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Research article

Decay estimate and blow-up for fractional Kirchhoff wave equations involving a logarithmic source

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Abstract: This paper was dedicated to researching a class of fractional Kirchhoff wave models involving a logarithmic source and strong damping term. We have proven the local existence and uniqueness of weak solutions through combining contraction mapping theory with Faedo-Galerkin's method. Based on the framework of a potential well and under suitable conditions, an exponential decay estimate of global weak solutions was established. Finally, the result of the finite time blow-up was obtained.

Keywords: decay estimate; blow-up; fractional Laplacian; Kirchhoff wave equations; logarithmic source
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1. Introduction

Fractional differential equations can better describe practical problems than classical differential equations. This has attracted the interest and attention of many scholars to the fractional p-Laplacian $(-\Delta)_p^s$ [1–5]. The fractional 2-Laplacian operator of the form $(-\Delta)^s$ $(p = 2)$ was first mentioned in physics when observing the Levy steady-state diffusion process, and later it was also used to depict abnormal plasma diffusion, fluid dynamics, and stochastic analysis [6–8]. Not only for mathematical purposes, but also for their importance in practical models, this paper will investigate the Kirchhoff-type wave models involving logarithmic nonlinearity and the fractional Laplacian operator as follows:

$$\begin{cases} u_{tt} + M\left([u]_s^2\right)(-\Delta)^s u + (-\Delta)^s u_t = |u|^{k-2} u \ln|u|, & x \in \Omega,\ t > 0; \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega; \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where, among them, $(-\Delta)^s (s \in (0,1))$ is the fractional Laplacian operator satisfying

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{2s+N}} dy.$$

The Kirchhoff term $M(\psi) = a + \psi^{\theta-1}$ is a function that satisfies local Lipschitz continuity, $a > 0$, $1 \le \theta < \frac{2_s^*}{2}$, where

$$2_s^* = \begin{cases} +\infty, & if\ N \le 2s, \\ \dfrac{2N}{N-2s}, & if\ N > 2s, \end{cases}$$

is the critical exponent of the fractional Sobolev embedding inequality. If $\psi = \psi(t)$, we impose the following assumption on $M(\psi)$:

$$\frac{d}{dt} M[\psi(t)] \le CM[\psi(t)]. \tag{1.2}$$

Moreover, $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a bounded domain and the boundary $\partial\Omega$ is the smooth and nonlinear index $2\theta < k \le 2_s^*$.

Recently, the research on Kirchhoff-type equations [9, 10] has received widespread attention. This kind of problem develops a major effect in the applications of nonlinear elasticity, electrorheological fluid, and image restoration [11, 12]. It is meaningful to investigate the nonexistence, existence, blow-up, extinction, and decay estimation of its solutions. Kirchhoff [13] first introduced the following equation:

$$\rho h u_{tt} + \delta u_t = \left\{ P_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x}(x,t) \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \quad t \ge 0, \quad 0 \le x \le L,$$

where u denotes lateral displacement, δ denotes the resistance modulus, ρ denotes mass density, h represents the cross-section area, P_0 denotes initial axial tension, L is the length, E is Young's modulus, and f represents external force. Since then, many researchers have become concerned with this kind of equation and have had excellent research results. In particular, many literatures have been devoted to discussing the Kirchhoff equation as follows:

$$u_{tt} + g(u_t) - M\left(\|\nabla u\|^2\right) \Delta u = f(u), \tag{1.3}$$

where $f(u)$ is a nonlinear function that satisfies appropriate conditions and $M \ge 0$ is a local Lipschitz function. When $g(u_t) = \Delta u_t$, Wu and Tsai [14] made a profound study for Problem (1.3), where they found the upper bound of the blow-up time of solutions by the direct energy method. Yang and Han [15] also discussed Problem (1.3), through the Banach fixed point theorem, where they proved uniqueness as well as local existence of weak solutions. Then, through constructing a potential well, the lifespan of solutions with arbitrary initial energy was established. When $g(u_t)$ is the non-linear dissipation term $|u_t|^{m-1} u_t$ or the linear dissipation term u_t, Ono studied Problem (1.3) involving $f(u) = |u|^p u$ in [16, 17], when the initial energy is negative, and proved the finite time blow-up. In addition, when the initial energy was positive, he provided sufficient conditions for the finite time blow-up of the solutions. More research on the problems of Kirchhoff-type can be found in references [18–22].

In 2017, Pan et al. [23] investigated the degenerate fractional Kirchhoff-type hyperbolic problems as follows:

$$u_{tt} + [u]_s^{2(\theta-1)} (-\Delta)^s u = |u|^{p-1} u.$$

Combining the potential wells theorem with the Galerkin method, they proved the global existence. Moreover, the vacuum isolating phenomenon and blow-up properties also were acquired. Additionally, the study of logarithmic source has a long history, appearing in different modules of physics [24].

Inspired by the above works, we investigate Problem (1.1) involving the fractional the Laplacian operator strong damping and logarithmic source, which is the first work that takes into account the blow-up property and decay estimate of weak solutions of Problem (1.1). We not only overcome the difficulty of logarithmic nonlinearity, but also deal with the fractional Laplacian operator. This work is extremely meaningful.

The structure of this article is as follows: In Section 2, we introduce important lemmas and basic definitions. In addition, potential wells and their properties are provided. Next, the local existence and uniqueness of the weak solutions are proved. Then we gain the global existence of weak solutions and establish an exponential decay estimate in Section 4. Finally, the finite time blow-up of the solutions is obtained.

2. Notations and primary lemmas

We introduce some symbols, lemmas, and basic definitions in this section. For convenience, we define the $L^k(\Omega)$ norm through $\|\cdot\|_k (1 \le k \le \infty)$. First, some definitions of Sobolev space are reviewed, which can be found in [25].

Let the fractional exponent $s \in (0,1)$, $H^s\left(\mathbb{R}^N\right)$ be the fractional Sobolev space satisfying

$$H^s\left(\mathbb{R}^N\right) = \left\{u \in L^2\left(\mathbb{R}^N\right) : \frac{u(x) - u(y)}{|x-y|^{s+\frac{N}{2}}} \in L^2\left(\mathbb{R}^N \times \mathbb{R}^N\right)\right\} \tag{2.1}$$

equipped with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{2s+N}} dxdy + \|u\|^2_{L^2(\mathbb{R}^N)}\right)^{\frac{1}{2}}. \tag{2.2}$$

We denote space $O = C(\Omega) \times C(\Omega) \subset \mathbb{R}^N$ where $C(\Omega) = \mathbb{R}^N \backslash \Omega$, and then denote $Q = \left(\mathbb{R}^N \times \mathbb{R}^N\right) \backslash O$. From nonlocal characteristics, we define the space

$$W = \left\{u \in L^2\left(\mathbb{R}^N\right) : \int\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{2s+N}} dxdy < \infty\right\}. \tag{2.3}$$

Let $W_0 = \{u \in W : u(x,t) = 0, x \in C(\Omega)\}$, which is a closed linear space, and $W_0 \subset W$. Moreover, $[u]_s$ is the Gagliardo seminorm satisfying

$$[u]_s = \left(\int\int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{2s+N}} dxdy\right)^{\frac{1}{2}}. \tag{2.4}$$

From the results of [26], it can be concluded that $[u]_s$ is equivalent to the norm of W_0, and it is clear that the main space W_0 is a Hilbert space. Moreover, we denote the inner product in L^2 as $(\cdot,\cdot)$, the inner product in W_0 as $(\cdot,\cdot)_{W_0}$, and the dual product of W_0 in Y_0 as $\langle\cdot,\cdot\rangle_{W_0}$. Y_0 is the dual space of W_0.

For $u \in W_0$, we denote the main energy functional of this paper:

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{a}{2}\|u\|_{W_0}^2 + \frac{1}{2\theta}\|u\|_{W_0}^{2\theta} - \frac{1}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k. \tag{2.5}$$

In addition, we define the potential energy functional

$$J(u) = \frac{a}{2}\|u\|_{W_0}^2 + \frac{1}{2\theta}\|u\|_{W_0}^{2\theta} - \frac{1}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k, \tag{2.6}$$

and Nehari functional

$$I(u) = a\|u\|_{W_0}^2 + \|u\|_{W_0}^{2\theta} - \int_\Omega |u|^k \ln|u|\,dx. \tag{2.7}$$

By direct computation, we have

$$E(t) = J(u) + \frac{1}{2}\|u_t\|_2^2, \tag{2.8}$$

and

$$J(u) = \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta} + \frac{1}{k}I(u). \tag{2.9}$$

We also define the depth of the potential well and the Nehari manifold, respectively, as

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.10}$$

$$\mathcal{N} = \{u \in W_0\backslash\{0\},\ I(u) = 0\}.$$

Further, we will introduce the sets

$$W^+ = \{u \in W_0 | I(u) > 0\} \cup \{0\},$$

$$W^- = \{u \in W_0 | I(u) < 0\}.$$

In this paper, to avoid confusion, we simply write $u(x,t)$ as $u(t)$ sometimes. Next, we give some definitions.

Definition 2.1. *The function $u = u(x,t)$ is a weak solution of Problem* (1.1) *on* $\Omega \times [0,T]$, *supposing that*

$$u \in C([0,T];W_0) \cap C^1\left([0,T];L^2(\Omega)\right) \cap C^2([0,T];Y_0)$$

and $u_t \in L^2(0,T;W_0)$ satisfying $u(0) = u_0, u_t(0) = u_1$, and it holds that

$$\langle u_{tt},\phi\rangle_{W_0} + M\left([u]_s^2\right)(u,\phi)_{W_0} + (u_t,\phi)_{W_0} = \left(|u|^{k-2}u\ln|u|,\phi\right),$$

for arbitrary $\phi \in W_0$, where the inner product

$$(u,v)_{W_0} = \int\int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2s+N}}dxdy.$$

Definition 2.2. *Let $u(x,t)$ be a weak solution of Problem* (1.1)*, and if the maximal existence time T_{max} is finite and*

$$\lim_{t\to T_{\max}^-}\left(\int_0^t \|u\|_{W_0}^2\,dt + \|u\|_2^2\right) = +\infty,$$

we say that $u(x,t)$ blows up in finite time.

Lemma 2.1. *Let $u \in W_0 \setminus \{0\}$, and we have*

(i) $\lim\limits_{\lambda\to+\infty} J(\lambda u) = -\infty$, $\lim\limits_{\lambda\to 0^+} J(\lambda u) = 0$;

(ii) *$J(\lambda u)$ is decreasing when $\lambda \in (\lambda_*, +\infty)$, and increasing when $\lambda \in (0, \lambda_*)$;*

(iii) *$I(\lambda u) < 0$ when $\lambda \in (\lambda_*, +\infty)$, and $I(\lambda u) > 0$ when $\lambda \in (0, \lambda_*)$.*

Proof. By (2.6), we have

$$J(\lambda u) = \frac{a\lambda^2}{2}\|u\|_{W_0}^2 + \frac{\lambda^{2\theta}}{2\theta}\|u\|_{W_0}^{2\theta} - \frac{\lambda^k}{k^2}\ln\lambda\,\|u\|_k^k - \frac{\lambda^k}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{\lambda^k}{k^2}\|u\|_k^k,$$

so the conclusion of (i) is obviously valid. For the derivation of the above formula, we can obtain

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda u) &= -\lambda^{k-1}\ln\lambda\,\|u\|_k^k - \lambda^{k-1}\int_\Omega |u|^k\ln|u|\,dx + a\lambda\|u\|_{W_0}^2 + \lambda^{2\theta-1}\|u\|_{W_0}^{2\theta}\\ &= \lambda\left(-\lambda^{k-2}\ln\lambda\,\|u\|_k^k - \lambda^{k-2}\int_\Omega |u|^k\ln|u|\,dx + a\|u\|_{W_0}^2 + \lambda^{2\theta-2}\|u\|_{W_0}^{2\theta}\right).\end{aligned}$$

Let

$$g(\lambda) = \lambda^{k-2}\left(-\ln\lambda\,\|u\|_k^k - \int_\Omega |u|^k\ln|u|\,dx + \lambda^{2\theta-k}\|u\|_{W_0}^{2\theta}\right),$$

and since $k > 2\theta$ and $\theta \geq 1$, we can obtain

$$\lim_{\lambda\to 0} g(\lambda) = 0 \textit{ and } \lim_{\lambda\to+\infty} g(\lambda) = -\infty. \tag{2.11}$$

Further, we have

$$\begin{aligned}g'(\lambda) &= \lambda^{k-3}\left[2\|u\|_{W_0}^{2\theta}(\theta-1)\lambda^{2\theta-k} - (k-2)\|u\|_k^k\ln\lambda - (k-2)\int_\Omega |u|^k\ln|u|\,dx - \|u\|_k^k\right]\\ &\equiv \lambda^{k-3}h(\lambda),\end{aligned}$$

where

$$h(\lambda) = 2\|u\|_{W_0}^{2\theta}(\theta-1)\lambda^{2\theta-k} - (k-2)\|u\|_k^k\ln\lambda - (k-2)\int_\Omega |u|^k\ln|u|\,dx - \|u\|_k^k,$$

which, together with $k > 2\theta \geq 2$ and $\theta \geq 1$, gives us $\lim\limits_{\lambda\to+\infty} h(\lambda) = -\infty$ and $\lim\limits_{\lambda\to 0} h(\lambda) = +\infty$. Taking the derivative of $h(\lambda)$, we obtain

$$h'(\lambda) = \frac{-2\|u\|_{W_0}^{2\theta}(k-2\theta)(\theta-1)\lambda^{2\theta-k} - (k-2)\|u\|_k^k}{\lambda} < 0.$$

So we infer that there is a unique λ_0 that satisfies $h(\lambda)|_{\lambda=\lambda_0} = 0$, which means that

$$g'(\lambda)\begin{cases} < 0, & \lambda_0 < \lambda < +\infty,\\ = 0, & \lambda = \lambda_0,\\ > 0, & 0 < \lambda < \lambda_0.\end{cases} \tag{2.12}$$

Combining (2.11) and (2.12), there is a unique λ_1 that satisfies $g(\lambda)|_{\lambda=\lambda_1} = 0$. Then we can get that there is a $\lambda_* > \lambda_1$ satisfying $a\|u\|_{W_0}^2 + g(\lambda) = 0$, which means that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda_*} = 0$, $\frac{d}{d\lambda}J(\lambda u)$ is negative on $(\lambda^*, +\infty)$, and $\frac{d}{d\lambda}J(\lambda u)$ is positive on $(0, \lambda^*)$. Therefore, it can be seen that the conclusion of (*ii*) is valid. By (2.6) and (2.7), we have

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} < 0, & \lambda_* < \lambda < +\infty, \\ = 0, & \lambda = \lambda_*, \\ > 0, & 0 < \lambda < \lambda_*. \end{cases}$$

Thus, the conclusion of (*iii*) holds. We have completed the proof of the properties of $J(\lambda u)$. □

Lemma 2.2. [27] *Suppose that μ is a positive constant. We can get*

$$\left|\Psi^k \ln \Psi\right| \leq (ek)^{-1}, \quad if\ 0 < \Psi < 1,$$

and

$$\Psi^k \ln \Psi \leq (e\mu)^{-1} \Psi^{k+\mu}, \quad if\ \Psi \geq 1,$$

where e is a natural constant.

Lemma 2.3. [28] *For $\forall r \in [1, 2_s^*]$ and $u \in W_0$, there is a constant $C_0(N, r, s) > 0$ that gives us*

$$\|u\|_r \leq C_0 \|u\|_{W_0}.$$

Lemma 2.4. [29] *Assume that W is a Banach space, and if $f \in L^k(0, T; W)$, $\frac{\partial f}{\partial t} \in L^k(0, T; W)$, then when the value is transformed in a suitable set of measure zero in $[0, T]$, f is a continuous injection from $[0, T]$ onto W.*

Lemma 2.5. [30] *Assume that (X, d) is a complete metric space, $F : X \to X$,and for any $x, y \in X$, we have*

$$d(F(x), F(y)) \leq \delta d(x, y),$$

for some constant $0 < \delta < 1$. Then F has a unique fixed point $\bar{x} \in X$ such that $F(\bar{x}) = \bar{x}$.

Lemma 2.6. *Assume that $u(x, t)$ is a weak solution of Problem (1.1), so the energy functional $E(t)$ is non-increasing about t.*

Proof. We multiply the first equation of (1.1) by u_t and integrate it on $\Omega \times [0, t)$, we can get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \frac{a}{2}\frac{d}{dt}\|u\|_{W_0}^2 + \frac{1}{2\theta}\frac{d}{dt}\|u\|_{W_0}^{2\theta} + \|u_t\|_{W_0}^2 = \frac{1}{k}\frac{d}{dt}\int_\Omega |u|^k \ln|u|\,dx - \frac{1}{k^2}\frac{d}{dt}\|u\|_k^k,$$

namely,

$$\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + E(t) = E(0). \tag{2.13}$$

Deriving $E(t)$ about t, and we get

$$E'(t) = -\|u_\tau\|_{W_0}^2 \leq 0.$$

Therefore, the proof of the properties of $E(t)$ has been completed. □

3. Local existence

Lemma 3.1. *For any* $2 \leq 2\theta < k \leq 2_s^*$, $T > 0$, $u \in \mathcal{H} = C([0,T];W_0) \cap C^1\left([0,T];L^2(\Omega)\right)$, *there is a unique*

$$v \in C([0,T];W_0) \cap C^1\left([0,T];L^2(\Omega)\right) \cap C^2([0,T];Y_0)$$

such that

$$\begin{cases} v_{tt} + M\left([u]_s^2\right)(-\Delta)^s v + (-\Delta)^s v_t = |u|^{k-2}u\ln|u|, & x \in \Omega,\ t > 0; \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega; \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t \geq 0. \end{cases} \tag{3.1}$$

Proof. (i) Proof of the existence.

According to literature [31], there is an eigenfunction sequence $\left\{e_j\right\}_j \subset C_0^\infty(\Omega)$ of fractional Laplacian operators, which is a completed orthogonal basis of W_0 and is an orthonormal basis in $L^2(\Omega)$. $\lambda_j > 0$ is defined as the corresponding eigenvalue satisfying $(-\Delta)^s e_j = \lambda_j e_j$. Taking $W_m = Span\{e_1, \cdots, e_m\}$ and constructing the approximate solutions

$$v_m(x,t) = \sum_{j=1}^{m} e_j(x)\, h_j^m(t),$$

for every $\eta \in W_m$ and $t \geq 0$, satisfies the equations

$$\begin{cases} \displaystyle\int_\Omega \left[\ddot{v}_m + M\left([u]_s^2\right)(-\Delta)^s v_m + (-\Delta)^s \dot{v}_m - |u|^{k-2}u\ln|u|\right]\eta dx = 0, \\ v_m(0) = u_0^m = \sum\limits_{j=1}^{m}\left(\displaystyle\int_\Omega u_0 \cdot e_j dx\right)e_j \to u_0 \ \ in\ \ W_0\ \ as\, m \to \infty, \\ \dot{v}_m(x) = u_1^m = \sum\limits_{j=1}^{m}\left(\displaystyle\int_\Omega u_1 \cdot e_j dx\right)e_j \to u_1 \ \ in\ \ L_2(\Omega)\ \ as\, m \to \infty. \end{cases} \tag{3.2}$$

For $j = 1, \cdots, m$, we make $\eta = e_j$ in the first equation in (3.2), and $\left\{h_j^m\right\}_{j=1}^m$ satisfies the Cauchy equations

$$\begin{cases} \ddot{h}_j^m(t) = F_j\left(t, h_1^m(t), h_2^m(t), \cdots, h_m^m(t)\right), \\ h_j^m(0) = \displaystyle\int_\Omega u_0 \cdot e_j dx,\ \dot{h}_j^m(0) = \int_\Omega u_1 \cdot e_j dx, \end{cases}$$

where

$$F_j = -\lambda_j h_j^m(t) - M\left([u]_s^2\right)\lambda_j \dot{h}_j^m(t) + \int_\Omega e_j(x) \cdot |u|^{k-2}u\ln|u|\, dx,$$

which is a linear ordinary differential equation about h_j^m. On the basis of Peano's theorem, a local solution $h_j^m \in C^1[0,T]$ has been obtained for the Cauchy problem mentioned above.

Now, we take $\eta = e_j$ and multiply two sides of the first equation (3.2) by $\dot{h}_{nj}^m(t)$, and then sum over j from 1 to m to get

$$\frac{d}{dt}\|v_{mt}\|_2^2 + \frac{d}{dt}\left[M\left([u]_s^2\right)\|v_m\|_{W_0}^2\right] + 2\|v_{m\tau}\|_{W_0}^2 = 2\int_\Omega v_{mt} \cdot |u|^{k-2}u\ln|u|\, dx + \|v_m\|_{W_0}^2 \frac{d}{dt}\left[M\left([u]_s^2\right)\right].$$

We integrate the above equation on $[0,t]$ to get

$$\begin{aligned}\|v_{mt}\|_2^2 + M\left([u]_s^2\right)\|v_m\|_{W_0}^2 + 2\int_0^t \|v_{m\tau}\|_{W_0}^2\,d\tau &= \|u_1\|_2^2 + M\left([u_0]_s^2\right)\left\|u_0^m\right\|_{W_0}^2 \\ &+ \int_0^t \|v_m\|_{W_0}^2 \frac{d}{dt}\left[M\left([u]_s^2\right)\right]d\tau + 2\int_0^t\int_\Omega v_{m\tau}\cdot|u|^{k-2}u\ln|u|\,dxd\tau.\end{aligned} \tag{3.3}$$

Recalling $u \in \mathcal{H}$, we see that $\|u\|_{W_0}$ is bounded. Through the definition and assumption of function $M(m)$, we arrive at

$$\begin{aligned}\int_0^t \|v_m\|_{W_0}^2 \frac{d}{dt}\left[M\left([u]_s^2\right)\right]d\tau &\le \int_0^t CM\left([u]_s^2\right)\|v_m\|_{W_0}^2\,d\tau \\ &= \int_0^t C\left(a + \|u\|_{W_0}^{2(\theta-1)}\right)\|v_m\|_{W_0}^2\,d\tau \\ &\le C_1\int_0^t \|v_m\|_{W_0}^2\,d\tau.\end{aligned} \tag{3.4}$$

Among them, C_1 is a positive number that only depends on T. Then we estimate the integral containing a logarithmic source on the right side of (3.3). Through Hölder's inequality, we can obtain

$$\begin{aligned}&2\int_0^t\int_\Omega v_{m\tau}\cdot|u|^{k-2}u\ln|u|\,dxd\tau \\ \le\, &2\int_0^t\left(\int_\Omega |v_{m\tau}|^{\frac{2N}{N-2s}}dx\right)^{\frac{N-2s}{2N}}\left(\int_\Omega \left||u|^{k-2}u\ln|u|\right|^{\frac{2N}{N+2s}}dx\right)^{\frac{N+2s}{2N}}d\tau \\ \le\, &2\int_0^t \|v_{m\tau}\|_{\frac{2N}{N-2s}}\left\||u|^{k-2}u\ln|u|\right\|_{\frac{2N}{N+2s}}d\tau.\end{aligned} \tag{3.5}$$

So next we deal with the term $\|v_{mt}\|_{\frac{2N}{N+2s}}$ in (3.5). According to Lemma 2.3, we have

$$\|v_{mt}\|_{\frac{2N}{N-2s}} \le C_0(N,s)\,\|v_{mt}\|_{W_0}. \tag{3.6}$$

Then let $\Omega_1 = \{x\in\Omega|\,|u_n(x)|<1\}$, $\Omega_2 = \{x\in\Omega|\,|u_n(x)|\ge 1\}$. Combining Lemmas 2.2 and 2.3, here we choose $0<\mu\le \frac{2N}{N-2s}-p$, and then we can obtain

$$\begin{aligned}\left\||u|^{k-2}u\ln|u|\right\|_{\frac{2N}{N+2s}}^{\frac{2N}{N+2s}} &= \int_{\Omega_1}\left||u|^{k-2}u\ln|u|\right|^{\frac{2N}{N+2s}}dx + \int_{\Omega_2}\left||u|^{k-2}u\ln|u|\right|^{\frac{2N}{N+2s}}dx \\ &\le \int_{\Omega_1}\left||u|^{k-1}\ln|u|\right|^{\frac{2N}{N+2s}}dx + \int_{\Omega_2}\left||u|^{-\mu}\ln|u|\cdot|u|^{k-1+\mu}\right|^{\frac{2N}{N+2s}}dx \\ &\le [e(k-1)]^{-\frac{2N}{N+2s}}|\Omega| + (e\mu)^{-\frac{2N}{N+2s}}\|u\|_{\frac{2N(k-1+\mu)}{N+2s}}^{\frac{2N(k-1+\mu)}{N+2s}} \\ &\le [e(k-1)]^{-\frac{2N}{N+2s}}|\Omega| + (e\mu)^{-\frac{2N}{N+2s}}C_0^{\frac{2N(k-1+\mu)}{N+2s}}\|u\|_{W_0}^{\frac{2N(k-1+\mu)}{N+2s}},\end{aligned} \tag{3.7}$$

so we can obtain

$$\begin{aligned}&\int_0^t \left\||u|^{k-2}u\ln|u|\right\|_{\frac{2N}{N+2s}}^2 dt \\ \le\, &\left([e(k-1)]^{-\frac{2N}{N+2s}}|\Omega| + (e\mu)^{-\frac{2N}{N+2s}}C_0^{\frac{2N(k-1+\mu)}{N+2s}}C^{\frac{2N(k-1+\mu)}{N+2s}}\right)^{\frac{N+2s}{N}} T = C_2T,\end{aligned} \tag{3.8}$$

where $C_2 = \left([e(k-1)]^{-\frac{2N}{N+2s}} |\Omega| + (e\mu)^{-\frac{2N}{N+2s}} C_0^{\frac{2N(k-1+\mu)}{N+2s}} C^{\frac{2N(k-1+\mu)}{N+2s}}\right)^{\frac{N+2s}{N}}$.

Utilizing Young's inequality, then combining (3.6) and (3.8), (3.5) can be written as

$$\begin{aligned} & 2\int_0^t \int_\Omega v_{m\tau} \cdot |u|^{k-2} u \ln |u| \, dx d\tau \\ \leq & 2\int_0^t C_0 \|v_{mt}\|_{W_0} \left\| |u|^{k-2} u \ln |u| \right\|_{\frac{2N}{N+2s}} d\tau \\ \leq & \int_0^t C_0^2 \left\| |u|^{k-2} u \ln |u| \right\|_{\frac{2N}{N+2s}}^2 d\tau + \int_0^t \|v_{m\tau}\|_{W_0}^2 \, d\tau \\ \leq & C_0^2 C_2 T + \int_0^t \|v_{m\tau}\|_{W_0}^2 \, d\tau. \end{aligned} \tag{3.9}$$

Due to the convergence of u_0^m and u_1^m, from (3.3), (3.4), and (3.9), we arrive at

$$\begin{aligned} & \|v_{mt}\|_2^2 + M\left([u]_s^2\right) \|v_m\|_{W_0}^2 + \int_0^t \|v_{m\tau}\|_{W_0}^2 \, d\tau \\ \leq & \|u_1\|_2^2 + M\left([u_0]_s^2\right) \left\|u_0^m\right\|_{W_0}^2 + C_0^2 C_2 T + C_1 \int_0^t \|v_m\|_{W_0}^2 \, d\tau \\ = & \tilde{C} + C_1 \int_0^t \|v_m\|_{W_0}^2 \, d\tau, \end{aligned} \tag{3.10}$$

where $\tilde{C} = \|u_1\|_2^2 + M\left([u_0]_s^2\right) \left\|u_0^m\right\|_{W_0}^2 + C_0^2 C_2 T > 0$ is independent of m. According to the definition of Kirchhoff function $M(m)$, we have

$$a \|v_m\|_{W_0}^2 \leq M\left([u]_s^2\right) \|v_m\|_{W_0}^2. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$a \|v_m\|_{W_0}^2 \leq \tilde{C} + C_1 \int_0^t \|v_m\|_{W_0}^2 \, d\tau. \tag{3.12}$$

Making use of the Gronwall inequality, we get

$$\|v_m\|_{W_0}^2 \leq \frac{\tilde{C}}{a} e^{\frac{C_1}{a} t}, \tag{3.13}$$

and integrating (3.13) on $[0, t]$, we arrive at

$$\int_0^t \|v_m\|_{W_0}^2 \, d\tau \leq \frac{\tilde{C}}{C_1} \left(e^{\frac{C_1}{a} t} - 1\right). \tag{3.14}$$

We substitute (3.14) into (3.10) to get

$$\|v_{mt}\|_2^2 + M\left([u]_s^2\right) \|v_m\|_{W_0}^2 + \int_0^t \|v_{m\tau}\|_{W_0}^2 \, d\tau \leq \frac{\tilde{C}}{C_1} \left(e^{\frac{C_1}{a} t} - 1\right) + \tilde{C} \leq C_T, \tag{3.15}$$

where C_T is a normal number that depends on T. From (3.15), we get

$$v_m \to v \quad weakly \quad star \quad in \quad L^\infty(0,T;W_0), \tag{3.16}$$

$$v_{mt} \to v_t \quad weakly \quad in \quad L^2(0,T;W_0), \tag{3.17}$$

$$v_{mt} \to v_t \quad weakly \quad star \quad in \quad L^\infty\left(0,T;L^2(\Omega)\right). \tag{3.18}$$

By (3.16), (3.17), and the Aubin-Lions-Simon lemma [32], we can obtain

$$v_m \to v \quad strongly \quad in \quad C([0,T],L^2(\Omega)).$$

Therefore, $v_m(x,0)$ makes sense, $v_m(x,0) \to v(x,0)$ *in* $L^2(\Omega)$, and $v_m(x,0) = u_0^m(x) \to u_0(x)$ *in* W_0. Thus, $v(x,0) = u_0(x)$.

Furthermore, dividing the two sides of the first equation in (3.2) by $\|\eta\|_{W_0}$, we have

$$\frac{\langle v_{mtt},\eta\rangle}{\|\eta\|_{W_0}} = \frac{-(v_t,\eta)_{W_0} - M\left([u]_s^2\right)(v,\eta)_{W_0} + \left(|u|^{k-2}u\ln|u|,\eta\right)}{\|\eta\|_{W_0}}. \tag{3.19}$$

By the Hölder inequality, (3.7), and (3.15), we get

$$\frac{\langle v_{mtt},\eta\rangle}{\|\eta\|_{W_0}} \le C_T. \tag{3.20}$$

For $\eta \in W_0\setminus\{0\}$, upper bounds are simultaneously taken on both sides of Eq (3.20), and we have

$$\|v_{mtt}\|_{Y_0} \le C_T, \tag{3.21}$$

namely

$$v_{mtt} \to v_{tt} \quad weakly \quad star \quad in \quad L^\infty(0,T;Y_0). \tag{3.22}$$

Combining $v_t \in L^\infty(0,T;L^2(\Omega))$ and $v_{tt} \in L^\infty(0,T;Y_0)$, through Lemma 2.4, we get

$$v_t \in C([0,T],Y_0).$$

Thus $v_{mt}(x,0)$ is meaningful and $v_{mt}(x,0) \to v_t(x,0)$ *in* Y_0. Owing to $v_{mt}(x,0) = u_1^m(x) \to u_1(x)$ in $L^2(\Omega)$, we have that $v_t(x,0) = u_1(x)$. We have completed the proof of the existence.

(*ii*) Proof of the uniqueness.

Assuming Problem (1.1) has two solutions v_1 and v_2 with the same starting conditions, substituting them into Problem (3.1), and then, by subtracting the obtained two equations, we can get

$$(v_1 - v_2)_{tt} + M\left([u]_s^2\right)(-\Delta)^s(v_1 - v_2) + (-\Delta)^s(v_1 - v_2)_t = 0. \tag{3.23}$$

Multiplying (3.23) by $v_{1t} - v_{2t}$ and integrating on $\Omega \times (0,T)$, we get

$$\frac{1}{2}\|v_{1t} - v_{2t}\|_2^2 + \frac{1}{2}M\left([u]_s^2\right)\|v_1 - v_2\|_{W_0}^2 + \int_0^t \|v_{1\tau} - v_{2\tau}\|_{W_0}^2\, d\tau = 0.$$

Obviously, this equality immediately yields $v_1 \equiv v_2$. This completes the proof. □

Based on the above lemma, we obtain the following theorem.

Theorem 3.1. *Let* $u_0 \in W_0$, $u_1 \in L^2(\Omega)$, *and* $2 \le 2\theta < k \le 2_s^*$. *Then there is a* $T > 0$ *that gives Problem* (1.1) *with a unique local solution* $u(x,t)$ *on* $[0,T]$ *satisfying*

$$u \in C([0,T]; W_0) \cap C^1\left([0,T]; L^2(\Omega)\right) \cap C^2([0,T]; Y_0).$$

Proof. For a given $T > 0$, we think over the important space $\mathcal{H} = C([0,T]; W_0) \cap C^1\left([0,T]; L^2(\Omega)\right)$ which has the following norm:

$$\|u\|_{\mathcal{H}}^2 = \max_{0 \le t \le T} \left(a\|u(t)\|_{W_0}^2 + \|u_t(t)\|_2^2 \right).$$

Let $R^2 = M\left([u_0]_s^2\right)\|u_0\|_{W_0}^2 + \|u_1\|_2^2$, and then we denote

$$\mathcal{M}_T = \{u \in \mathcal{H} : u_t(0) = u_1, u(0) = u_0, \|u\|_{\mathcal{H}} \le R\}.$$

We first prove that $\mathcal{M}_T$ is a complete metric space. Let $\{u_n\}$ be the Cauchy-Schwarz sequence in $\mathcal{M}_T$. Thus, for any $\varepsilon > 0$, there exists μ_ε such that if $n, m \ge \mu_\varepsilon$, then

$$\|u_n - u_m\|_{\mathcal{H}}^2 = \max_{0 \le t \le T} \left(\|u_n - u_m\|_{W_0}^2 + \|u_n - u_m\|_2^2 \right) \le \varepsilon,$$

and by the completeness of $L^2(\Omega)$ and W_0, there exist $u \in L^2(\Omega)$ such that $u_m \to u$ in $L^2(\Omega)$ and $u \in W_0$ such that $u_m \to u$ in W_0 when $m \to \infty$, namely

$$\|u_n - u\|_{\mathcal{H}}^2 = \max_{0 \le t \le T} \left(\|u_n - u\|_{W_0}^2 + \|u_n - u\|_2^2 \right) \le \varepsilon.$$

Therefore, $\mathcal{M}_T$ is a complete metric space.

Next, using the conclusion of Lemma 3.1, we denote $v = \Phi(u)$ for any $u \in \mathcal{M}_T$ as the unique solution to Problem (3.1). We will prove the mapping Φ is a contraction mapping satisfying $\Phi(\mathcal{M}_T) \subset \mathcal{M}_T$. We multiply the first equation of the Problem (3.1) with v_t and integrate it on $\Omega \times (0,t)$, and we obtain

$$\begin{aligned} &\|v_t\|_2^2 + M\left([u]_s^2\right)\|v\|_{W_0}^2 + 2\int_0^t \|v_\tau\|_{W_0}^2 \, d\tau \\ =\ & \|u_1\|_2^2 + M\left([u_0]_s^2\right)\|u_0\|_{W_0}^2 + \int_0^t \frac{d}{dt}\left[M\left([u]_s^2\right)\right]\|v\|_{W_0}^2 \, d\tau \\ &+ 2\int_0^t \int_\Omega v_\tau \cdot |u|^{k-2}u \ln|u| \, dx d\tau. \end{aligned} \tag{3.24}$$

Using a calculation method similar to the processes in (3.5) and (3.7), we find

$$\begin{aligned} & 2\int_0^t \int_\Omega v_\tau \cdot |u|^{k-2}u \ln|u| \, dx d\tau \\ \le\ & 2\int_0^t \|v_\tau\|_{\frac{2N}{N-2s}} \left\||u|^{k-2}u \ln|u|\right\|_{\frac{2N}{N+2s}} d\tau \\ \le\ & \int_0^t \frac{C_0^2}{2}\left\||u|^{k-2}u \ln|u|\right\|_{\frac{2N}{N+2s}}^2 d\tau + 2\int_0^t \|v_\tau\|_{W_0}^2 \, d\tau \\ \le\ & \frac{C_0^2 T}{2}\left\{ (e\mu)^{-\frac{2N}{N+2s}} \left(\frac{C_0^2 R^2}{a}\right)^{\frac{N(k-1+\mu)}{N+2s}} + [e(k-1)]^{-\frac{2N}{N+2s}}|\Omega| \right\}^{\frac{N+2s}{N}} + 2\int_0^t \|v_\tau\|_{W_0}^2 \, d\tau. \end{aligned} \tag{3.25}$$

Then by a similar computation to that of (3.4) and (3.14), we can derive that

$$\int_0^t \|v\|_{W_0}^2 \frac{d}{dt}\left[M\left([u]_s^2\right)\right] d\tau \leq \tilde{C}\left(e^{\frac{C_1}{a}T}-1\right). \tag{3.26}$$

Combining (3.25) and (3.26), (3.24) becomes

$$\begin{aligned}&\|v_t\|_2^2 + M\left([u]_s^2\right)\|v\|_{W_0}^2\\ &\leq R^2+\tilde{C}\left(e^{\frac{C_1}{a}T}-1\right)+\frac{C_0^2T}{2}\left\{(e\mu)^{-\frac{2N}{N+2s}}\left(\frac{C_0^2R^2}{a}\right)^{\frac{N(k-1+\mu)}{N+2s}}+\left[e\left(k-1\right)\right]^{-\frac{2N}{N+2s}}|\Omega|\right\}^{\frac{N+2s}{N}}.\end{aligned}$$

Further,

$$\begin{aligned}\|v\|_{\mathcal{H}}^2 &= \|v_t\|_2^2 + a\|v\|_{W_0}^2 \leq \|v_t\|_2^2 + M\left([u]_s^2\right)\|v\|_{W_0}^2\\ &\leq R^2+\tilde{C}\left(e^{\frac{C_1}{a}T}-1\right)+\frac{C_0^2T}{2}\left\{(e\mu)^{-\frac{2N}{N+2s}}\left(\frac{C_0^2R^2}{a}\right)^{\frac{N(k-1+\mu)}{N+2s}}+\left[e\left(k-1\right)\right]^{-\frac{2N}{N+2s}}|\Omega|\right\}^{\frac{N+2s}{N}}.\end{aligned}$$

So we can choose a $T>0$ to make it small enough so that $\|v\|_{\mathcal{H}}^2 \leq R^2$.

Next, we will prove that Φ is a contraction mapping. Let $v_1=\Phi(w_1)$, $v_2=\Phi(w_2)$ where $w_1,\ w_2\in\mathcal{M}_T$. Then if $v=v_1-v_2$, v satisfies

$$\begin{cases} v_{tt}+M\left([w_1]_s^2\right)(-\Delta)^s v+(-\Delta)^s v_t = +|w_1|^{k-2}w_1\ln|w_1|-|w_2|^{k-2}w_2\ln|w_2| \\ \qquad -\left[M\left([w_1]_s^2\right)-M\left([w_2]_s^2\right)\right](-\Delta)^s v_2, & x\in\Omega,\ t>0;\\ v(x,0)=v_t(x,0)=0, & x\in\Omega;\\ v(x,t)=0, & x\in\mathbb{R}^N\setminus\Omega,\ t\geq 0.\end{cases}$$

We will multiply the first equation of the above problem by v_t, and then integrate on $\Omega\times(0,t)$, and we have

$$\begin{aligned}&\|v_t\|_2^2+M\left([w_1]_s^2\right)\|v\|_{W_0}^2+2\int_0^t\|v_\tau\|_{W_0}^2\,d\tau\\ \leq&\int_0^t\frac{d}{dt}\left[M\left([w_1]_s^2\right)\right]\|v\|_{W_0}^2\,d\tau+2\int_0^t\int_\Omega\left|M\left([w_1]_s^2\right)-M\left([w_2]_s^2\right)\right|(-\Delta)^s v_2 v_\tau dxd\tau\\ &+2\int_0^t\int_\Omega\left(|w_1|^{k-2}w_1\ln|w_1|-|w_2|^{k-2}w_2\ln|w_2|\right)v_\tau dxd\tau.\end{aligned} \tag{3.27}$$

Next, we estimate the terms on the right side of (3.27) one by one. First, by performing calculations similar to (3.4), we obtain

$$\begin{aligned}\int_0^t\frac{d}{dt}\left[M\left([w_1]_s^2\right)\right]\|v\|_{W_0}^2\,d\tau &\leq \int_0^t CM\left([w_1]_s^2\right)\|v\|_{W_0}^2\,d\tau\\ &\leq C\int_0^t\left[M\left([w_1]_s^2\right)\|v\|_{W_0}^2+\|v_\tau\|_2^2\right]d\tau.\end{aligned} \tag{3.28}$$

Due to the function $M(m)$ being locally Lipschitz continuous, we arrive at

$$
\begin{aligned}
&2\int_0^t\int_\Omega\left|M\left([w_1]_s^2\right)-M\left([w_2]_s^2\right)\right|(-\Delta)^s v_2 v_\tau dxd\tau\\
=&\ 2\int_0^t\int_\Omega\left|\frac{M\left([w_1]_s^2\right)-M\left([w_2]_s^2\right)}{[w_1]_s^2-[w_2]_s^2}\right|\left|[w_1]_s^2-[w_2]_s^2\right|(-\Delta)^s v_2 v_\tau dxd\tau\\
\leq&\ 2C_L\int_0^t\int_\Omega\left|\|w_1\|_{W_0}^2-\|w_2\|_{W_0}^2\right|(-\Delta)^s v_2 v_\tau dxd\tau\\
=&\ 2C_L\left|\|w_1\|_{W_0}+\|w_2\|_{W_0}\right|\left|\|w_1\|_{W_0}-\|w_2\|_{W_0}\right|\int_0^t\int_\Omega(-\Delta)^s v_2 v_\tau dxd\tau\\
\leq&\ 2C_L\left|\|w_1\|_{W_0}+\|w_2\|_{W_0}\right|\|w_1-w_2\|_{W_0}\int_0^t\int_\Omega(-\Delta)^s v_2 v_\tau dxd\tau.
\end{aligned}
$$

Next we use Young's inequality and Hölder's inequality to obtain

$$
\begin{aligned}
&2\int_0^t\int_\Omega\left|M\left([w_1]_s^2\right)-M\left([w_2]_s^2\right)\right|(-\Delta)^s v_2 v_\tau dxd\tau\\
\leq&\ 2C_L\left|\|w_1\|_{W_0}+\|w_2\|_{W_0}\right|\|w_1-w_2\|_{W_0}\int_0^t\|v_2\|_{W_0}\|v_\tau\|_{W_0}d\tau\\
\leq&\ 4C_L\frac{R}{\sqrt{a}}\|w_1-w_2\|_{W_0}\int_0^t\|v_2\|_{W_0}\|v_\tau\|_{W_0}d\tau\\
\leq&\ \frac{\left(4C_L\frac{R}{\sqrt{a}}\|w_1-w_2\|_{W_0}\right)^2}{4}\int_0^t\|v_2\|_{W_0}^2\,d\tau+\int_0^t\|v_\tau\|_{W_0}^2\,d\tau\\
\leq&\ 4C_L^2\frac{R^4}{a^2}T\,\|w_1-w_2\|_{W_0}^2+\int_0^t\|v_\tau\|_{W_0}^2\,d\tau\\
\leq&\ 4C_L^2\frac{R^4}{a^3}T\,\|w_1-w_2\|_{\mathcal{H}}^2+\int_0^t\|v_\tau\|_{W_0}^2\,d\tau.
\end{aligned}
\tag{3.29}
$$

Similarly, taking advantage of the Lipschitz continuity of $f(x)=|x|^{k-2}x\ln|x|$ in $\mathbb{R}^+\to\mathbb{R}^+$, and from Lemma 2.3, we can see that

$$
\begin{aligned}
&2\int_0^t\int_\Omega\left(|w_1|^{k-2}w_1\ln|w_1|-|w_2|^{k-2}w_2\ln|w_2|\right)v_t dxd\tau\\
=&\ 2\int_0^t\int_\Omega\left(\frac{f(w_1)-f(w_2)}{w_1-w_2}\right)(w_1-w_2)\,v_t dxd\tau\\
\leq&\ 2\tilde{C}_L\int_0^t\|w_1-w_2\|_2\|v_t\|_2 d\tau\\
\leq&\ 2\tilde{C}_L C_0^2\int_0^t\|w_1-w_2\|_{W_0}\|v_t\|_{W_0}d\tau\\
\leq&\ \frac{\left(2\tilde{C}_L C_0^2\|w_1-w_2\|_{W_0}\right)^2 T}{4}+\int_0^t\|v_t\|_{W_0}^2\,d\tau\\
\leq&\ \frac{\tilde{C}_L^2 C_0^4}{a}T\,\|w_1-w_2\|_{\mathcal{H}}^2+\int_0^t\|v_t\|_{W_0}^2\,d\tau.
\end{aligned}
\tag{3.30}
$$

Substitute (3.28)–(3.30) into (3.27), and we can infer that

$$\begin{aligned}&\|v_t\|_2^2 + M\left([w_1]_s^2\right)\|v\|_{W_0}^2 + 2\int_0^t \|v_\tau\|_{W_0}^2\, d\tau \\ \le\, & C\int_0^t \left[M\left([w_1]_s^2\right)\|v\|_{W_0}^2 + \|v_\tau\|_2^2\right] d\tau + \left(4C_L^2\frac{R^4}{a^3}T + \frac{\tilde{C}_L^2 C_0^4}{a}T\right)\|w_1 - w_2\|_{\mathcal{H}}^2 .\end{aligned}$$

Applying the Gronwall inequality to the above inequality, we have

$$\|v_t\|_2^2 + M\left([w_1]_s^2\right)\|v\|_{W_0}^2 \le \left(4C_L^2\frac{R^4}{a^3}T + \frac{\tilde{C}_L^2 C_0^4}{a}T\right)\left(1 + CTe^{CT}\right)\|w_1 - w_2\|_{\mathcal{H}}^2 .$$

We can take $T > 0$ to make it small enough so that

$$\left(4C_L^2\frac{R^4}{a^3}T + \frac{\tilde{C}_L^2 C_0^4}{a}T\right)\left(1 + CTe^{CT}\right) < 1,$$

and we conclude that there exists $\delta \in (0, 1)$ satisfying

$$\|\Phi(w_1) - \Phi(w_2)\|_{\mathcal{H}} = \|v\|_{\mathcal{H}}^2 \le \delta\|w_1 - w_2\|_{\mathcal{H}}.$$

Therefore, the above processes make certain the existence of weak solutions. □

4. Global existence and decay estimate

First of all, we will prove the invariance of the set W^+.

Lemma 4.1. *Let u be the solution to Problem (1.1). In the case of $u_0 \in W^+$, $u_1 \in L^2(\Omega)$, and $0 < E(0) < d$, then for all $t \ge 0$, we have $u(t) \in W^+$.*

Proof. First, according to Lemma 2.6, we can get $E(t) \le E(0) < d$. Next, we will prove through the method of contradiction that there is a minimum time $t_* \in (0, T_{max})$ that satisfies $u(t_*) \in \partial W^+$, i.e., $I(u(t)) > 0$ for $t \in [0, t_*)$ and $I(u(t_*)) = 0$. From (2.8), (2.9), and (2.13), we have

$$0 < \frac{1}{2}\|u_t(t_*)\|_2^2 + \frac{a(k-2)}{2p}\|u(t_*)\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u(t_*)\|_{W_0}^{2\theta} + \int_0^t \|u_\tau(t_*)\|_{W_0}^2\, d\tau = E(0) < d,$$

which means that $u(t_*) \ne 0$ and $u_t(t_*) \ne 0$. Thus according to (2.10), we get

$$d > E(0) \ge E(t_*) \ge J(u(t_*)) \ge \inf_{u\in\mathcal{N}} J(u) = d,$$

which is not valid. Hence, we obtain $u(t) \in W^+$. □

Theorem 4.1. *Let $u_0 \in W_0$, $u_1 \in L^2(\Omega)$, and $2 \le 2\theta < k \le 2_s^*$. If $E(0) \le d$, $I(u_0) > 0$, and then Problem* (1.1) *possesses a global weak solution $u \in L^\infty(0, +\infty; W_0)$ and $u_t \in L^\infty\left(0, +\infty; L^2(\Omega)\right) \cap L^2(0, +\infty; W_0)$. In addition, if*

$$E(0) \le \left\{d, \frac{a(k-2)}{2p}\left(\frac{ae\mu}{C_0^{p+\mu}}\right)^{\frac{2}{k+\mu-2}}\right\} \text{ and } I(u_0) > 0,$$

there exist positive constants κ and γ so that $E(t)$ satisfies the exponential decay estimate for $\forall\ t \ge 0$ as follows:

$$E(t) \le \kappa e^{-\gamma t}.$$

Proof. We will prove the global existence and decay estimate.

Step 1. Global existence.

We will divide into two cases to prove the global existence.

Case 1: $I(u_0) > 0$ **and** $E(0) < d$

First of all, we need to declare that for the cases where $E(0) < d$ and $I(u_0) > 0$, we have:

(i) If $I(u_0) > 0$ and $E(0) < 0$, this contradicts (2.8) and (2.9).

(ii) If $I(u_0) > 0$ and $E(0) = 0$, by (2.8) and (2.9), it is evident that $u_0 \equiv 0$ and $u_1 \equiv 0$, which is an ordinary situation.

As a result, we only need to think about the cases where $0 < E(0) < d$ and $I(u_0) > 0$.

From Lemma 4.1, we can see that $u(t) \in W^+$, which means that $I(u(t)) > 0$. Combining (2.5), (2.9), and (2.13), we get

$$\frac{1}{k}I(u) + \frac{1}{2}\|u_t\|_2^2 + \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta} + \int_0^t \|u_\tau\|_{W_0}^2 d\tau = E(0) < d,$$

namely

$$\frac{1}{2}\|u_t\|_2^2 + \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta} + \int_0^t \|u_\tau\|_{W_0}^2 d\tau < E(0) < d. \tag{4.1}$$

The right end of Eq (4.1) is a constant unrelated to t. This estimate enables us to take $T_{max} = +\infty$. As a consequence, we are able to get that Problem (1.1) has a unique global weak solution $u(t)$.

Case 2: $I(u_0) > 0$ **and** $E(0) = d$

Above all, we can select a sequence $\{\theta_n\}_{n=1}^{\infty} \subset (0,1)$ satisfying $\lim_{n\to\infty} \theta_n = 1$. Next we think about the problem as follows:

$$\begin{cases} u_{tt} + M\left([u]_s^2\right)(-\Delta)^s u + (-\Delta)^s u_t = |u|^{k-2}u\ln|u|, & x \in \Omega,\ t > 0; \\ u(x,0) = u_{0n} = \theta_n u_0(x), u_t(x,0) = u_{1n} = \theta_n u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R}^N \setminus \Omega,\ t \geq 0. \end{cases}$$

We claim that $I(u_{0n}) > 0$ and $0 < E(u_{0n}) < d$. In fact, from $I(u_0) > 0$ and $\theta_n \subset (0,1)$, we get

$$\begin{aligned} I(u_{0n}) &= -\theta_n^k \int_\Omega |u_0|^k \ln|u_0| dx + a\theta_n^2 \|u\|_{W_0}^2 + \theta_n^{2\theta}\|u\|_{W_0}^{2\theta} - \theta_n^k \ln\theta_n \|u_0\|_k^k \\ &\geq -\theta_n^k \int_\Omega |u_0|^k \ln|u_0| dx + a\theta_n^2 \|u\|_{W_0}^2 + \theta_n^{2\theta}\|u\|_{W_0}^{2\theta} \\ &\geq \theta_n^k I(u_0) > 0. \end{aligned}$$

On the other hand, combining the above inequality and Lemma 2.1, we have

$$\frac{d}{d\theta_n}J(\theta_n u_0) = \frac{1}{\theta_n}I(u_{0n}) > 0,$$

which indicates that $J(\theta_n u_0)$ is strictly increasing relative to θ_n and

$$J(u_{0n}) = J(\theta_n u_0) < J(u_0).$$

Further, we have

$$0 < E(u_{0n}, u_{1n}) = J(u_{0n}) + \frac{1}{2}\|u_{1n}\|_2^2 < J(u_0) + \frac{1}{2}\|u_1\|_2^2 = E(0) = d.$$

Since $u_{0n} \to u_0$ and $u_{1n} \to u_1$ as $n \to +\infty$, we are able to get that there is a global weak solution $u(t)$ to the above problem through a method similar to Case 1.

Step 2. Decay estimate.

We establish an exponential energy decay estimate of Problem (1.1) when $I(u_0) > 0$ as well as $E(0) \le \left\{d, \frac{a(k-2)}{2p}\left(\frac{ae\mu}{C_0^{p+\mu}}\right)^{\frac{2}{k+\mu-2}}\right\}$.

It follows from (4.1) that

$$\|u\|_{W_0}^2 \le \frac{2kE(0)}{a(k-2)}. \tag{4.2}$$

Next, we define an auxiliary function as follows:

$$K(t) = \varepsilon \int_\Omega u_t \cdot u dx + \frac{\varepsilon}{2}\|u\|_{W_0}^2 + E(t),$$

where $\varepsilon > 0$ will be specified later. Through Lemma 4.1, we get $I(u(t)) > 0$. Then by (2.5), (2.8), and (2.9), we obtain

$$E(t) > \frac{1}{2}\|u_t\|_2^2 + \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta} > 0. \tag{4.3}$$

Next, utilizing Young's inequality, and combining (4.3) and Lemma 2.3, we arrive at

$$\begin{aligned} K(t) &\le \frac{\varepsilon}{2}\|u\|_2^2 + \frac{\varepsilon}{2}\|u\|_{W_0}^2 + \frac{\varepsilon}{2}\|u_t\|_2^2 + E(t) \\ &\le \frac{\varepsilon}{2}\|u_t\|_2^2 + \left(\frac{\varepsilon C_0^2}{2} + \frac{\varepsilon}{2}\right)\|u\|_{W_0}^2 + E(t) \\ &\le \varepsilon E(t) + \left(\frac{\varepsilon C_0^2}{2} + \frac{\varepsilon}{2}\right)\frac{2k}{a(k-2)}E(t) + E(t) \\ &\le \eta_2 E(t), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} K(t) &\ge -\frac{\varepsilon}{4\xi}\|u_t\|_2^2 + \frac{\varepsilon}{2}\|u\|_{W_0}^2 - \varepsilon\xi\|u\|_2^2 + E(t) \\ &\ge -\frac{\varepsilon}{4\xi}\|u_t\|_2^2 + \left(\frac{\varepsilon}{2} - \varepsilon\xi C_0^2\right)\|u\|_{W_0}^2 + E(t). \end{aligned} \tag{4.5}$$

Taking ξ small enough such that $\xi \le \frac{1}{2C_0^2}$, then we can obtain

$$K(t) \ge -\frac{\varepsilon}{4\xi}\|u\|_{W_0}^2 + E(t) \ge -\frac{\varepsilon}{2\xi}E(t) + E(t),$$

and we fix ξ and choose a sufficiently small normal number ε such that $\varepsilon < 2\delta$, so (4.4) can be written as

$$K(t) \ge \left[1 - \varepsilon(2\xi)^{-1}\right]E(t) = \eta_1 E(t). \tag{4.6}$$

Combining (4.4) and (4.6), it is easy to conclude that $L(t)$ is equivalent to $E(t)$, because there exist two constants $\eta_1 > 0$ and $\eta_2 > 0$ about ε satisfying

$$\eta_1 E(t) \leq K(t) \leq \eta_2 E(t), \quad for\ t \geq 0. \tag{4.7}$$

Next, we derive $K(t)$ about t and choose $0 < M < 2\theta$. From (2.5) and Lemma 2.3, we arrive at

$$\begin{aligned}
K'(t) &= E'(t) + \varepsilon(u, u_{tt}) + \varepsilon(u, u_t)_{W_0} + \varepsilon \|u_t\|_2^2 \\
&= -\|u_t\|_{W_0}^2 + \varepsilon\left(\int_\Omega |u|^k \ln|u|\,dx - a\|u\|_{W_0}^2 - \|u\|_{W_0}^{2\theta}\right) + \varepsilon\|u_t\|_2^2 \\
&= -\varepsilon ME(t) - \|u_t\|_{W_0}^2 + \left(\frac{\varepsilon M}{2} + \varepsilon\right)\|u_t\|_2^2 + \left(\frac{a\varepsilon M}{2} - a\varepsilon\right)\|u\|_{W_0}^2 \\
&\quad + \left(\frac{\varepsilon M}{2\theta} - \varepsilon\right)\|u\|_{W_0}^{2\theta} + \left(\varepsilon - \frac{\varepsilon M}{k}\right)\int_\Omega |u|^k \ln|u|\,dx + \frac{\varepsilon M}{k^2}\|u\|_k^k \\
&\leq -\varepsilon ME(t) + \left(\frac{\varepsilon M}{2} + \varepsilon - \frac{1}{C_0^2}\right)\|u_t\|_2^2 + \left(\frac{a\varepsilon M}{2} - a\varepsilon\right)\|u\|_{W_0}^2 \\
&\quad + \left(\varepsilon - \frac{\varepsilon M}{k}\right)\int_\Omega |u|^k \ln|u|\,dx + \frac{\varepsilon M}{k^2}\|u\|_k^k.
\end{aligned} \tag{4.8}$$

Next we will estimate each term of (4.8). For the term $\|u\|_k^k$, by (4.2), we obtain

$$\|u\|_k^k \leq C_0^k\|u\|_{W_0}^k = C_0^k\|u\|_{W_0}^{k-2}\|u\|_{W_0}^2 \leq C_0^k\left(\frac{2kE(0)}{a(k-2)}\right)^{\frac{k-2}{2}}\|u\|_{W_0}^2. \tag{4.9}$$

From (4.3) and Lemmas 2.2 and 2.3, for the logarithmic source term, we have

$$\begin{aligned}
\int_\Omega |u|^k \ln|u|\,dx &\leq \int_{\Omega_2} |u|^k \frac{1}{e\mu}|u|^\mu dx \\
&\leq \frac{1}{e\mu}\|u\|_{k+\mu}^{k+\mu} \leq \frac{C_0^{k+\mu}}{e\mu}\|u\|_{W_0}^{k+\mu} \\
&\leq \frac{C_0^{k+\mu}}{e\mu}\left(\frac{2kE(0)}{a(k-2)}\right)^{\frac{k+\mu-2}{2}}\|u\|_{W_0}^2,
\end{aligned} \tag{4.10}$$

where μ satisfies $0 < \mu < 2_s^* - k$. Substituting (4.9) and (4.10) into (4.8), we can derive that

$$\begin{aligned}
K'(t) &- \varepsilon ME(t) + \left(\frac{\varepsilon M}{2} + \varepsilon - \frac{1}{C_0^2}\right)\|u_t\|_2^2 + \left(\frac{a\varepsilon M}{2} - a\varepsilon\right)\|u\|_{w_0}^2 \\
&+ \left(\varepsilon - \frac{\varepsilon M}{k}\right)\frac{C_0^{k+\mu}}{e\mu}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k+\mu-2}{2}}\|u\|_{w_0}^2 + \frac{\varepsilon MC_0^k}{k^2}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k-2}{2}} \\
&= -\varepsilon ME(t) + \left(\frac{\varepsilon M}{2} + \varepsilon - \frac{1}{C_0^2}\right)\|u_t\|_2^2 + \varepsilon\left\{\frac{aM}{2} - a + \frac{C_0^{k+\mu}}{e\mu}\left[\frac{2pE(0)}{a(p-2)}\right]^{\frac{k+\mu-2}{2}}\right. \\
&\left. - \frac{C_0^{k+\mu}M}{ke\mu}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k+\mu-2}{2}} + \frac{C_0^k M}{k^2}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k-2}{2}}\right\}\|u\|_{w_0}^2,
\end{aligned} \tag{4.11}$$

where we require $E(0) \leq \left\{ d, \frac{a(k-2)}{2k}\left(\frac{ae\mu}{C_0^{k+\mu}}\right)^{\frac{2}{k+\mu-2}} \right\}$ such that

$$\frac{C_0^{k+\mu}}{e\mu}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k+\mu-2}{2}} - a \leq \frac{C_0^{k+\mu}}{e\mu}\left[\frac{2kd}{a(k-2)}\right]^{\frac{k+\mu-2}{2}} - a < 0.$$

Now, we choose a small enough M to make

$$\frac{aM}{2} - a + \frac{C_0^{k+\mu}}{e\mu}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k+\mu-2}{2}} + \frac{C_0^k M}{k^2}\left[\frac{2kE(0)}{a(k-2)}\right]^{\frac{k-2}{2}} < 0. \tag{4.12}$$

We fix M and then select a sufficiently small ε to make

$$\frac{\varepsilon M}{2} + \varepsilon - \frac{1}{C_0^2} < 0. \tag{4.13}$$

Through (4.11)–(4.13), we arrive at

$$K'(t) \leq -\varepsilon M E(t). \tag{4.14}$$

Further, by (4.7), let $\gamma = \frac{\varepsilon M}{\eta_2}$ and (4.14) becomes

$$K'(t) \leq -\gamma K(t). \tag{4.15}$$

Eventually, by integrating (4.15) with $(0, t)$, we can infer that $K(t) \leq K(0)e^{-\gamma t}$, and combining with (4.7), we can get

$$0 < E(t) \leq \kappa e^{-\gamma t}, \quad \forall\, t > 0,$$

where $\kappa = \frac{L(0)}{\eta_1}$. □

5. Blow-up

First, we will prove the invariance of the set W^-.

Lemma 5.1. *Let $u(x,t)$ be a weak solution to Problem (1.1). Assuming $u_0 \in W^-$ and $E(0) < d$, for all $t \geq 0$, we have $u(t) \in W^-$. In addition, we have the following inequality:*

$$d < \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta}. \tag{5.1}$$

Proof. It follows from $u_0 \in W^-$ that $I(u_0) < 0$. Next, we discuss it in two situations.

When $E(0) \leq 0$, from (2.8) and (2.9), we arrive at

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{k}I(u) + \frac{a(k-2)}{2k}\|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k}\|u\|_{W_0}^{2\theta} < E(0) \leq 0 < d,$$

which means that $I(u(t)) < 0$, i.e., $u(t) \in W^-$.

When $0 < E(0) < d$, by (2.13), we have

$$0 < E(t) + \int_0^t \|u_\tau\|_{W_0}^2\, d\tau = E(0) < d, \tag{5.2}$$

which implies that $u(x,t) \neq 0$. For $t \in [0, T_{max})$, we prove that $I(u(t)) < 0$. If not, there is a $t_1 \in (0, T_{max})$ to make $u(t_1) \in \partial W^-$, i.e., $I(u(t_1)) = 0$ and $I(u(t)) < 0$, $t \in [0, t_1)$. Looking back at (2.10), it is obvious that

$$E(0) \geq E(t_1) \geq J(u(t_1)) \geq \inf_{u \in \mathcal{N}} J(u) = d,$$

which is opposite to (5.2). Therefore, for $t \in [0, T_{max}]$, we get $u(t) \in W^-$.

By Lemma 2.1, due to $I(u(t)) < 0$, there is a $\lambda_* < 1$ satisfying $I(\lambda_* u(t)) = 0$, and combining with (2.9) gives

$$\begin{aligned} d \leq J(\lambda^* u) &= \frac{1}{k} I(\lambda_* u) + \frac{a(k-2)}{2k} \lambda_*^2 \|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k} \lambda_*^{2\theta} \|u\|_{W_0}^{2\theta} \\ &= \frac{a(k-2)}{2k} \lambda_*^2 \|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k} \lambda_*^{2\theta} \|u\|_{W_0}^{2\theta} \\ &< \frac{a(k-2)}{2k} \|u\|_{W_0}^2 + \frac{k-2\theta}{2\theta k} \|u\|_{W_0}^{2\theta}. \end{aligned}$$

Therefore, Lemma 4.1 is proven. □

Theorem 5.1. *Let $2 \leq 2\theta < k \leq 2_s^*$, $u_0 \in W^-$, and $u_1 \in L^2(\Omega)$. Assuming $E(0) < d$, the solution of Problem* (1.1) *blows up in finite time.*

Proof. First of all, by $E(0) < d$, $u_0 \in W^-$, and Lemma 4.1, we get $u \in W^-$, which means that $I(u) < 0$.

We will prove that $u(t)$ blows up in finite time. If this is not established, we assume the global existence of u, i.e., $T_{max} = +\infty$. For any $T_0 > 0$, we define the positive function

$$Q(t) = (T_0 - t) \|u_0\|_{W_0}^2 \int_0^t \|u\|_{W_0}^2 d\tau + \|u\|_2^2 .$$

We calculate the first-order and second-order derivatives of $Q(t)$, respectively, as

$$Q'(t) = \|u\|_{W_0}^2 + 2(u, u_t) - \|u_0\|_{W_0}^2 = 2\int_0^t (u, u_\tau)_{W_0} d\tau + 2(u, u_t), \tag{5.3}$$

and

$$Q''(t) = 2\|u_t\|_2^2 + 2(u, u_t)_{W_0} + 2(u, u_{tt}) = 2\|u_t\|_2^2 - 2I(u) > 0. \tag{5.4}$$

Through direct calculation, we arrive at

$$\begin{aligned} &Q(t) Q''(t) - \frac{k+2}{4} [Q'(t)]^2 \\ &= Q(t) \left(2\|u_t\|_2^2 - 2I(u)\right) + (k+2) \cdot \\ &\quad \left\{\Phi(t) - \left[Q(t) - (T-t)\|u_0\|_{W_0}^2\right] \left(\int_0^t \|u_\tau\|_{W_0}^2 \, d\tau + \|u_t\|_2^2\right)\right\}, \end{aligned} \tag{5.5}$$

where the definition of $\psi(t)$ is as follows:

$$\psi(t) = \left(\int_0^t \|u\|_{W_0}^2 \, d\tau + \|u\|_2^2\right) \cdot \left(\int_0^t \|u_\tau\|_{W_0}^2 \, d\tau + \|u_t\|_2^2\right) - \left[\int_0^t (u, u_\tau)_{W_0} d\tau + (u, u_t)\right]^2.$$

Using Hölder's inequality and the Cauchy-Schwarz inequality, for any $t \in (0, T_0)$, it is clear that $\psi(t) \geq 0$.

Further, (5.5) becomes

$$\begin{aligned}
&Q(t)\,Q''(t) - \frac{k+2}{4}[Q'(t)]^2 \\
&\geq Q(t)\left(2\|u_t\|_2^2 - 2I(u)\right) - (k+2)\left[Q(t) - (T-t)\|u_0\|_{W_0}^2\right]\left(\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + \|u_t\|_2^2\right) \\
&\geq Q(t)\left[2\|u_t\|_2^2 - 2I(u) - (k+2)\left(\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + \|u_t\|_2^2\right)\right] \\
&= Q(t)\left[-k\|u_t\|_2^2 - 2I(u) - (k+2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau\right] \\
&= Q(t)\,\varphi(t),
\end{aligned} \tag{5.6}$$

where $\varphi(t) = -k\|u_t\|_2^2 - 2I(u) - (k+2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau$. Moreover, by (2.5), (2.13), and (5.1), we have

$$\begin{aligned}
\varphi(t) &= -2kE(t) + a(k-2)\|u\|_{W_0}^2 + \left(\frac{k}{\theta} - 2\right)\|u\|_{W_0}^{2\theta} + \frac{2}{k}\|u\|_k^k - (k+2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau \\
&\geq -2kE(0) + 2k\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + a(k-2)\|u\|_{W_0}^2 + \left(\frac{k}{\theta} - 2\right)\|u\|_{W_0}^{2\theta} \\
&\quad + \frac{2}{k}\|u\|_k^k - (k+2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau \\
&= -2kE(0) + (k-2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + a(k-2)\|u\|_{W_0}^2 + \left(\frac{k}{\theta} - 2\right)\|u\|_{W_0}^{2\theta} + \frac{2}{k}\|u\|_k^k \\
&\geq 2kd - 2kE(0) + (k-2)\int_0^t \|u_\tau\|_{W_0}^2\,d\tau + \frac{2}{k}\|u\|_k^k.
\end{aligned} \tag{5.7}$$

Since $E(0) < d$, we can get $\varphi(t) > 0$. The combination of (5.6) and (5.7) means that for all $t \in [0, T_0]$,

$$Q(t)\,Q''(t) - \frac{k+2}{4}[Q'(t)]^2 > 0.$$

Let

$$q(t) := Q(t)^{-\frac{k-2}{4}},$$

and then we arrive at

$$\begin{aligned}
q''(t) &= \left(-\frac{k-2}{4} - 1\right)\left(-\frac{k-2}{4}\right)Q(t)^{-\frac{k-2}{4}-2}[Q'(t)]^2 - Q''(t)\,\frac{k-2}{4}Q(t)^{-\frac{k-2}{4}-1} \\
&= \left\{Q(t)\,Q''(t) - \frac{k+2}{4}[Q'(t)]^2\right\}\left(-\frac{k-2}{4}\right)Q(t)^{-\frac{k-2}{4}-2} < 0.
\end{aligned}$$

As a result, it can be obtained that

$$\lim_{t\to T_0} q(t) = 0,$$

namely

$$\lim_{t \to T_0} Q(t) = +\infty.$$

Therefore, the hypothesis does not hold. We have completed the proof. □

6. Conclusions

We have investigated the initial boundary value problem that includes the fractional laplacian operator, strong damping term, and logarithmic source. To our knowledge, Kirchhoff proposed groundbreaking work on Kirchhoff-type problems, and we consider Problem (1.1), which has not been studied before. We proved the local existence by the contraction mapping principle and Faedo-Galerkin's method, namely, Theorem 3.1. Furthermore, global existence and properties of decay and blow-up when the initial value meets certain conditions are obtained, namely, Theorems 4.1 and 5.1. In future work, we will attempt to study the qualitative analysis of some interesting new models.

Author contributions

A. H. Sun: Methodology, writing–original draft, writing–review and editing; H. Xu: Methodology, writing–original draft. Both of authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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