



Research article

Algebraic invariants of edge ideals of some bristled circulant graphs

Ibad Ur Rehman¹, Mujahid Ullah Khan Afridi¹, Muhammad Ishaq^{1,*}, Asim Asiri² and Aftab Hussain^{2,*}

¹ Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, Sector H-12, Islamabad, Pakistan

² Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

* **Correspondence:** Email: aniassuirathka@kau.edu.sa; mishaq@sns.nust.edu.pk.

Abstract: Let S be a polynomial ring over a field K and I be the edge ideal associated with the bristled graph of some four or five regular circulant graph. We discuss the depth, projective dimension, regularity and Stanley depth of S/I .

Keywords: Stanley depth; depth; regularity; monomial ideal; projective dimension; bristled graph

Mathematics Subject Classification: 05C38, 05E99, 13C15, 13F20

1. Introduction

Let $S = K[e_1, \dots, e_l]$ be the polynomial ring over the field K , equipped with the standard grading. Let M be a finitely generated graded S -module and suppose that M has the following minimal free resolution:

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{r,j}(M)} \longrightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{r-1,j}(M)} \longrightarrow \dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

Let us denote the *Castelnuovo-Mumford regularity* of a module M as $\text{reg}(M)$. Then, $\text{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}$. The *projective dimension* of a module M , denoted as $\text{pdim}(M)$, is defined as $\text{pdim}(M) = \max\{i : \beta_{i,j}(M) \neq 0\}$.

Let $G = (\mathcal{V}(G), E(G))$, be a graph with vertex set $\mathcal{V}(G) = \{e_1, \dots, e_l\}$ and edge set $E(G)$. All graphs considered in this paper are simple, finite, and undirected. The *degree* of a vertex is the number of edges incident on it. The *edge ideal* of a graph G is the squarefree monomial ideal $I(G) = (e_i e_j : \{e_i, e_j\} \in E(G))$. If I is a monomial ideal and $\mathcal{G}(I)$ represents the minimal system of monomial generators of I , then $\text{supp}(I) := \{e_i : e_i | a \text{ for some } a \in \mathcal{G}(I)\}$. For $m \in \mathbb{R}$, $\lfloor m \rfloor := \max\{b \in \mathbb{Z} : b \leq m\}$ and $\lceil m \rceil := \min\{b \in \mathbb{Z} : b \geq m\}$. Let $l \geq 2$ be an integer and A be a subset of $\{1, \dots, \lfloor \frac{l}{2} \rfloor\}$. A *circulant graph* $C_l(A)$

is a graph with vertex set $\{e_1, \dots, e_l\}$ such that $\{e_i, e_j\} \in E(C_l(A))$ if and only if $|i - j| \in A$ or $l - |i - j| \in A$. It is worth noting that $C_l(A)$ can be considered as a generalized cycle since $C_l = C_l(\{1\})$. For simplicity, the graph $C_l(\{b_1, \dots, b_m\})$ is denoted as $C_l(b_1, \dots, b_m)$. A circulant graph $C_l(b_1, \dots, b_m)$ is $2m$ regular, except in the case when $2b_m = l$, in which case it is $(2m - 1)$ regular. Let $t \geq 1$, and G be a graph. Then, the graph G' is called the t -fold bristled graph of G if we add t pendants to each vertex of the graph G . For a graph G , the t -fold bristled graph of G is denoted by $Br_t(G)$. There are many applications of circulant graphs in group theory [1] and network theory [2, 3].

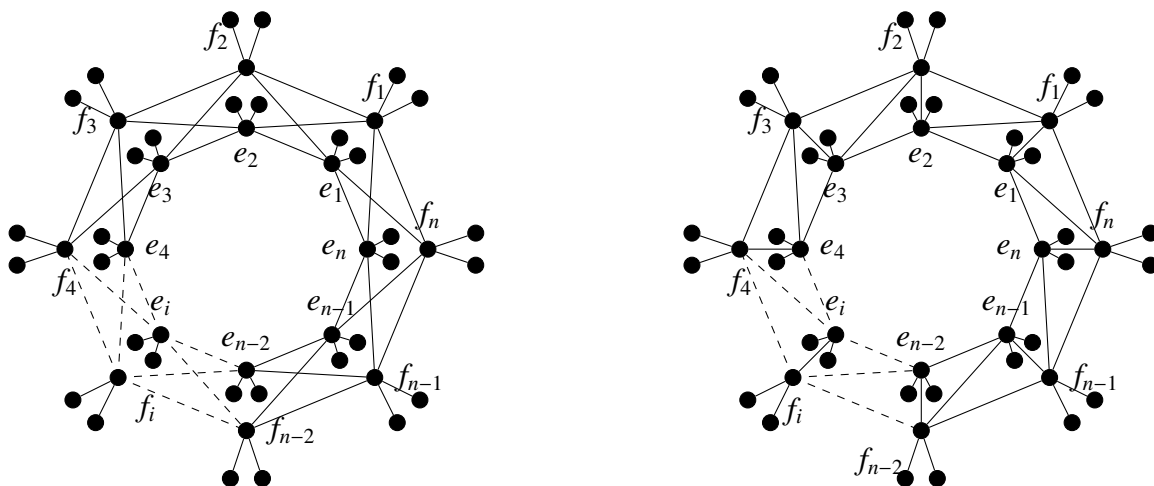


Figure 1. From left to right $\mathcal{D}_{n,2}$ and $\mathcal{E}_{n,2}$.

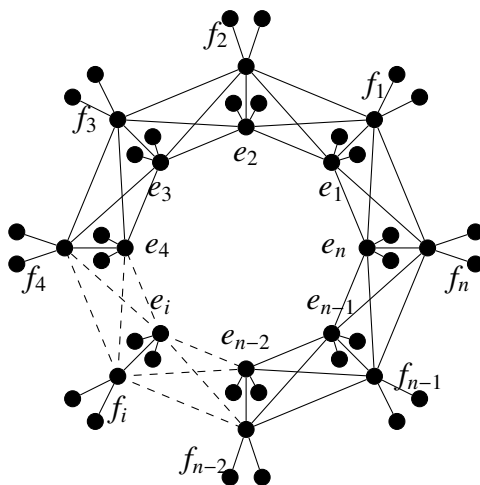


Figure 2. $\mathcal{F}_{n,2}$.

This paper draws inspiration from the recent work of Shaukat et al. [4], which specifically explores the algebraic invariants of the residue class rings of the edge ideals of some four and five regular circulant graphs, namely, $C_{2n}(1, n-1)$, $C_{2n}(1, 2)$, and $C_{2n}(1, n-1, n)$, where $n \geq 3$. For $n \geq 3$ and $t \geq 1$, if $\mathcal{D}_{n,t}$, $\mathcal{E}_{n,t}$, and $\mathcal{F}_{n,t}$ represent the bristled graphs of these graphs, respectively, then $\mathcal{D}_{n,t} := Br_t(C_{2n}(1, n-1))$, $\mathcal{E}_{n,t} := Br_t(C_{2n}(1, 2))$, and $\mathcal{F}_{n,t} := Br_t(C_{2n}(1, n-1, n))$. This study aims to investigate the regularity,

depth, and Stanley depth of the residue class rings of the edge ideals of $\mathcal{D}_{n,t}$, $\mathcal{E}_{n,t}$, and $\mathcal{F}_{n,t}$. These graphs are shown in Figures 1 and 2. Before proving our main results, we further require investigating the said algebraic invariants of the residue class rings of the edge ideals of various subgraphs of these graphs. These supporting results are obtained in Section 3. We prove our main results in Section 4. We prove the following main theorems for regularity.

Theorem 1.1. *Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{D}_{n,t})]$. Then,*

$$\text{reg}(S/I(\mathcal{D}_{n,t})) = \begin{cases} n, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.2. *Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{E}_{n,t})]$. Then,*

$$\text{reg}(S/I(\mathcal{E}_{n,t})) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil, & \text{if } n \equiv 0, 1 \pmod{3}; \\ 2\lceil \frac{n-1}{3} \rceil - 1, & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

Theorem 1.3. *Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{F}_{n,t})]$. Then,*

$$\text{reg}(S/I(\mathcal{F}_{n,t})) = \lceil \frac{n-1}{2} \rceil.$$

We prove the following main result for depth and Stanley depth.

Theorem 1.4. *Let $n \geq 3$, and $S = K[\mathcal{V}(\mathcal{D}_{n,t})]$. Then,*

$$\text{sdepth}(S/I(\mathcal{D}_{n,t})) \geq \text{depth}(S/I(\mathcal{D}_{n,t})) = \begin{cases} n(t+1), & \text{if } n \text{ is odd;} \\ n(t+1) + t - 1, & \text{if } n \text{ is even.} \end{cases}$$

We acknowledge the use of CoCoA [5] and Macaulay2 [6] during our experimental calculations. These software tools have limitations when generators of the edge ideal increases. In such scenarios, additional mathematical calculations and theoretical frameworks are necessary to compute the algebraic invariants that cannot be computed by these software packages.

2. Preliminaries

In this section, we present some definitions and findings that are extensively used in the subsequent sections of this paper. Let M be a finitely generated $\mathbb{Z}^l$ -graded S -module. A Stanley decomposition of M is a presentation of the K -vector space M as a finite direct sum:

$$\mathcal{T} : M = \bigoplus_{j=1}^z u_j K[W_j],$$

where $u_j \in M$ is a homogeneous element, $W_j \subseteq \{e_1, \dots, e_l\}$, and $u_j K[W_j]$ is a K -subspace of M generated by $u_j b$, where b is a monomial in $K[W_j]$. The $\mathbb{Z}^l$ -graded K -subspace $u_j K[W_j]$ of M is called a Stanley space of dimension $|W_j|$ if $u_j K[W_j]$ is a free $K[W_j]$ -module. The *Stanley depth* of $\mathcal{T}$ is defined as $\text{sdepth}(\mathcal{T}) = \min\{|W_j| : j = 1, \dots, z\}$, and the *Stanley depth* of M is defined as

$\text{sdepth}(M) = \max\{\text{sdepth}(\mathcal{T}) : \mathcal{T} \text{ is a Stanley decomposition of } M\}$. In 1982, Stanley conjectured in [7] that for a $\mathbb{Z}^l$ -graded module M , $\text{sdepth}(M) \geq \text{depth}(M)$. In 2016, Duval et al. disproved this conjecture in [8] by providing a counter example. Stanley depth has gained attention, particularly when Herzog et al. presented an algorithm in [9] for computing $\text{sdepth}(M)$ for modules of the form $M = Q_1/Q_2$, where $Q_2 \subset Q_1 \subset S$ are monomial ideals. This algorithm is useful for studying Stanley depth in certain special cases. However, calculating Stanley depth using this algorithm is generally a challenging combinatorial problem. Therefore, finding Stanley depth for particular classes of modules is useful, see for instance [10–12].

Lemma 2.1 ([13, Lemma 3.3]). *Let $I \subset S$ be a squarefree monomial ideal such that $\text{supp}(I) = \{e_1, e_2, \dots, e_l\}$. Suppose $\beta := e_{i_1}e_{i_2} \dots e_{i_r} \in S/I$ and $e_j\beta \in I$ for all $e_j \in \{e_1, e_2, \dots, e_l\} \setminus \text{supp}(\beta)$. Then, $\text{sdepth}(S/I) \leq r$.*

The following lemmas are important for finding lower bounds for depth and Stanley depth of modules.

Lemma 2.2 ([14]). *If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of modules over a Noetherian graded ring with S_0 local, or a local ring S , then*

- (a) $\text{depth}(M) \geq \min\{\text{depth}(M''), \text{depth}(M')\}$.
- (b) $\text{depth}(M') \geq \min\{\text{depth}(M), \text{depth}(M'') + 1\}$.
- (c) $\text{depth}(M'') \geq \min\{\text{depth}(M') - 1, \text{depth}(M)\}$.

Lemma 2.3 ([15, Lemma 2.2]). *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of a $\mathbb{Z}^l$ -graded S -module. Then, $\text{sdepth}(M) \geq \min\{\text{sdepth}(M'), \text{sdepth}(M'')\}$.*

Lemma 2.4 ([16, Theorem 4.7]). *If I is a monomial ideal and e is a variable in S , then*

- (a) $\text{reg}(S/I) = 1 + \text{reg}(S/(I : e))$, if $\text{reg}(S/(I, e)) < \text{reg}(S/(I : e))$.
- (b) $\text{reg}(S/I) = \text{reg}(S/(I, e))$ if $\text{reg}(S/(I : e)) < \text{reg}(S/(I, e))$.
- (c) $\text{reg}(S/I) \in \{\text{reg}(S/(I, e)), \text{reg}(S/(I, e)) + 1\}$, if $\text{reg}(S/(I : e)) = \text{reg}(S/(I, e))$.

Lemma 2.5 ([17, Proposition 2.2.20]). *Let I_1 and I_2 be monomial ideals, $I_1 \subset S' = K[e_1, \dots, e_p]$ and $I_2 \subset S'' = K[e_{p+1}, \dots, e_l]$, where $1 \leq p < l$, and $S = S' \otimes_K S''$. Then, $S/(I_1 + I_2) \cong S'/I_1 \otimes_K S''/I_2$.*

Lemma 2.6 ([17, Proposition 2.2.21]). *Let $I_1 \subset S' = K[e_1, \dots, e_p]$, $I_2 \subset S'' = K[e_{p+1}, \dots, e_l]$ be monomial ideals, where $1 \leq p < l$. If $S = S' \otimes_K S''$, then*

$$\text{depth}(S'/I_1 \otimes_K S''/I_2) = \text{depth}(S/(I_1S + I_2S)) = \text{depth}_{S'}(S'/I_1) + \text{depth}_{S''}(S''/I_2).$$

A vertex of degree one in a graph is termed as a *pendant vertex or leaf*. A vertex that is not a leaf is called an *internal vertex*. For $r \geq 2$, a *star graph* denoted as S_r is a graph with one internal vertex and r leaves connected to it. For $p \geq 2$, a path on p vertices is denoted as P_p , and for $p \geq 3$ a cycle on p vertices is denoted as C_p . Let $S_{p,t} := Br_t(P_p)$ and $C_{p,t} := Br_t(C_p)$.

Lemma 2.7 ([18]). *Let $S = K[\mathcal{V}(S_r)]$ and $I = I(S_r)$. Then,*

$$\text{depth}(S/I) = \text{reg}(S/I) = \text{sdepth}(S/I) = 1.$$

Lemma 2.8 ([19, Theorem 2.28]). Let $S = K[\mathcal{V}(S_{p,t})]$ and $I = I(S_{p,t})$. Then,

- (a) $\text{reg}(S/I) = \lceil \frac{p}{2} \rceil$.
- (b) $\text{depth}(S/I) = \text{sdepth}(S/I) = \lceil \frac{p}{2} \rceil + \lceil \frac{p-1}{2} \rceil t$.

Lemma 2.9 ([19, Theorem 2.30 and Theorem 2.9]). Let $S = K[\mathcal{V}(C_{p,t})]$ and $I = I(C_{p,t})$. Then,

- (a) $\text{reg}(S/I) = \lceil \frac{p-1}{2} \rceil$.
- (b) $\text{depth}(S/I) = \text{sdepth}(S/I) = \lceil \frac{p-1}{2} \rceil + \lceil \frac{p}{2} \rceil t$.

Lemma 2.10 ([15, Corollary 1.3] and [20, Proposition 2.7]). Let $I \subset S$ be a monomial ideal. Then, for all monomials $a \notin I$,

- (a) $\text{depth}(S/(I : a)) \geq \text{depth}(S/I)$.
- (b) $\text{sdepth}(S/I) \leq \text{sdepth}(S/(I : a))$.

Lemma 2.11 ([21, Lemma 2.13]). Let $I_1 \subset S' = K[e_1, \dots, e_p]$, $I_2 \subset S'' = K[e_{p+1}, \dots, e_l]$ be monomial ideals, where $1 \leq p < l$ and $S = S' \otimes_K S''$. Then,

$$\text{sdepth}(S'/I_1 \otimes_K S''/I_2) = \text{sdepth}(S/(I_1 S + I_2 S)) \geq \text{sdepth}_{S'}(S'/I_1) + \text{sdepth}_{S''}(S''/I_2).$$

Lemma 2.12 ([9, Lemma 3.6] and [22, Lemma 3.5]). Let $I \subset S$ be a monomial ideal. If $S' = S \otimes_K K[e_{l+1}] \cong S[e_{l+1}]$, then

- (a) $\text{depth}(S'/IS') = \text{depth}(S/I) + 1$.
- (b) $\text{sdepth}(S'/IS') = \text{sdepth}(S/I) + 1$.
- (c) $\text{reg}(S'/IS') = \text{reg}(S/I)$.

The following lemma is proved by Kalai et al. [23, Theorem 1.4] for squarefree monomial ideals. Herzog further extended this result to any monomial ideal [24, Corollary 3.2].

Lemma 2.13. For monomial ideals I_1 and I_2 of S , $\text{reg}(S/(I_1 + I_2)) \leq \text{reg}(S/I_1) + \text{reg}(S/I_2)$.

A graph G is a *weakly chordal graph* if neither G nor its complement graph G^c contain an induced cycle of length $n \geq 5$. A *matching*, denoted by $\mathcal{M}$, in a graph G is a subset of $E(G)$ where there is no common vertex between any pair of edges. A matching that forms an induced subgraph of G is referred to as a *induced matching* in G . Furthermore, $\text{indmat}(G)$ represents the induced matching number of graph G , which is defined as follows: $\text{indmat}(G) = \max\{|\mathcal{M}| : \mathcal{M} \text{ is an induced matching in } G\}$.

Lemma 2.14 ([25, Lemma 2.2] and [26, Corollary 6.9]). For a finite simple graph G , $\text{reg}(S/I(G)) \geq \text{indmat}(G)$. Furthermore, for a chordal graph G , $\text{indmat}(G) = \text{reg}(S/I(G))$.

Lemma 2.15 ([27, Theorem 14]). For a weakly chordal graph G , $\text{indmat}(G) = \text{reg}(S/I(G))$.

Lemma 2.16 ([28, Lemma 3.2]). Let $1 \leq p < l$. If $I_1 \subset S' = K[e_1, \dots, e_p]$ and $I_2 \subset S'' = K[e_{p+1}, \dots, e_l]$ are non-zero homogeneous ideals of S' and S'' and we regard $I_1 + I_2$ as a homogeneous ideal of $S = S' \otimes_K S''$, then

$$\text{reg}(S/(I_1 + I_2)) = \text{reg}(S'/I_1) + \text{reg}(S''/I_2).$$

For $n \geq 1$, let $\mathbb{A}_n$, $\mathbb{B}_n$, and $\mathbb{C}_n$ be subgraphs of $C_{2n}(1, n-1)$, $C_{2n}(1, 2)$, and $C_{2n}(1, n-1, n)$, respectively. Subgraphs $\mathbb{A}_n$, $\mathbb{B}_n$, and $\mathbb{C}_n$ are shown in Figures 3 and 4. Let us denote the t -fold bristled graphs of $\mathbb{A}_n$, $\mathbb{B}_n$, and $\mathbb{C}_n$ as follows: $\mathbb{A}_{n,t} := Br_t(\mathbb{A}_n)$, $\mathbb{B}_{n,t} := Br_t(\mathbb{B}_n)$, and $\mathbb{C}_{n,t} := Br_t(\mathbb{C}_n)$.



Figure 3. From left to right $\mathbb{A}_n$ and $\mathbb{B}_n$.

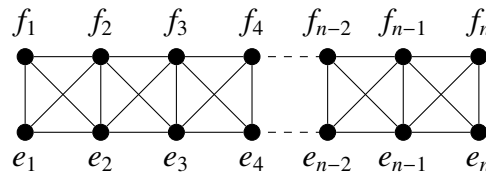


Figure 4. $\mathbb{C}_n$.

3. Algebraic invariants of cyclic modules associated to $\mathbb{A}_{n,t}$, $\mathbb{B}_{n,t}$, and $\mathbb{C}_{n,t}$

In this section, we compute the regularity of the cyclic modules $K[\mathcal{V}(\mathbb{A}_{n,t})]/I(\mathbb{A}_{n,t})$, $K[\mathcal{V}(\mathbb{B}_{n,t})]/I(\mathbb{B}_{n,t})$, and $K[\mathcal{V}(\mathbb{C}_{n,t})]/I(\mathbb{C}_{n,t})$. We also compute depth, Stanley depth, and projective dimension of $K[\mathcal{V}(\mathbb{A}_{n,t})]/I(\mathbb{A}_{n,t})$. These results are crucial for our main findings in the next section. Let $I \subset S$ be a squarefree monomial ideal that is minimally generated by monomials of degree at most 2. We define a graph G_I associated with the ideal I , where $\mathcal{V}(G_I) = \text{supp}(I)$ and $E(G_I) = \{e_i, e_j\} : e_i e_j \in \mathcal{G}(I)\}$. Examples of $G_{(I(\mathbb{A}_{6,2}), f_6)}$ and $G_{(I(\mathbb{A}_{6,2}), f_6, e_6)}$ are given in Figure 5. These graphs help us to understand the following isomorphisms:

$$K[\mathcal{V}(\mathbb{A}_{6,2})]/(I(\mathbb{A}_{6,2}) : f_6) \cong K[\mathcal{V}(\mathbb{A}_{4,2})]/I(\mathbb{A}_{4,2}) \otimes_K K[\mathcal{V}(S_2)]/I(S_2) \otimes_K K[f_6, f_{5,1}, f_{5,2}, e_{5,1}, e_{5,2}],$$

$$K[\mathcal{V}(\mathbb{A}_{6,2})]/(I(\mathbb{A}_{6,2}), f_6, e_6) \cong K[\mathcal{V}(\mathbb{A}_{5,2})]/I(\mathbb{A}_{5,2}) \otimes_K K[f_{6,1}, f_{6,2}, e_{6,1}, e_{6,2}].$$



Figure 5. From left to right $G_{(I(\mathbb{A}_{6,2}), f_6)}$ and $G_{(I(\mathbb{A}_{6,2}), f_6, e_6)}$.

Remark 3.1. We may have the following special cases in our proofs. These special cases are fixed as follows:

- $K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t}) \cong K.$ Thus,
 $\text{reg}(K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t})) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t})) = \text{sdepth}(K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t})) = 0$
- $K[\mathcal{V}(\mathbb{B}_{0,t})]/I(\mathbb{B}_{0,t}) \cong K.$ Thus,
 $\text{reg}(K[\mathcal{V}(\mathbb{B}_{0,t})]/I(\mathbb{B}_{0,t})) = \text{depth}(K[\mathcal{V}(\mathbb{B}_{0,t})]/I(\mathbb{B}_{0,t})) = \text{sdepth}(K[\mathcal{V}(\mathbb{B}_{0,t})]/I(\mathbb{B}_{0,t})) = 0$
- $K[\mathcal{V}(\mathbb{C}_{0,t})]/I(\mathbb{C}_{0,t}) \cong K.$ Thus,
 $\text{reg}(K[\mathcal{V}(\mathbb{C}_{0,t})]/I(\mathbb{C}_{0,t})) = \text{depth}(K[\mathcal{V}(\mathbb{C}_{0,t})]/I(\mathbb{C}_{0,t})) = \text{sdepth}(K[\mathcal{V}(\mathbb{C}_{0,t})]/I(\mathbb{C}_{0,t})) = 0.$

If $n = 1$, then $K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t}) \cong K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[\mathcal{V}(S_t)]/I(S_t)$. By Lemmas 2.7 and 2.16, $\text{reg}(K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t})) = 2$.

Lemma 3.2 ([11, Lemma 3.6]). *Let G be a graph. Then, $Br_t(G)$ is weakly chordal if and only if G is weakly chordal.*

Let G be a graph and $Q \subset \mathcal{V}(G)$, where Q is called an *independent set* if no two vertices in Q are adjacent in G . A *maximum independent set* is an independent set of the largest possible size. The cardinality of the maximum independent set is called the *independence number* of G and is denoted by $\alpha(G)$.

Lemma 3.3 ([11, Lemma 3.7]). *If G is a graph, then $\text{indmat}(Br_t(G)) = \alpha(G)$.*

Lemma 3.4. *Let $n, t \geq 1$, $S = K[\mathcal{V}(\mathbb{A}_{n,t})]$ and $I = I(\mathbb{A}_{n,t})$. Then,*

$$\text{reg}(S/I) = \begin{cases} n+1, & \text{if } n \text{ is odd;} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $E \subset \mathcal{V}(\mathbb{A}_n)$ such that $E = \{e_1, f_1, e_3, f_3, \dots, e_{n-1}, f_{n-1}\}$ if n is even, and $E = \{e_1, f_1, e_3, f_3, \dots, e_n, f_n\}$ if n is odd. One can easily see that E is a maximum independent set of $\mathbb{A}_n$, $|E| = n$ when n is even, and $|E| = n+1$ when n is odd. Since $\mathbb{A}_n$ is a weakly chordal graph, then by Lemma 3.2, $\mathbb{A}_{n,t}$ is a weakly chordal graph. Thus, by applying Lemmas 3.3 and 2.15, the required result follows. $\square$

Lemma 3.5. *For $n, t \geq 1$, $S = K[\mathcal{V}(\mathbb{B}_{n,t})]$ and $I = I(\mathbb{B}_{n,t})$. Then,*

$$\text{reg}(S/I) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil + 1, & \text{if } n \equiv 1 \pmod{3}; \\ 2\lceil \frac{n}{3} \rceil, & \text{if } n \equiv 0, 2 \pmod{3}. \end{cases}$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.4. If $|E|$ is a maximum independent set, then E is of the form:

- 1) $E = \{f_1, e_2, f_4, e_5, \dots, f_{n-2}, e_{n-1}\}$, when $n \equiv 0 \pmod{3}$; and so $|E| = 2\lceil \frac{n}{3} \rceil$.
- 2) $E = \{f_1, e_2, f_4, e_5, \dots, e_{n-2}, f_n\}$, when $n \equiv 1 \pmod{3}$; and so $|E| = 2\lceil \frac{n-1}{3} \rceil + 1$.
- 3) $E = \{f_1, e_2, f_4, e_5, \dots, f_{n-1}, e_n\}$, when $n \equiv 2 \pmod{3}$, and so $|E| = 2\lceil \frac{n}{3} \rceil$.

$\square$

Lemma 3.6. Let $n, t \geq 1$, $S = K[\mathcal{V}(\mathbb{C}_{n,t})]$ and $I = I(\mathbb{C}_{n,t})$. Then, $\text{reg}(S/I(\mathbb{C}_{n,t})) = \lceil \frac{n}{2} \rceil$.

Proof. The proof is similar to the proof of Lemma 3.4. $\square$

If G is a graph and e_i is a vertex of G , then in $Br_t(G)$ the newly added t pedants to vertex e_i are labeled as $e_{i,1}, e_{i,2}, \dots, e_{i,t}$.

Lemma 3.7. Let $n \geq 2$, $t \geq 1$, and $S = K[\mathcal{V}(\mathbb{A}_{n,t})]$. Then,

$$\text{sdepth}(S/I(\mathbb{A}_{n,t})) \geq \text{depth}(S/I(\mathbb{A}_{n,t})) = \begin{cases} n(t+1), & \text{if } n \text{ is even;} \\ n(1+t) - t + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If $n = 2$, then we have the following short exact sequence:

$$0 \longrightarrow S/(I(\mathbb{A}_{2,t}) : f_2) \xrightarrow{f_2} S/I(\mathbb{A}_{2,t}) \longrightarrow S/(I(\mathbb{A}_{2,t}), f_2) \longrightarrow 0, \quad (3.1)$$

and

$$\begin{aligned} S/(I(\mathbb{A}_{2,t}) : f_2) &\cong K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[f_2, f_{1,1}, \dots, f_{1,t}, e_{1,1}, \dots, e_{1,t}], \\ S/(I(\mathbb{A}_{2,t}), f_2) &\cong K[\mathcal{V}(S_{3,t})]/I(S_{3,t}) \otimes_K K[f_{2,1}, \dots, f_{2,t}]. \end{aligned}$$

By Lemmas 2.6 and 2.12, $\text{depth}(S/(I(\mathbb{A}_{2,t}) : f_2)) = \text{depth}(\mathcal{V}(S_t)/I(S_t)) + 1 + t + t$, and $\text{depth}(S/(I(\mathbb{A}_{2,t}), f_2)) = \text{depth}(K[\mathcal{V}(S_{3,t})]/I(S_{3,t})) + t$. By applying Lemmas 2.7 and 2.8, $\text{depth}(S/(I(\mathbb{A}_{2,t}) : f_2)) = 2t + 2$, and $\text{depth}(S/(I(\mathbb{A}_{2,t}), f_2)) = 2 + t + t = 2t + 2$. By Lemmas 2.2 and 2.10 along with the use of the short exact sequence 3.1, $\text{depth}(S/I(\mathbb{A}_{2,t})) = 2t + 2$. Let $n \geq 3$. Considering the short exact sequence:

$$0 \longrightarrow S/(I(\mathbb{A}_{n,t}) : f_n) \xrightarrow{f_n} S/I(\mathbb{A}_{n,t}) \longrightarrow S/(I(\mathbb{A}_{n,t}), f_n) \longrightarrow 0,$$

we have

$$\begin{aligned} S/(I(\mathbb{A}_{n,t}) : f_n) &\cong K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t}) \otimes_K K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[f_n, f_{n-1,1}, \dots, f_{n-1,t}, \\ &\quad e_{n-1,1}, \dots, e_{n-1,t}]. \end{aligned} \quad (3.2)$$

Let

$$L := (I(\mathbb{A}_{n,t}), f_n) = (I(\mathbb{A}_{n-1,t}), f_n, f_{n-1}e_n, e_ne_{n-1}, e_ne_{n,1}, \dots, e_ne_{n,t}).$$

We consider another short exact sequence

$$0 \longrightarrow S/(L : e_n) \xrightarrow{e_n} S/L \longrightarrow S/(L, e_n) \longrightarrow 0,$$

and it is easy to see that

$$S/(L : e_n) \cong K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t}) \otimes_K K[e_n, f_{n-1,1}, \dots, f_{n-1,t}, e_{n-1,1}, \dots, e_{n-1,t}], \quad (3.3)$$

$$S/(L, e_n) \cong K[\mathcal{V}(\mathbb{A}_{n-1,t})]/I(\mathbb{A}_{n-1,t}) \otimes_K K[e_{n,1}, \dots, e_{n,t}, f_{n,1}, \dots, f_{n,t}]. \quad (3.4)$$

Case 1: If n is odd, then applying Lemmas 2.6 and 2.12 on Eqs (3.2)–(3.4),

$$\text{depth}(S/(I(\mathbb{A}_{n,t}) : f_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t})) + \text{depth}(K[\mathcal{V}(S_t)]/I(S_t)) + 1 + 2t,$$

$$\text{depth}(S/(L, e_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-1,t})]/I(\mathbb{A}_{n-1,t})) + 2t,$$

$$\text{depth}(S/(L : e_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t})) + 3t.$$

By mathematical induction on n along with the use of Lemma 2.7,

$$\text{depth}(S/(I(\mathbb{A}_{n,t}) : f_n)) = (n-2)(1+t) - t + 1 + 1 + 1 + 2t = n(t+1) - t + 1, \quad (3.5)$$

$$\text{depth}(S/(L, e_n)) = (n-1)(t+1) + 2t = n(1+t) + t - 1, \quad (3.6)$$

$$\text{depth}(S/(L : e_n)) = (t+1)(n-2) - t + 2 + t + 2t = n(1+t). \quad (3.7)$$

Using Lemmas 2.2 and 2.10 on Eqs 3.6 and 3.7,

$$\text{depth}(S/I(\mathbb{A}_{n,t}), f_n) = n(t+1). \quad (3.8)$$

By applying Lemmas 2.2 and 2.10 on Eqs (3.5) and (3.8), we get

$$\text{depth}(S/I(\mathbb{A}_{n,t})) = n(t+1) - t + 1.$$

Case 2: If n is even, then by applying Lemmas 2.6 and 2.12 on Eqs (3.2)–(3.4), we have

$$\text{depth}(S/(I(\mathbb{A}_{n,t}) : f_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t})) + \text{depth}(K[\mathcal{V}(S_t)]/I(S_t)) + 1 + t + t,$$

$$\text{depth}(S/(L : e_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-2,t})]/I(\mathbb{A}_{n-2,t})) + 1 + 3t = t - 1 + n(1+t),$$

$$\text{depth}(S/(L, e_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-1,t})]/I(\mathbb{A}_{n-1,t})) + 2t.$$

By mathematical induction on n along with the use of Lemma 2.7,

$$\text{depth}(S/(I(\mathbb{A}_{n,t}) : f_n)) = (n-2)(t+1) + 2 + 2t = n(t+1), \quad (3.9)$$

$$\text{depth}(S/(L : e_n)) = (n-2)(t+1) - t + 1 + 2t = n(1+t), \quad (3.10)$$

$$\text{depth}(S/(L, e_n)) = (n-2)(t+1) + t + 1 + 2t = n(1+t) + t - 1. \quad (3.11)$$

By applying Lemma 2.2 on Eqs (3.10) and (3.11), we have

$$\text{depth}(S/(I(\mathbb{A}_{n,t}), f_n)) \geq n(1+t). \quad (3.12)$$

Since $e_{n-1} \notin L$,

$$\begin{aligned} \text{depth}(S/(L : e_{n-1})) &\cong K[\mathcal{V}(\mathbb{A}_{n-3})]/I(\mathbb{A}_{n-3}) \otimes_K K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[e_{n-1}, e_{n-2,1}, \dots, \\ &\quad e_{n-2,t}, f_{n-2,1}, \dots, f_{n-2,t}, e_{n,1}, \dots, e_{n,t}]. \end{aligned} \quad (3.13)$$

By induction and applying Lemma 2.7,

$$\text{depth}(S/(L : e_{n-1})) = 3t + (n-3)(t+1) - t + t + 3 = n(t+1). \quad (3.14)$$

Now, using Lemma 2.10 on Eq (3.14),

$$\text{depth}(S/(I(\mathbb{A}_{n,t}), f_n)) \leq \text{depth}(S/((I(\mathbb{A}_{n,t}), f_n) : e_{n-1})) \leq n(1+t). \quad (3.15)$$

From Eqs (3.12)–(3.15), we get

$$\text{depth}(S/(I(\mathbb{A}_{n,t}), f_n)) = n(t+1). \quad (3.16)$$

By using the depth lemma on Eqs (3.9) and (3.16),

$$\text{depth}(S/I(\mathbb{A}_{n,t})) = n(t+1). \quad (3.17)$$

To find the bound for Stanley depth, the proof is similar, and we use Lemma 2.3 instead of Lemma 2.2. $\square$

Corollary 3.8. *Let $t, n \geq 1$ and $S = K[\mathcal{V}(\mathbb{A}_{n,t})]$. Then,*

$$\text{pdim}(S/I(\mathbb{A}_{n,t})) = \begin{cases} n(1+t), & \text{if } n \text{ is even;} \\ n(1+t) + t - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The desired result can be obtained by applying Lemma [14, Theorem 1.3.3] and Lemma 3.7. $\square$

4. Algebraic invariants of cyclic modules associated with bristled graphs of some circulant graphs

In this section, we prove our main results. We compute the regularity of the cyclic modules, namely $K[\mathcal{V}(\mathcal{D}_{n,t})]/I(\mathcal{D}_{n,t})$, $K[\mathcal{V}(\mathcal{E}_{n,t})]/I(\mathcal{E}_{n,t})$, and $K[\mathcal{V}(\mathcal{F}_{n,t})]/I(\mathcal{F}_{n,t})$. Furthermore, we compute a lower bound for the Stanley depth and exact values for depth of $K[\mathcal{V}(\mathcal{D}_{n,t})]/I(\mathcal{D}_{n,t})$.

Theorem 4.1. *Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{D}_{n,t})]$. Then,*

$$\text{reg}(S/I(\mathcal{D}_{n,t})) = \begin{cases} n, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $n = 3$. Then,

$$S/(I(\mathcal{D}_{3,t}) : e_3) \cong K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t}) \otimes_K K[\mathcal{V}(S_q)]/I(S_q) \otimes_K K[e_3, e_{1,1}, \dots, e_{1,t}, e_{2,1}, \dots, e_{2,t}, f_{1,1}, \dots, f_{1,t}, f_{2,1}, \dots, f_{2,t}], \quad (4.1)$$

$$S/((I(\mathcal{D}_{3,t}), e_3) : f_3) \cong K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t}) \otimes_K K[f_3, f_{1,1}, \dots, f_{1,t}, f_{2,1}, \dots, f_{2,t}, e_{1,1}, \dots, e_{1,t}, e_{2,1}, \dots, e_{2,t}, e_{3,1}, \dots, e_{3,t}], \quad (4.2)$$

$$S/((I(\mathcal{D}_{3,t}), e_3), f_3) \cong K[\mathcal{V}(\mathbb{A}_{2,t})]/I(\mathbb{A}_{2,t}) \otimes_K K[e_{3,1}, \dots, e_{3,t}, f_{3,1}, \dots, f_{3,t}]. \quad (4.3)$$

By applying Lemmas 2.12 and 2.16 on Eqs (4.1)–(4.3), we obtain

$$\text{reg}(S/(I(\mathcal{D}_{3,t}) : e_3)) = \text{reg}(K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t})) + \text{reg}(K[\mathcal{V}(S_t)]/I(S_t)), \quad (4.4)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3) : f_3)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{0,t})]/I(\mathbb{A}_{0,t})), \quad (4.5)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3), f_3)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{2,t})]/I(\mathbb{A}_{2,t})). \quad (4.6)$$

Applying Lemma 2.7 and Remark 3.1, we have

$$\operatorname{reg}(S/(I(\mathcal{D}_{3,t}) : e_3)) = 1,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3) : f_3)) = 0,$$

and

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3), f_3)) = 2.$$

Since

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3) : f_3)) < \operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_3), f_3)),$$

now, using Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{D}_{3,t}), e_3)) = 2 > \operatorname{reg}(S/(I(\mathcal{D}_{3,t}) : e_3)).$$

Again, by applying Lemma 2.4(b), $\operatorname{reg}(S/I(\mathcal{D}_{3,t})) = 2$. If $n = 4$, then

$$S/(I(\mathcal{D}_{4,q}) : e_4) \cong K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t}) \otimes_K K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[e_4, e_{1,1}, \dots, e_{1,t}, e_{3,1}, \dots, e_{3,t}, f_{1,1}, \dots, f_{1,t}, f_{3,1}, \dots, f_{3,t}], \quad (4.7)$$

$$S/((I(\mathcal{D}_{4,t}), e_4) : f_4) \cong K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t}) \otimes_K K[f_4, f_{1,1}, \dots, f_{1,t}, f_{3,1}, \dots, f_{3,t}, e_{1,1}, \dots, e_{1,t}, e_{3,1}, \dots, e_{3,t}, e_{4,1}, \dots, e_{4,t}], \quad (4.8)$$

$$S/((I(\mathcal{D}_{4,t}), e_4), f_4) \cong K[\mathcal{V}(\mathbb{A}_{3,t})]/I(\mathbb{A}_{3,t}) \otimes_K K[e_{4,1}, \dots, e_{4,t}]. \quad (4.9)$$

By applying Lemma 2.12 on Eqs (4.7)–(4.9),

$$\operatorname{reg}(S/(I(\mathcal{D}_{4,t}) : e_4)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t})) + \operatorname{reg}(K[\mathcal{V}(S_q)]/I(S_q)), \quad (4.10)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{4,t}), e_4) : f_4)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{1,t})]/I(\mathbb{A}_{1,t})), \quad (4.11)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{4,t}), e_4), f_4)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{3,t})]/I(\mathbb{A}_{3,t})), \quad (4.12)$$

By applying Remark 3.1 and Lemma 2.7,

$$\operatorname{reg}(S/(I(\mathcal{D}_{4,t}) : e_4)) = 3,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{3,t}), e_4) : f_4)) = 2,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{4,t}), e_4), f_4)) = 4.$$

Since

$$\operatorname{reg}(S/((I(\mathcal{D}_{4,t}), e_4) : f_4)) < \operatorname{reg}(K[\mathcal{V}(\mathcal{D}_{4,t})]/((I(\mathcal{D}_{4,t}), e_4), f_4)),$$

now, using Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{D}_{4,t}), e_4)) = 4 > \operatorname{reg}(K[\mathcal{V}(\mathcal{D}_{4,t})]/(I(\mathcal{D}_{4,t}) : e_4)).$$

Again, by applying Lemma 2.4(b), $\operatorname{reg}(S/I(\mathcal{D}_{4,t})) = 4$. Now we consider two cases.

Case 1: Let n be even. Then,

$$S/((I(\mathcal{D}_{n,t}) : e_n)) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] \otimes_K K[\mathcal{V}(S_t)/I(S_t)] \otimes_K K[e_n, e_{1,1}, \dots, e_{1,t}, \\ e_{n-1,1}, \dots, e_{n-1,t}, f_{1,1}, \dots, f_{1,t}, f_{n-1,1}, \dots, f_{n-1,t}], \quad (4.13)$$

$$S/((I(\mathcal{D}_{n,t}), e_n) : f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] \otimes_K K[f_n, f_{1,1}, \dots, f_{1,t}, f_{n-1,1}, \dots, f_{n-1,t}, \\ e_{1,1}, \dots, e_{1,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{n,1}, \dots, e_{n,t}], \quad (4.14)$$

$$S/((I(\mathcal{D}_{n,t}), e_n), f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-1,t})/I(\mathbb{A}_{n-1,t})] \otimes_K K[e_{n,1}, \dots, e_{n,t}, f_{n-1,1}, \dots, f_{n-1,t}], \quad (4.15)$$

by applying Lemmas 2.12 and 2.16 on Eqs (4.13)–(4.15),

$$\operatorname{reg}(S/(I(\mathcal{D}_{n,t}) : e_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] + K[\mathcal{V}(S_t)/I(S_t)]), \quad (4.16)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n) : f_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})]), \quad (4.17)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-1,t})/I(\mathbb{A}_{n-1,t})]). \quad (4.18)$$

By using induction on n and applying Lemmas 3.4 and 2.7 on Eqs (4.16)–(4.18),

$$\operatorname{reg}(S/(I(\mathcal{D}_{n,t}) : e_n)) = n - 3 + 1 + 1 = n - 1,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n) : f_n)) = n - 3 + 1 = n - 2,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = n - 1 + 1 = n.$$

As, $\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n) : f_n)) < \operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n))$, by applying Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n)) = n.$$

Also, $\operatorname{reg}(S/(I(\mathcal{D}_{n,t}) : e_n)) < \operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n))$, and again by applying Lemma 2.4(b),

$$\operatorname{reg}(S/I(\mathcal{D}_{n,t})) = n.$$

Case 2: When n is odd.

$$S/(I(\mathcal{D}_{n,t}) : e_n) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] \otimes_K K[\mathcal{V}(S_t)/I(S_t)] \otimes_K K[e_n, e_{1,1}, \dots, e_{1,t}, \\ e_{n-1,1}, \dots, e_{n-1,t}, f_{1,1}, \dots, f_{1,t}, f_{n-1,1}, \dots, f_{n-1,t}], \quad (4.19)$$

$$S/((I(\mathcal{D}_{n,t}), e_n) : f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] \otimes_K K[f_n, f_{1,1}, \dots, f_{1,t}, f_{n-1,1}, \dots, f_{n-1,t}, \\ e_{1,1}, \dots, e_{1,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{n,1}, \dots, e_{n,t}], \quad (4.20)$$

$$S/((I(\mathcal{D}_{n,t}), e_n), f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-1,t})/I(\mathbb{A}_{n-1,t})] \otimes_K K[e_{n,1}, \dots, e_{n,t}, f_{n-1,1}, \dots, f_{n-1,t}], \quad (4.21)$$

By Lemmas 2.12 and 2.16 on Eqs (4.19)–(4.21),

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n))) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})] + K[\mathcal{V}(S_t)]/I(S_t), \quad (4.22)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-3,t})/I(\mathbb{A}_{n-3,t})]), \quad (4.23)$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{A}_{n-1,t})/I(\mathbb{A}_{n-1,t})]). \quad (4.24)$$

By applying induction on n along with the use of Lemmas 3.4 and 2.7 on Eqs (4.22)–(4.24),

$$\operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n)) = n - 3 + 1 = n - 2,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = n - 3,$$

$$\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) = n - 1.$$

As $\operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n)) < \operatorname{reg}(S/((I(\mathcal{D}_{n,t}), e_n), f_n))$, by applying Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n)) = n - 1.$$

Also, $\operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n)) < \operatorname{reg}(S/(I(\mathcal{D}_{n,t}), e_n))$, so applying Lemma 2.4(b),

$$\operatorname{reg}(S/I(\mathcal{D}_{n,t})) = n - 1.$$

□

Lemma 4.2 ([29, Corollary 4.4]). *Let G be a graph, and $S = K[\mathcal{V}(G)]$. Then, $\operatorname{reg}(S/I(\operatorname{Br}_t(G))) = \operatorname{indmat}(\operatorname{Br}_t(G))$.*

Theorem 4.3. *Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{E}_{n,t})]$. Then,*

$$\operatorname{reg}(S/I(\mathcal{E}_{n,t})) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil - 1, & \text{if } n \equiv 2 \pmod{3}; \\ 2\lceil \frac{n-1}{3} \rceil, & \text{if } n \equiv 0, 1 \pmod{3}; \end{cases}$$

Proof. Let $n = 3$. Then, clearly $\operatorname{indmat}(\mathcal{E}_{3,t}) = 2$, so by using Lemma 4.2 we have

$$\operatorname{reg}(S/I(\mathcal{E}_{3,t})) = 2.$$

For $n \geq 4$, we have the following isomorphisms:

$$S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-2}) \cong K[\mathcal{V}(\mathbb{B}_{n-4,t})/I(\mathbb{B}_{n-4,t})] \otimes_K K[e_{n-1}, f_{n-2}, e_{n,1}, \dots, e_{n,t}, e_{n-2,1}, \dots, e_{n-2,t}, f_{n-1,1}, \dots, f_{n-1,t}, f_{n,1}, \dots, f_{n,t}, f_{n-3,1}, \dots, f_{n-3,t}], \quad (4.25)$$

$$S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-2}) \cong K[\mathcal{V}(\mathbb{B}_{n-3,t})/I(\mathbb{B}_{n-3,t})] \otimes_K K[e_{n-1}, e_{n,1}, \dots, e_{n,t}, e_{n-2,1}, \dots, e_{n-2,t}, f_{n-1,1}, \dots, f_{n-1,t}, f_{n-2,1}, \dots, f_{n-2,t}], \quad (4.26)$$

$$S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1}) \cong K[\mathcal{V}(\mathbb{B}_{n-1,t})/I(\mathbb{B}_{n-1,t})] \otimes_K K[e_{n-1,1}, \dots, e_{n-1,t}, f_{n-1,1}, \dots, f_{n-1,t}], \quad (4.27)$$

$$S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n) \cong K[\mathcal{V}(\mathbb{B}_{n-3,t})]/I(\mathbb{B}_{n-3,t}) \otimes_K K[f_{n-1}, e_{n,1}, \dots, e_{n,t}, f_{n,1}, \dots, f_{n,t}, f_{n-2,1}, \dots, f_{n-2,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{n-2,1}, \dots, e_{n-2,t}], \quad (4.28)$$

$$S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n) \cong K[\mathcal{V}(\mathbb{B}_{n-4,t})]/I(\mathbb{B}_{n-4,t}) \otimes_K K[f_{n-1}, e_{n-1}, e_{n-1,1}, \dots, e_{n-1,t}, e_{1,1}, \dots, e_{1,t}, f_{1,1}, \dots, f_{1,t}, f_{n,1}, \dots, f_{n,t}, e_{n-2,1}, \dots, e_{n-2,t}, f_{n-2,1}, \dots, f_{n-2,t}]. \quad (4.29)$$

Now consider the following cases:

Case 1: Let $n \equiv 2 \pmod{3}$.

We apply Lemmas 2.12 and 3.5 on Eqs (4.25)–(4.29). Since $n - 4 \equiv 1 \pmod{3}$, so

$$\text{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) = \text{reg}(K[\mathcal{V}(\mathbb{B}_{n-4,t})]/I(\mathbb{B}_{n-4,t})) = 2\lceil \frac{n-1}{3} \rceil - 3.$$

We have $n - 3 \equiv 2 \pmod{3}$, so

$$\text{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2})) = \text{reg}(K[\mathcal{V}(\mathbb{B}_{n-3,t})]/I(\mathbb{B}_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil - 2.$$

We have $n - 1 \equiv 1 \pmod{3}$, so

$$\text{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1})) = \text{reg}(K[\mathcal{V}(\mathbb{B}_{n-1,t})]/I(\mathbb{B}_{n-1,t})) = 2\lceil \frac{n-1}{3} \rceil - 1.$$

Since $n - 3 \equiv 2 \pmod{3}$,

$$\text{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n)) = \text{reg}(K[\mathcal{V}(\mathbb{B}_{n-3,t})]/I(\mathbb{B}_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil - 2,$$

Also,

$$\text{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n)) \cong K[\mathcal{V}(\mathbb{B}_{n-4,t})]/I(\mathbb{B}_{n-4,t}) = 2\lceil \frac{n-1}{3} \rceil - 3.$$

As $\text{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) < \text{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2}))$, so applying Lemma 2.4(b),

$$\text{reg}(S/(I(\mathcal{E}_{n,t}) : e_{n-1})) = 2\lceil \frac{n-1}{3} \rceil - 2,$$

and $\text{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n)) < \text{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n))$. By using Lemma 2.4(b),

$$\text{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1})) = 2\lceil \frac{n-1}{3} \rceil - 2.$$

Also, $\text{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1})) < \text{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1}))$. By using Lemma 2.4(b),

$$\text{reg}(S/(I(\mathcal{E}_{n,t}), e_{n-1})) = 2\lceil \frac{n-1}{3} \rceil - 1.$$

Also, $\text{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1})) < \text{reg}(S/(I(\mathcal{E}_{n,t}), e_{n-1}))$. By using Lemma 2.4(b),

$$\text{reg}(S/(I(\mathcal{E}_{n,t}))) = 2\lceil \frac{n-1}{3} \rceil - 1.$$

Case 2: Let $n \equiv 0, 1 \pmod{3}$.

Applying Lemmas 2.12 and 3.5 on Eqs (4.25)–(4.29), we have

$$\operatorname{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) = \operatorname{reg}(K[\mathcal{V}(\mathbb{B}_{n-4,t})]/I(\mathbb{B}_{n-4,t})) = 2\lceil \frac{n-4}{3} \rceil,$$

$$\operatorname{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2})) = \operatorname{reg}(K[\mathcal{V}(\mathbb{B}_{n-3,t})]/I(\mathbb{B}_{n-3,t})) = 2\lceil \frac{n-3}{3} \rceil,$$

$$\operatorname{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1})) = \operatorname{reg}(K[\mathcal{V}(\mathbb{B}_{n-1,t})]/I(\mathbb{B}_{n-1,t})) = 2\lceil \frac{n-1}{3} \rceil,$$

$$\operatorname{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{B}_{n-3,t})]/I(\mathbb{B}_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil,$$

$$\operatorname{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n)) = \operatorname{reg}(K[\mathcal{V}(\mathbb{B}_{n-1,t})]/I(\mathbb{B}_{n-1,t})) = 2\lceil \frac{n-4}{3} \rceil.$$

As $\operatorname{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) < \operatorname{reg}(S/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2}))$, applying Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{E}_{n,t}) : e_{n-1})) = 2\lceil \frac{n-4}{3} \rceil,$$

and $\operatorname{reg}(S/(((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n)) < \operatorname{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n))$. By using Lemma 2.4(b)

$$\operatorname{reg}(S/(I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1})) = 2\lceil \frac{n-3}{3} \rceil.$$

Since $\operatorname{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1})) < \operatorname{reg}(S/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1}))$, by using Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{E}_{n,t}), e_{n-1})) = 2\lceil \frac{n-1}{3} \rceil.$$

Also, $\operatorname{reg}(S/(I(\mathcal{E}_{n,t}) : e_{n-1})) < \operatorname{reg}(S/(I(\mathcal{E}_{n,t}), e_{n-1}))$. By using Lemma 2.4(b)

$$\operatorname{reg}(S/I(\mathcal{E}_{n,t})) = 2\lceil \frac{n-1}{3} \rceil.$$

□

Theorem 4.4. Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{F}_{n,t})]$. Then,

$$\operatorname{reg}(S/I(\mathcal{F}_{n,t})) = \lceil \frac{n-1}{2} \rceil.$$

Proof. We have the following isomorphism:

$$S/(I(\mathcal{F}_{n,t}) : f_n) \cong K[\mathcal{V}(\mathbb{C}_{n-3,t})]/I(\mathbb{C}_{n-3,t}) \otimes_K K[f_n, f_{n-1,1}, \dots, f_{n-1,t}, f_{1,1}, \dots, f_{1,t}, e_{n,1}, \dots, e_{n,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{1,1}, \dots, e_{1,t}], \quad (4.30)$$

$$S/((I(\mathcal{F}_{n,t}), f_n), e_n) \cong K[\mathcal{V}(\mathbb{C}_{n-1,t})]/I(\mathbb{C}_{n-1,t}) \otimes_K K[f_{n,1}, \dots, f_{n,t}, e_{n,1}, \dots, e_{n,t}], \quad (4.31)$$

$$S/((I(\mathcal{F}_{n,t}), f_n) : e_n) \cong K[\mathcal{V}(\mathbb{C}_{n-3,t})]/I(\mathbb{C}_{n-3,t}) \otimes_K K[e_n, f_{n,1}, \dots, f_{n,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{1,1}, \dots, e_{1,t}]. \quad (4.32)$$

Applying Lemmas 3.6 and 2.12 on Eqs (4.30)–(4.32), we have

$$\begin{aligned} \operatorname{reg}(S/(I(\mathcal{F}_{n,t}) : f_n)) &= \operatorname{reg}(K[\mathcal{V}(\mathbb{C}_{n-3,t})]/I(\mathbb{C}_{n-3,t})) = \lceil \frac{n-3}{2} \rceil, \\ \operatorname{reg}(S/((I(\mathcal{F}_{n,t}), f_n), e_n)) &= \operatorname{reg}(K[\mathcal{V}(\mathbb{C}_{n-1,t})]/I(\mathbb{C}_{n-1,t})) = \lceil \frac{n-1}{2} \rceil, \\ \operatorname{reg}(S/((I(\mathcal{F}_{n,t}), f_n) : e_n)) &= \operatorname{reg}(K[\mathcal{V}(\mathbb{C}_{n-3,t})]/I(\mathbb{C}_{n-3,t})) = \lceil \frac{n-3}{2} \rceil. \end{aligned}$$

Since, $\operatorname{reg}(S/((I(\mathcal{F}_{n,t}), f_n) : e_n)) < \operatorname{reg}(S/((I(\mathcal{F}_{n,t}), f_n), e_n))$, by applying Lemma 2.4(b),

$$\operatorname{reg}(S/(I(\mathcal{F}_{n,t}), f_n)) = \lceil \frac{n-1}{2} \rceil.$$

Moreover, $\operatorname{reg}(S/(I(\mathcal{F}_{n,t}) : f_n)) < \operatorname{reg}(S/(I(\mathcal{F}_{n,t}), f_n))$, and by applying Lemma 2.4(b),

$$\operatorname{reg}(S/I(\mathcal{F}_{n,t})) = \lceil \frac{n-1}{2} \rceil.$$

□

Theorem 4.5. Let $n \geq 3$, $t \geq 1$, and $S = K[\mathcal{V}(\mathcal{D}_{n,t})]$. Then,

$$\operatorname{sdepth}(S/I(\mathcal{D}_{n,t})) \geq \operatorname{depth}(S/I(\mathcal{D}_{n,t})) = \begin{cases} n(t+1), & \text{if } n \text{ is odd;} \\ n(t+1) + t - 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Consider the following short exact sequences:

$$0 \longrightarrow S/(I(\mathcal{D}_{n,t}) : e_n) \xrightarrow{\cdot e_n} S/I(\mathcal{D}_{n,t}) \longrightarrow S/(I(\mathcal{D}_{n,t}), e_n) \longrightarrow 0,$$

and

$$0 \longrightarrow S/(L : f_n) \xrightarrow{\cdot f_n} S/L \longrightarrow S/(L, f_n) \longrightarrow 0,$$

where $L := (I(\mathcal{D}_{n,t}), e_n)$. Then, we have the following isomorphisms:

$$S/(I(\mathcal{D}_{n,t}) : e_n) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})]/I(\mathbb{A}_{n-3,t}) \otimes_K K[\mathcal{V}(S_t)]/I(S_t) \otimes_K K[e_n, e_{n-1,1}, \dots, e_{n-1,t}, e_{1,1}, \dots, e_{1,t}, f_{n-1,1}, \dots, f_{n-1,t}, f_{1,1}, \dots, f_{1,t}] \quad (4.33)$$

$$S/(L : f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-3,t})]/I(\mathbb{A}_{n-3,t}) \otimes_K K[f_n, f_{n-1,1}, \dots, f_{n-1,t}, f_{1,1}, \dots, f_{1,t}, e_{n-1,1}, \dots, e_{n-1,t}, e_{1,1}, \dots, e_{1,t}], \quad (4.34)$$

$$S/(L, f_n) \cong K[\mathcal{V}(\mathbb{A}_{n-1})]/I(\mathbb{A}_{n-1}) \otimes_K K[e_{n,1}, \dots, e_{n,t}, f_{n,1}, \dots, f_{n,t}]. \quad (4.35)$$

Case 1: When n is odd. By applying Lemmas 2.6 and 2.12 on Eqs (4.33)–(4.35),

$$\operatorname{depth}(S/(I(\mathcal{D}_{n,t}) : e_n)) = \operatorname{depth}(K[\mathcal{V}(\mathbb{A}_{n-3,t})]/I(\mathbb{A}_{n-3,t})) + \operatorname{depth}(K[\mathcal{V}(S_t)]/I(S_t)) + 1 + 4t, \quad (4.36)$$

$$\text{depth}(S/(L : f_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-3,t})]/I(\mathbb{A}_{n-3,t})) + 1 + 5t, \quad (4.37)$$

$$\text{depth}(S/(L, f_n)) = \text{depth}(K[\mathcal{V}(\mathbb{A}_{n-1})]/I(\mathbb{A}_{n-1})) + t + t. \quad (4.38)$$

Now, by applying Lemmas 2.7 and 3.7 on Eqs (4.36)–(4.38),

$$\text{depth}(S/(I(\mathcal{D}_{n,t}) : e_n)) = t - 1 + n(1 + t),$$

$$\text{depth}(S/(L : f_n)) = n(1 + t) + 2t - 2,$$

$$\text{depth}(S/(L, f_n)) = t - 1 + n(1 + t),$$

Now, applying Lemma 2.2,

$$\text{depth}(S/L) \geq t - 1 + n(t + 1). \quad (4.39)$$

Also, as $x_1 \notin L$, and $\text{depth}(S/(L : x_1)) = t - 1 + n(1 + t)$. Now, applying Lemma 2.10,

$$\text{depth}(S/L) \leq t - 1 + n(t + 1).$$

So,

$$\text{depth}(S/L) = n(t + 1) + t - 1. \quad (4.40)$$

Now, by using the depth lemma, we have

$$\text{depth}(S/(I(\mathcal{D}_{n,t}))) = n(t + 1) + t - 1.$$

Case 2: If n is even, then the proof is similar to Case 1.

The proof for Stanley depth similar we use Lemma 2.3 instead of Lemma 2.2.

□

Corollary 4.6. Let $t \geq 1$, $n \geq 3$, and $S = K[\mathcal{V}(\mathcal{D}_{n,t})]$. Then,

$$\text{pdim}(S/I(\mathcal{D}_{n,t})) = \begin{cases} n(1 + t), & \text{if } n \text{ is even;} \\ n(1 + t) - t + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The desired result can be obtained by applying Theorem 4.5 and [14, Theorem 1.3.3].

□

5. Conclusions

In this paper, we study the algebraic invariants, namely the regularity, projective dimension, depth, and Stanley depth, of the quotient rings of the edge ideals associated to the bristled graphs of various classes of circulant graphs. We give precise values of the said invariants for the quotient rings we considered, except Stanley depth. For the Stanley depth, we give lower bounds that are good enough to verify Stanley's inequality.

Author contributions

Every author made an equal contribution to this paper. Conceptualization, validation, and review by M. Ishaq, A. Asiri, and A. Hussain; formal analysis, writing original draft, and editing by I. U. Rehman and M. U. K. Afridi.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

The authors thanks the employer of King Abdulaziz University Jeddah Saudi Arabia for their encouragement and financial support.

Conflict of interest

The authors declare that there is no conflict of interest in this article.

References

1. B. Alspach, T. D. Parsons, Isomorphism of circulant graphs and digraphs, *Discrete Math.*, **25** (1979), 97–108. [https://doi.org/10.1016/0012-365X\(79\)90011-6](https://doi.org/10.1016/0012-365X(79)90011-6)
2. F. K. Hwang, A survey on multi-loop networks, *Theor. Comput. Sci.*, **299** (2003), 107–121. [https://doi.org/10.1016/S0304-3975\(01\)00341-3](https://doi.org/10.1016/S0304-3975(01)00341-3)
3. J. C. Bermond, F. Comellas, D. F. Hsu, Distributed loop computer-networks: A survey, *J. Parallel Distr. Com.*, **24** (1995), 2–10. <https://doi.org/10.1006/jpdc.1995.1002>
4. B. Shaukat, M. Ishaq, A. U. Haq, Algebraic invariants of edge ideals of some circulant graphs, *AIMS Mathematics*, **9** (2023), 868–895. <https://doi.org/10.3934/math.2024044>
5. CoCoATeam, CoCoA: A system for computations in commutative algebra. Available from: <http://cocoa.dima.unige.it/>.
6. D. R. Grayson, M. E. Stillman, Macaulay2: A software system for research in algebraic geometry, 2002. Available from: <https://www.unimelb-macaulay2.cloud.edu.au/home>.
7. R. P. Stanley, Linear Diophantine equations and local cohomology, *Invent. Math.*, **68** (1982), 175–193. <https://doi.org/10.1007/BF01394054>
8. A. M. Duval, B. Goeckner, C. J. Klivans, J. L. Martin, A non-partitionable Cohen–Macaulay simplicial complex, *Adv. Math.*, **299** (2016), 381–395. <https://doi.org/10.1016/j.aim.2016.05.011>
9. J. Herzog, M. Vladoiu, X. Zheng, How to compute the Stanley depth of a monomial ideal, *J. Algebra*, **322** (2009), 3151–3169. <https://doi.org/10.1016/j.jalgebra.2008.01.006>
10. J. Herzog, A survey on Stanley depth, In: *Monomial ideals, computations and applications*, Berlin, Heidelberg: Springer, 2013, 3–45. https://doi.org/10.1007/978-3-642-38742-5_1
11. M. U. K. Afridi, I. Ur Rehman, M. Ishaq, Algebraic invariants of the edge ideals of whisker graphs of cubic circulant graphs. *Heliyon*, **11** (2025), e41783. <https://doi.org/10.1016/j.heliyon.2025.e41783>
12. M. M. S. Shahid, M. Ishaq, A. Jirawattanapanit, K. Subkrajang, Depth and Stanley depth of the edge ideals of multi triangular snake and multi triangular ouroboros snake graphs, *AIMS Mathematics*, **7** (2022), 16449–16463. <https://doi.org/10.3934/math.2022900>

13. Z. Iqbal, M. Ishaq, M. Aamir, Depth and Stanley depth of the edge ideals of square paths and square cycles, *Commun. Algebra*, **46** (2018), 1188–1198. <https://doi.org/10.1080/00927872.2017.1339068>
14. W. Bruns, J. Herzog, Cohen-Macaulay rings, *Cambridge university press*, 1998. <https://doi.org/10.1017/CBO9780511608681>
15. A. Rauf, Depth and Stanley depth of multigraded modules *Commun. Algebra*, **38** (2010), 773–784. <https://doi.org/10.1080/00927870902829056>
16. G. Caviglia, H. T. Hà, J. Herzog, M. Kummini, N. Terai, N. V. Trung, Depth and regularity modulo a principal ideal, *J. Algebr. Comb.*, **49** (2019), 1–20. <https://doi.org/10.1007/s10801-018-0811-9>
17. R. H. Villarreal, *Monomial algebras, monographs and textbooks in pure and applied mathematics*, New York: Marcel Dekker, Inc., 2001.
18. A. Alipour, A. Tehranian, Depth and Stanley depth of edge ideals of star graphs, *Int. J. Appl. Math. Stat.*, **56** (2017), 63–69.
19. B. Shaukat, M. Ishaq, A. U. Haq, Z. Iqbal, Algebraic properties of edge ideals of corona product of certain graphs, 2022, arXiv: 2211.05721. <https://doi.org/10.48550/arXiv.2211.05721>
20. M. Cimpoeas, Several inequalities regarding Stanley depth, *Rom. J. Math. Comput. Sci.*, **2** (2012), 28–40.
21. N. U. Din, M. Ishaq, Z. Sajid, Values and bounds for depth and Stanley depth of some classes of edge ideals, *AIMS Mathematics*, **6** (2021), 8544–8566. <https://doi.org/10.3934/math.2021496>
22. S. Morey, R. H. Villarreal, Edge ideals: Algebraic and combinatorial properties, *Progress in Commut. Algebr.*, **1** (2012), 85–126.
23. G. Kalai, R. Meshulam, Intersections of Leray complexes and regularity of monomial ideals, *J. Comb. Theory A*, **113** (2006), 1586–1592. <https://doi.org/10.1016/j.jcta.2006.01.005>
24. J. Herzog, A generalization of the Taylor complex construction, *Commun. Algebra*, **35** (2007), 1747–1756. <https://doi.org/10.1080/00927870601139500>
25. M. Katzman, Characteristic-independence of Betti numbers of graph ideals, *J. Comb. Theory A*, **113** (2006), 435–454. <https://doi.org/10.1016/j.jcta.2005.04.005>
26. H. T. Hà, A. V. Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, *J. Algebr. Comb.*, **27** (2008), 215–245. <https://doi.org/10.1007/s10801-007-0079-y>
27. R. Woodroffe, Matchings, coverings, and Castelnuovo-Mumford regularity, *J. Commut. Algebra*, **6** (2014), 287–304.
28. L. T. Hoa, N. D. Tam, On some invariants of a mixed product of ideals, *Arch. Math.*, **94** (2010), 327–337. <https://doi.org/10.1007/s00013-010-0112-6>
29. Y. Muta, M. R. Pournaki, N. Terai, A local cohomological viewpoint on edge rings associated with multi-whisker graphs, *Commun. Algebra*, **53** (2024), 1856–1865. <https://doi.org/10.1080/00927872.2024.2423888>



AIMS Press

© 2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)