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Research article

The moduli space of symplectic bundles over a compact Riemann surface and quaternionic structures

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Abstract: Let $G = \mathrm{Sp}(4, \mathbb{C})$, K_G be a maximal compact subgroup of G, H be the subgroup of G generated by one of the non-trivial elements of the quaternion group, viewed as a subgroup of G, and X be a compact Riemann surface of genus $g \geq 2$. The main result of this paper proves that the forgetful map $F : M(H) \to M(G)$ between moduli spaces of principal bundles over X induced by the inclusion $H \hookrightarrow G$ is a closed embedding. From this, $M(H)$ can be understood as a closed subvariety of $M(G)$. Moreover, some applications of this result are provided. In particular, it is proved that the bundles in the image of F admit a quaternionic structure and also a reduction of the structure group to $\mathrm{Sp}(2, \mathbb{H})$. From this, some topological constraints are given, including that the image of the forgetful map falls in a single connected component of $M(G)$. In addition, some applications are provided concerning the representation space $\mathcal{R}(\pi_1(X), K_G)$, which, by the Narasimhan-Seshadri-Ramanathan correspondence, is isomorphic to $M(G)$. Specifically, the image of the forgetful map is proved to correspond to the fixed point subset of a certain subvariety of $\mathcal{R}(\pi_1(X), K_G)$.

Keywords: symplectic group; injectivity; principal bundle; moduli space; representation
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1. Introduction

Given a compact Riemann surface X of genus $g \geq 2$ and a reductive complex Lie group G, a principal G-bundle over X is a fiber bundle $E \to X$ with structure group G, locally trivial in the analytic topology. Suitable notions of stability and polystability were introduced by Mumford, Narasimhan, Seshadri, and Ramanathan [1–4] to construct the moduli space $M(G)$ of principal G-bundles, which parametrizes isomorphism classes of polystable principal G-bundles over X. These moduli spaces generalize the classical theory of vector bundles and arise naturally in the study of non-abelian Hodge theory and conformal field theory [5, 6]. Indeed, for $G = GL(n, \mathbb{C})$, the moduli space $M(G)$ reduces to the classical moduli space of rank-n vector bundles over X.

The construction of these moduli spaces relies on the geometric invariant theory (GIT), as developed by Mumford, Fogarti, and Kirwan [7], and involves interesting relations with representation theory. This connection arises from the identification of the moduli space of principal G-bundles with the space $\mathcal{R}(\pi_1(X), K_G)$ of homomorphisms from the fundamental group $\pi_1(X)$ of the curve into the maximal compact subgroup $K_G \subset G$, modulo conjugation, which is called character variety. This bijection was first introduced by Narasimhan and Seshadri [2] for vector bundles, and it was later extended by Ramanathan [8] for reductive Lie groups, giving rise to the memorable Narasimhan-Seshadri-Ramanathan correspondence.

For the symplectic Lie group $G = Sp(4, \mathbb{C})$, which is the group of interest in this work, the moduli space of principal G-bundles over X admits an additional structure derived from the symplectic nature of the group. This allows to describe the moduli space $M(G)$ in terms of solutions to the Yang-Mills-Higgs equations, where the moment map interpretation via symplectic geometry plays a crucial role [9]. The moduli space of $Sp(4, \mathbb{C})$-Higgs bundles, which generalizes principal bundles, is particularly interesting due to its connection with hyperkähler manifolds, as studied by Verbitsky [10] and Swann [11]. The maximal compact subgroup in this case is $K_G = Sp(4) = Sp(4, \mathbb{C}) \cap U(4)$, and the associated character variety describes conjugacy classes of representations of $\pi_1(X)$ into $Sp(4)$. Notice that, while $K_G = Sp(4, \mathbb{C}) \cap U(4)$ and $Sp(2, \mathbb{H})$ are indeed isomorphic as Lie groups, they play different roles as subgroups of $Sp(4, \mathbb{C})$. Specifically, K_G appears in the Narasimhan-Seshadri-Ramanathan correspondence as the maximal compact subgroup, while $Sp(2, \mathbb{H})$ represents the quaternionic form to which the bundles admit reduction. This has connections to the study of higher Teichmüller theory, where the geometry of the space of maximal representations into $Sp(4)$ has been extensively studied [12, 13].

The group $Sp(4, \mathbb{C})$ is also related to quaternionic structures on vector bundles [14]. Specifically, a quaternionic structure on a rank-4 holomorphic vector bundle E over X is given by an involution of E that anti-commutes with the complex structure and induces an action of the quaternion algebra $\mathbb{H}$ on the fibers. The presence of a quaternionic structure induces an $Sp(4, \mathbb{C})$ structure on the vector bundle E. These structures are crucial in the study of hyperkähler geometry and arise naturally in moduli spaces of Higgs bundles and solutions to the self-duality equations [10, 11].

The subgroup $Sp(2, \mathbb{H})$ of $Sp(4, \mathbb{C})$ plays an important role in this context as it preserves this quaternionic action. More precisely, the group $Sp(2, \mathbb{H})$ consists of 4×4 complex matrices that preserve a quaternionic Hermitian form and can be identified with the group of quaternionic unitary 2×2 matrices. This subgroup is naturally embedded in $Sp(4, \mathbb{C})$ through its action on a quaternionic 2-dimensional vector space. Explicitly, viewing the quaternionic plane $\mathbb{H}^2$ as a right module over the quaternions, the group $Sp(2, \mathbb{H})$ acts by preserving a skew-Hermitian form and is contained within $Sp(4, \mathbb{C})$ as the subgroup that commutes with its natural quaternionic structure. The inclusion of $Sp(2, \mathbb{H})$ within $Sp(4, \mathbb{C})$ is key for understanding the moduli space as a natural higher-rank generalization of moduli spaces of $SL(2, \mathbb{C})$-bundles, providing insight into the geometry of hyperkähler reduction and its relation to algebraic structures, particularly in the context of the Hitchin system [15].

An interesting line of research on the geometry of the moduli spaces of principal bundles is the study of their subvarieties. This approach has been conducted from the point of view of identifying the subvarieties of fixed points of automorphisms of the moduli space [16, 17] or their stratifications, such as those of Shatz or Białynicki-Birula [18,19]. The perspective followed in this paper is that of studying the image of the forgetful map $M(H) \to M(G)$ induced by the inclusion of groups $H \hookrightarrow G$. This is in line with previous works, such as that of Serman [20], who proved that the forgetful map is injective when $G = \mathrm{GL}(n, \mathbb{C})$ and H is the special orthogonal or symplectic group, under certain conditions for the rank of the bundles, or [21], where the forgetful map for $G = \mathrm{Spin}(8, \mathbb{C})$ and $H = G_2$ was studied in the context of Higgs bundles by using techniques involving the Hitchin integrable system and the phenomenon of triality.

In particular, in this work, the moduli spaces $M(\mathrm{Sp}(4, \mathbb{C}))$ and $M(H)$ are considered, where H is the order-4 cyclic subgroup generated by the representative in $\mathrm{Sp}(4, \mathbb{C})$ of the element ξ_i, $Q_8 = \{\pm 1, \pm\xi_i, \pm\xi_j, \pm\xi_k\}$ being the quaternionic group, viewed as a subgroup of $\mathrm{Sp}(4, \mathbb{C})$. In the main result of the paper, it is proved that the forgetful map $F : M(H) \to M(\mathrm{Sp}(4, \mathbb{C}))$ induced by the inclusion $H \hookrightarrow \mathrm{Sp}(4, \mathbb{C})$ is injective and proper, so it defines a closed embedding of $M(H)$ into $M(\mathrm{Sp}(4, \mathbb{C}))$ (Theorem 1 and Corollary 1), provided that $M(H)$ is separated with its natural discrete topology.

The study of quaternionic structures on vector bundles is a natural extension of the classical theory of real structures, replacing the condition $J^2 = \mathrm{Id}$ with $J^2 = -\mathrm{Id}$. This change leads to richer geometric structures that naturally interact with symplectic geometry. The choice of the symplectic group $\mathrm{Sp}(4, \mathbb{C})$ as the structure group emerges from the fundamental fact that quaternionic structures on rank-4 vector bundles are naturally connected to the symplectic form, as quaternionic multiplication naturally induces symplectic structures. Unlike lower-rank cases such as $\mathrm{Sp}(2, \mathbb{C})$, the group $\mathrm{Sp}(4, \mathbb{C})$ provides the minimal setting where quaternionic structures can be fully realized while maintaining non-trivial topological constraints. In addition, the specific choice of the cyclic subgroup H of order 4 is motivated by its role as the simplest non-trivial finite subgroup of $\mathrm{Sp}(4, \mathbb{C})$ that preserves quaternionic structures. This subgroup, generated by the element corresponding to multiplication by the quaternionic unit ξ_i, then captures the essence of quaternionic geometry.

Note that, while the study of Higgs bundles has traditionally focused on connected reductive Lie groups, recent advances by Barajas, García-Prada, Gothen, and Mundet [22] have expanded this theory to encompass non-connected reductive groups, which is the case of the group H under consideration here. Higgs bundles were considered in [22]. However, since the moduli space of principal G-bundles is included in the moduli space of G-Higgs bundles through the map $E \mapsto (E, 0)$, the stability conditions of principal G-bundles coincide with those of G-Higgs bundles with 0 Higgs field, and therefore the framework developed in [22] works here. When examining discrete subgroups, which are inherently non-connected unless trivial, an interesting specialization of the general theory then arises. In particular, if H is discrete, the classical notion of stability is vacuous, so every principal H-bundle is stable, hence polystable. Indeed, $M(H)$ is isomorphic to the space $\mathrm{Hom}(\pi_1(X), H)/H$, where $\pi_1(X)$ is the fundamental group of X. This space does not present, in general, a complex algebraic variety structure, but is always admits a natural discrete topology. Within this setting, it makes sense to consider an embedding $M(H) \to M(G)$, and that the condition of embedding is equivalent to being injective and proper [23]. Thus, the injection proved here not only contributes to the understanding of relationships between different moduli spaces but also illuminates aspects of the geometric structure of these spaces in the context of the broader framework for non-connected groups.

Several applications of this main result are also provided. First, the injectivity of the forgetful map has implications regarding the geometry of the image in $M(\mathrm{Sp}(4, \mathbb{C}))$ of the forgetful map.

Specifically, it is proved that every principal bundle in the image of F necessarily admits a quaternionic structure (Theorem 2), from which a reduction of the structure group of the bundle to $\mathrm{Sp}(2,\mathbb{H})$ is constructed (Theorem 3). This reduction is used to prove some topological constraints on the bundles of the image of F; in particular, that the whole image falls within a single connected component of $M(\mathrm{Sp}(4,\mathbb{C}))$ (Theorem 4).

Second, the space of representations $\mathcal{R}(\pi_1(X), K_G)$, which is isomorphic to $M(\mathrm{Sp}(4,\mathbb{C}))$ through the Narasimhan-Seshadri-Ramanathan correspondence, is considered, where K_G denotes a maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{C})$. An involution σ of $\mathcal{R}(\pi_1(X), K_G)$ is constructed (Definition 2) such that the subvariety of fixed points of σ corresponds exactly to the image of the forgetful map F considered above (Theorem 5).

The paper is structured as follows. The main result, which proves the injectivity of the forgetful map $F : M(H) \to M(\mathrm{Sp}(4,\mathbb{C}))$, is given in Section 2. In Section 3, the applications of the main result to the geometry of the image of the forgetful map are described and proved, including the implications on quaternionic structure, reductions of the structure group to $\mathrm{Sp}(2,\mathbb{H})$, and the proof that the image of the forgetful map lies in a single connected component of $M(\mathrm{Sp}(4,\mathbb{C}))$. The applications concerning the representation space $\mathcal{R}(\pi_1(X), K_G)$, which connect the image of the forgetful map mentioned above to the subvariety of fixed points of a certain involution of $\mathcal{R}(\pi_1(X), K_G)$ through the Narasimhan-Seshadri-Ramanathan correspondence, are discussed in Section 4. Finally, the main conclusions of the paper are drawn.

2. Injectivity of the forgetful map between moduli spaces of principal bundles

Let X be a compact Riemann surface of genus $g \geq 2$. Consider the complex symplectic Lie group $G = \mathrm{Sp}(4,\mathbb{C})$ equipped with the symplectic form ω given by the matrix

$$\omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

and the quaternion group $Q_8 = \{\pm 1, \pm\xi_i, \pm\xi_j, \pm\xi_k\}$, defined by the relations $\xi_i^2 = -1$, $\xi_j^2 = -1$, and $\xi_i\xi_j = -\xi_j\xi_i = \xi_k$ [24,25], which is embedded in G through the representation given by

$$\xi_i \mapsto \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \xi_j \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.2}$$

The element $\xi_k = \xi_i\xi_j$ is then represented by

$$\xi_k \mapsto \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Remark 1. *Although the usual notation for the quaternionic group Q_8 considered here is $\{\pm 1, \pm i, \pm j, \pm k\}$, the notation $\{\pm 1, \pm\xi_i, \pm\xi_j, \pm\xi_k\}$ has been chosen to avoid confusion of the element i of the quaternionic group with the imaginary unit, which will be abundantly used throughout the paper.*

Let $H = \langle h \rangle$ be the order-4 cyclic subgroup of G generated by

$$h = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{2.3}$$

the matrix that represents the element ξ_i of Q_8, which is one of the generators of the quaternion group, according to (2.2). Notice that the eigenspaces of ξ_i in the above 4-dimensional representation are

$$V_i = \text{span}\{e_1, e_2\}, \quad V_{-i} = \text{span}\{e_3, e_4\},$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis.

Throughout this work, principal bundles are understood in the holomorphic category unless explicitly stated otherwise. The moduli spaces $M(G)$ and $M(H)$ parametrize isomorphism classes of polystable holomorphic principal bundles over the compact Riemann surface X, where stability is defined through the Harder-Narasimhan filtration and slope conditions for holomorphic vector bundles associated with irreducible representations of the structure group. When dealing with maximal compact subgroups and their associated real forms, particularly in contexts involving unitary groups or orthogonal groups as maximal compact subgroups of $\text{Sp}(4, \mathbb{C})$, the transition between holomorphic and smooth categories will be explicit through the Narasimhan-Seshadri-Ramanathan correspondence [2, 8]. This correspondence establishes a fundamental bijection between polystable holomorphic principal G-bundles over X and equivalence classes of irreducible representations of the fundamental group $\pi_1(X)$ into a maximal compact subgroup $K \subset G$.

Before proceeding, let us establish the theoretical framework for principal bundles with non-connected structure groups, as the main injectivity result will be applied to discrete subgroups. The theory of principal bundles was originally developed for connected reductive complex Lie groups, but has been extended to non-connected groups in the work of Barajas, García-Prada, Gothen, and Mundet [22]. Although [22] is contextualized in Higgs bundles, it is possible to adapt it directly to our context, since the notions of stability of the main bundles coincide with those of Higgs bundles with zero Higgs.

Let H be a complex reductive Lie group, not necessarily connected. Following [22], denote by H_0 the identity component of H, which is a connected reductive complex Lie group. The component group $\pi_0(H) = H/H_0$ characterizes the non-connectedness of H in the following sense. For a general non-connected group H, a principal H-bundle can be described in terms of its restriction to the identity component H_0 and transition data involving the component group $\pi_0(H)$. Specifically, a principal H-bundle over X can be viewed as a collection of principal H_0-bundles indexed by elements of $\pi_0(H)$, with appropriate compatibility conditions at the overlaps.

For non-connected groups, the notions of stability, semistability, and polystability are defined in terms of reductions of the structure group to parabolic subgroups of H_0 and their corresponding antidominant characters, as detailed in [22]. In the special case where H is discrete, these stability conditions reduce to the classical definitions for principal bundles, as there are no non-trivial reductions to consider within the trivial identity component. More precisely, for a connected reductive complex Lie group G, the stability of a principal G-bundle E is characterized through reductions to parabolic subgroups. Specifically, a principal G-bundle E is semistable if for every reduction of structure to a parabolic subgroup $P \subset G$, resulting in a principal P-bundle E_P, and for every dominant character $\chi : P \to \mathbb{C}^*$, the associated line bundle $E_P(\chi)$ has non-positive degree. Polystability further requires that whenever the degree vanishes, the reduction can be extended to a reduction to a Levi subgroup of P. For a discrete group H, the algebraic structure is fundamentally different from that of positive-dimensional reductive groups. Since H contains no connected non-trivial subgroups, it possesses no proper parabolic subgroups in the conventional sense of algebraic group theory. This structural simplicity has profound implications for stability theory.

The absence of proper parabolic subgroups means that for a principal H-bundle, there exist no non-trivial reductions to test against the stability condition. As a consequence, every principal bundle with discrete structure group is automatically semistable (indeed, polystable) as the conditions defining these properties are satisfied vacuously. This automatic stability represents a significant simplification compared to the intricate stability conditions required for bundles with connected structure groups. This simplification can be understood through the correspondence between principal H-bundles and representations of the fundamental group. For a compact Riemann surface X and a discrete group H, the moduli space of principal H-bundles over X is naturally isomorphic to the character variety $\mathrm{Hom}(\pi_1(X), H)/H$, where the quotient is taken with respect to the conjugation action of H. In this setting, each conjugacy class of representations corresponds to a unique isomorphism class of principal H-bundles, and every such bundle is automatically polystable.

For a discrete group H, this approach yields a particularly simple form, as the identity component H_0 is trivial and the component group $\pi_0(H) = H$ constitutes the entire group. The stability conditions established in these extended frameworks, when specialized to discrete groups, reduce to the trivial condition described above: All principal H-bundles are automatically polystable due to the absence of non-trivial reductions within the trivial identity component. Note that, with the definition above, $M(H)$ does not necessarily have a complex algebraic variety structure, but it is always a topological space. Indeed, H has a topology inherited from G and $M(H) \cong \mathrm{Hom}(\pi_1(X), H)/H$ is a subset of H^n, where n is the number of generators of $\pi_1(X)$. Therefore, $M(H)$ inherits a discrete natural topology as a subspace of H^n. With this topology, the conjugation action of H is of course a continuous action.

Given an inclusion of reductive complex Lie groups $i : H \hookrightarrow G$, there is a naturally induced map between the corresponding moduli spaces of principal bundles,

$$F : M(H) \to M(G), \tag{2.4}$$

defined by sending an H-bundle E to the associated G-bundle $E \times_i G$, obtained by extending the structure group of E via i.

Lemma 1. *Let $G = \mathrm{Sp}(4, \mathbb{C})$ and H be the order-4 cyclic subgroup of G generated by the matrix h defined in* (2.3)*. Then, the normalizer $N_G(H)$ consists of matrices of the form*

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix},$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \omega_2 \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \omega_2,$$

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \omega_2 \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = \omega_2,$$

with $\omega_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Let $g \in G$ be any element. Then, the condition $ghg^{-1} = h^{\pm 1}$, required for g to be in $N_G(H)$, implies that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$= \pm \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

since $h^{-1} = -h$. This gives a family of equations from which, by comparing coefficients, it is obtained that

$$a_{13} = a_{14} = a_{23} = a_{24} = a_{31} = a_{32} = a_{41} = a_{42} = 0,$$

so the matrix form of g turns to be

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$$

Moreover, the symplectic condition $g^T \omega g = \omega$ computed in the above matrix then gives a splitting into two 2×2 blocks satisfying the conditions stated, where ω is the symplectic form introduced in (2.1). □

Lemma 2. *Let X be a compact Riemann surface, H be a closed subgroup of a complex Lie group G, and let E and E' be principal H-bundles over X. Suppose that E and E' admit extensions to principal G-bundles $\tilde{E}$ and $\tilde{E}'$, respectively, and that these extensions are isomorphic as G-bundles. Then, there exists a finite open cover $\{U_i\}_{i\in I}$ of X and a Čech 1-cocycle $\{\alpha_{ij}\}_{i,j\in I}$ with values in the sheaf $\mathcal{O}_X(N_G(H)/H)$ of holomorphic maps from X to $N_G(H)/H$ such that the transition functions $\{\phi_{ij}\}_{i,j\in I}$ and $\{\phi'_{ij}\}_{i,j\in I}$ of E and E', respectively, are related by gauge transformations induced by elements of the normalizer $N_G(H)$.*

Proof. Since $\tilde{E}$ and $\tilde{E}'$ are isomorphic as holomorphic principal G-bundles, there exists a holomorphic bundle isomorphism $\Phi : \tilde{E} \to \tilde{E}'$. The compactness of X ensures that any open cover admits a finite refinement, and since principal bundles are locally trivial, there exists a finite open cover $\{U_i\}_{i\in I}$ of X such that both $\tilde{E}$ and $\tilde{E}'$ admit local trivializations over each U_i.

Let $\{\psi_{ij}\}_{i,j\in I}$ and $\{\psi'_{ij}\}_{i,j\in I}$ denote the holomorphic transition functions of $\tilde{E}$ and $\tilde{E}'$, respectively, with respect to this cover. The isomorphism Φ can be represented locally by holomorphic maps $g_i : U_i \to G$ such that

$$\psi'_{ij} = g_i \psi_{ij} g_j^{-1}$$

on each intersection $U_i \cap U_j$. Since $\tilde{E}$ and $\tilde{E}'$ are extensions of the principal H-bundles E and E', respectively, the H-bundle transition functions ϕ_{ij} and ϕ'_{ij} can be obtained as the restrictions of ψ_{ij} and ψ'_{ij} to the H-bundle structures. More precisely, the transition functions satisfy $\phi_{ij} = \psi_{ij}|_H$ and $\phi'_{ij} = \psi'_{ij}|_H$ in an appropriate sense that respects the bundle extensions.

For the isomorphism Φ to be compatible with the H-bundle structures of E and E', the local representatives g_i must preserve the H-action on fibers. This compatibility condition requires that for each $i \in I$, the map g_i satisfies $g_i \cdot H \cdot g_i^{-1} \subseteq H$, which is equivalent to $g_i \in N_G(H)$, the normalizer of H in G. With $g_i \in N_G(H)$ for all i, the relationship between the H-bundle transition functions becomes

$$\phi'_{ij} = g_i \phi_{ij} g_j^{-1}$$

on each intersection $U_i \cap U_j$. Define $\alpha_{ij} = g_i g_j^{-1} \in N_G(H)$. Then the relation can be expressed as

$$\phi'_{ij} = \alpha_{ij} \phi_{ij} \alpha_{ij}^{-1},$$

which shows that the transition functions of E' are obtained from those of E by conjugation with elements of $N_G(H)$.

The collection $\{\alpha_{ij}\}_{i,j\in I}$ satisfies the Čech cocycle condition. On triple intersections $U_i \cap U_j \cap U_k$, the associativity of the group operation and the definition $\alpha_{ij} = g_i g_j^{-1}$ yield

$$\alpha_{ij}\alpha_{jk} = (g_i g_j^{-1})(g_j g_k^{-1}) = g_i g_k^{-1} = \alpha_{ik}.$$

Since the maps g_i take values in $N_G(H)$ and are holomorphic, the quotient maps α_{ij} define holomorphic functions from $U_i \cap U_j$ to $N_G(H)/H$, where the quotient is taken in the sense of complex analytic spaces. The coherence of the sheaf $\mathcal{O}_X(N_G(H)/H)$ follows from the fact that $N_G(H)/H$ has the structure of a complex analytic space when G is a complex Lie group, and the compactness of X ensures that the associated cohomology groups are finite-dimensional. Therefore, $\{\alpha_{ij}\}_{i,j\in I}$ defines a Čech 1-cocycle with values in $\mathcal{O}_X(N_G(H)/H)$, and the transition functions of the two H-bundles are related by the action of this cocycle through conjugation by elements of the normalizer. □

Remark 2. *The specific case where* $G = \mathrm{Sp}(4, \mathbb{C})$ *and* H *is the cyclic subgroup of order* 4 *generated by a matrix* h *fits naturally into this framework. The symplectic group* $\mathrm{Sp}(4, \mathbb{C})$ *is a complex Lie group, and any finite subgroup is automatically closed. The cyclic subgroup* $H = \langle h \rangle$ *embeds as a discrete closed subgroup of* $\mathrm{Sp}(4, \mathbb{C})$*. The normalizer* $N_G(H)$ *inherits a complex analytic structure from* $\mathrm{Sp}(4, \mathbb{C})$*, and since* H *is finite, the quotient* $N_G(H)/H$ *is a complex analytic space. When* X *is a compact Riemann surface, the existence of holomorphic* G*-extensions for principal* H*-bundles is governed by obstruction theory. The inclusion* $H \hookrightarrow G$ *induces a map of classifying spaces, and the obstruction to extending a principal* H*-bundle* E *to a principal* G*-bundle lies in* $H^2(X, \pi_1(G/H))$*.*

Since $\mathrm{Sp}(4, \mathbb{C})$ *is simply connected, the fibration* $H \to G \to G/H$ *yields the exact sequence*

$$\pi_1(G) \to \pi_1(G/H) \to H \to \pi_0(G) \to \pi_0(G/H).$$

With $\pi_1(G) = 0$ *and* $\pi_0(G) = 0$*, this gives* $\pi_1(G/H) \cong H$*. However, for finite groups H, the cohomology group* $H^2(X, H)$ *is finite, and when X is a compact Riemann surface of genus g, this group has order dividing* $|H|^{2g}$*. The vanishing of obstruction classes depends on the specific geometry of X and the particular element of* $H^1(X, H)$ *representing the original H-bundle.*

The holomorphic transition functions ϕ_{ij} *and* ϕ'_{ij} *taking values in the finite group H are indeed locally constant functions. However, when two H-bundles admit G-extensions that are isomorphic as G-bundles, the lemma shows that their relationship is encoded by a cocycle* $\{\alpha_{ij}\}$ *with values in* $N_G(H)/H$*. Since* $N_G(H)/H$ *is generally not discrete (even when H is finite), these maps* $\alpha_{ij} : U_i \cap U_j \to N_G(H)/H$ *can vary non-trivially and holomorphically over the intersections. This holomorphic variation captures the geometric difference between the two G-extensions while preserving the underlying discrete H-bundle structure.*

Lemma 3. *Let X be a compact Riemann surface of genus* $g \geq 2$*,* $G = \mathrm{Sp}(4, \mathbb{C})$ *and H be the order-*4 *cyclic subgroup of G generated by the matrix h defined in* (2.3)*. Then, the cohomology group* $H^1(X, N_G(H)/H)$ *vanishes through the exact sequence*

$$1 \to H \to N_G(H) \to N_G(H)/H \to 1.$$

Proof. The long exact sequence in cohomology induced by the short exact sequence of the statement gives

$$\cdots \to H^1(X, H) \to H^1(X, N_G(H)) \to H^1(X, N_G(H)/H) \to H^2(X, H) \to \cdots$$

By Lemma 1, $N_G(H)/H$ is isomorphic to

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathrm{GL}(2, \mathbb{C}) \right\} / \{\pm 1, \pm i\}.$$

Clearly, this quotient has dimension 8 as an algebraic group. For any principal $N_G(H)$-bundle E over X, the adjoint bundle $\mathrm{ad}(E)$ has degree 0. Then, by the Riemann-Roch formula,

$$h^1(X, \mathrm{ad}(E)) - h^0(X, \mathrm{ad}(E)) = \deg(\mathrm{ad}(E)) - \dim(N_G(H)/H)(g-1) = -8(g-1).$$

Since, under the hypotheses, $g \geq 2$, this implies $H^1(X, N_G(H)) = 0$. □

Remark 3. *The techniques developed in Lemma 3 could be extended to more general settings. For arbitrary connected reductive complex Lie groups G and finite subgroups* $H \subset G$*, a similar result could be established, provided that the normalizer* $N_G(H)$ *satisfies appropriate conditions that here are guaranteed by the specific form that* $N_G(H)$ *admits for* $G = \mathrm{Sp}(4, \mathbb{C})$ *provided by Lemma 1. Notice that Lemma 3 strongly uses the concrete form given in Lemma 1, and Lemma 3 is crucial in the proof of the following main theorem. Consequently, the specific choice of* $\mathrm{Sp}(4, \mathbb{C})$ *and the cyclic group of order* 4 *is necessary here. A proper generalization of these results would necessarily require a careful study of* $N_G(H)$ *in a more general context.*

Theorem 1. *(Injectivity of the forgetful map) Let X be a compact Riemann surface of genus* $g \geq 2$*,* $G = \mathrm{Sp}(4, \mathbb{C})$*, and H be the order-*4 *cyclic subgroup of G generated by the element h defined in* (2.3)*. Then, the forgetful map* $F : M(H) \to M(G)$ *defined in* (2.4) *between moduli spaces of principal bundles over X induced by the inclusion of groups* $H \hookrightarrow G$ *is injective.*

Proof. Let E and E' be principal H-bundles over X with isomorphic G-extensions. By Lemma 2 and Remark 2, their difference is measured by an element of the cohomology $H^1(X, N_G(H)/H)$. But this group vanishes, by Lemma 3, so $E \cong E'$ as principal H-bundles over X, proving that the map F is injective, as stated. □

Lemma 4. *Let X be a compact Riemann surface, H a finite group, and $M(H)$ the moduli space of principal H-bundles over X. Then, $M(H)$ is separated.*

Proof. Since H is finite, every principal H-bundle on X is given by a monodromy representation

$$\rho\colon \pi_1(X) \to H.$$

Because $\pi_1(X)$ is finitely generated (indeed, when $g \geq 2$, it is generated by $2g$ elements with one relation) and H is finite, the set $\mathrm{Hom}(\pi_1(X), H)$ is finite. The moduli space is defined as the quotient

$$M(H) \cong \mathrm{Hom}(\pi_1(X), H)/H,$$

where H acts by conjugation. Since the number of representations is finite and the conjugation action partitions this finite set into a finite number of orbits, $M(H)$ is a finite space. When endowed with the discrete topology, or the induced topology from an algebraic structure as a zero-dimensional scheme, any finite set is separated, since any two distinct points can be separated by disjoint openings. This completes the proof. □

Corollary 1. *Let X be a compact Riemann surface of genus $g \geq 2$, $G = \mathrm{Sp}(4, \mathbb{C})$, and let*

$$H = \langle h \rangle \subset G$$

be the cyclic subgroup of order 4 *generated by the element h defined in* (2.3)*. Denote by*

$$F : M(H) \longrightarrow M(G)$$

the forgetful map between the moduli spaces of principal H-bundles and G-bundles over X. Then, F is a closed embedding.

Proof. By Theorem 1, the map F is injective. Since, by Lemma 4, $M(H)$ is separated, and $M(G)$ is also separated, to check that F is a closed embedding, it suffices to prove that F is proper, so that its image is closed and F induces an isomorphism onto its image [23, Proposition 4.4.9(c)].

Let R be a discrete valuation ring with fraction field K, maximal ideal, $\mathfrak{m}$ and residue field $s = R/\mathfrak{m}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, S & \longrightarrow & M(H) \\ \big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathrm{Spec}\, R & \longrightarrow & M(G). \end{array}$$

This diagram corresponds to the following situation: there is a principal H-bundle E_S over $X \times \mathrm{Spec}\, K$ whose associated G-bundle extends to a G-bundle Q_R over $X \times \mathrm{Spec}\, R$. To verify the valuative criterion of properness for F, it must be proved that E_S extends to a principal H-bundle E_R over $X \times \mathrm{Spec}\, R$ and that such an extension is unique up to an isomorphism.

Since H is a finite group, any principal H-bundle over X is given by a representation $\rho : \pi_1(X) \to H$. Hence, the moduli space

$$M(H) \cong \mathrm{Hom}(\pi_1(X), H)/H$$

is a finite set and, endowed with its natural (discrete) topology, is separated by Lemma 4. Cover X by a sufficiently fine open cover $\{U_i\}$ such that E_S is trivial on each U_i. Then the bundle E_S is given by transition functions

$$h_{ij} : U_i \cap U_j \times \mathrm{Spec}\, K \longrightarrow H,$$

satisfying the cocycle condition

$$h_{ij}\, h_{jk} = h_{ik} \quad \text{on } U_i \cap U_j \cap U_k \times \mathrm{Spec}\, K.$$

An extension of E_S to an H-bundle E_R over $X \times \mathrm{Spec}\, R$ amounts to extending the h_{ij} to maps

$$\tilde{h}_{ij} : U_i \cap U_j \times \mathrm{Spec}\, R \longrightarrow H,$$

such that $\tilde{h}_{ij}|_{U_i \cap U_j \times \mathrm{Spec}\, K} = h_{ij}$ and the cocycle condition holds on triple overlaps.

In general, the obstruction to extending the transition functions lies in the cohomology group

$$H^2(X,\ E_S \times_H \mathfrak{h}),$$

where $\mathfrak{h}$ denotes the Lie algebra of H. Since H is finite, its Lie algebra is trivial, i.e., $\mathfrak{h} = \{0\}$, and hence

$$H^2(X,\ E_S \times_H \mathfrak{h}) = 0.$$

Thus, there is no obstruction to extending E_S to an H-bundle E_R over $X \times \mathrm{Spec}\, R$.

Next, we prove that such an extension is unique. Let E_R and E'_R be two extensions of E_S to $X \times \mathrm{Spec}\, R$. Their isomorphism classes form a torsor under the group

$$H^1(X,\ E_S \times_H \mathfrak{h}).$$

Again, since $\mathfrak{h} = \{0\}$, we have

$$H^1(X,\ E_S \times_H \mathfrak{h}) = 0,$$

which implies that the two extensions are isomorphic.

Therefore, the valuative criterion of properness is satisfied for the map F. Combined with the injectivity of F and the fact that $M(H)$ is separated, it follows that F is a closed embedding. □

3. Application to quaternionic structures on principal bundles

Throughout this section, X denotes a compact Riemann surface of genus $g \geq 2$. The study of quaternionic structures in geometry has attracted the attention of recent research [26, 27], including research on quaternionic structures on bundles [28, 29]. Following Atiyah's seminal work on K-theory [30], quaternionic structures were first studied systematically in the context of real vector bundles.

This section recalls the notion of quaternionic structure, and some properties about the principal bundles that admit quaternionic structures are proved. Particularly, a decomposition result and another one that establishes topological constraints, expressed through the Chern classes of the bundles, caused by the presence of a quaternionic structure, are provided. In the following, it is proved that all bundles that are in the image of the forgetful map F defined in (2.4) admit a quaternionic structure. From this result, it is also shown that these bundles falling in the image of F admit a reduction of their structure group to the subgroup $\mathrm{Sp}(2,\mathbb{H})$ of $\mathrm{Sp}(4,\mathbb{C})$, from which some applications are demonstrated. Particularly, that the image of the forgetful map falls into a single connected component of $M(\mathrm{Sp}(4,\mathbb{C}))$.

3.1. Quaternionic structures on $\mathrm{Sp}(4,\mathbb{C})$*-bundles*

Here, the notion of quaternionic structures is recalled and applied to rank-4 vector bundles over X. A symplectic form is naturally induced by the presence of a quaternionic structure on the vector bundle. Next, the restriction results concerning the decomposition and characteristic classes of principal $\mathrm{Sp}(4,\mathbb{C})$-bundles admitting a quaternionic structure are presented and proved.

Definition 1. *Let E be a holomorphic rank-4 vector bundle over X with structure group $\mathrm{Sp}(4,\mathbb{C})$. A quaternionic structure on E is defined through the associated principal $\mathrm{Sp}(4,\mathbb{C})$-bundle P_E. Specifically, a quaternionic structure on E corresponds to a reduction of the structure group of P_E to a subgroup that preserves a quaternionic structure on the fiber. More precisely, it is an antilinear bundle automorphism $J : E \to E$ satisfying $J^2 = -\mathrm{Id}_E$, where antilinearity is understood in the sense that the automorphism J is compatible with the action of the structure group through the involution τ defined by conjugation with the matrix*

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In terms of the principal bundle P_E, this means that for any section s of P_E and any element $g \in \mathrm{Sp}(4,\mathbb{C})$, the quaternionic structure satisfies the compatibility condition $J(s \cdot g) = J(s) \cdot \tau(g)$, where $\tau(g) = J_0 g J_0^{-1}$. The vector bundle E equipped with a quaternionic structure is called a quaternionic vector bundle.

Remark 4. *The condition $J^2 = -\mathrm{Id}_E$ distinguishes quaternionic structures from real structures, which requires $J^2 = \mathrm{Id}_E$. The quaternionic structure J induces a natural action of the quaternion algebra $\mathbb{H}$ on the vector bundle E as follows: The quaternionic units $1, \xi_i, \xi_j, \xi_k$ act on E through the maps Id_E, J, J_0, and $J \circ J_0$, respectively, where J_0 corresponds to the action of ξ_j as defined by the matrix representation in Eq (2.2). The element $\xi_k = \xi_i \xi_j$ then acts as the composition $J \circ J_0$, and the element -1 acts as $-\mathrm{Id}_E = J^2$. This construction makes E into an $\mathbb{H}$-module bundle, with the quaternionic multiplication rules naturally satisfied through the composition of these bundle automorphisms.*

Remark 5. *Notice that a quaternionic bundle E always admits a reduction of the structure group to the symplectic group, since the quaternionic structure induces a symplectic form on E. More precisely, if h is a Hermitian metric on E, then $\omega(v, w) = h(v, \xi_j \cdot w) + ih(v, \xi_k \cdot w)$ gives an explicit definition of the symplectic form, where ξ_j and ξ_k are generators of the quaternionic group. Then, E can be naturally understood as a symplectic bundle. However, the converse is not true. Indeed, not every symplectic manifold admits a quaternionic structure, since a quaternionic structure imposes more conditions than* *those imposed by the symplectic structure. To check this, take the complex torus $M = \mathbb{C}^n/\Lambda$, where Λ is a lattice and n is odd. The cotangent bundle T^*M admits a natural symplectic structure, defined by the Liouville canonical symplectic form $\omega = d\lambda$, where λ is the tautological 1-form (this is true for the cotangent bundle of every complex variety). However, since the real dimension of T^*M is $2n$ and n is odd, this dimension is not a multiple of 4. This implies that T^*M does not admit a quaternionic structure. Indeed, for T^*M to admit a quaternionic structure, it should admit three complex structures satisfying the quaternionic rule, which would imply that the real dimension of T^*M is a multiple of 4.*

Lemma 5. *Let E be a quaternionic bundle over X with quaternionic structure J. Then, E admits a decomposition of the form $E = V \oplus J(V)$, where V is a holomorphic subbundle of E.*

Proof. At any point $x \in X$, the fiber E_x becomes a quaternionic vector space through J_x. Choose a complex subspace $V_x \subset E_x$ such that $E_x = V_x \oplus J_x(V_x)$. It suffices to check that this decomposition varies holomorphically with x.

For that, take a trivializing open subset $U \subset X$ of E. On it, the quaternionic structure J is represented by a matrix-valued function $J(z)$ satisfying $J(z)^2 = -I$, where I is the identity matrix. By solving the local equation $J(z)v(z) = iv(z)$, holomorphic sections are obtained generating $V|_U$. This proves the holomorphic variation. □

The presence of a quaternionic structure on a complex vector bundle imposes severe topological restrictions, proved in the following result, that constrain the bundle's characteristic classes. These constraints arise from the interplay between the antilinear nature of the quaternionic structure and the underlying complex geometry of the base manifold. Understanding these restrictions is interesting for classifying quaternionic bundles and determining which topological data can be realized by bundles with quaternionic structure, providing essential obstructions in moduli problems involving quaternionic geometry.

Lemma 6. *Let E be a complex vector bundle over X admitting a quaternionic structure. Then,* $\operatorname{rk} E$ *is even,* $\deg E = 0$*, and the first Chern class $c_1(E)$ vanishes in $H^2(X, \mathbb{Z})$.*

Proof. Let J be a quaternionic structure that E admits. At any point $x \in X$, the fiber E_x of E over x is a complex vector space equipped with the antilinear map $J_x : E_x \to E_x$ satisfying $J_x^2 = -\operatorname{Id}_{E_x}$, which makes E_x into a quaternionic vector space. Since the quaternion algebra $\mathbb{H}$ has dimension 4 over $\mathbb{R}$, and hence dimension 2 over $\mathbb{C}$, the complex dimension of E_x must be even.

Moreover, J induces an antilinear isomorphism between E and its dual E^*. This implies that for any connection ∇ on E compatible with J (meaning that J is invariant under the covariant differentiation, that is, $\nabla J = 0$),

$$\deg(E) = -\frac{i}{2\pi}\int_X \operatorname{tr}(F_\nabla) = -\deg(E),$$

where F_∇ is the curvature of ∇. Therefore, $\deg(E) = 0$.

The vanishing of $c_1(E)$ follows from the degree computation above, as $c_1(E)$ is represented by $\frac{i}{2\pi}\operatorname{tr}(F_\nabla)$ for any connection ∇. □

3.2. The forgetful map and quaternionic structures

This subsection focuses on holomorphic principal bundles and their moduli spaces, following the framework established by Narasimhan-Seshadri [2] and Ramanathan [8] for the study of stability conditions on holomorphic vector bundles and principal bundles over compact Riemann surfaces. The principal bundles considered here are holomorphic objects, and the moduli spaces $M(H)$ and $M(G)$ parametrize isomorphism classes of polystable holomorphic principal bundles with the respective structure groups.

From the above background, the injectivity of the forgetful map given in Theorem 1 will lead to several interesting consequences on the geometry of quaternionic Higgs bundles over X. These will be established in the next results. Specifically, the structure group $G = \mathrm{Sp}(4, \mathbb{C})$ and a discrete subgroup H of it will be considered, and it will be proved that the principal G-bundles, which are the image of the forgetful map $F : M(H) \to M(G)$ defined in (2.4), admit a quaternionic structure, in the sense explained above.

The following main result provides a complete characterization of the geometric content carried by principal bundles in the image of the forgetful map, revealing that these bundles possess a rich additional structure beyond their symplectic nature. This result has significant implications for the study of moduli spaces of principal bundles, as it identifies a distinguished subfamily of symplectic bundles that can be parametrized through discrete group data. Furthermore, this characterization enables the application of techniques from quaternionic geometry to problems in the moduli theory of principal bundles. It is shown that the reduction of structure group from $\mathrm{Sp}(4, \mathbb{C})$ to H naturally induces the antilinear bundle automorphism required for a quaternionic structure, with the compatibility conditions arising from the embedding of the quaternion group into the symplectic group.

Theorem 2. *(Quaternionic structure theorem) Let $G = \mathrm{Sp}(4, \mathbb{C})$ and H be the cyclic subgroup of G generated by the element h defined in* (2.3). *Then, every principal G-bundle over X lying in the image of the forgetful map $F : M(H) \to M(G)$ defined in* (2.4) *between the moduli spaces of holomorphic principal bundles over X admits a quaternionic structure.*

Proof. Let E be a principal H-bundle lying in the image of the forgetful map F. A quaternionic structure J will be now constructed for E, viewed as a principal G-bundle over X.

For any point $x \in X$, there is a local trivialization for which the corresponding transition functions take values in H, since the G-Higgs bundle admits a reduction of structure group to H. In such a trivialization, define $J_x(v) = hv$, where h is defined in (2.3). If $\{\phi_{ij}\}_{ij}$ is a family of transition functions, then

$$J_x(\phi_{ij}v) = h(\phi_{ij}v) = \phi_{ij}h(v) = \phi_{ij}J_x(v)$$

for $v \in E$, where the second equality uses that h commutes with ϕ_{ij}, since $\phi_{ij} \in H$, as E is an H-bundle and H is an abelian group. This proves that J is well-defined, proving the result. □

3.3. Reduction to the real form $\mathrm{Sp}(2, \mathbb{H})$ *of* $\mathrm{Sp}(4, \mathbb{C})$

Consider the symplectic complex group $G = \mathrm{Sp}(4, \mathbb{C})$ and let H be the cyclic subgroup generated by the element h defined in (2.3). It will be now proved that every principal G-bundle over X lying in the image of the forgetful map $F : M(H) \to M(G)$ defined in (2.4) admits a reduction of structure group to the subgroup $\mathrm{Sp}(2, \mathbb{H})$ of G.

Here, it is important to note the transition from the holomorphic to the smooth category. While the principal $\mathrm{Sp}(4, \mathbb{C})$-bundles are initially considered as holomorphic objects, the reduction to $\mathrm{Sp}(2, \mathbb{H})$ necessarily involves the smooth category since $\mathrm{Sp}(2, \mathbb{H})$ is a real Lie group, not a complex Lie group. This reduction is understood in the sense that the holomorphic $\mathrm{Sp}(4, \mathbb{C})$-bundle admits a smooth reduction of structure group to the real form $\mathrm{Sp}(2,\mathbb{H}) \subset \mathrm{Sp}(4,\mathbb{C})$. Such reductions are canonical consequences of the quaternionic structure and can be constructed explicitly using the quaternionic automorphism J. The moduli space $M(\mathrm{Sp}(2,\mathbb{H}))$ mentioned in the subsequent remark should be understood as parametrizing smooth principal $\mathrm{Sp}(2,\mathbb{H})$-bundles over X, in contrast to the holomorphic moduli spaces $M(H)$ and $M(G)$ considered earlier.

Recall that the group $\mathrm{Sp}(2,\mathbb{C})$ can be understood as the subgroup of $\mathrm{Sp}(4,\mathbb{C})$ of matrices that preserve the diagonal Hermitian form with entries $(i, i, -i, -i)$ [31–33]. More precisely, consider the standard quaternionic form defined in $\mathbb{H}^2$ by

$$(v, w) = \bar{v}_1 w_1 + \bar{v}_2 w_2.$$

Through the natural embedding of $\mathbb{H}^2$ in $\mathbb{C}^4$, this form induces the Hermitian form with matrix

$$\begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. Therefore, a matrix in $\mathrm{Sp}(4,\mathbb{C})$ falls in the image of this embedding if and only if it preserves the above diagonal form $(i, i, -i, -i)$.

The next result identifies a canonical geometric feature of principal $\mathrm{Sp}(4,\mathbb{C})$-bundles arising from the reduction to a cyclic subgroup. Specifically, it is shown that any such bundle admits a smooth reduction of structure group to the real form $\mathrm{Sp}(2,\mathbb{H})$. This connects complex symplectic geometry with quaternionic geometry through a topological mechanism. The reduction arises directly from the internal symmetries of the bundle and reflects the quaternionic structure previously established.

Theorem 3. *(Structure group reduction) Let $G = \mathrm{Sp}(4,\mathbb{C})$, H be the cyclic subgroup of G generated by the element h defined in* (2.3)*, and E be a principal G-bundle over X lying in the image of the forgetful map $F : M(H) \to M(G)$ defined in* (2.4)*. Then, E admits a smooth reduction of structure group to the subgroup* $\mathrm{Sp}(2,\mathbb{H})$ *of G.*

Proof. First, the generator h of H given in (2.3) is fixed by the involution τ of Definition 1. Indeed, using the matrix representation of h and the action of τ, it follows that

$$\tau(h) = J_0 h J_0^{-1} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = h,$$

where J_0 is also introduced in Definition 1.

Consider the subgroup G' of G that preserves the action of the quaternionic structure J that E admits by Theorem 2. Then the presence of J implies that E must admit a reduction of the structure group to G'. By the definition of J given in the proof of Theorem 2 ($J(v) = v \cdot h$ for $v \in E$) and since h is fixed by the involution τ, a matrix $A \in G$ preserves J if and only if it commutes with h, which can be explicitly written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

This implies that A must preserve the $(+i, +i, -i, -i)$ eigenspace decomposition, which precisely characterizes $\mathrm{Sp}(2, \mathbb{H})$ as a subgroup of $\mathrm{Sp}(4, \mathbb{C})$. Therefore, E has naturally the structure of a principal $\mathrm{Sp}(2, \mathbb{H})$-bundle, which gives the desired reduction. □

Remark 6. *The reduction of the structure group given by Theorem 3 is canonical in the sense that it is constructed using only the quaternionic structure guaranteed by the quaternionic structure theorem (Theorem 2), so this gives a well-defined map of moduli spaces* $M(H) \to M(\mathrm{Sp}(2, \mathbb{H}))$*, where H is the cyclic group of order* 4 *generated by h, which is defined in* (2.3)*. Note that this map connects the holomorphic moduli space* $M(H)$ *to the smooth moduli space* $M(\mathrm{Sp}(2, \mathbb{H}))$*, reflecting the categorical transition from holomorphic to smooth principal bundles given by the reduction process.*

3.4. Applications of the structure group reduction to the geometry of the image of the forgetful map

In this section, some geometric applications of the structure group reduction theorem (Theorem 3) are provided. Throughout, let X be a compact Riemann surface of genus $g \geq 2$. In the first result, it will be proved that a principal $\mathrm{Sp}(4, \mathbb{C})$-bundle lying in the image of the forgetful map defined in (2.4) has an even second Chern class. From this topological constraint, it is proved that the image of the forgetful map falls in a unique connected component of the moduli space of principal $\mathrm{Sp}(4, \mathbb{C})$-bundles over X.

Proposition 1. *(Topological constraint) Let* $G = \mathrm{Sp}(4, \mathbb{C})$*, H be the cyclic subgroup of G generated by the element h defined in* (2.3)*,* $F : M(H) \to M(G)$ *be the forgetful map defined in* (2.4)*, and E be a principal* $\mathrm{Sp}(4, \mathbb{C})$*-bundle lying in the image of F. Then, the second Chern class* $c_2(E)$ *is even.*

Proof. By Theorem 3, E admits a reduction of the structure group $E_{\mathbb{H}}$ to the subgroup $\mathrm{Sp}(2, \mathbb{H})$ of G. Take the Lie algebra $\mathfrak{sp}(4, \mathbb{C})$. A general element $Z \in \mathfrak{sp}(4, \mathbb{C})$ satisfies

$$Z^t J + JZ = 0,$$

where ω is the matrix of the symplectic form defined in (2.1). Computing this explicitly,

$$\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0,$$

for

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where I_2 denotes the 2×2 identity matrix. This yields the conditions

$$\begin{aligned} B^t &= B, \\ C^t &= C, \\ D &= -A^t. \end{aligned}$$

Therefore, every element of $\mathfrak{sp}(4, \mathbb{C})$ can be written uniquely as

$$Z = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

where $B = B^t$ and $C = C^t$ are symmetric 2×2 matrices.

The action of the quaternionic structure J with which E is equipped by Theorem 2 on $\mathfrak{sp}(4, \mathbb{C})$ will be now explicitly computed. This action is given by conjugation with the element h defined in (2.3). Specifically, if $Z \in \mathfrak{sp}(4, \mathbb{C})$ is a general element as above, then

$$\begin{aligned} J(Z) &= h \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} h^{-1} \\ &= \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \end{aligned}$$

Writing $A = (a_{ij})_{ij}$, $B = (b_{ij})_{ij}$, and $C = (c_{ij})_{ij}$, and computing explicitly, it follows that

$$J(Z) = \begin{pmatrix} A & -B \\ -C & -A^t \end{pmatrix}.$$

Then, the action above has ± 1 as the unique two eigenvalues. The $+1$ eigenspace of the J-action (corresponding to the subalgebra $\mathfrak{sp}(2, \mathbb{H}) \otimes \mathbb{C}$) consists of matrices where $B = C = 0$.

The characteristic classes can be now computed. The quaternionic structure gives an isomorphism

$$\mathrm{Ad}(E) \cong \mathrm{Ad}(E_{\mathbb{H}}) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is the -1 eigenspace of the action of J described above. Computing the total Chern class,

$$\begin{aligned} c(\mathrm{Ad}(E)) &= c(\mathrm{Ad}(E_{\mathbb{H}}))c(\mathfrak{m}) \\ &= (1 + c_1(\mathrm{Ad}(E_{\mathbb{H}})) + c_2(\mathrm{Ad}(E_{\mathbb{H}})))(1 + c_1(\mathfrak{m}) + c_2(\mathfrak{m})). \end{aligned}$$

The presence of the quaternionic structure J on E implies that

$$c_1(\mathfrak{m}) = -c_1(\mathrm{Ad}(E_{\mathbb{H}})).$$

Therefore,

$$c_2(\mathrm{Ad}(E)) = c_2(\mathrm{Ad}(E_{\mathbb{H}})) + c_2(\mathfrak{m}) - c_1(\mathrm{Ad}(E_{\mathbb{H}}))c_1(\mathfrak{m}).$$

Now, the characteristic classes of the adjoint bundle $\mathrm{Ad}(E_{\mathbb{H}})$ can be explicitly computed using the fact that the principal bundle $E_{\mathbb{H}}$ is classified by a map

$$f : X \to \mathrm{BSp}(2, \mathbb{H}),$$

where $\mathrm{BSp}(2, \mathbb{H})$ denotes the classifying space for principal $\mathrm{Sp}(2, \mathbb{H})$-bundles over X,

$$\mathrm{BSp}(2, \mathbb{H}) = \mathrm{ESp}(2, \mathbb{H}) / \mathrm{Sp}(2, \mathbb{H}),$$

$\mathrm{ESp}(2, \mathbb{H})$ being the universal covering space of $\mathrm{Sp}(2, \mathbb{H})$, which is a contractible space with a free action of the group $\mathrm{Sp}(2, \mathbb{H})$ that serves as the total space of a principal fibration with fiber $\mathrm{Sp}(2, \mathbb{C})$ [34]. It is also used that the homotopy groups of $\mathrm{Sp}(2, \mathbb{H})$ in low degrees are

$$\begin{aligned} \pi_1(\mathrm{Sp}(2, \mathbb{H})) &= 0, \\ \pi_2(\mathrm{Sp}(2, \mathbb{H})) &\cong 2\mathbb{Z}, \\ \pi_3(\mathrm{Sp}(2, \mathbb{H})) &\cong \mathbb{Z}, \end{aligned}$$

since $\mathrm{Sp}(2, \mathbb{H})$ is isomorphic to the compact simple group $\mathrm{Sp}(2)$ of unitary 4×4 matrices over the quaternions that preserves certain Hermitian inner product [35, 36].

The Serre spectral sequence for the fibration,

$$\mathrm{Sp}(2, \mathbb{H}) \to \mathrm{ESp}(2, \mathbb{H}) \to \mathrm{BSp}(2, \mathbb{H}),$$

gives then that $H^4(\mathrm{BSp}(2, \mathbb{H}), \mathbb{Z}) \cong 2\mathbb{Z}$. Therefore,

$$c_2(\mathrm{Ad}(E_{\mathbb{H}})) \in 2\mathbb{Z}.$$

Similarly, the quaternionic structure J forces

$$c_2(\mathfrak{m}) = c_2(\mathrm{Ad}(E_{\mathbb{H}})).$$

Therefore,

$$c_2(\mathrm{Ad}(E)) = 2c_2(\mathrm{Ad}(E_{\mathbb{H}})) + c_1(\mathrm{Ad}(E_{\mathbb{H}}))^2 \in 2\mathbb{Z},$$

which completes the proof that $c_2(E)$ is even. □

As a consequence of Proposition 1, in the next result, it will be proved that the image of the forgetful map $F : M(H) \to M(G)$ lies in a single connected component of $M(G)$, when G is the symplectic group $\mathrm{Sp}(4, \mathbb{C})$ and H is the group generated by h.

Theorem 4. *(Moduli space component) Let $G = \mathrm{Sp}(4, \mathbb{C})$, H be the cyclic subgroup of G generated by the element h defined in* (2.3)*, and $F : M(H) \to M(G)$ be the forgetful map between the moduli spaces of principal bundles defined in* (2.4)*. Then, the image of F lies in a single connected component of the moduli space $M(G)$.*

Proof. Recall first that, for any principal G-bundle E, $c_1(E) = 0$, since $G = \mathrm{Sp}(4, \mathbb{C})$ is simply connected. More precisely, as in the proof of Proposition 1, the principal bundle E is classified by a map $f : X \to \mathrm{BSp}(4, \mathbb{C})$. Since $\pi_1(G) = 0$, it is satisfied that

$$H^1(\mathrm{BSp}(4, \mathbb{C}), \mathbb{Z}) = 0.$$

Therefore,

$$f^* : H^2(\mathrm{BSp}(4,\mathbb{C}),\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

must vanish, which implies $c_1(E) = 0$.

Now, for any principal G-bundle E in the image of the forgetful map F, Theorem 3 gives a reduction of the structure group $E_{\mathbb{H}}$ to $\mathrm{Sp}(2,\mathbb{H})$. Consider the associate vector bundle defined by

$$E' = E_{\mathbb{H}} \times_{\mathrm{Sp}(2,\mathbb{H})} \mathbb{C}^4,$$

whose Harder-Narasimhan filtration will be now explicitly computed. The quaternionic structure J that E admits by Theorem 2 gives a decomposition of the form

$$E' = E'_1 \oplus E'_2,$$

where E'_1 and E'_2 are rank-2 vector subbundles whose Chern classes are related in the following way with those of E':

$$\begin{aligned} c_1(E') &= c_1(E'_1) + c_1(E'_2), \\ c_2(E') &= c_2(E'_1) + c_2(E'_2) + c_1(E'_1)c_1(E'_2). \end{aligned}$$

The quaternionic structure induces an isomorphism $E'_1 \cong E'^*_2$, so that $c_1(E'_1) = -c_1(E'_2)$ and $c_2(E_1) = c_2(E'_2)$. Therefore,

$$\begin{aligned} c_1(E') &= 0, \\ c_2(E') &= 2c_2(E'_1) - c_1(E'_1)^2. \end{aligned}$$

Notice also that, by Proposition 1, $c_2(E')$ must be even.

Now, let E and V be two principal G-bundles over X falling in the image of the forgetful map F, whose associated vector bundles defined above are denoted by E' and S'. Then, it is satisfied that

$$\begin{aligned} c_1(E') &= c_1(S') = 0, \\ c_2(E') &\equiv c_2(S') \pmod 2. \end{aligned}$$

Since, by Proposition 1, all principal G-bundles in the image of the forgetful map F have an even second Chern class and the same first Chern class, they must lie in the same connected component, as these components are labeled by pairs of the form $(c_1, c_2 \pmod 2)$ [37]. □

Remark 7. *There is a stratification of $M(G)$ into locally closed subvarieties labeled by the second Chern class of the principal bundles used in the proof of Theorem 4, which measures the deviation from semistability. This stratification is described in terms of the Harder-Narasimhan-type filtrations. More precisely, given a strictly polystable principal* $\mathrm{Sp}(4,\mathbb{C})$*-bundle E over X, the associated rank-*4 *holomorphic vector bundle admits a Harder-Narasimhan filtration*

$$0 \subset E_1 \subset E_2 \subset E_3 = E. \tag{3.1}$$

Then, the invariants of the graded pieces E_i/E_{i-1} determine the stratum. The topological constraints provided by Proposition 1 condition the strata into which the bundles in the image of the forgetful map can fall. In addition to being a result of interest in itself, this is the key to understanding how the entire forgetful map image falls in a single connected component of $M(\mathrm{Sp}(4,\mathbb{C}))$, as stated in Theorem 4.

4. Applications to representations of $\pi_1(X)$

Let X be a compact Riemann surface of genus $g \geq 2$ and G be a complex reductive group with maximal compact subgroup K_G. The group K_G acts by conjugation of the space $\mathrm{Hom}(\pi_1(X), K_G)$ of isomorphism classes of representations of the fundamental group of X into K_G. The representation space $\mathcal{R}(\pi_1(X), K_G)$ is defined as the quotient of $\mathrm{Hom}(\pi_1(X), K_G)$ by the conjugation action of K_G. Thus defined, there is an isomorphism between the moduli space of principal G-bundles over X and the space $\mathcal{R}(\pi_1(X), K_G)$, called the Narasimhan-Seshadri-Ramanathan correspondence [2, 8].

The seminal work of Narasimhan and Seshadri [2] established that stable holomorphic vector bundles of degree 0 over X correspond bijectively to irreducible unitary representations of a certain central extension of the fundamental group of X. Later, Ramanathan [8] extended the correspondence to arbitrary reductive groups. This result was later reformulated by Atiyah and Bott [9] in terms of connections and gauge theory, providing a new geometric perspective on the correspondence, which was extended later by Donaldson [38] and Simpson [39].

This correspondence can be realized explicitly through flat connections. Given a representation $\rho : \pi_1(X) \to K_G$, a flat K_G-bundle with monodromy ρ is obtained. The complexification of this bundle yields a holomorphic G-bundle, which is polystable. Conversely, every polystable G-bundle admits a unique flat connection with K_G-monodromy, up to conjugation.

For the symplectic group $G = \mathrm{Sp}(4, \mathbb{C})$, the Narasimhan-Seshadri-Ramanathan correspondence takes the following specific form. The maximal compact subgroup K_G of G is obtained as

$$K_G = \mathrm{Sp}(4, \mathbb{C}) \cap \mathrm{U}(4), \tag{4.1}$$

which is isomorphic to the compact symplectic group $\mathrm{USp}(4)$ [40]. Moreover, it is known that $\dim_{\mathbb{R}} K_G = 10$, $\dim_{\mathbb{C}} G = 10$, and the Cartan decomposition of $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ with respect to this maximal compact subgroup is

$$\mathfrak{sp}(4, \mathbb{C}) = \mathfrak{k} \oplus i\mathfrak{k},$$

where $\mathfrak{k}$ is the Lie algebra of K_G. For a compact Riemann surface X of genus $g \geq 2$, the Narasimhan-Seshadri-Ramanathan correspondence then gives a homeomorphism

$$M(G) \cong \mathcal{R}(\pi_1(X), K_G),$$

the dimension of this moduli space being

$$\dim_{\mathbb{C}} M(G) = (2g - 2) \dim_{\mathbb{C}} G = 10(2g - 2),$$

by the Riemann-Roch theorem.

In this section, the previous results on the geometry of $M(G)$, where $G = \mathrm{Sp}(4, \mathbb{C})$, derived from the injectivity of the forgetful map (2.4) are used to provide some consequences on the space $\mathcal{R}(\pi_1(X), K_G)$. Specifically, an involution σ of $\mathcal{R}(\pi_1(X), K_G)$ is defined so that the image of the forgetful map is proved to be exactly the fixed point subset of σ.

Lemma 7. *Let X be a compact Riemann surface of genus $g \geq 2$, $G = \mathrm{Sp}(4, \mathbb{C})$, and K_G be a maximal compact subgroup of G. Then, the matrix J_0 of Definition 1 satisfies the following properties:*

(1) $J_0^2 = -I_4$,

(2) $J_0^{-1} = -J_0$,

(3) $J_0 G J_0^{-1} = G$ *and* $J_0 K_G J_0^{-1} = K_G$,

where I_4 *denotes the* 4×4 *identity matrix.*

Proof. The two first properties can be checked by direct computation. For the first,

$$J_0^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -I_4,$$

and similarly for the second property.

For the third, notice that J_0 preserves both the standard symplectic form ω defined in (2.1) (this can be checked by direct computation, as above, by taking the expression given in (2.1) for ω and checking that $J_0 \omega J_0^{-1} = \omega$) and the Hermitian form defining K_G. Indeed, K_G can be realized as the intersection $K_G = \mathrm{Sp}(4, \mathbb{C}) \cap \mathrm{U}(4)$ given in (4.1), and J_0 lies in U(4), thus conjugation by J_0 preserves K. □

Definition 2. *Let X be a compact Riemann surface of genus* $g \geq 2$, $G = \mathrm{Sp}(4, \mathbb{C})$, K_G *be a maximal compact subgroup of G, and* J_0 *be the matrix*

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

introduced in Definition 1. Then, the involution σ *of the representation space* $\mathcal{R}(\pi_1(X), K_G)$ *is defined by*

$$\sigma([\rho]) = [J_0 \rho J_0^{-1}],$$

where $[\rho]$ *denotes the conjugacy class of a representation* $\rho : \pi_1(X) \to K_G$.

Remark 8. *The action of* σ *is well-defined. Indeed, if* ρ_1 *and* ρ_2 *are conjugated, then* $J_0 \rho_1 J_0^{-1}$ *and* $J_0 \rho_2 J_0^{-1}$ *are also conjugated. This is a consequence of Lemma 7, since* $J_0 \in G$ *and conjugation by* J_0 *is a group automorphism, so, if* $\rho_2 = g \rho_1 g^{-1}$ *for some* $g \in K_G$, *then*

$$J_0 \rho_2 J_0^{-1} = J_0 g \rho_1 g^{-1} J_0^{-1} = (J_0 g J_0^{-1})(J_0 \rho_1 J_0^{-1})(J_0 g J_0^{-1})^{-1}.$$

As, by Lemma 7, $J_0 K_G J_0^{-1} = K_G$, *the conjugating element* $J_0 g J_0^{-1}$ *remains in* K_G, *ensuring well-definedness.*

Lemma 8. *Let X be a compact Riemann surface of genus* $g \geq 2$, $G = \mathrm{Sp}(4, \mathbb{C})$, K_G *be a maximal compact subgroup of G, and* $\rho : \pi_1(X) \to K_G$ *be any representation. If* $[\rho] \in \mathrm{Fix}(\sigma)$, *where* $[\rho]$ *denotes the conjugacy class of* ρ *by elements of G and* σ *is the involution of* $\mathcal{R}(\pi_1(X), K_G)$ *given in Definition 2, then the conjugating element* $g \in G$ *satisfying* $J_0 \rho J_0^{-1} = g \rho g^{-1}$ *can be chosen to lie in* K_G.

Proof. If $J_0 \rho J_0^{-1} = g \rho g^{-1}$ for some $g \in G$, then $g J_0$ defines a quaternionic structure on the flat K_G-bundle E_ρ induced by $[\rho]$ thorugh the Narasimhan-Seshadri-Ramanathan correspondence. By the compactness of K_G, there exists a K_G-invariant Hermitian metric on E_ρ, and the quaternionic structure can be made compatible with this metric through averaging, yielding a new conjugating element $g' \in K_G$. □

Finally, the image of the forgetful map F is interpreted in the context of representation varieties. Using the Narasimhan-Seshadri-Ramanathan correspondence [2, 8], the following result proves that the image corresponds precisely to the fixed-point locus of an involution σ on the representation variety of the surface group into a maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{C})$. This identification not only provides a topological characterization of the moduli image but also links it to symmetry phenomena in gauge theory.

Theorem 5. *(Fixed-point structure) Let X be a compact Riemann surface of genus $g \geq 2$, $G = \mathrm{Sp}(4,\mathbb{C})$, K_G be a maximal compact subgroup of G, H be the order-4 cyclic subgroup of G generated by the element h defined in (2.3), and $F : M(H) \to M(G)$ be the forgetful map defined in (2.4). Then, through the Narasimhan-Seshadri-Ramanathan correspondence, the image of F corresponds to the fixed-point subset $\mathrm{Fix}(\sigma)$ of $\mathcal{R}(\pi_1(X), K_G)$.*

Proof. Under the homeomorphism $M(G) \cong \mathcal{R}(\pi_1(X), K_G)$ given by the Narasimhan-Seshadri-Ramanathan correspondence, a polystable principal G-bundle E corresponds to a conjugacy class of some representation ρ of $\pi_1(X)$ into K_G. Let E_ρ be the associated flat K_G-bundle. It will be proved that $[\rho] \in \mathrm{Fix}(\sigma)$ if and only if E_ρ admits a quaternionic structure.

Suppose first that $[\rho] \in \mathrm{Fix}(\sigma)$. By Lemma 8, there exists $g \in K_G$ such that

$$J_0\rho(\gamma)J_0^{-1} = g\rho(\gamma)g^{-1}$$

for all $\gamma \in \pi_1(X)$. Define $J = gJ_0$. Then,

$$J^2 = gJ_0gJ_0 = g(J_0gJ_0^{-1})J_0^2 = g(g^{-1})(-I_4) = -I_4,$$

where I_4 is the 4×4 identity matrix, and the second equality follows from Lemma 7. Moreover, for any $\gamma \in \pi_1(X)$,

$$J\rho(\gamma)J^{-1} = gJ_0\rho(\gamma)J_0^{-1}g^{-1} = g(g^{-1}\rho(\gamma)g)g^{-1} = \rho(\gamma).$$

Since $g \in K_G$, the quaternionic structure J is compatible with the unitary structure of E_ρ.

Conversely, if E_ρ admits a quaternionic structure J compatible with its unitary structure, then $J = gJ_0$ for some $g \in K$, and the same computations show that $[\rho] \in \mathrm{Fix}(\sigma)$.

Now, by the quaternionic structure theorem (Theorem 2) and the Narasimhan-Seshadri-Ramanathan correspondence, every principal G-bundle in the image of F corresponds to a representation class in $\mathrm{Fix}(\sigma)$.

For the reverse inclusion, if $[\rho] \in \mathrm{Fix}(\sigma)$, then the corresponding flat K_G-bundle E_ρ admits a quaternionic structure, as above. The existence of this structure and the reduction to $\mathrm{Sp}(2,\mathbb{H})$ given by Theorem 3 implies that E_ρ lies in the image of F. □

5. Conclusions

Let X be a compact Riemann surface of genus $g \geq 2$, $G = \mathrm{Sp}(4,\mathbb{C})$, $Q_8 = \{\pm 1, \pm\xi_i, \pm\xi_j, \pm\xi_k\}$ be the quaternionic group, which is embedded in G through a given representation $Q_8 \to G$, and H be the order-4 cyclic subgroup of G generated by the representative of ξ_i in G. Let $M(G)$ and $M(H)$ be the respective moduli spaces of principal bundles, and $F : M(H) \to M(G)$ be the forgetful map induced by the inclusion $H \hookrightarrow G$. The main contribution of the article proves that F is injective and proper.

Therefore, it induces a proper embedding. This injectivity result implies that a principal G-bundle admits at most one reduction to an H-bundle, and allows to understand $M(H)$ as a subvariety of $M(G)$.

As an application of the injectivity result, it is proved that every G-bundle falling in the image of F admits a quaternionic structure and, from this, a reduction of the structure group of the bundle to the real form $\mathrm{Sp}(2, \mathbb{H})$ of G is provided. As a consequence of this reduction theorem, some topological constraints of the G-bundles lying in the image of F are checked, including that the second Chern class vanishes for them. The above reduction of the structure group, that every element of the image of F admits, also allows to prove that the entire image lies in a single connected component of $M(G)$.

In addition to the above, other applications of the main result of the paper are derived concerning the representation space $\mathcal{R}(\pi_1(X), K_G)$, which parametrizes conjugation classes of representations of $\pi_1(X)$ in the maximal compact subgroup K_G of G. In particular, an involution σ of $\mathcal{R}(\pi_1(X), K_G)$ is constructed, whose subvariety of fixed points is proved to coincide, through the Narasimhan-Seshadri-Ramanathan correspondence, with the image of the forgetful map F, contained in $M(G)$.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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