



*Research article***Combinatorial analysis of line graphs: domination, chromaticity, and Hamiltonianity****Yubin Zhong¹, Sakander Hayat^{2,*}, Suliman Khan³, Vito Napolitano³ and Mohammed J. F. Alenazi⁴**

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Abstract: Line graphs are a fundamental class of graphs extensively studied for their structural properties and applications in diverse fields such as network design, optimization, and algorithm development. Pan and lollipop graphs, with their distinctive hybrid structures, offer a fertile ground for exploring combinatorial properties in their line graphs. Motivated by the need to better understand domination, chromaticity, and Hamiltonian properties in line graphs, this study examined the line graphs of pan and lollipop graphs. These investigations were inspired by their potential applications in connectivity analysis and optimization in networks. We derived analytical formulas for the domination and chromatic numbers of these line graphs, established relationships between these parameters and their corresponding original graphs, and proved that the line graph of a pan graph is Hamiltonian while that of a lollipop graph is traceable. The methodology combines established theoretical results and inequalities, including domination bounds and chromaticity relations, with rigorous combinatorial analysis. Our results not only contribute to the theoretical understanding of line graphs but also have implications for practical problems in network optimization and graph algorithm design, opening avenues for further research into hybrid graph structures.

Keywords: line graph; pan graph; lollipop graph; domination number; Hamiltonianity; chromatic number; traceability

Mathematics Subject Classification: 05C09, 05C25, 05C50

1. Introduction

All the graphs in the paper are finite, simple, connected, and without loops.

Graphs and their derived structures play a pivotal role in various domains, ranging from computer science and network analysis to chemistry and social sciences. Among these, domination and chromaticity are fundamental concepts with broad applications, including morphological analysis, communication networks, social network theory, and facility placement. In 1958, it appeared as the coefficient of external stability [2], and later on it was named the domination number. The study of domination began in the late 1950s, with its formalization as a “domination number” emerging in the 1960s through foundational work by Ore [23] and others. Since then, significant results have been established, such as the domination properties of planar graphs by MacGillivray and Seyffarth [22], and later refined by Henning and Goddard [10]. Liu et al. [20] studied network coherence analysis of n -polygon networks. Moreover, Liu et al. [19] conducted a graphical analysis of hierarchical networks. For more on chromaticity of graphs, we refer the reader to [9, 24]. For more applications of graph-theoretic parameters in communication systems and network design, we refer the reader to [7, 11, 28, 33].

Line graphs, an essential graph family, have received considerable attention due to their applications in diverse fields such as optimization [26], coding theory, and Hamiltonian graph studies [14]. These graphs transform the edge set of an original graph into vertices, preserving the adjacency relationships, thereby creating a powerful framework for analyzing structural and combinatorial properties. Harary and Nash-Williams [15] provided foundational insights into Eulerian and Hamiltonian line graphs, while Clark [6] and Brualdi and Shanny [4] contributed further to understanding their Hamiltonian properties. Recent studies, such as those by Zhong et al. [32], highlighted the significance of line graphs in the context of Hamiltonian connectivity and related metrics. Liu et al. [21] studied Zagreb indices of Eulerian graphs. For the extensive study of the traceable and Hamiltonian graphs, we refer the reader to, for instance, [18].

This paper focuses on two specific graph families—the pan graph and the lollipop graph—and explores their line graphs with respect to domination number, chromatic number, and Hamiltonian properties. Pan and lollipop graphs, characterized by their hybrid structures, have been extensively used in modeling problems where connectivity and domination play critical roles. Notably, their line graphs exhibit unique combinatorial features that merit detailed exploration. Siddiqui and James [27] provided initial insights into the chromatic and domination numbers of pan and lollipop graphs, which serve as a basis for extending these results to their line graphs.

Our contributions are threefold. First, we derive analytical formulas for the domination and chromatic numbers of the line graphs of pan and lollipop graphs, using established inequalities and structural properties [5, 25]. Second, we examine the relationships between the chromatic and domination numbers of these graphs and their line graphs, offering insights into how these parameters influence each other. Finally, we analyze Hamiltonian and traceable properties, demonstrating that the line graph of a pan graph is Hamiltonian, while that of a lollipop graph is traceable, extending prior work on Hamiltonian structures [29]. These findings not only deepen the understanding of these graph families but also provide potential applications in network design, algorithm optimization, and combinatorial modeling.

In this paper, we obtain the analytical formulas for the domination and chromatic numbers of these

graphs by using a well-known inequality $\gamma(G - e) - 1 \leq \gamma(G) \leq \gamma(G - e)$, with the help of some known results. Then, we discuss the relation between the chromatic and domination number of pan graphs with the line graphs of pan graphs. Finally, we elaborate the Hamiltonianity for these line graphs by showing that the line graph of a pan graph is Hamiltonian, while the line graph of a lollipop graph is traceable.

2. Preliminaries

Let us start by recalling some useful definitions and results (we refer the reader to [30] for other notions not recalled). Let $G = (V, E)$ be a graph. A set D of vertices of a simple graph G is a dominating set if every vertex $u \in V(G) \setminus D$ is adjacent to a vertex $v \in D$. The dominating number (domination number) $\gamma(G)$ of a graph G is the size of the smallest dominating set of G . A dominating set D with $|D| = \gamma(G)$ is called a minimum dominating set, see Haynes and Henning [12]. A γ -set in G is a dominating set of cardinality γ in G .

The line graph $L'(G)$ [13] of a graph G is the graph whose vertex set is $E(G)$ and two vertices $u, v \in V(L'(G))$ are adjacent if they are adjacent edges in G . A coloring or proper coloring for a graph G is the process of assigning colors to the vertex set V of G in such a way that any two adjacent vertices of G do not have the same color. The chromatic number $\chi(G)$ for a graph G is the smallest or minimum number of colors necessary for a graph coloring. Recall that P_n (resp. C_n) denotes a path (resp. cycle) of order n .

Since we are mainly interested in the domination number, we recall some known results on it.

Lemma 2.1. [12] *If a graph G of order n has a vertex of degree $n - 1$, then $\gamma(G) = 1$.*

We refer to the following result for the domination number of the path and cycle graphs.

Proposition 2.1. [5, 8] *For $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$ (that is, $\gamma = \frac{n}{3}$ if $n \equiv 0 \pmod{3}$ and $\gamma = \lceil \frac{n}{3} \rceil$ if $n \geq 6$).*

Next, we give the following lower bound on the domination number of a graph in terms of its diameter.

Proposition 2.2. [17] *For a connected graph G with diameter $\text{diam}(G)$, we have*

$$\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1).$$

The next classical result bounds the domination number of a graph in terms of its order, size, and the maximum degree.

Theorem 2.1. [5] *If $\Delta(G)$ is the maximum vertex-degree of a graph G of order p and size q , then*

$$\left\lceil \frac{p}{1 + \Delta(G)} \right\rceil \leq \gamma(G) \leq p - \Delta(G).$$

The next result delivers relations between the domination number of a graph in terms of its smallest degree $\delta(G)$.

Theorem 2.2. [5] (1) If G is a graph of order n and minimum degree δ , then $\gamma(G) \leq \frac{n}{\delta+1} \sum_{j=1}^{\delta+1} \frac{1}{j}$.

(2) If G is a graph of order n with $\delta(G) = 1$, then $\gamma(G) \leq n/2$.

(3) If G is a connected graph of order n with $\delta(G) = 2$ and $n \geq 8$, then $\gamma(G) \leq 2n/5$.

For more informations, see [3].

In [16], the following result is proved. Note that, by $G \cup G'$, we denote the disjoint union of two graphs, and the phrase *below dominating set* does not mean a minimal dominating set.

Proposition 2.3. If uv is an edge of a connected graph G and $G - uv = G_1 \cup G_2$, then $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$.

Proof. Let uv be an edge of a connected graph such that $G - uv = G_1 \cup G_2$. Let S_i , $i = 1, 2$, be a dominating set of G_i . Let $S = S_1 \cup S_2$, and then $N[S] = N[S_1 \cup S_2] = N[S_1] \cup N[S_2] = V(G)$, and so S is a dominating set for G , hence, $\gamma(G) \leq |S| = |S_1| + |S_2| = \gamma(G_1) + \gamma(G_2)$. \square

In [31], the following result is proved for an edge-deleted graph.

Theorem 2.3. [31] If $e \in E(G)$, then $\gamma(G - e) - 1 \leq \gamma(G) \leq \gamma(G - e)$.

A cycle which covers all the vertices of a graph G is known as a Hamiltonian cycle. Similarly, a path covering all the vertices of a graph G is a Hamiltonian path. A graph G is Hamiltonian, if there exists a Hamiltonian cycle in G . A graph G is traceable if there exists a Hamiltonian path in it. Obviously, if a graph is Hamiltonian then it is traceable. The converse is not true, in general.

Finally, let us discuss the domination number of the line graph of a complete graph on k vertices. Recall that the line graph G_k of a complete graph, also known as the triangular graph $T(k)$ (or T_k), is the graph whose vertices are the pairs of the 2-subsets of $[k] = \{1, 2, \dots, k\}$ and let us denote these $\binom{k}{2}$ vertices with $v_1, v_2, \dots, v_{\binom{k}{2}}$. Two vertices v_i, v_j of G_k are adjacent if and only if $v_i \cap v_j \neq \emptyset$. These graphs have received widespread attention. More recently, the codes generated by the rows of the adjacency matrix of these graphs have also received attention. It is known that

$$\gamma(G_k) = \left\lceil \frac{k-1}{2} \right\rceil.$$

If k is even, a γ -set is given by $\{1, 2\}, \{3, 4\}, \dots, \{k-1, k\}$ and $\gamma(G_k) = \left\lceil \frac{k-1}{2} \right\rceil$. If k is odd, a γ -set of G_k is $\{1, 2\}, \{3, 4\}, \dots, \{k-2, k-1\}$ and $\gamma(G_k) = \frac{k-1}{2}$.

Let G be a graph of order n , $m \geq 0$ be an integer, and $P_{m+1} = y_0 y_1 \dots y_m$ be a path of length m . Pick a vertex v_0 in G , and the graph $G_{v_0}(m)$ (or simply $G(m)$) is the graph obtained from G by identifying the vertex v_0 with the end vertex y_0 of P_{m+1} , that is, its vertex set consists of all the vertices of G with those of P_{m+1} with $v_0 = y_0$, so its order is $n + m$. Its edge set consists of all the edges of G with those of P_{m+1} .

Thus, if $G = P_2$, then $G(m) = P_{m+2}$. When $G = C_v$, we have the graph $C_v(\mu)$, and in [1] the following result is proved.

Proposition 2.4. For every $v \geq 3$ and $\mu \geq 0$, $\gamma(C_v(\mu)) = \lceil \frac{v}{3} \rceil + \lceil \frac{\mu-1}{3} \rceil$.

Definition 2.1. If $m = 1$, the graph $C_v(1)$ is the pan graph Π_n . If $G = K_n$ is the complete graph on n vertices, then $G(m) = K_n(m)$ is the lollipop graph $\mathcal{L}'_{n,m}$.

Thus, a lollipop graph $\mathcal{L}'_{v,\mu}$ is a graph obtained by joining the complete graph (K_v) to the μ -order path graph (P_μ) with a bridge. A pan graph Π_μ is a graph obtained by joining a μ -cycle C_μ to K_1 (singleton graph) with a bridge.

By Proposition 2.4, it follows that the domination number of the pan graph Π_n is $\lceil \frac{n}{3} \rceil$.

Siddiqui and James, in [27], showed the following results for the pan and lollipop graphs.

Lemma 2.2. [27] Let $G = \Pi_\mu$, and the domination number of G is

$$\gamma(G) = \begin{cases} \frac{\mu}{3}, & \text{for } \mu = 3k, \\ \lceil \frac{\mu}{3} \rceil, & \text{otherwise.} \end{cases}$$

Lemma 2.3. [27] Let $G = \Pi_\mu$, and the chromatic number of G is

$$\chi(G) = \begin{cases} 3, & \text{if } \mu \text{ is odd,} \\ 2, & \text{if } \mu \text{ is even.} \end{cases}$$

Lemma 2.4. [27] Let $G = \mathcal{L}_{v,\mu}$. The domination number of G is $\gamma(G) = \lceil \frac{\mu+2}{3} \rceil$.

Lemma 2.5. [27] Let $G = \mathcal{L}_{v,\mu}$. The chromatic number of G is $\chi(G) = v$.

Here, we mention some more known results which play an important rule in our proofs. In 1979, Walikar and Acharya [31] showed that if $e \in E(G)$, then the following inequality holds for the domination number of G .

Theorem 2.4. [31] If $e \in E(G)$, then $\gamma(G - e) - 1 \leq \gamma(G) \leq \gamma(G - e)$.

In 2006, Chartrand computed important results in his book [5] for the domination number of some well-known graphs. For the μ -th order path graph, we have the following:

Lemma 2.6. [5] $\gamma(P_\mu) = \lceil \frac{\mu}{3} \rceil$, for $\mu \geq 1$.

In [8], Chartrand and Zhang computed the domination number for C_μ , a cyclic graph of order μ , as given in the following lemma.

Lemma 2.7. [8] $\gamma(C_\mu) = \lceil \frac{\mu}{3} \rceil$, for $\mu \geq 3$.

Here, we extend the result of Chartrand and Zhang [8] for $C_{\mu+1}$, because of their implication in our next results.

Lemma 2.8. Let $C_{\mu+1}$ be the $(\mu + 1)$ -th order cyclic graph. Then, the domination number of $C_{\mu+1}$ is

$$\gamma(C_{\mu+1}) = \begin{cases} \lceil \frac{\mu+1}{3} \rceil, & \text{if } \mu = 3k, \text{ for all } k \geq 1, \\ \lceil \frac{\mu}{3} \rceil, & \text{if } \mu = 3k + 1, \text{ or } 3k + 2, \text{ for all } k \geq 1. \end{cases}$$

The proof of Lemma 2.8 is similar to the proof of Chartrand and Zhang [8].

Next, we present the main results of this work.

3. The domination number of the line graphs

In this paper, we study the domination number in the line graphs of lollipop and pan graphs. The next subsection investigates minimizing dominating sets in the line graphs of lollipop graphs.

3.1. Minimal dominating sets in the line graph of $C_v(\mu)$

Note that a cycle of length n is isomorphic to its line graph, and so $L'(C_n) \simeq C_n$. Let $G = \Pi_n = C_v(1)$. Let v_1, \dots, v_v, u be its vertices, and then $L'(G)$ is a graph of order $v + 1$ and it may be seen as a cycle C_v with an extra vertex, say, w plus two edges connecting w with two adjacent vertices of C_v . Next, we prove that this graph is a subgraph of the line graph of $C_v(\mu)$.

Let us denote the vertices of $C_v(\mu)$ with $v_1, \dots, v_v, u_{v+1}, \dots, u_{v+\mu}$. Denote the vertices of its line graph with p_{ij} , where i and j are the pairs of indices of the endpoints of the edges of $C_v(\mu)$:

$$p_{12}, p_{23}, \dots, p_{n-1,n}, p_{n1}, p_{n,n+1}, \dots, p_{n+m-1,n+m},$$

so it consists of the cycle

$$C_n : p_{12}p_{23} \dots p_{n-1,n}p_{n1},$$

the cycle

$$C_3 : p_{n-1,n}p_{n1}p_{n,n+1},$$

and the path

$$p_{n,n+1}p_{n+1,n+2} \dots p_{n+m-1,n+m}.$$

Consider the edge $e = p_{n1}p_{n,n+1}$ of $G = L'(C_v(\mu))$ and let $G' = L'(C_v(\mu)) - e$. Then,

$$\gamma(G') - 1 \leq \gamma(G) \leq \gamma(G'). \quad (3.1)$$

Since G' is isomorphic to $C_v(\mu)$, by Proposition 2.4 and Eq (3.1), we obtain

$$\left\lceil \frac{v}{3} \right\rceil + \left\lceil \frac{\mu - 1}{3} \right\rceil - 1 \leq \gamma(G) \leq \left\lceil \frac{v}{3} \right\rceil + \left\lceil \frac{\mu - 1}{3} \right\rceil. \quad (3.2)$$

Now, consider the edge $e' = p_{n-1,n}p_{n1}$ of G and let $G'' = G - e'$. Then,

$$\gamma(G'') - 1 \leq \gamma(G) \leq \gamma(G''). \quad (3.3)$$

Thus, G'' is isomorphic to $C_{v+1}(\mu - 1)$. Note that one has to remark that for $m = 1$, we have $G'' = C_{v+1}$. By Proposition 2.4 and Eq (3.3), considering $m \geq 2$ implies that

$$\left\lceil \frac{v+1}{3} \right\rceil + \left\lceil \frac{\mu-2}{3} \right\rceil - 1 \leq \gamma(G) \leq \left\lceil \frac{v+1}{3} \right\rceil + \left\lceil \frac{\mu-2}{3} \right\rceil. \quad (3.4)$$

Thus, if $\mu = 1$, we obtain the domination number for the line graph of the pan graph $L'(C_v(1))$ from Eq (3.4). Next, we conjecture that the domination of $L'(C_v(\mu))$ for general μ is as follows:

Conjecture 3.1. *The domination number of $G = L'(C_v(\mu))$ is computed by the following formula:*

$$\gamma(G) = \left\lceil \frac{v}{3} \right\rceil + \left\lceil \frac{\mu - 1}{3} \right\rceil.$$

3.2. Minimal dominating sets in the line graph of Π_n

In this subsection, we discuss the main results of our study. Before jumping to the main results, we start with some known results which perform a crucial rule in our proofs. Alikhani et al. [1] computed the domination number for $C_v(\mu)$, where $C_v(\mu)$ is shown in Figure 1. Moreover, Figures 2 and 3 deliver the pan graph Π_3 and lollipop graphs $\mathcal{L}'_{3,1}$ and $\mathcal{L}'_{4,1}$, respectively.

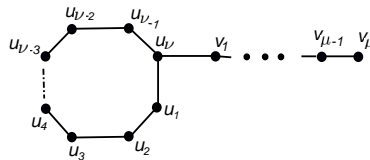


Figure 1. The graph $C_v(\mu)$.

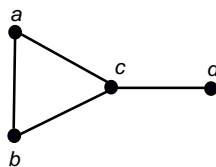


Figure 2. The pan graph Π_3 (for $v = 3$).

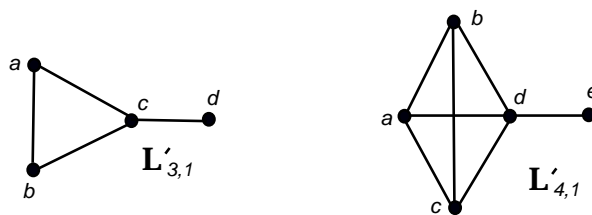


Figure 3. The lollipop graphs $\mathcal{L}'_{3,1}$ and $\mathcal{L}'_{4,1}$, respectively.

As a special case, we obtain the domination number of $C_3(\mu)$, see Figure 4.

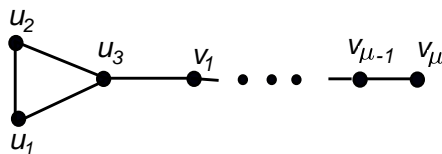


Figure 4. The graph $C_3(\mu)$.

Corollary 3.1. For $v = 3$ and $\mu \geq 0$, as shown in Figure 4, $\gamma(C_3(\mu)) = 1 + \lceil \frac{\mu-1}{3} \rceil$.

Proof. The proof of this corollary is immediately understood from Proposition 2.4 by considering $v = 3$. \square

Corollary 3.2. For $v = 4$ and $\mu \geq 0$, as shown in Figure 5, the domination number of $\gamma(C_4(\mu))$ is

$$\gamma(C_4(\mu)) = 2 + \left\lceil \frac{\mu - 1}{3} \right\rceil,$$

for all $k \geq 1$.

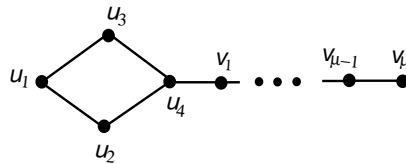


Figure 5. The graph $C_4(\mu)$.

Proof. The proof of this corollary is immediately from understood Proposition 2.4 by considering $v = 4$. \square

Theorem 3.1. For $\mu \geq 3$, let $G_\mu = L'(\Pi_\mu)$. Then,

$$\left\lceil \frac{\mu}{3} \right\rceil - 1 \leq \gamma(G_\mu) \leq \left\lceil \frac{\mu}{3} \right\rceil.$$

If $\mu = 3k$, then the domination number of G_μ is

$$\gamma(G_\mu) = \left\lceil \frac{\mu}{3} \right\rceil.$$

Proof. Consider the μ -th-order line graph of the pan graph $L'(\Pi_\mu)$, with vertices labeled as in Figure 6.

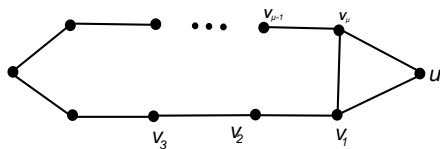


Figure 6. The line graph $G_\mu = L'(\Pi_\mu)$.

In any case, by Theorem 2.4, we have that $\gamma(G_\mu) \leq \left\lceil \frac{\mu}{3} \right\rceil$. Moreover, a minimal dominating set of the cycle C_μ with a vertex in v_1 or v_μ is a dominating set in the pan graph and so we have the result.

We prove this result by using the inequality $\gamma(G - e) - 1 \leq \gamma(G) \leq \gamma(G - e)$ given in Theorem 2.4. Let us apply Theorem 2.4 to $e = v_\mu u_1$ in G_μ , and we obtain the following inequality:

$$\begin{aligned} \gamma(\Pi_\mu) - 1 &\leq \gamma(G_\mu) \leq \gamma(\Pi_\mu) \\ \Rightarrow \left\lceil \frac{\mu}{3} \right\rceil - 1 &\leq \gamma(G_\mu) \leq \left\lceil \frac{\mu}{3} \right\rceil. \end{aligned} \quad (3.5)$$

Assume $\mu = 3k$ and let $\gamma(G_\mu) = \left\lceil \mu/3 \right\rceil - 1$. Thus, $\gamma(G_\mu) = k - 1$.

Now, consider $e = v_1 v_\mu$ in Figure 6, and by Theorem 2.4 we get

$$\gamma(C_{\mu+1}) - 1 \leq \gamma(G_\mu) \leq \gamma(C_{\mu+1}) \quad (3.6)$$

and so

$$\lceil \frac{\mu+1}{3} \rceil - 1 \leq \gamma(G_\mu) \leq \lceil \frac{\mu+1}{3} \rceil. \quad (3.7)$$

It follows that

$$k - 1 = \gamma(G_\mu) \geq \lceil \frac{\mu+1}{3} \rceil - 1 > k - 1,$$

which is a contradiction, and so for $\mu = 3k$ we have $\gamma(G_\mu) = \lceil \mu/3 \rceil$. \square

In the next result, we investigate the domination number of the join of two graphs H and K .

Theorem 3.2. *Let H and K be any two graphs with vertex sets $V(H)$ and $V(K)$, respectively. Let G' be a graph obtained by joining H and K through a bridge $e = uv$, where $u \in V(H)$ and $v \in V(K)$. Then $\gamma(G') = \gamma(H) + \gamma(K)$ or $\gamma(G') = \gamma(H) + \gamma(K) - 1$.*

Proof. Let H and K be any two graphs and G' be a graph obtained by connecting H and K through a bridge $e = uv$, where $u \in V(H)$ and $v \in V(K)$ as shown in Figure 7.

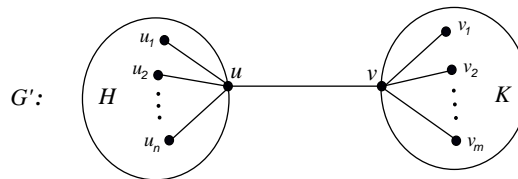


Figure 7. A graph G' connected by a bridge $e = uv$.

Therefore, the graph G' has $V(G') = V(H) \cup V(K)$ with $V(H) \cap V(K) = \emptyset$ and $E(G') = E(H) \cup E(K) \cup \{e\}$. Now, for any dominating set \mathcal{D} of the graph G' ,

$$\mathcal{D} = \mathcal{D} \cap V(G') = \mathcal{D} \cap (V(H) \cup V(K)) = (\mathcal{D} \cap V(H)) \cup (\mathcal{D} \cap V(K)), \quad (3.8)$$

and $(\mathcal{D} \cap V(H)) \cap (\mathcal{D} \cap V(K)) = \emptyset$.

Moreover,

$$|\mathcal{D}| = |\mathcal{D} \cap V(H)| + |\mathcal{D} \cap V(K)|. \quad (3.9)$$

If \mathcal{D}_1 is a minimum dominating set of H and \mathcal{D}_2 is a minimum dominating set of K , then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a dominating set of G' and so

$$\gamma(G') \leq |\mathcal{D}| = |\mathcal{D}_1| + |\mathcal{D}_2| = \gamma(H) + \gamma(K).$$

Let \mathcal{D} be a minimum dominating set of G' such that $\gamma(G') = \gamma(H) + \gamma(K) - q$, where $q \geq 2$. Then by Eq (3.9), $|\mathcal{D}| = |\mathcal{D} \cap V(H)| + |\mathcal{D} \cap V(K)| = \gamma(G') = \gamma(H) + \gamma(K) - q$, where $q \geq 2$. Therefore, $|\mathcal{D} \cap V(H)| = \gamma(H) - q_1$ and $|\mathcal{D} \cap V(K)| = \gamma(K) - q_2$ with $q_1 + q_2 = q$.

Suppose if neither $q_1 = 0$ nor $q_2 = 0$, then $\mathcal{D} \cap V(H)$ will not be the dominating set of H . But as \mathcal{D} is the dominating set of G' , some of the vertices of $\mathcal{D} \cap V(K)$ must dominate the vertices of H which are not dominated by the vertices of $\mathcal{D} \cap V(H)$. But as $e = uv$ is the only edge that connects H with K , we

must have that $v \in \mathfrak{D} \cap V(K)$ and $u \notin \mathfrak{D} \cap V(H)$. Similarly if we consider the set $\mathfrak{D} \cap V(K)$, then it is not the dominating set of K , and hence by above the argument we have $u \in \mathfrak{D} \cap V(H)$ and $v \notin \mathfrak{D} \cap V(K)$. This contradiction shows that both q_1 and q_2 cannot be non-zero simultaneously. Hence, either $q_1 = 0$ or $q_2 = 0$.

Without loss of generality, we take $q_1 = 0$, and then $|\mathfrak{D} \cap V(H)| = \gamma(H)$ and $|\mathfrak{D} \cap V(K)| = \gamma(K) - q$, where $q \geq 2$. This implies that $K - v$ has a dominating set $\mathfrak{D} \cap V(K)$ with cardinality less than or equal to $\gamma(K) - q$. Hence, $(\mathfrak{D} \cap V(K)) \cup \{v\}$ is a dominating set of K with cardinality less than or equal to $\gamma(K) - (q - 1)$, where $q > 1$. This contradicts the hypothesis of the dominating set. Hence, $q \leq 1$, thus, $\gamma(H) + \gamma(K) - 1 \leq \gamma(G') \leq \gamma(H) + \gamma(K)$. \square

Regarding Theorem 3.2, we refer to the work of Rajasekar and Nagarajan [25].

Corollary 3.3. For $\mu = 3k$, the domination number of Π_μ and $L'(\Pi_\mu)$ satisfy the following relation:

$$\gamma(\Pi_\mu) = \gamma(L'(\Pi_\mu)).$$

Proof. Comparing Lemma 2.2 and Theorem 3.1 completes the proof of this corollary. \square

Theorem 3.3. For $\nu = 3$, let $G_{3,\mu} = L'(\mathcal{L}'_{3,\mu})$, where $\mu \geq 1$. The domination number of $G_{3,\mu}$ is

$$\gamma(G_{3,\mu}) = 1 + \left\lceil \frac{\mu - 1}{3} \right\rceil, \quad \text{for all } \mu \geq 1.$$

Proof. For the fixed value of $\nu = 3$ and $\mu \geq 1$, the line graph of the lollipop graph $L'(\mathcal{L}'_{3,\mu})$ is given in Figure 8.

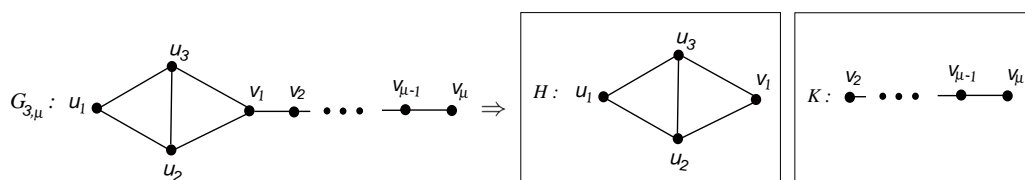


Figure 8. $G_{3,\mu} = L'(\mathcal{L}'_{3,\mu})$.

From Figure 8, it is clear that the line graph $G_{3,\mu} = L'(\mathcal{L}'_{3,\mu})$ is the combination of two graphs H and K , connected by a bridge $e = v_1v_2$. Now, to prove this result, we use Theorem 3.2, which reveals the domination number of two graphs connected by a bridge. Thus, from Theorem 3.2, we have

$$\gamma(G_{3,\mu}) = \gamma(H) + \gamma(K). \quad (3.10)$$

Here, the domination number of the connected graph H is 1, i.e., $\gamma(H) = 1$. The connected graph K is a $(\mu - 1)$ -th-order path graph $P_{\mu-1}$ with domination number $\gamma(P_{\mu-1}) = \lceil \frac{\mu-1}{3} \rceil$.

Equation (3.10) implies that

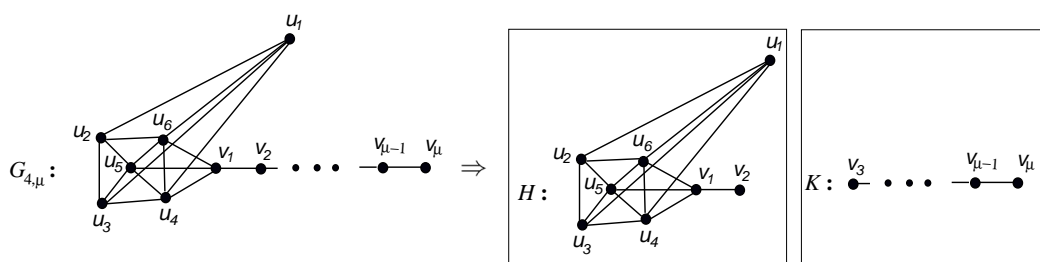
$$\gamma(G_{3,\mu}) = L'(\mathcal{L}'_{3,\mu}) = 1 + \left\lceil \frac{\mu - 1}{3} \right\rceil.$$

This completes the proof of this theorem. \square

Lemma 3.1. *For a given graph $G_{4,2} = L'(\mathcal{L}'_{4,2})$ as shown in Figure 9, the domination number is $G_{4,2} = 2$.*

Theorem 3.4. *For $v = 4$, let $G_{4,\mu} = L'(\mathcal{L}'_{4,\mu})$, where $\mu \geq 1$. The domination number of $G_{4,\mu}$ is*

Proof. For the fixed value of $\nu = 4$ and $\mu \geq 1$, the line graph of the lollipop graph $L'(\mathcal{L}'_{4,\mu})$ is given in Figure 10.



From Figure 10, it is clear that the line graph $G_{3,\mu} = L'(\mathcal{L}'_{3,\mu})$ is the combination of two graphs H and K , connected by a bridge $e = v_2v_3$. Now, to prove this result, we use Theorem 3.2, which reveals the domination number of two graphs connected by a bridge. Thus, from Theorem 3.2, we have

In Lemma 3.1, we showed that the domination number of a connected graph H is 2, i.e., $\gamma(H) = 2$. The connected graph K is a $(\mu - 2)$ -th-order path graph such that $P_{\mu-2}$ has the domination number $\gamma(P_{\mu-2}) = \lceil \frac{\mu-2}{3} \rceil$.

$$\gamma(G_{4,\mu}) = L'(\mathcal{L}'_{4,\mu}) = 2 + \left\lceil \frac{\mu - 2}{3} \right\rceil.$$
☐

Here, we have to present a short result on the domination number in terms of a lemma.

Lemma 3.2. For a given graph $G_{5,1} = L'(\mathcal{L}'_{5,1})$ as shown in Figure 11, the domination number is $\gamma(G_{5,1}) = 2$.

Proof. From Figure 11, the minimal possible dominating set is $\{u_1, u_{10}\}$ and hence $\gamma(G_{5,1}) = 2$.

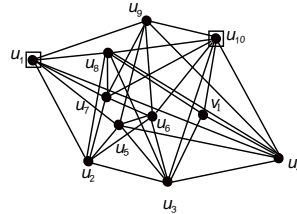


Figure 11. $G_{5,1} = L'(\mathcal{L}'_{5,1})$.

Theorem 3.5. For $\nu = 5$, let $G_{5,\mu} = L'(\mathcal{L}'_{5,\mu})$, where $\mu \geq 1$. The domination number of $G_{5,\mu}$ is

$$\gamma(G_{5,\mu}) = 2 + \left\lceil \frac{\mu - 1}{3} \right\rceil, \quad \text{for all } \mu \geq 1.$$

Proof. For the fixed value of $\nu = 5$ and $\mu \geq 1$, the line graph of the lollipop graph $L'(\mathcal{L}'_{5,\mu})$ is given in Figure 12.

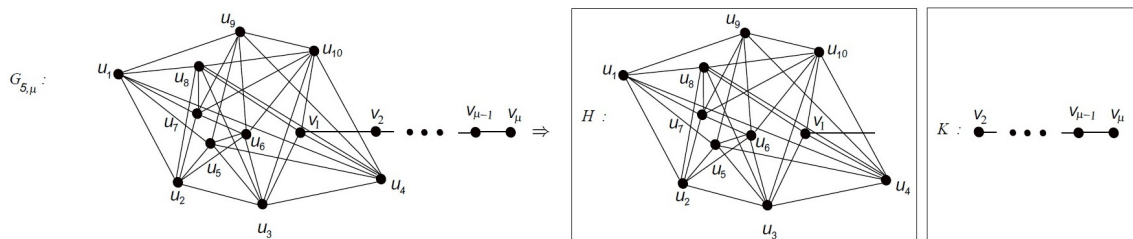


Figure 12. $G_{5,\mu} = L'(\mathcal{L}'_{5,\mu})$.

From Figure 12 it is clear that the line graph $G_{5,\mu} = L'(\mathcal{L}'_{5,\mu})$ is the combination of two graphs H and K , connected by a bridge $e = v_1v_2$. Now, to prove this result, we use Theorem 3.2, which reveals about the domination number of two graphs connected by a bridge. Thus, from Theorem 3.2, we have

$$\gamma(G_{5,\mu}) = \gamma(H) + \gamma(K). \quad (3.12)$$

In Lemma 3.2, we showed that the domination number of a connected graph H is 2, i.e., $\gamma(H) = 2$. The connected graph K is a $(\mu - 1)$ -th-order path graph such that $P_{\mu-1}$ has the domination number $\gamma(P_{\mu-1}) = \left\lceil \frac{\mu-1}{3} \right\rceil$.

Equation (3.12) implies that

$$\gamma(G_{5,\mu}) = L'(\mathcal{L}'_{5,\mu}) = 2 + \left\lceil \frac{\mu - 1}{3} \right\rceil.$$

This completes the proof of this theorem. \square

Tables 1–3 give us the direction for Conjecture 3.1, and hence the idea to generalize the results for all $\mu \geq 1$ and $\nu \geq 3$.

Table 1. The line graphs of the lollipop graph $L'(\mathcal{L}'_{\nu,\mu})$.

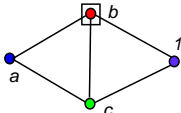
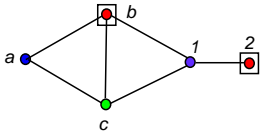
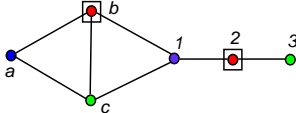
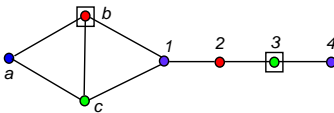
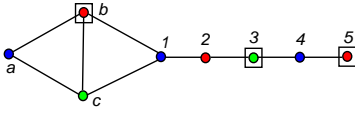
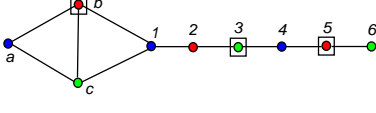
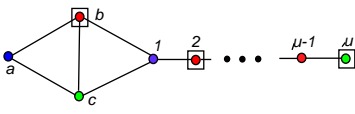
$L'(\mathcal{L}'_{\nu,\mu})$	Line graph	$\gamma(G)$	$\chi(G)$
$L'(\mathcal{L}'_{3,1})$		1	3
$L'(\mathcal{L}'_{3,2})$		2	3
$L'(\mathcal{L}'_{3,3})$		2	3
$L'(\mathcal{L}'_{3,4})$		2	3
$L'(\mathcal{L}'_{3,5})$		3	3
$L'(\mathcal{L}'_{3,6})$		3	3
\vdots	\vdots	\vdots	\vdots
$L'(\mathcal{L}'_{3,\mu})$		$1 + \lceil \frac{\mu-1}{3} \rceil$	ν

Table 2. The line graphs of the lollipop graph $L'(\mathcal{L}'_{v,\mu})$.

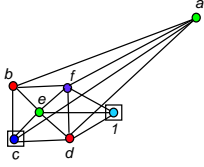
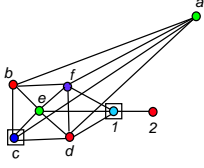
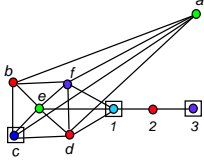
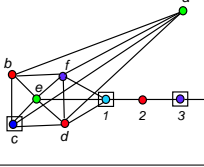
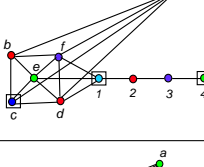
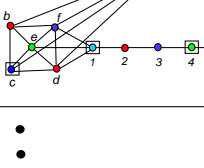
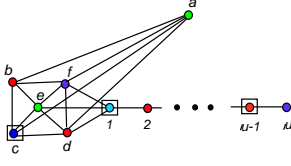
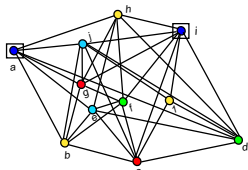
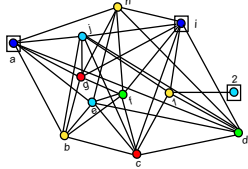
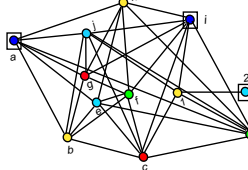
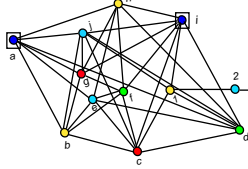
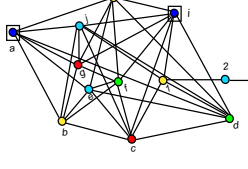
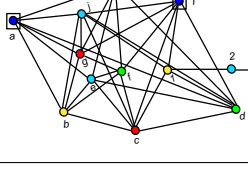
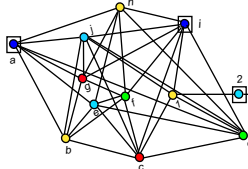
$L'(\mathcal{L}'_{v,\mu})$	Line graph	$\gamma(G)$	$\chi(G)$
$L'(\mathcal{L}'_{4,1})$		2	4
$L'(\mathcal{L}'_{4,2})$		2	4
$L'(\mathcal{L}'_{4,3})$		3	4
$L'(\mathcal{L}'_{4,4})$		3	4
$L'(\mathcal{L}'_{4,5})$		3	4
$L'(\mathcal{L}'_{4,6})$		4	4
\vdots	\vdots	\vdots	\vdots
$L'(\mathcal{L}'_{4,\mu})$		$2 + \lceil \frac{\mu-2}{3} \rceil$	ν

Table 3. The line graphs of the lollipop graph $L'(\mathcal{L}'_{v,\mu})$.

$L'(\mathcal{L}'_{v,\mu})$	Line graph	$\gamma(G)$	$\chi(G)$
$L'(\mathcal{L}'_{5,1})$		2	5
$L'(\mathcal{L}'_{5,2})$		3	5
$L'(\mathcal{L}'_{5,3})$		3	5
$L'(\mathcal{L}'_{5,4})$		3	5
$L'(\mathcal{L}'_{5,5})$		4	5
$L'(\mathcal{L}'_{5,6})$		4	5
\vdots	\vdots	\vdots	\vdots
$L'(\mathcal{L}'_{5,\mu})$		$2 + \lceil \frac{\mu-1}{3} \rceil$	ν

Next, we generalize these results from to $L'(\mathcal{L}'_{v,\mu})$ by getting idea from above theorems and those of Tables 1–3. We present the general form for the domination number of $L'(\mathcal{L}'_{v,\mu})$ in the following

conjecture.

Conjecture 3.2. For any $v \geq 3$, let $G_{v,\mu} = L'(\mathcal{L}'_{v,\mu})$, where $\mu \geq 1$. The domination number of $G_{v,\mu}$ is

$$\gamma(G_{v,\mu}) = \begin{cases} k + \lceil \frac{\mu-1}{3} \rceil, & v = \text{odd} (v = 2k + 1 \text{ for all } k \geq 1), \\ k + \lceil \frac{\mu-2}{3} \rceil, & v = \text{even} (v = 2k \text{ for all } k \geq 2). \end{cases}$$

4. Chromaticity

This section is dedicated for studying the chromaticity in the line graphs of pan and lollipop graphs.

From our observation and theory of chromatic numbers, the chromatic number for the union of two graphs G and G' is always the maximum one, that is, $\chi(G \cup G') = \max\{\chi(G), \chi(G')\}$.

Theorem 4.1. Let $G_\mu = L'(\Pi_\mu)$. The chromatic number of G_μ is $\chi(G_\mu) = 3$.

Proof. We know that, for μ -order cyclic graph, the chromatic number is given by

$$\chi(C_\mu) = \begin{cases} 3, & \text{if } n \text{ is odd,} \\ 2, & \text{otherwise.} \end{cases}$$

It is also clear that $G_\mu = C_\mu \cup C_3$, as shown in Figure 6. From the theory of chromatic numbers and our observation we have $\chi(G \cup G') = \max\{\chi(G), \chi(G')\}$. Clearly, $\max\{\chi(C_\mu), \chi(C_3)\} = \chi(C_3) = 3$, and hence $\chi(G_\mu) = 3$, which completes the proof. \square

Proposition 4.1. Let $G_{v,\mu} = L'(\mathcal{L}'_{v,\mu})$, where $3 \leq v \leq 5$, and $\mu \geq 1$. Then, the chromatic number of $G_{v,\mu}$ is

$$\chi(G_{v,\mu}) = v, \text{ where } 3 \leq v \leq 5.$$

Proof. To prove this result, we assign colors to the vertices of $G_{v,\mu}$ in such a way that if any two vertices are adjacent they will be assign different colors. The colors and the notations that we use in this theorem are red “r”, blue “b”, yellow “y”, sky blue “s”, green “g”, and so on. First, we prove some base results and then we generalize those results. Consider $V(G_{v,\mu})$ to be the set of vertices and $D(G_{v,\mu})$ to be the set of colors for $V(G_{v,\mu})$. Then we discuss the following cases.

Case 1. $v = 3$ and $\mu \geq 1$

$$V(G_{3,\mu}) = \{v_1, v_2, v_3\} \cup \{1, 2, \dots, \mu - 1, \mu\},$$

$$D(G_{3,\mu}) = \{r, b, g\} \cup \{r, b, \dots, r, b\},$$

$$\min(D(G_{3,\mu})) = \{r, b, g\} = 3$$

$$\Rightarrow \chi(G_{3,\mu}) = 3.$$

See Figure 13 for the line graph $L'(\mathcal{L}'_{3,\mu})$.

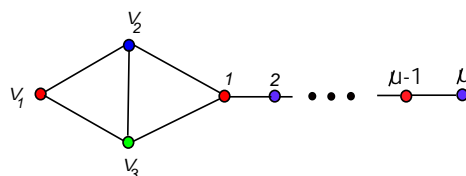


Figure 13. The line graph $L'(\mathcal{L}'_{3,\mu})$.

Case 2. $\nu = 4$ and $\mu \geq 1$

$$V(G_{4,\mu}) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{1, 2, 3, \dots, \mu - 1, \mu\},$$

$$D(G_{4,\mu}) = \{b, r, g, s, b, g\} \cup \{r, b, \dots, r, b\},$$

$$\min(D(G_{4,\mu})) = \{b, r, g, s\} = 4$$

$$\Rightarrow \chi(G_{4,\mu}) = 4.$$

See Figure 14 for the line graph $L'(\mathcal{L}'_{4,\mu})$.

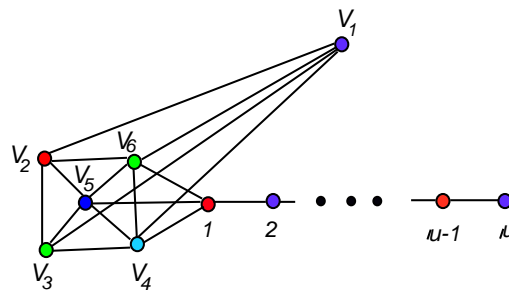


Figure 14. The line graph $L'(\mathcal{L}'_{4,\mu})$.

Case 3. $\nu = 5$ and $\mu \geq 1$

$$V(G_{5,\mu}) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\} \cup \{1, 2, \dots, \mu - 1, \mu\},$$

$$D(G_{5,\mu}) = \{b, r, g, s, y, b, r, g, s, y\} \cup \{r, b, \dots, r, b\},$$

$$\min(D(G_{5,\mu})) = \{b, r, g, s, y\} = 5$$

$$\Rightarrow \chi(G_{5,\mu}) = 5.$$

See Figure 15 for the line graph $L'(\mathcal{L}'_{5,\mu})$.

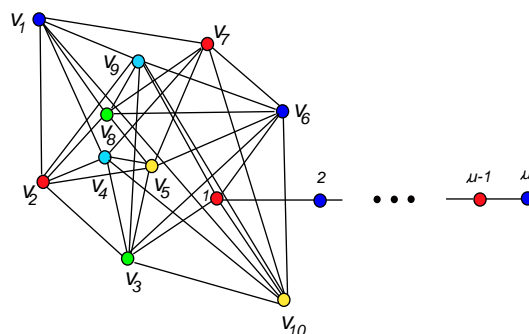


Figure 15. The line graph $L'(\mathcal{L}'_{5,\mu})$.

Since there are $(\frac{\nu(\nu-1)}{2} + \mu)$ vertices in the line graph of the lollipop graph, we think that such a result may be generalized, that is, the following conjecture is true.

Conjecture 4.1. Let $G_{v,\mu} = L'(\mathcal{L}'_{v,\mu})$ where $v \geq 3$ and $\mu \geq 1$. Then, the chromatic number of $G_{v,\mu}$ is

$$\chi(G_{v,\mu}) = v, \text{ where } v \geq 3.$$

Corollary 4.1. Let $G = \mathcal{L}'_{v,\mu}$, and $H' = L'(\mathcal{L}'_{v,\mu})$, with $3 \leq v \leq 5$. Then, for the chromatic number of G and H' , the following relation holds:

$$\chi(G) = \chi(H'). \quad (4.1)$$

Proof. Comparing Lemma 2.5 and Proposition 4.1 completes the proof of this corollary. \square

The following proposition delivers the chromatic number of the line graph of a pan graph.

Proposition 4.2. For $\mu \geq 3$, the chromatic numbers of Π_μ and $L'(\Pi_\mu)$ satisfy the following relation:

- (1) $\chi(\Pi_\mu) = \chi(L'(\Pi_\mu))$, if n is odd.
- (2) $\chi(\Pi_\mu) = 1 + \chi(L'(\Pi_\mu))$, if n is even.

Proof. From the Siddiqui and James results in Lemma 2.3, we say that

$$\chi(\Pi_\mu) = \begin{cases} 3, & \text{if } \mu \text{ is odd,} \\ 2, & \text{if } \mu \text{ is even.} \end{cases} \quad (4.2)$$

In Theorem 4.1, we showed that

$$\chi(L'(\Pi_\mu)) = 3, \text{ for all } n \geq 3. \quad (4.3)$$

Comparing Eqs (4.2) and (4.3) completes the proof of this corollary. \square

5. Hamiltoniancity and traceability

In this section, we discuss the Hamiltoniancity of line graphs of pan graphs and lollipop graphs. Here, we prove that $L'(\Pi_\mu)$ is Hamiltonian, while $L'(\mathcal{L}'_{v,\mu})$ is traceable. Now, we introduce some undefined notations and symbols, which we will use in this part. Let H_c (resp. H_p) be a Hamiltonian cycle (resp. path). The symbol \sim represents the adjacency relation of two vertices, while the symbol \circ represents the adjacency relation of a vertex to the set of vertices.

Theorem 5.1. Let $G = L'(\Pi_\mu)$, where $\mu \geq 3$. Then G is Hamiltonian.

Proof. To show that G is Hamiltonian, we construct a Hamiltonian cycle in G . Let y denote an arbitrary vertex of G . Consider the labeling of the vertices of $G = L'(\Pi_\mu)$ in Figure 16, and following is the Hamiltonian cycle with the same initial and end vertex y :

$$H_c(G) := y = v_1 v_2 v_3 \circ \{v_{j+1} : 3 \leq j \leq \mu\} \circ v_1 = y.$$

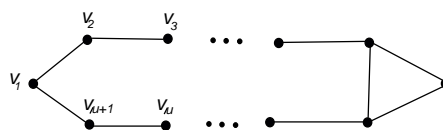


Figure 16. The line graph of the pan graph $L'(P_\mu)$.

Theorem 5.2. Let $G = L'(\mathcal{L}'_{v,\mu})$, where $v \geq 3$. Then, G is traceable.

Proof. To show that G is traceable, we have to construct a Hamiltonian path in $L'(\mathcal{L}'_{v,\mu})$. Let x_q and y_q denote the arbitrary vertices of $L'(\mathcal{L}'_{v,\mu})$. Consider the vertex labeling of $G = L'(\mathcal{L}'_{v,\mu})$ in Figure 17, and the following three cases embrace the Hamiltonian path between x_q and y_q .

Case 1. For $v = 3$ and $\mu \geq 1$,

$$P_h(x_q, y_q) : x_q = u_1 \sim u_2 \sim u_3 \circ \{v_r : 1 \leq r \leq \mu\} = y_q.$$

Case 2. For $v = 4$ and $\mu \geq 1$,

$$P_h(x_q, y_q) : x_q = u_1 \sim u_2 \sim u_3 \sim u_4 \sim u_5 \sim u_6 \circ \{v_r : 1 \leq r \leq \mu\} = y_q.$$

Case 3. For $v \geq 5$ and $\mu \geq 1$,

$$P_h(x_q, y_q) : x_q = u_1 \circ \{u_r : 2 \leq r \leq v\} \circ \left\{ u_t : v+1 \leq t \leq \frac{v(v-1)}{2} - v \right\} \circ \{v_j : 1 \leq j \leq \mu\} = y_q.$$

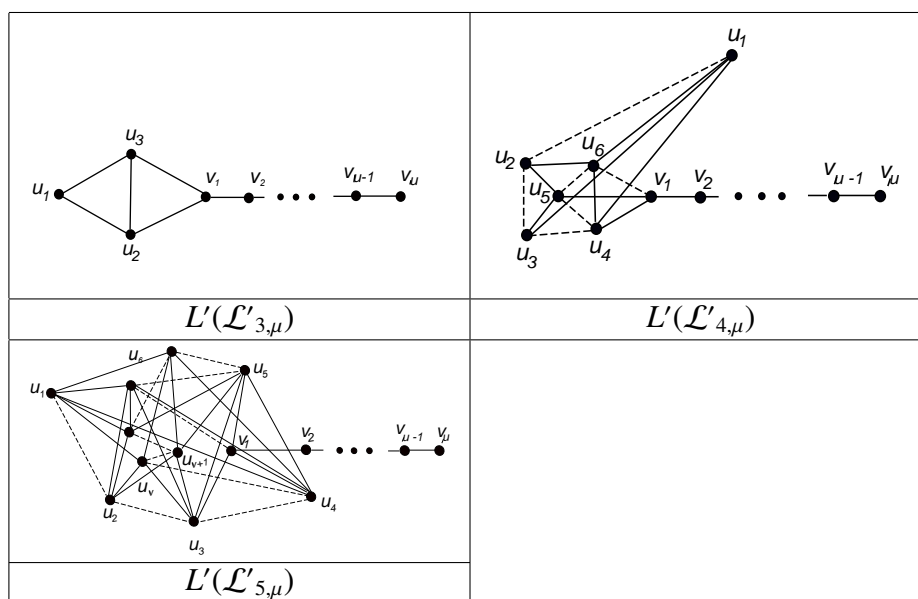


Figure 17. The line graphs of the lollipop graph $L'(\mathcal{L}'_{v,\mu})$.

6. Conclusions

In this study, we investigated the domination number, chromatic number, and Hamiltonian properties of the line graphs of pan and lollipop graphs. Analytical formulas for the domination and chromatic numbers were derived, and their interrelationships with the original graphs were established. We demonstrated that the line graph of a pan graph is Hamiltonian, while that of a lollipop graph is traceable.

These findings enrich the theoretical understanding of line graphs and highlight their structural uniqueness. The contributions are significant for practical applications in network design, optimization, and graph algorithm development, where properties like domination and Hamiltonian paths are critical.

However, the study is limited to specific families of graphs (pan and lollipop) and their line graphs, leaving other graph families unexplored. Future research could extend this work by examining similar properties in broader classes of graphs or exploring algorithmic applications based on these structural insights. Additionally, computational methods could complement theoretical approaches to validate and generalize the findings further.

Author contributions

Yubin Zhong and Sakander Hayat: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing—original draft preparation, Writing—review and editing, Visualization; Suliman Khan and Vito Napolitano: Formal analysis, Investigation, Writing—original draft preparation, Editing; Mohammed J. F. Alenazi: Methodology, Validation, Resources, Writing—original draft preparation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests.

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