^{m} \varphi_k(t) \int_{t_{i-1}}^{t_i} \psi_k(\tau) d\tau, \quad t \in [t_{r-1}, t_r],$$

$$\frac{\partial v_r(t_{r-1}, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i} = 0, \quad r, i = \overline{1, N},$$

where

$$\sigma_{ri} = \begin{cases} I, & r = i, \quad I \text{ is the } 2 \times 2 \text{ identity matrix}, \\ O, & r \neq i, \quad O \text{ is the } 2 \times 2 \text{ zero matrix}. \end{cases}$$

Hence, if we denote by $z_{ri}(t, \lambda_1, \dots, \lambda_N)$ the partial derivative $\dfrac{\partial v_r(t, \lambda_1, \dots, \lambda_N)}{\partial \lambda_i}$, $r, i = \overline{1, N}$, then for each $i = \overline{1, N}$, the function system $z_i[t, \lambda] = (z_{1i}(t, \lambda), \dots, z_{Ni}(t, \lambda))$ is a solution to the linear special matrix Cauchy problem

$$\frac{dz_{ri}}{dt} = \Big(A(t) + f'_x(t, v_r(t, \lambda_1, \dots, \lambda_N) + \lambda_r)\Big)[z_{ri} + \sigma_{ri}]$$

$$+ \varphi(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi(\tau) z_{ji}(\tau) d\tau + \varphi(t) \int_{t_{i-1}}^{t_i} \psi(\tau) d\tau, \quad t \in [t_{r-1}, t_r], \tag{3.17}$$

$$z_{ri}(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{3.18}$$

4. Solvability of problems (2.1), (2.2) and a method of finding its solution

Let the function system $v[t,\lambda] \in S(v^{(0)}[t],\rho_u)$ be a unique solution to the special Cauchy problems (3.1) and (3.2) for $\lambda \in S(\lambda^{(0)},\rho_\lambda)$. Employing (2.10), (2.11), and taking into consideration (3.3) and (3.4), we obtain the system of nonlinear algebraic equations in parameters

$$\lambda_1 = d, \tag{4.1}$$

$$\lambda_p + v_p(t_p,\lambda_1,\dots,\lambda_N) - \lambda_{p+1} = 0, \quad p = \overline{1,N-1}. \tag{4.2}$$

Rewrite (4.1) and (4.2) in the form:

$$Q_*(\Delta_N,\lambda) = 0, \quad \lambda \in R^{2N}. \tag{4.3}$$

Theorem 3. *Let $\lambda^* \in S(\lambda^{(0)},\rho_\lambda)$ be a solution to Eq (4.2), and let $u[t,\lambda^*] \in S(u^{(0)}[t],\rho_u)$ be the corresponding solution to the special Cauchy problems (3.1) and (3.2) for $\lambda = \lambda^*$. Then, the function $y^*(t)$, defined by*

$$y^*(t) = \lambda_r^* + u_r(t,\lambda^*), \quad t \in [t_{r-1},t_r), \quad r = \overline{1,N},$$

and

$$y^*(T) = \lambda_N^* + \lim_{t\to T-0} u_N(t,\lambda^*),$$

is a solution to problems (2.1), (2.2), and $y^*(t) \in S(y^{(0)}(t),\rho)$.

Application of Theorem 2 [28, p.45] to Eq (4.3) yields the ensuing result.

Theorem 4. *Assume the following conditions hold:*
(i) the Jacobi matrix $\partial Q_*(\Delta_N,\lambda)/\partial\lambda$ is uniformly continuous in $S(\lambda^{(0)},\rho_\lambda)$;
(ii) $\partial Q_*(\Delta_N,\lambda)/\partial\lambda$ is invertible and $\left\|[\partial Q_*(\Delta_N,\lambda)/\partial\lambda]^{-1}\right\| \le \gamma^*$ for all $\lambda \in S(\lambda^{(0)},\rho_\lambda)$, γ^* is const;
(iii) $\gamma^*\left\|Q_*(\Delta_N,\lambda^{(0)})\right\| < \rho_\lambda$.

Then, there exists $\alpha_0 \ge 1$ such that for any $\alpha \ge \alpha_0$, the sequence $\{\lambda^{(k+1)}\}$, generated by the iterative process

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\alpha}\left[\frac{\partial Q_*(\Delta_N,\lambda^{(k)})}{\partial\lambda}\right]^{-1} Q_*(\Delta_N,\lambda^{(k)}), \quad k = 0,1,2,\dots, \tag{4.4}$$

converges to λ^*, an isolated solution to equation (4.2) in $S(\lambda^{(0)},\rho_\lambda)$, and

$$\left\|\lambda^* - \lambda^{(0)}\right\| \le \gamma^*\left\|Q_*(\Delta_N,\lambda^{(0)})\right\|. \tag{4.5}$$

The explicit expression for $Q_*(\Delta_N,\lambda)$ is generally difficult to obtain and can be derived only in exceptional cases. However, if $v[t,\hat\lambda] = (v_1(t,\hat\lambda),\dots,v_N(t,\hat\lambda))$ is the solution to problems (3.1) and (3.2) for $\lambda = \hat\lambda = (d,\hat\lambda_2,\dots,\hat\lambda_N) \in S(\lambda^{(0)},\rho_\lambda)$, then

$$Q_*(\Delta_N,\hat\lambda) = \begin{pmatrix} d \\ d + v_1(t_1,d,\dots,\hat\lambda_N) - \hat\lambda_2 \\ \dots \\ \hat\lambda_{N-1} + v_{N-1}(t_{N-1},d,\dots,\hat\lambda_N) - \hat\lambda_N \end{pmatrix}. \tag{4.6}$$

The conditions stated in Theorem 2 ensure that the Jacobi matrix

$$\frac{\partial Q_*(\Delta_N,\hat{\lambda})}{\partial\lambda} = \begin{pmatrix} q_{1,1}(\hat{\lambda}) & \dots & q_{1,N-1}(\hat{\lambda}) & q_{1,N}(\hat{\lambda}) \\ q_{2,1}(\hat{\lambda}) & \dots & q_{2,N-1}(\hat{\lambda}) & q_{2,N}(\hat{\lambda}) \\ \dots & \dots & \dots & \dots \\ q_{N,1}(\hat{\lambda}) & \dots & q_{N,N-1}(\hat{\lambda}) & q_{N,N}(\hat{\lambda}) \end{pmatrix} \tag{4.7}$$

for any $\hat{\lambda} \in S(\lambda^{(0)},\rho_\lambda)$, and that it is uniformly continuous throughout the set $S(\lambda^{(0)},\rho_\lambda)$. Here the components of $\dfrac{\partial Q_*(\Delta_N,\hat{\lambda})}{\partial\lambda}$ are the 2×2 matrices

$$q_{1,s}(\hat{\lambda}) = O, \quad s=\overline{1,N},$$

$$q_{p,r}(\hat{\lambda}) = z_{p-1,r}(t_{p-1},d,\dots,\hat{\lambda}_N), \quad p\neq r, \quad p\neq r+1,$$

$$q_{p,p}(\hat{\lambda}) = -I + z_{p-1,p}(t_{p-1},d,\dots,\hat{\lambda}_N),$$

$$q_{p,p-1}(\hat{\lambda}) = I + z_{p-1,p-1}(t_{p-1},d,\dots,\hat{\lambda}_N), \quad p=\overline{2,N}, \ r=\overline{1,N},$$

where $z_i[t,\hat{\lambda}] = (z_{1,i}(t,d,\dots,\hat{\lambda}_N),\dots,z_{N,i}(t,d,\dots,\hat{\lambda}_N))$, $i=\overline{1,N}$, is the solution to the special Cauchy problems (3.17) and (3.18) for $\lambda=\hat{\lambda}$.

An algorithm will be proposed to solve problems (1.1) and (1.2), based on the parameter continuation method [29, p.230]. This method is effective in addressing nonlinear problems by gradually varying parameters. To implement this approach, we first consider the following IVP:

$$\frac{dz}{dt} = \delta f(t,z) + A(t)z + f_0(t) + \varphi(t)\int_0^T \psi(\tau)z(\tau)d\tau, \quad t\in(0,T), \quad z\in R^2, \quad \delta\in[0,1], \tag{4.8}$$

$$z(0)=d, \quad d\in R^2. \tag{4.9}$$

It is clear that if $z(\delta,t)$ denotes a solution to IVP (4.8), (4.9), then the function $y^*(t)=z(1,t)$ satisfies problems (2.1) and (2.2).

Suppose that the vector $\lambda^{(\delta,0)}\in R^{2N}$ is an initial guess solution to Eq (4.3) and the function system $v^{(\delta,0,0)}[t]$ is an initial guess solution to the special Cauchy problems (3.1) and (3.2) for $\lambda=\lambda^{(\delta,0)}$.

Algorithm A. It is proposed for determining a solution to the IVP specified by Eqs (4.8) and (4.9).
Step 0. Using the iterative process (3.11), find the solution $v[t,\lambda^{(\delta,0)}]$ to problems (3.1) and (3.2).
Step 1. *(a)* By the elements of the function system $v[t,\lambda^{(\delta,0)}]$, construct the vector $Q_*(\Delta_N,\lambda^{(\delta,0)})$ by (4.6).
(b) Compose the 2×2 matrices

$$A_r^{(\delta,0)}(t) = A(t) + f'_x(t, v_r(t,\lambda_1^{(\delta,0)},\dots,\lambda_N^{(\delta,0)}) + \lambda_r^{(\delta,0)}),$$

$$P_{ri}^{(\delta,0)}(t) = A_r^{(\delta,0)}(t)\sigma_{ri} + \varphi(t)\int_{t_{i-1}}^{t_i}\psi(\tau)d\tau, \quad t\in[t_{r-1},t_r], \quad r,i=\overline{1,N}.$$

Then, by solving N special matrix Cauchy problems for the system of linear IDEs

$$\frac{dz_{ri}}{dt} = A_r^{(\delta,0)}(t)z_{ri} + \varphi(t)\sum_{j=1}^N\int_{t_{j-1}}^{t_j}\psi(\tau)z_{ji}(\tau)d\tau + P_{ri}^{(\delta,0)}(t), \quad t\in[t_{r-1},t_r],$$

$$z_{ri}(t_{r-1}) = 0, \quad r, i = \overline{1, N},$$

find the function systems

$$z_i[t, \lambda^{(\delta,0)}] = (z_{1i}(t, \lambda_1^{(\delta,0)}, \ldots, \lambda_N^{(\delta,0)}), \ldots, z_{Ni}(t, \lambda_1^{(\delta,0)}, \ldots, \lambda_N^{(\delta,0)})), \quad i = \overline{1, N}.$$

(c) Construct the Jacobi matrix $\dfrac{\partial Q_*(\Delta_N, \lambda^{(\delta,0)})}{\partial \lambda}$ by formula (4.7), where $\hat{\lambda} = \lambda^{(\delta,0)}$.

Solve the system of linear algebraic equations

$$\frac{\partial Q_*(\Delta_N; \lambda^{(\delta,0)})}{\partial \lambda} \Delta\lambda = -\frac{1}{\alpha} Q_*(\Delta_N; \lambda^{(\delta,0)}), \quad \Delta\lambda \in R^{2N},$$

for some $\alpha \geq 1$ and find $\Delta\lambda^{(\delta,0)}$. Determine $\lambda^{(\delta,1)}$ as follows:

$$\lambda^{(\delta,1)} = \lambda^{(\delta,0)} + \Delta\lambda^{(\delta,0)}.$$

(d) Choose the function system $v[t, \lambda^{(\delta,0)}]$ as an initial guess solution to problems (3.1) and (3.2) for $\lambda = \lambda^{(\delta,1)}$, and using iterative process (3.11), find the function system $v[t, \lambda^{(\delta,1)}]$.

At the kth step of Algorithm A, this process yields the pair $(\lambda^{(\delta,k)}, v[t, \lambda^{(\delta,k)}])$ for $k = 2, 3, \ldots$. Let us compose the sequence of pairs $(\lambda^{(\delta,k)}, u[t, \lambda^{(\delta,k)}])$, where $u_r(t, \lambda^{(\delta,k)}) = v_r(t, \lambda^{(\delta,k)})$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$. The conditions of Theorems 2 and 4 ensure the convergence of this sequence to $(\lambda^{\delta}, u[t, \lambda^{\delta}])$, the solution to problems (2.8)–(2.11), as $k \to \infty$. Define the solution $z(\delta, t)$ to the problems (4.8) and (4.9) as follows: $z(\delta, t) = \lambda_r^{(\delta)} + u_r(t, \lambda^{(\delta)})$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, $z(\delta, T) = \lambda_N^{(\delta)} + u_N(T, \lambda^{(\delta)})$.

With the aim of finding solution to problems (2.1), (2.2) and relying on Algorithm A, we propose:

Algorithm B. *Step 0. (a)* Choose the values of parameter δ as folows: $0 = \delta_0 < \delta_1 < \ldots < \delta_\nu = 1$.
(b) Solve problems (4.8) and (4.9) with $\delta = \delta_0 = 0$ by employing the algorithm proposed in [21] and find $z(0, t)$.
(c) By the equalities $\lambda_r^{(0)} = z(0, t_{r-1})$, $v_r^{(0)}(t) = z(0, t) - \lambda_r^{(0)}$, $t \in [t_{r-1}, t_r]$, $r = \overline{1, N}$, identify $\lambda^{(0)} \in R^{2N}$ and $v^{(0)}[t] \in C([0, T], \Delta_N, R^{2N})$.
Step 1. (a) Choose a parameter $\delta = \delta_1$, and initial guesses $\lambda^{(\delta_1,0)} = \lambda^{(0)}$, $v^{(\delta,0,0)}[t] = v^{(0)}[t]$.
(b) Solve problems (4.8) and (4.9) with $\delta = \delta_1$ by employing Algorithm A and find the solution $z(\delta_1, t)$.
Step ν. (a) Choose a parameter $\delta = \delta_\nu$, and initial guesses $\lambda^{(\delta_\nu,0)} = \lambda^{(\delta_{\nu-1})}$, $v^{(\delta_\nu,0,0)}[t] = v^{(\delta_{\nu-1})}[t]$.
(b) Solve problems (4.8) and (4.9) with $\delta = \delta_\nu$ by employing Algorithm A and find the solution $z(\delta_\nu, t)$.

Proceeding to the νth step of the algorithm, when $\delta = \delta_\nu = 1$, the solution to problems (2.1) and (2.2), $y^*(t)$, is obtained. The first component of this vector function is the solution to problems (1.1) and (1.2).

5. Numerical experiments

In this section, we conduct numerical experiments that examine the efficacy of the proposed algorithms and validate the underlying theoretical approach.

Example 1. *We consider problems* (2.1) *and* (2.2) *with the data*

$$\alpha = 1, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{1}{3}, \quad \eta = 3, \quad \omega = 2, \quad \varphi_1(t) = t^2, \quad \varphi_2(t) = t + 1, \quad \psi_1(t) = 2t, \quad \psi_2(t) = t^2,$$

$$a = 1, \quad p(t) = \frac{39t}{10} + \frac{21t^2}{10} + 6sin^2 t + \frac{(t^3 - t^2 + 1)^3}{2} + \frac{(t^3 - t^2 + 1)^5}{3} - \frac{51}{10} + \eta cos(\omega t).$$

Δ_4 is the regular partition, so we choose the partition points: $t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$, $t_4 = 1$. *Solving this problem by algorithms A, B, with the values of parameter*

$$\delta : \ 0, \quad , \frac{1}{20}, \quad \frac{1}{10}, \quad \frac{1}{8}, \quad \frac{1}{4}, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{4}{5}, \quad 1,$$

we get the following results which are shown in Figures 1–5.

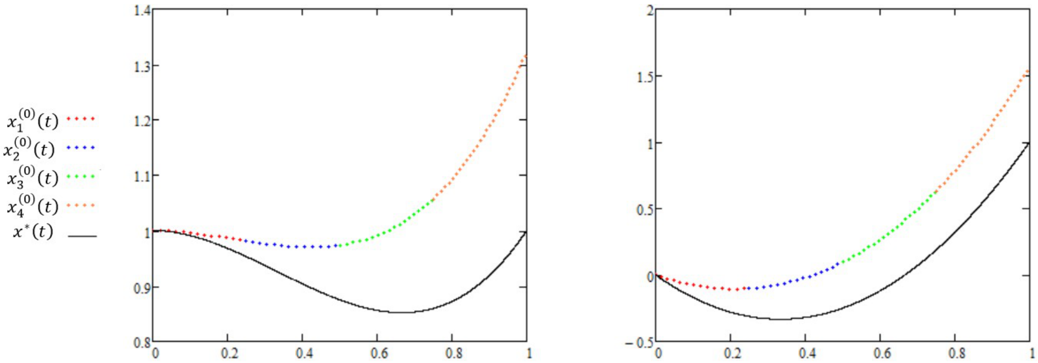

Figure 1. Graphs of the exact and approximate solutions, $\delta_0 = 0$, for Example 1.

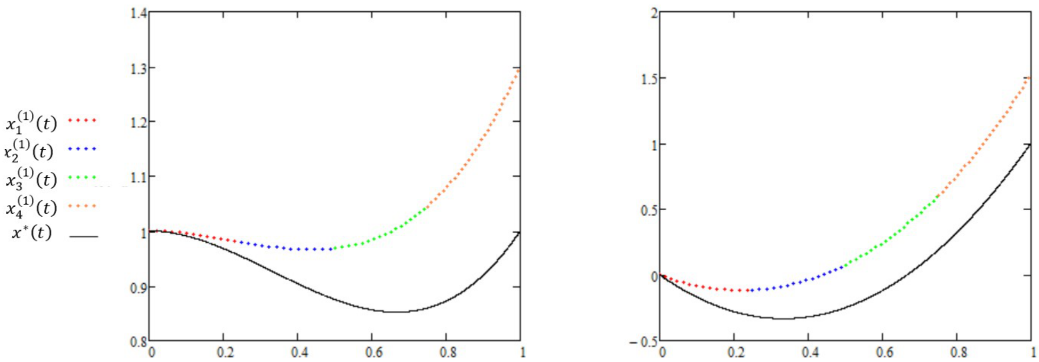

Figure 2. Graphs of the exact and approximate solutions, $\delta_1 = \frac{1}{20}$, for Example 1.

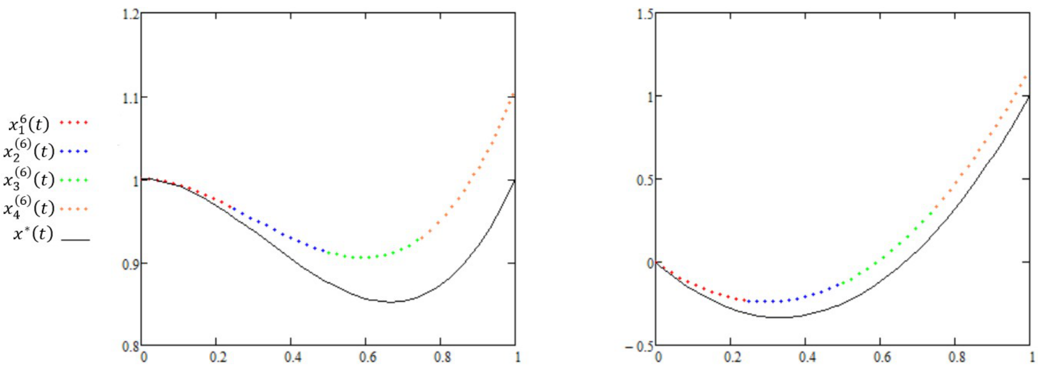

Figure 3. Graphs of the exact and approximate solutions, $\delta_6 = \frac{3}{5}$, for Example 1.

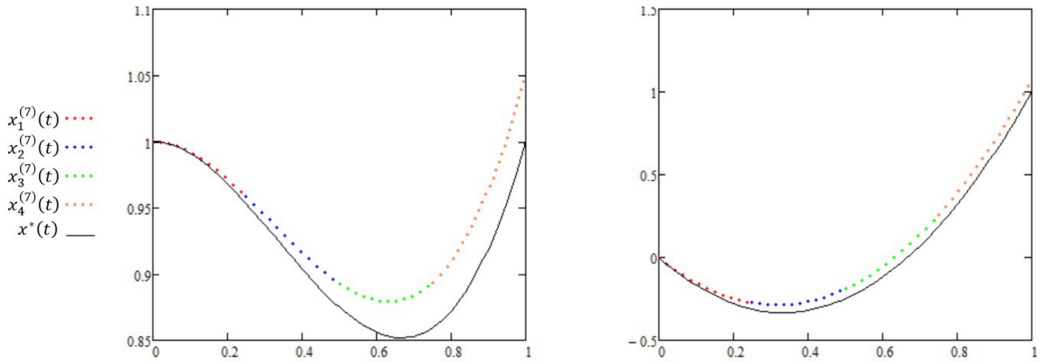

Figure 4. Graphs of the exact and approximate solutions, $\delta_7 = \frac{4}{5}$, for Example 1.

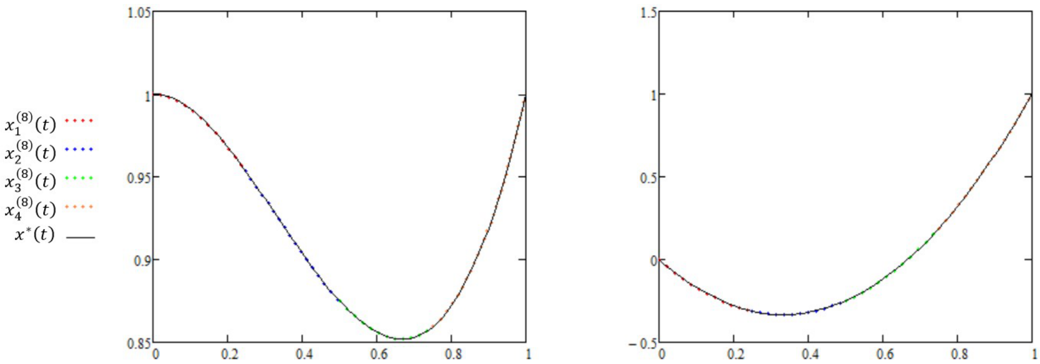

Figure 5. Graphs of the exact and approximate solutions, $\delta_8 = 1$, for Example 1.

The approximation error does not exceed 10^{-8}.

Example 2. *Since the Fredholm IDE is not always solvable, let us consider the equation in problem 2 ([11, p. 1486]) with the Fredholm integral term involving the initial condition*

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x\left(1 + \frac{dx}{dt}\right) + \int_0^1 s^2 x(s)ds = f(t), \quad t \in (0,1), \tag{5.1}$$

$$x(0) = \frac{4e-7}{2(1+e)}, \quad x(0) = \frac{4e-7}{1+e}, \tag{5.2}$$

where $f(t) = \frac{1}{24(e+1)^2}(18t + 726e^{2t} + 455e + 244e^2 - 462e^t - 288xe + 480xe^2 - 1188e^{t+1} + 198te^t - 528te^{t+1} + 211)$.

In problems (5.1)*,* (5.2) Δ_4 *is also the regular partition, so we select the partitioning points and the values of the parameter* δ *in the same way as in the previous example. By solving the problem using algorithms A, B, we get the results, which are shown in Figures 6–10.*

The approximation error does not exceed 10^{-5}.

We observed quadratic convergence when solving nonlinear special Cauchy problems by the iterative method at fixed parameter values, and performed 5 iterations to find a solution with a given accuracy of 10^{-15}*. To solve Cauchy problems for ordinary differential equations, the fourth-order Runge-Kutta method was used; to evaluate definite integrals, Simpson's formula was used. The* *realization of these approaches does not cause difficulties as they are characterized by simplicity and clarity, and they were performed using parallel calculations. The iterative method exhibited quadratic convergence when solving the parameter-dependent nonlinear algebraic system, with five iterations performed to find a solution with a given accuracy. Linear convergence is observed when the continuation method used on the parameter δ. The calculations were performed using MathCad Prime on a 6-core computer with 16 GB of RAM.*

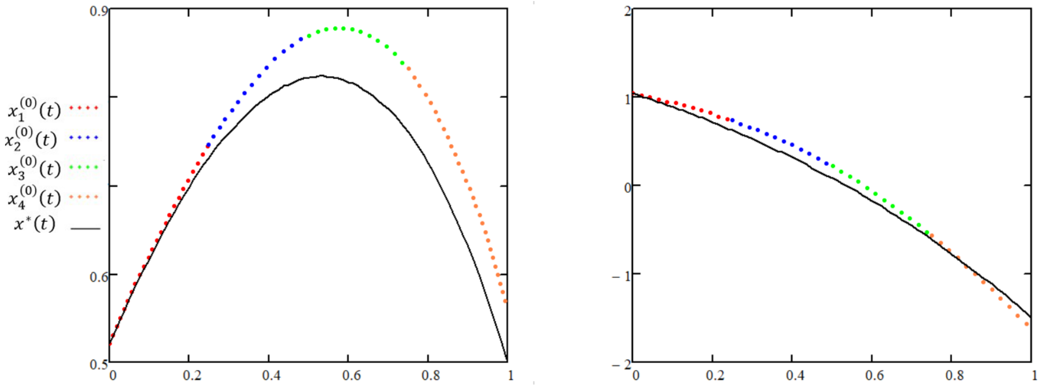

Figure 6. Graphs of the exact and approximate solutions, $\delta_0 = 0$, for Example 2.

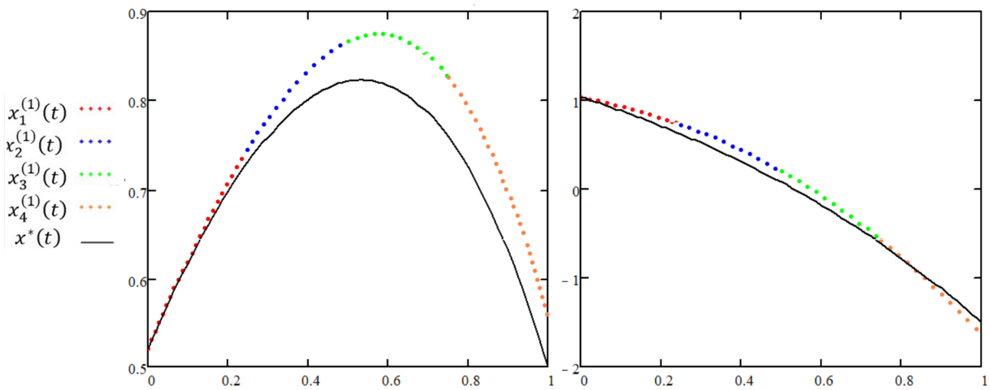

Figure 7. Graphs of the exact and approximate solutions, $\delta_1 = \frac{1}{20}$, for Example 2.

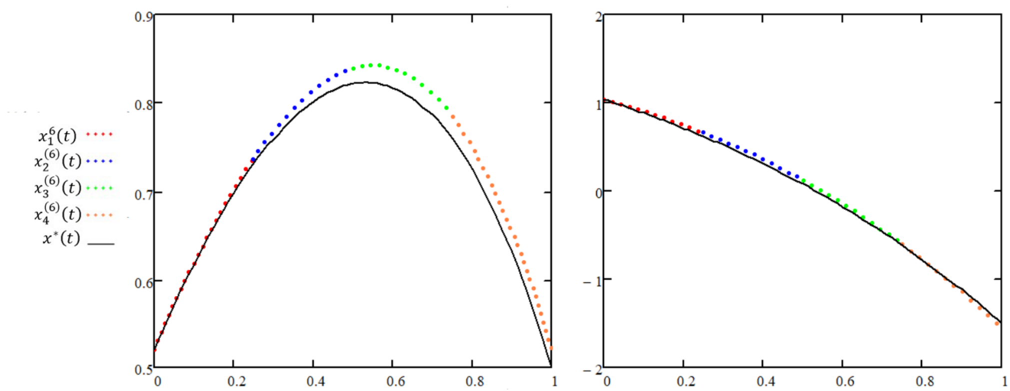

Figure 8. Graphs of the exact and approximate solutions, $\delta_6 = \frac{3}{5}$, for Example 2.

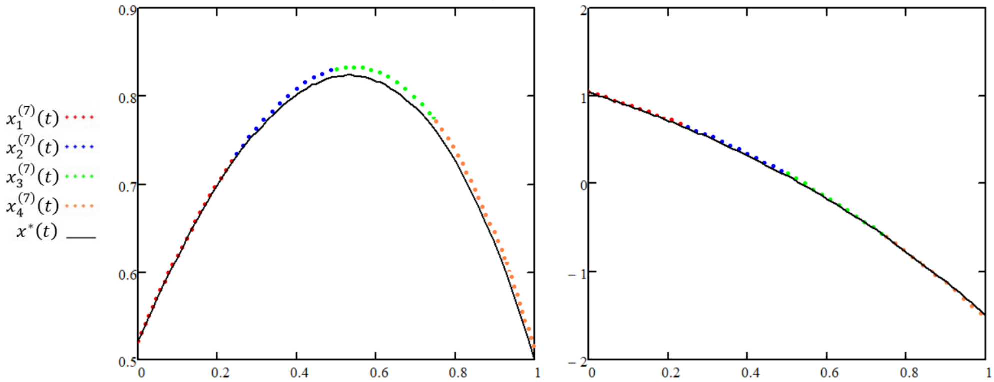

Figure 9. Graphs of the exact and approximate solutions, $\delta_7 = \frac{4}{5}$, for Example 2.

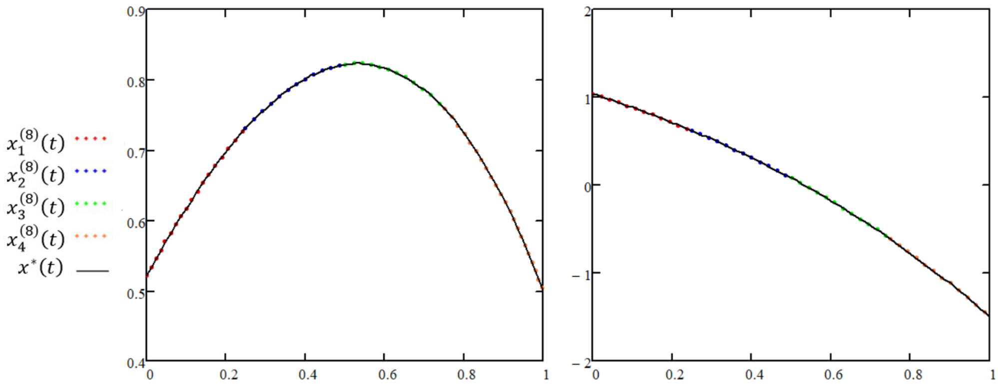

Figure 10. Graphs of the exact and approximate solutions, $\delta_8 = 1$, for Example 2.

6. Conclusions

In this study, the Dzhumabaev parametrization technique is employed for solving an initial value problem for the Duffing equation involving an integral forcing term. Numerical examples demonstrate that the proposed method is highly efficient and offers improved accuracy. Furthermore, in the future, we will study boundary value problems for Duffing type integrodifferential equations using the proposed method, and we will make a comparative analysis using the results obtained from other known methods.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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