AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 29319–29341.
DOI: 10.3934/math.20251288
Received: 12 October 2025
Revised: 22 November 2025
Accepted: 02 December 2025
Published: 12 December 2025

Research article

New oscillation criteria for third-order functional differential equations with general delay argument

Asma Al-Jaser[1], **Inas Ibrhim**[2], **Faizah Alharbi**[3], **Belgees Qaraad**[4] **and Dimplekumar Chalishajar**[5,*]

[1] Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Omar Al-Mukhtar University, Libya
[3] Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah 24227, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[5] Department of Applied Mathematics, Virginia Military Institute (VMI), Lexington, VA 24450, USA

* **Correspondence:** Email: chalishajardn@vmi.edu.

Abstract: This study explores the asymptotic and oscillatory behavior of solutions to third-order functional differential equations. By employing Riccati transformation, we effectively eliminate the possibility of nonoscillatory solutions, allowing for the development of oscillation criteria that are applicable to a broad range of equation models. A key objective of this work is to relax traditional constraints commonly imposed on these criteria, thereby enhancing their general applicability. The results presented not only refine and extend existing theories but also contribute to a deeper understanding of the subject. Practical implications of the theoretical findings are demonstrated through several illustrative examples, highlighting their relevance and potential applications.

Keywords: functional differential equations; neutral term; oscillation; Riccati transformation
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

This paper deals with the oscillation of all solutions of the third-order delay differential equation

$$\left(\varrho(\top)\, Q''(\top)\right)' + \eta(\top)\, x(b(\top)) = 0, \quad \top \geq \top_0 > 0, \tag{1.1}$$

where

$$Q(\top) = x(\top) + R(\top)\, x(\iota(\top)), \tag{1.2}$$

and the following hypotheses are satisfied:

($\mathbf{I}_1$) $\varrho, \eta, R \in C([\top_0, \infty), \mathbb{R})$, $b \in C^1([\top_0, \infty), \mathbb{R})$, $\iota \in C([\top_0, \infty), (0, \infty))$, and $\eta(\top) > 0$;
($\mathbf{I}_2$) $\varrho > 0$,

$$\int_{\top_0}^{\infty} \varrho^{-1}(s)\, \mathrm{d}s = \infty$$

and

$$0 \leq R(\top) \leq R_0 < \infty; \tag{1.3}$$

($\mathbf{I}_3$) $b(\top) \leq \top, \iota(\top) \leq \top$, $\iota'(\top) \geq \iota_0 > 0$, $\lim_{\top \to \infty} \iota(\top) = \lim_{\top \to \infty} b(\top) = \infty$, and

$$\iota \circ b = b \circ \iota. \tag{1.4}$$

Definition 1.1. *A solution of (1.1) is defined as nontrivial solutions of (1.1)* $x \in C([\top_x, \infty), [0, \infty))$ *with* $\top_x \geq \top_0$*, which satisfies (1.1) and* $\varrho(Q'') \in C([\top_x, \infty), [0, \infty))$ *on* $[\top_x, \infty)$*. Those solutions of (1.1) exist on some half-line* $[\top_x, \infty)$ *and satisfy* $\sup\{|x(\top)| : \top_a \leq \top < \infty\} > 0$ *for all* $\top_a \geq \top_x$*. A solution x is called oscillatory if it becomes either positive or negative. Otherwise, it is called a nonoscillatory solution.*

Definition 1.2. *We say that (1.1) has property F if any solution x of (1.1) is either oscillatory or satisfies*

$$x(\top) \to 0 \text{ as } \top \to \infty. \tag{1.5}$$

Control theory provides a rigorous framework for analyzing the dynamic behavior of systems governed by differential equations, particularly in determining whether solutions remain bounded, converge to equilibrium, or exhibit oscillatory patterns. Through techniques such as Lyapunov stability analysis, frequency-domain methods, and feedback control design, it becomes possible to predict and regulate the stability of solutions under varying initial conditions and system parameters. Moreover, control-theoretic approaches enable the systematic suppression or amplification of oscillatory behavior, ensuring that solutions align with desired performance criteria. This deep connection highlights how control theory not only offers a set of analytical tools for understanding the qualitative behavior of differential equations but also establishes practical strategies for engineering stable and robust systems across diverse scientific and technological domains [1,2].

Third-order delayed differential equations (DDEs) are mathematical models that describe systems where the rate of change of a variable depends not only on its current state but also on its past behavior, incorporating a time delay in the system's response. These equations find broad applications in diverse fields, including biology, engineering, economics, and physics, where delays play a crucial role in shaping system dynamics. Unlike lower-order DDEs, third-order delayed equations involve derivatives up to the third order, leading to more intricate and complex dynamical behaviors. The presence of time delays introduces a non-local character to the system, as its future behavior is influenced by past states. Third-order delayed DDEs are particularly valuable in modelling systems such as control mechanisms with feedback, population dynamics with delayed effects, and electrical circuits with delayed responses. For instance, the following equation:

$$(\varrho(\top)(x''(\top)))' + \eta(\top)\, x(b(\top)) = 0$$

can be used to model oscillatory mechanical systems that exhibit time-dependent properties and delays in their responses. For instance, it is applicable in describing the vibrations of structures subjected to external forces, where the system's reaction is influenced by its previous states [3–6].

The oscillation theory of differential equations is a fundamental area of mathematical analysis, with extensive applications across both science and engineering. Differential equations are foundational in modelling dynamic systems, and understanding their oscillatory behavior is crucial for analyzing complex phenomena such as mechanical vibrations, electrical circuits, and biological rhythms. By studying the oscillatory properties of solutions, researchers can enhance the design of more efficient, stable, and reliable systems. In electrical engineering, second-order linear differential equations are used to describe the dynamics of RLC circuits, where oscillations in current and voltage significantly impact the performance of signal processing devices, communication systems, and power networks. Similarly, in mechanical engineering, oscillation theory plays a vital role in evaluating the stability of structures subjected to periodic forces, such as bridges, buildings, and mechanical resonators.

Additionally, the study of oscillatory behavior is essential in control theory, where uncontrolled oscillations—such as those observed in aircraft autopilots or robotic systems—can lead to instability and degraded performance. Engineers utilise differential equations to analyze and mitigate these undesirable oscillations, ensuring the stable and efficient operation of automated systems [7–12].

A powerful method for studying oscillatory properties is the comparison principle, which allows the derivation of oscillation criteria by comparing a given equation with a well-established auxiliary equation. This approach simplifies the analysis by utilizing known results from lower-order or related differential equations. The oscillation of third-order differential equations, in particular, has been extensively explored using this principle, resulting in significant theoretical advancements. Notably, Agarwal et al. [13] developed foundational comparison theorems for third-order linear differential equations, introducing oscillation criteria based on corresponding second-order equations.

The condition (1.3) is critical for ensuring both the stability of the model and its relevance to real-world systems, such as delayed feedback control mechanisms and population dynamics. This more flexible constraint serves as a generalization of stricter conditions, such as $R(\top) \in [0,1)$, which is often difficult to implement or overly restrictive in practical applications. In the context of population dynamics, a similar degree of flexibility is essential. Ecological models often involve feedback mechanisms where interaction strengths can vary depending on factors like environmental conditions or population densities. The condition (1.3) ensures that these interactions remain within a plausible range, enabling the model to capture both immediate and delayed effects of population changes while avoiding unrealistic outcomes, such as unbounded growth or decay [14–16].

In particular, the oscillatory properties of the differential equation

$$\left(\varrho(\top)\left(Q''(\top)\right)\right)' + f\left(x\left(b(\top)\right)\right) = 0$$

were tested by using the comparison principle in [17–21]. Also, the authors in [22] and [23] studied different oscillation criteria of the equation

$$\left(\varrho(\top)\left(\widetilde{Q}(\top)\right)''\right)' + \eta(\top)\,x\left(b(\top)\right) = 0, \tag{1.6}$$

where

$$\widetilde{Q}(\top) = x(\top) + R(\top)\,x\left(\iota(\top)\right)$$

in the case where

$$0 \leq R(\top) \leq R_0 < 1. \tag{1.7}$$

Building upon these foundational results, Özdemir and Kaya [24] introduced novel comparison theorems for third-order functional differential equations with mixed neutral terms, thereby broadening the oscillation criteria to encompass equations with deviating arguments

$$\left(\varrho(\top)\left(\widetilde{Q}(\top)\right)''\right)' + \eta(\top)\,x(b(\top)) = 0,$$

where

$$\widetilde{Q}(\top) = x(\top) + R_1(\top)\,x(\iota(\top)) + R_2(\top)\,x(\iota(\top)).$$

On the other hand, Al Themairi et al. [25] studied third-order nonlinear delay differential equations with distributed arguments

$$(\varrho(\top)((x(\top) + R(\top)\,x(\iota(\top)))'')^{\alpha})' + \int_m^n \eta(\top, s)\,x(b(\top, s))\,\mathrm{d}s = 0,$$

where $\alpha > 0$ is the ratio of odd positive integers. By applying the comparison principle, they derived sufficient conditions to guarantee the nonexistence of positive decreasing solutions, thus establishing new oscillation criteria for this class of equations. Moreover, Grace [26] studied the delay differential equations

$$(\varrho(\top)(x''(\top))^{\alpha})' - \eta_1(\top)\,x^{\alpha}(b(\top)) - \eta_2(\top)\,x^{\alpha}(w(\top)) = 0,$$

where $\alpha \geq 1$ is the ratio of odd positive integers and $w(\top) \geq \top$. The author established new oscillation criteria through comparison with first-order equations whose oscillatory behavior is known.

For the neutral delay equation

$$(\varrho(Q'')^{\mu})'(\top) + \eta(\top)\,x^{\mu}(b(\top)) = 0, \tag{1.8}$$

where μ is a quotient of odd positive integers, and

$$0 \leq R(\top) \leq R < 1,$$

the oscillation and asymptotic properties were discussed in [27, Corollary 1].

Theorem 1.1. *[27, Corollary 1] Let $x(\top)$ be a solution of (1.8). Assume that $\varrho'(\top) \geq 0$, and*

$$\int_{\top_0}^{\infty}\int_{\upsilon}^{\infty}\left(\varrho^{-1}(u)\int_u^{\infty}\widetilde{\eta}(s)\,\mathrm{d}s\right)^{1/\mu}\mathrm{d}u\,\mathrm{d}\upsilon = \infty. \tag{1.9}$$

If

$$\liminf_{\top\to\infty}\frac{\top^{\mu}}{\varrho(\top)}\int_{\top}^{\infty}\eta(s)\,\frac{b^{2\mu}(s)}{s^{\mu}}\mathrm{d}s > \frac{(2\mu)^{\mu}}{(\mu+1)^{\mu+1}(1-R(s))^{\mu}}, \tag{1.10}$$

then (1.1) has property F.

On the other hand, several studies have established oscillation criteria for equation (1.1) and its special cases using Riccati transformation techniques (see for example [28, 29]).

This paper introduces new insights into the oscillatory behavior of solutions of third-order differential equations by extending classical results using advanced analytical techniques. We derive precise oscillation criteria that apply to a broader range of equations, thus contributing to the advancement of both the theoretical framework of oscillation theory and its practical applications in various scientific disciplines. Specifically, our work refines the inequalities relating the function $Q(\top)$ to its higher derivatives, providing more precise criteria that ensure the absence of non-oscillatory solutions. Furthermore, we present criteria that guarantee oscillation without relying on the additional constraints typically required in the existing literature, such as condition (1.7). The examples mentioned in the final section highlight the novelty and effectiveness of our results.

2. Preliminary results

The following lemmas play a fundamental role in proving the subsequent results.

Lemma 2.1. *[30, Lemma 3] Suppose that* $g_1, g_2, g_3 \in C([\top_0,\infty),\mathbb{R})$, $\lim_{\top\to\infty} g_1(\top)$ *exists*, $g_3(\top) \le \top$,

$$\lim_{\top\to\infty} g_3(\top) = \infty \ \textit{for all}\ \top \in [\top_0,\infty),$$

and

$$\liminf_{\top\to\infty} g_2(\top) > -1.$$

Moreover, assume that there is a function $y \in C([\top_*,\infty),\mathbb{R}^+)$*, where* $\top_* := \min_{\top\in[\top_0,\infty)}\{g_3(\top)\}$*, such that*

$$y(\top) - g_1(\top) - g_2(\top)\, y(g_3(\top)) = 0, \ \textit{for all}\ \top \in [\top_0,\infty).$$

If $\limsup_{\top\to\infty} y(\top) > 0$*, then*

$$\lim_{\top\to\infty} g_1(\top) > 0.$$

Lemma 2.2. *[31] Assume that x is a nonoscillatory solution of (1.1). Then the corresponding function Q satisfies*

$$Q(\top) > 0,\ Q''(\top) > 0,\ (\varrho(\top)(Q''(\top)))' \le 0.$$

Moreover, there exist only two possible cases for the first derivative of Q:

$$Q'(\top) < 0 \tag{2.1}$$

and

$$Q'(\top) > 0. \tag{2.2}$$

3. Main results

In the section, we will adopt the following notation:

$$\beta_1(\top,\top_1) = \int_{\top_1}^{\top} \varrho^{-1}(s)\,\mathrm{d}s,$$

$$\beta_2(\top, \top_1) = \int_{\top_1}^{\top} \int_{\top_1}^{s} \varrho^{-1}(u)\, \mathrm{d}u\mathrm{d}s,$$

and

$$\widetilde{\eta}(\top) = \min\{\eta(\top), \eta(\iota(\top))\}. \tag{3.1}$$

Lemma 3.1. *Suppose that $x(\top) > 0$ is a solution of Eq (1.1), and*

$$\lim_{\top\to\infty} x(\top) \neq 0. \tag{3.2}$$

If

$$\int_{\top_0}^{\infty} \int_{\upsilon}^{\infty} \left(\varrho^{-1}(\iota(u)) \int_{u}^{\infty} \widetilde{\eta}(s)\, \mathrm{d}s\right) \mathrm{d}u\mathrm{d}\upsilon = \infty, \tag{3.3}$$

then (2.2) holds.

Proof. Let $x > 0$ be a solution of (1.1). From (1.2), we see that

$$(\varrho(\top)(Q''(\top)))' = -\eta(\top)\, x(b(\top)) \leq 0.$$

Using $\iota(\top) \leq \top$, we have

$$\varrho(\top)(Q''(\top)) \leq \varrho(\iota(\top))(Q''(\iota(\top))). \tag{3.4}$$

Hence, $(\varrho(\top)(Q''(\top)))' \leq 0$ and has one sign. Also, $Q''(\top)$ has one sign, and so either $Q''(\top) > 0$ or $Q''(\top) < 0$ for $\top \geq \top_1$. Let $Q''(\top) < 0$. Then there is a constant $M > 0$ such that

$$\varrho(\top)(Q''(\top)) + M \leq 0. \tag{3.5}$$

Integrating (3.5) from $\top_1$ to $\top$, we get

$$Q'(\top) - Q'(\top_1) \leq -M \int_{\top_1}^{\top} \varrho^{-1}(s)\, \mathrm{d}s.$$

Therefore, $\lim_{\top\to\infty} Q'(\top) = -\infty$. Using the fact that $Q'(\top)$ and $Q''(\top)$ are negative, we note that $\lim_{\top\to\infty} Q(\top) = -\infty$. This contradiction leads to $Q''(\top) > 0$.

Now, Let $Q'(\top) < 0$. According to (1.1), we obtain

$$(\varrho(\top)(Q''(\top)))' + \frac{R_0(\varrho(\iota(\top))(Q''(\iota(\top))))'}{\iota_0} \leq -\eta(\top)\, x(b(\top)) - R_0\eta(\iota(\top))\, x(b(\iota(\top))).$$

Set

$$\widetilde{Q}(\top) = (\varrho(\top)(Q''(\top)))' + \frac{R_0(\varrho(\iota(\top))(Q''(\iota(\top))))'}{\iota_0}.$$

That is,

$$\begin{aligned} \widetilde{Q}(\top) &\leq -(\eta(\top)\, x(b(\top)) + R_0\eta(\iota(\top))\, x(b(\iota(\top)))) \\ &\leq -\widetilde{\eta}(\top)(x(b(\top)) + R_0 x(b(\iota(\top)))). \end{aligned}$$

In view of (3.1), we have

$$\widetilde{Q}(\top) + Q(b(\top))\widetilde{\eta}(\top) \leq 0. \tag{3.6}$$

Integrating (3.6) from $\top$ to ∞, we get

$$\varrho(\top)\left(Q''(\top)\right)+\frac{R_0\varrho(\iota(\top))\left(Q''(\iota(\top))\right)}{\iota_0}-\int_{\top}^{\infty}Q(b(s))\widetilde{\eta}(s)\,\mathrm{d}s\leq 0.$$

Using (3.4), we have

$$\varrho(\iota(\top))\left(Q''(\iota(\top))\right)-\frac{\iota_0}{\iota_0+R_0}\int_{\top}^{\infty}Q(b(s))\widetilde{\eta}(s)\,\mathrm{d}s\geq 0.$$

Based on (3.2) and Lemma 2.1, and since $Q(b(\top))\geq L$ and $\lim_{\top\to\infty}Q(\top)=L>0$, then we obtain

$$Q''(\iota(\top))-\frac{L\iota_0}{\iota_0+R_0}\varrho^{-1}(\iota(u))\int_{\top}^{\infty}\int_{a}^{b}\widetilde{\eta}(s)\,\mathrm{d}s\geq 0.$$

Integrating from $\top$ to ∞, we get

$$-\frac{Q'(\iota(\top))}{\iota_0}-\frac{L\iota_0}{\iota_0+R_0}\int_{\top}^{\infty}\varrho^{-1}(\iota(u))\int_{\top}^{\infty}\widetilde{\eta}(s)\,\mathrm{d}s\mathrm{d}u\geq 0. \tag{3.7}$$

Integrating again (3.7) from $\top_1$ to ∞, we find

$$-\frac{Q(\iota(\top_1))}{\iota_0}-\frac{L\iota_0}{\iota_0+R_0}\int_{\top_1}^{\infty}\int_{v}^{\infty}\varrho^{-1}(\iota(u))\left(\int_{u}^{\infty}\widetilde{\eta}(s)\,\mathrm{d}s\right)\mathrm{d}u\mathrm{d}v\geq 0,$$

which implies that $Q'(\top)$ is positive. This completes the proof. □

Lemma 3.2. *Suppose that Q is satisfied (2.2) for $\top\geq\top_1$. Then*

$$\varrho(\top)\beta_1(\top,\top_1)\leq\frac{Q'(\top)}{Q''(\top)}, \tag{3.8}$$

and

$$\varrho(\top)\beta_2(\top,\top_1)\leq\frac{Q(\top)}{Q''(\top)}. \tag{3.9}$$

Proof. Since $\varrho(\top)\left(Q''(\top)\right)$ is nonincreasing, we have

$$Q'(\top)\geq\int_{\top_1}^{\top}\frac{1}{\varrho(s)}\left(\varrho(s)\left(Q''(s)\right)\right)\mathrm{d}s\geq\varrho(s)\left(Q''(s)\right)\int_{\top_1}^{\top}\frac{1}{\varrho(s)}\mathrm{d}s,$$

which implies that

$$Q(\top)-\varrho(s)Q''(s)\int_{\top_1}^{\top}\left(\int_{\top_1}^{s}\frac{1}{\varrho(u)}\mathrm{d}u\right)\mathrm{d}s\geq 0.$$

This completes the proof. □

In the following results, we will assume that there exists a function $\rho\in C^1([\top_0,\infty),(0,\infty))$, for all $\top_1\geq\top_0$.

Theorem 3.1. *Suppose that (3.3) is satisfied,* $b(\top) \leq \iota(\top)$, *and* $b'(\top) > 0$. *If*

$$\limsup_{\top\to\infty} \int_{\top_2}^{\top} \left[\rho(s)\widetilde{\eta}(s) - \frac{(\iota_0 + R_0)}{4\iota_0 b'(s)\rho(s)\beta_1(b(s),\top_1)}\left((\rho'(s))_+\right)^2\right] \mathrm{d}s = \infty, \tag{3.10}$$

for $\top_2 > \top_1$, *then (1.1) has property* F.

Proof. Assume that $x > 0$ is a solution of (1.1). As in the proof of Lemma 3.1, from (2.2) in Lemma 2.2 and (3.6), and by using Lemma 3.2, we get (3.8). Set

$$\omega(\top) = \frac{\varrho(\top) Q''(\top)}{Q(b(\top))}\rho(\top) > 0. \tag{3.11}$$

From Lemma 3.1, we see that

$$\begin{aligned} \omega'(\top) &= \frac{\varrho(\top) Q''(\top)}{Q(b(\top))}\rho'(\top) + \left(\frac{\varrho(\top)(Q''(\top))}{Q(b(\top))}\right)'\rho(\top) \\ & \frac{\varrho(\top) Q''(\top)}{Q(b(\top))}\rho'(\top) + \frac{(\varrho(\top)(Q''(\top)))'}{Q(b(\top))}\rho(\top) \\ & -\frac{\varrho(\top)(Q''(\top)) Q'(b(\top))\rho(\top) b'(\top)}{Q^2(b(\top))} \\ &= \frac{\varrho(\top) Q''(\top)}{Q(b(\top))}\rho'(\top) + \frac{(\varrho(\top)(Q''(\top)))'}{Q(b(\top))}\rho(\top) \\ & -\frac{\varrho(\top) Q''(\top) Q'(b(\top))}{Q^2(b(\top))}\rho(\top) b'(\top). \end{aligned} \tag{3.12}$$

In view of (2.2), (3.8), and $b(\top) \leq \top$, we obtain

$$\begin{aligned} Q'(b(\top)) &\geq \varrho(b(\top)) Q''(b(\top))\beta_1(b(\top),\top_1) \\ &\geq (\varrho(\top) Q''(\top))\beta_1(b(\top),\top_1). \end{aligned}$$

Using (3.11) and (3.12), we get

$$\omega'(\top) \leq \rho(\top)\frac{(\varrho(\top)(Q''(\top)))'}{Q(b(\top))} + \frac{1}{\rho(\top)}\rho'(\top)\omega(\top) - \frac{\beta_1(b(\top),\top_1)}{\rho(\top)}b'(\top)\omega^2(\top). \tag{3.13}$$

Define another function as follows:

$$\nu(\top) = \frac{\varrho(\iota(\top)) Q''(\iota(\top))}{Q(b(\top))}\rho(\top). \tag{3.14}$$

That is, according to Lemma 3.1, $\nu(\top) > 0$, and

$$\begin{aligned} \nu'(\top) &= \rho'(\top)\frac{\varrho(\iota(\top))(Q''(\iota(\top)))}{Q(b(\top))} + \rho(\top)\left(\frac{\varrho(\iota(\top))(Q''(\iota(\top)))}{Q(b(\top))}\right)' \\ &= \rho'(\top)\frac{\varrho(\iota(\top))(Q''(\iota(\top)))}{Q(b(\top))} + \frac{(\varrho(\iota(\top))(Q''(\iota(\top))))'}{Q(b(\top))}\rho(\top) \\ & -\frac{\varrho(\iota(\top))(Q''(\iota(\top))) Q'(b(\top)) b'(\top)\rho(\top)}{Q^2(b(\top))} \end{aligned}$$

$$= \rho'(\top)\varrho(\iota(\top))\frac{(Q''(\iota(\top)))}{Q(b(\top))} + \frac{(\varrho(\iota(\top))(Q''(\iota(\top))))'}{Q(b(\top))}\rho(\top)$$
$$-\rho(\top)\varrho(\iota(\top))\left(\frac{(Q''(\iota(\top)))Q'(b(\top))}{Q^2(b(\top))}\right)b'(\top). \quad (3.15)$$

Taking (2.2) and (3.8) with the fact that $b(\top) \le \iota(\top)$ into account, we have

$$Q'(b(\top)) \ge (\varrho(b(\top))Q''(b(\top)))\beta_1(b(\top),\top_1) \ge (\varrho(\iota(\top))Q''(\iota(\top)))\beta_1(b(\top),\top_1),$$

which from (3.14) and (3.15) implies that

$$\nu'(\top) \le \frac{(\varrho(\iota(\top))(Q''(\iota(\top))))'}{Q(b(\top))}\rho(\top) + \nu(\top)\frac{(\rho'(\top))_+}{\rho(\top)} - b'(\top)\nu^2(\top)\frac{\beta_1(b(\top),\top_1)}{\rho(\top)}. \quad (3.16)$$

Using (3.13) and (3.16), we obtain

$$\begin{aligned}\omega'(\top) + \upsilon'(\top)\frac{R_0(\top)}{\iota_0} \le\ & \frac{\rho(\top)}{Q(b(\top))}(\varrho(\top)(Q''(\top)))' + \frac{R_0(\varrho(\iota(\top))(Q''(\iota(\top))))'}{\iota_0}\\ &+\omega(\top)\frac{(\rho'(\top))}{\rho(\top)} - \omega^2(\top)\frac{\beta_1(b(\top),\top_1)}{\rho(\top)}b'(\top)\\ &+\frac{R_0}{\iota_0}\left[\upsilon(\top)\frac{(\rho'(\top))}{\rho(\top)} - \upsilon^2(\top)\frac{\beta_1(b(\top),\top_1)}{\rho(\top)}b'(\top)\right]. \quad (3.17)\end{aligned}$$

By (3.6) and (3.17), we have

$$\begin{aligned}\omega'(\top) + \upsilon'(\top)\frac{R_0}{\iota_0} \le\ & -\rho(\top)\widetilde{\eta}(s) + \omega(\top)\frac{(\rho'(\top))_+}{\rho(\top)} - \omega^2(\top)\frac{\beta_1(b(\top),\top_1)b'(\top)}{\rho(\top)}\\ &+\frac{R_0}{\iota_0}\left[\upsilon(\top)\frac{(\rho'(\top))_+}{\rho(\top)} - \upsilon^2(\top)\frac{\beta_1(b(\top),\top_1)b'(\top)}{\rho(\top)}\right]. \quad (3.18)\end{aligned}$$

By applying the inequality stated in [32], namely,

$$\frac{B^2}{4A} \ge Bu - Au^2,\ A > 0. \quad (3.19)$$

That is

$$\frac{\rho(\top)}{4\beta_1(b(\top),\top_1)b'(\top)}\left(\frac{(\rho'(\top))_+}{\rho(\top)}\right)^2 \ge \omega(\top)\left(\frac{(\rho'(\top))_+}{\rho(\top)}\right) - \omega^2(\top)\left(\frac{\beta_1(b(\top),\top_1)b'(\top)}{\rho(\top)}\right)$$

and

$$\frac{\rho(\top)}{4\beta_1(b(\top),\top_1)b'(\top)}\left(\frac{(\rho'(\top))_+}{\rho(\top)}\right)^2 \ge \upsilon(\top)\left(\frac{(\rho'(\top))_+}{\rho(\top)}\right) - \upsilon^2(\top)\left(\frac{\beta_1(b(\top),\top_1)b'(\top)}{\rho(\top)}\right),$$

which implies that

$$\omega'(\top) + \upsilon'(\top)\frac{R_0}{\iota_0} \le -\rho(\top)\widetilde{\eta}(s) + \frac{1}{4}\frac{((\rho'(\top))_+)^2}{\rho(\top)b'(\top)\beta_1(b(\top),\top_1)}$$

$$+\frac{R_0}{4\iota_0}\frac{((\rho'(\top))+)^2}{\rho(\top)b'(\top)\beta_1(b(\top),\top_1)}. \tag{3.20}$$

Integrating from $\top_2$ to $\top$, we see that

$$\begin{aligned}&\int_{\top_2}^{\top}\left(\rho(s)\widetilde{\eta}(s)-\frac{\iota_0+R_0}{4\iota_0}\frac{((\rho'(\top))_+)^2}{\rho(\top)b'(s)\beta_1(b(s),\top_1)}\right)\mathrm{d}s\\ \leq\ & \omega(\top_2)+v(\top_2)\frac{R_0}{\iota_0},\end{aligned}$$

which is a contradiction with (3.10). The proof is complete. □

Theorem 3.2. *Let (3.3) be satisfied and $b(\top)\geq\iota(\top)$. If*

$$\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left[\rho(s)\widetilde{\eta}(s)-\frac{(\iota_0+\rho_0)((\rho'(s))_+)^2}{4\iota_0\iota'(s)\rho(s)\beta_1(\iota(s),\top_1)}\right]\mathrm{d}s=\infty, \tag{3.21}$$

for $\top_2>\top_1$, then (1.1) has property F.

Proof. Assume that $x>0$ is a solution of (1.1). As in the proof of Lemma 3.1, we get (2.2) in Lemma 2.2 and (3.6). By Lemma 3.2, we obtain (3.8). Define

$$\omega(\top)=\frac{\varrho(\top)Q''(\top)}{Q(\iota(\top))}\rho(\top). \tag{3.22}$$

That is,

$$\begin{aligned}\omega'(\top) &= \frac{\varrho(\top)Q''(\top)}{Q(\iota(\top))}\rho'(\top)+\left(\frac{\varrho(\top)(Q''(\top))}{Q(\iota(\top))}\right)'\rho(\top)\\ &= \frac{\varrho(\top)Q''(\top)}{Q(\iota(\top))}\rho'(\top)+\frac{(\varrho(\top)(Q''(\top)))'\rho(\top)}{Q(\iota(\top))}\\ &\quad-\frac{\varrho(\top)(Q''(\top))Q'(\iota(\top))\rho(\top)\iota'(\top)}{Q^2(\iota(\top))}\\ &= \frac{\varrho(\top)Q''(\top)}{Q(\iota(\top))}\rho'(\top)+\frac{(\varrho(\top)(Q''(\top)))'}{Q(\iota(\top))}\rho(\top)\\ &\quad-\frac{\varrho(\top)Q''(\top)Q'(\iota(\top))}{Q^2(\iota(\top))}\rho(\top)\iota'(\top).\end{aligned} \tag{3.23}$$

It follows from (2.2), (3.8), and $\iota(\top)\leq\top$ that

$$\begin{aligned}Q'(\iota(\top)) &\geq \varrho(\iota(\top))Q''(\iota(\top))\beta_1(\iota(\top),\top_1)\\ &\geq \varrho(\iota(\top))Q''(\top)\beta_1(\iota(\top),\top_1).\end{aligned}$$

Using (3.22) and (3.23) gives

$$\omega'(\top) \leq \frac{(\varrho(\top)(Q''(\top)))'}{Q(\iota(\top))}\rho(\top)+\omega(\top)\frac{\rho'(\top)}{\rho(\top)}$$

$$-\omega^{2}(\top)\frac{\iota'(\top)\beta_{1}(\iota(\top),\top_{1})}{\rho(\top)}. \tag{3.24}$$

Now, define the following positive function:

$$v(\top)=\frac{\varrho(\iota(\top))Q''(\iota(\top))}{Q(\iota(\top))}\rho(\top). \tag{3.25}$$

That is

$$\begin{aligned}
v'(\top) &= \frac{(Q''(\iota(\top)))}{Q(b(\top))}\rho'(\top)\varrho(\iota(\top))+\left(\frac{\varrho(\iota(\top))(Q''(\iota(\top)))}{Q(\iota(\top))}\right)'\rho(\top)\\
&= \frac{\varrho(\iota(\top))(Q''(\iota(\top)))}{Q(b(\top))}\rho'(\top)+\frac{(\varrho(\iota(\top))(Q''(\iota(\top))))'\rho(\top)}{Q(\iota(\top))}\\
&\quad -\frac{\varrho(\iota(\top))(Q''(\iota(\top)))Q'(\iota(\top))}{Q^{2}(\iota(\top))}\rho(\top)\iota'(\top)\\
&= \frac{\varrho(\iota(\top))Q''(\iota(\top))}{Q(\iota(\top))}\rho'(\top)+\frac{(\varrho(\iota(\top))(Q''(\iota(\top))))'}{Q(\iota(\top))}\rho(\top)\\
&\quad -Q''(\iota(\top))\frac{Q'(\iota(\top))}{Q^{2}(\iota(\top))}\rho(\top)\varrho(\iota(\top))\iota'(\top).
\end{aligned} \tag{3.26}$$

From (2.2) and (3.8), we have

$$Q'(\iota(\top))\geq\beta_{1}(\iota(s),\top_{1})\varrho(\iota(\top))Q''(\iota(\top)),$$

which from (3.25) and (3.26) implies

$$\begin{aligned}
v'(\top) &\leq \frac{(\varrho(\iota(\top))(Q''\iota(\top)))'}{Q(\iota(\top))}\rho(\top)+\frac{\rho'(\top)}{\rho(\top)}v(\top)\\
&\quad -\beta_{1}(\iota(\top),\top_{1})\frac{\iota'(\top)}{\rho(\top)}v^{2}(\top).
\end{aligned} \tag{3.27}$$

Due to (3.24) and (3.27), we have

$$\begin{aligned}
\omega'(\top)+\upsilon'(\top)\frac{R_{0}}{\iota_{0}} &\leq \frac{(\varrho(\top)(Q''(\top)))'}{Q(\iota(\top))}\rho(\top)+\frac{R_{0}(\varrho(\iota(\top))(Q''(\iota(\top))))'}{\iota_{0}Q(\iota(\top))}\\
&\quad +\omega(\top)\frac{(\rho'(\top))_{+}}{\rho(\top)}-\omega^{2}(\top)\frac{\iota'(\top)}{\rho(\top)}\beta_{1}(\iota(\top),\top_{1})\\
&\quad +\frac{R_{0}}{\iota_{0}}\left[\upsilon(\top)\frac{(\rho'(\top))_{+}}{\rho(\top)}-v^{2}(\top)\beta_{1}(\iota(\top),\top_{1})\frac{\iota'(\top)}{\rho(\top)}\right].
\end{aligned} \tag{3.28}$$

From (2.2), (3.6), (3.28), and $b(\top)\geq\iota(\top)$, we obtain

$$\begin{aligned}
\omega'(\top)+\upsilon'(\top)\frac{R_{0}}{\iota_{0}} &\leq -\rho(\top)\widetilde{\eta}(s)+\omega(\top)\frac{(\rho'(\top))_{+}}{\rho(\top)}-\omega^{2}(\top)\frac{\iota'(\top)\beta_{1}(\iota(\top),\top_{1})}{\rho(\top)}\\
&\quad +\frac{R_{0}}{\iota_{0}}\left(v(\top)\frac{(\rho'(\top))_{+}}{\rho(\top)}-v^{2}(\top)\frac{\iota'(\top)\beta_{1}(\iota(\top),\top_{1})}{\rho(\top)}\right).
\end{aligned} \tag{3.29}$$

By (3.29) and (3.19), we get

$$\omega'(\top)+\upsilon'(\top)\frac{R_0}{\iota_0} \le -\rho(\top)\widetilde{\eta}(s)+\frac{((\rho'(\top))_+)^2}{4\iota'(\top)\rho(\top)\beta_1(\iota(\top),\top_1)} +\frac{R_0}{\iota_0}\frac{(\rho'(\top))_+^2}{4\iota'(\top)\rho(\top)\beta_1(\iota(\top),\top_1)}. \tag{3.30}$$

Integrating (3.30) from $\top_2$ to $\top$, it is clear that

$$\int_{\top_2}^{\top}\left(\rho(s)\widetilde{\eta}(s)-\frac{(\iota_0+R_0)((\rho'(s))_+)^2}{4\iota_0(\rho(s)\beta_1(\iota(s),\top_1)\iota'(s))}\right)\mathrm{d}s \le \omega(\top_2)+\frac{R_0}{\iota_0}v(\top_2),$$

which contradicts (3.21). The proof is complete. □

Remark 3.1. *From Theorems 3.1 and 3.2, we can get some oscillation criteria for (1.1) with different choices of ρ.*

3.1. Absence of solutions in class (2.1)

Theorem 3.3. *If there exists a function $\zeta(\top)\in C([\top_0,\infty),(0,\infty))$, $b(\top)<\zeta(\top)$, and $\iota^{-1}(\zeta(\top))<\top$ such that the first-order delay differential equation*

$$w'(\top)+\frac{\iota_0\widetilde{\eta}(\top)\beta_2(\zeta(\top),b(\top))}{\iota_0+R_0}w\left(\iota^{-1}(\zeta(\top))\right)=0 \tag{3.31}$$

is oscillatory, then case (2.1) cannot hold.

Proof. Assume that $x>0$ is a solution of (1.1). Let (2.1) in Lemma 2.2 hold. By (3.6), we obtain

$$\widetilde{Q}(\top)+Q(b(\top))\widetilde{\eta}(\top)\le 0. \tag{3.32}$$

It follows from $\varrho(\top)(Q''(\top))\ge 0$ that

$$\begin{aligned} -Q'(u) &\ge Q'(v)-Q'(u)=\int_u^v Q''(s)\frac{\varrho(s)}{\varrho(s)}\mathrm{d}s \\ &\ge \varrho(v)Q''(v)\int_u^v\frac{1}{\varrho(s)}\mathrm{d}s,\ \text{for } v\ge u\ge\top_1. \end{aligned}$$

From (3.9), we have

$$Q(b(\top))\ge\varrho(\zeta)(Q''(\zeta))\beta_2(\zeta,b(\top)),$$

which from (3.32) implies that

$$\widetilde{Q}(\top)\le-\widetilde{\eta}(\top)\beta_2(\zeta,b(\top))\varrho(\zeta)Q''(\zeta). \tag{3.33}$$

Since $\varrho(\top)Q''(\top)$ is nonincreasing, we get

$$\widetilde{Q}(\top)\le\frac{(\iota_0+R_0)\varrho(\iota(\top))Q''(\iota(\top))}{\iota_0}. \tag{3.34}$$

Hence, we have

$$\varrho(\iota(\top))\,Q''(\iota(\top)) \geq \frac{\iota_0\widetilde{Q}\left(\iota^{-1}(\zeta(\top))\right)}{\iota_0+R_0}. \tag{3.35}$$

Combining (3.34) and (3.33), we note that w is a positive solution of the first-order delay differential inequality

$$\left(\widetilde{Q}(\top)\right)' \leq \frac{\iota_0\widetilde{\eta}(\top)\beta_2(\zeta(\top),b(\top))}{\iota_0+R_0}\widetilde{Q}\left(\iota^{-1}(\zeta(\top))\right).$$

According to [33, Theorem 1], however, the Eq (3.31) has a positive solution, which is a contradiction. The proof is complete. □

Corollary 3.1. *If there exists a function* $\zeta(\top) \in C([\top_0,\infty),(0,\infty))$ *satisfying* $b(\top) < \zeta(\top)$ *and* $\iota^{-1}(\zeta(\top)) < \top$ *such that*

$$\liminf_{\top\to\infty}\int_{\iota^{-1}(\zeta(\top))}^{\top}\widetilde{\eta}(s)\beta_2(\zeta(s),b(s))\,\mathrm{d}s > \frac{\iota_0+R_0}{\iota_0 e}, \tag{3.36}$$

then case (2.1) cannot hold.

Proof. Based on the well-known criterion in [34, Theorem 2], we find that the delay differential equation (3.31) is oscillatory if the condition (3.36) is satisfied. The proof is complete. □

Theorem 3.4. *If there exists a function* $\eta(\top) \in C([\top_0,\infty),(0,\infty))$*,* $\eta(\top) < \top$ *and* $b(\top) < \iota(\eta(\top))$ *such that*

$$\liminf_{\top\to\infty}\beta_2(\iota(\eta(\top)),b(\top))\int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,\mathrm{d}s > \frac{\iota_0+R_0}{\iota_0}, \tag{3.37}$$

then case (2.1) cannot hold.

Proof. Following the same method as in the proof of Theorem 3.3 by integrating (3.32) from $\eta(\top)$ to $\top$ and using the fact that Q is decreasing, we see that

$$\begin{aligned} U'(\eta(\top)) &\geq \varrho(\top)\,Q''(\top) \\ &\quad +\iota_0^{-1}R_0\varrho(\iota(\top))\left(Q''(\iota(\top))\right)\int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,Q(b(s))\,\mathrm{d}s \\ &\geq \int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,Q(b(s))\,\mathrm{d}s \geq Q(b(\top))\int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,\mathrm{d}s, \end{aligned}$$

where

$$U(\eta(\top)) = \left(\varrho(\eta(\top))\left(Q''(\eta(\top))\right) + \iota_0^{-1}R_0\varrho(\iota(\eta(\top)))\left(Q''(\iota(\eta(\top)))\right)\right)'.$$

Since $\iota(\varrho(\top)\,Q''(\top))' \leq 0$ and $\eta(\top) < \iota(\top)$, we have

$$\varrho(\iota(\eta(\top)))\,Q''(\iota(\eta(\top)))\left(\iota_0^{-1}\iota_0+R_0\right) \geq Q(b(\top))\int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,\mathrm{d}s. \tag{3.38}$$

Using (3.9) with $u = b(\top)$ and $v = \iota(\eta(\top))$ in (3.38), we see that

$$\frac{\varrho(\iota(\eta(\top)))\left(Q''(\iota(\eta(\top)))\right)}{(\iota_0+R_0)^{-1}\iota_0} \geq \varrho(\iota(\eta(\top)))\left(Q''(\iota(\eta(\top)))\right)\beta_2(\iota(\eta(\top)),b(\top))\int_{\eta(\top)}^{\top}\widetilde{\eta}(s)\,\mathrm{d}s,$$

that is,

$$\iota_0^{-1}(\iota_0 + R_0) \geq \beta_2(\iota(\eta(\top)), b(\top)) \int_{\eta(\top)}^{\top} \widetilde{\eta}(s)\,\mathrm{d}s.$$

The proof is complete. □

Letting $\eta(\top) = \iota(\top)$ in Theorem 3.4, the following result is immediate.

Corollary 3.2. *If $b(\top) < \iota(\iota(\top))$ such that*

$$\limsup_{\top\to\infty} \beta_2(\iota(\iota(\top)), b(\top)) \int_{\eta(\top)}^{\top} \widetilde{\eta}(s)\,\mathrm{d}s > \iota_0^{-1}(\iota_0 + R_0),$$

then case (2.1) cannot hold.

Now, note that Theorems 3.1 and 3.2 guarantee that any solution of class (2.2) converges to zero when $\top \to \infty$, while conditions (3.36) and (3.37) remove solutions from class (2.1). Thus, by combining Theorems 3.1 and 3.2 with Conditions (3.36) and (3.37), respectively, one can easily provide new oscillation criteria for (1.1).

3.2. Philos-type oscillation criteria

In this section, we will establish some Philos-type oscillation results for (1.1).

Definition 3.1. *Assume that $D_0 = \{(\top, s) : \top > s \geq \top_0\}$ and $D = \{(\top, s) : \top \geq s \geq \top_0\}$, and $F \in C([D, \mathbb{R})$ satisfies the following:*
(I) $F(\top, s) > 0$, $(\top, s) \in D_0$, and $F(\top, \top) = 0$, for $\top \geq \top_0$;
(II) $\frac{\partial F}{\partial s}(\top, s)$, and $\frac{\partial F}{\partial s}(\top, s) \leq 0$.
Under these assumptions, the function F satisfies the property P.

We introduce the following functions:

$$G_1(\top, s) =: \frac{\theta(\top, s)}{\beta_1(b(\top), \top_1)\, b'(\top)}, \quad G_2(\top, s) =: \frac{\theta(\top, s)}{\beta_2(b(\top), \top_1)\, b'(\top)}$$

and

$$G_3(\top, s) = \frac{\theta(\top, s)}{\beta_1(\iota(\top), \top_1)\, \iota'(\top)}, \quad G_4(\top, s) = \frac{\theta(\top, s)}{\beta_2(\iota(\top), \top_1)\, \iota'(\top)},$$

where $\theta(\top, s) = \frac{(\iota_0 + R_0)(h(\top, s))^2}{4\iota_0 \rho(\top)}$.

Furthermore, in this section, suppose that $F \in C(D, \mathbb{R})$ has the property P and there exists a function $\rho \in C^1([\top_0, \infty), (0, \infty))$, for all $\top_1 \geq \top_0$.

Theorem 3.5. *Assume that (3.3) is satisfied, $b'(\top) > 0$, and $b(\top) \leq \iota(\top)$. If*

$$h(\top, s)\frac{(F(\top, s))^{\frac{1}{2}}}{\rho(s)} = -\frac{\partial}{\partial s}F(\top, s) - \frac{\rho'(s)}{\rho(s)}F(\top, s), \quad (\top, s) \in D_0, \tag{3.39}$$

and

$$\limsup_{\top\to\infty} F^{-1}(\top, \top_2) \int_{\top_2}^{\top} (F(\top, s)\rho(s)\widetilde{\eta}(s) - G_1(\top, s))\,\mathrm{d}s = \infty, \tag{3.40}$$

for $\top_2 > \top_1$, then (1.1) has property F.

Proof. Assume that $x > 0$ be a solution of (1.1). As in Theorem 3.1, by defining ω and υ, we get (3.18). From (3.18) and by replacing $(\rho'(\top))_+$ by $\rho'(\top)$, we obtain

$$\begin{aligned}\rho(\top)\widetilde{\eta}(s) \leq\ & -\omega'(\top) - \frac{R_0\upsilon'(\top)}{\iota_0} + \frac{\rho'(\top)\omega(\top)}{\rho(\top)} \\ & -\frac{\beta_1(b(s),\top_1)b'(s)}{\rho(\top)}\omega^2(\top) \\ & +\frac{R_0}{\iota_0}\left(\frac{\rho'(\top)\upsilon(\top)}{\rho(\top)} - \frac{\beta_1(b(s),\top_1)b'(s)}{\rho(\top)}\upsilon^2(\top)\right).\end{aligned} \tag{3.41}$$

Multiply both sides by $F(\top, s)$, then integrate from $\top_2$ to $\top$. It follows that

$$\begin{aligned}\int_{\top_2}^{\top}\rho(s)F(\top,s)\widetilde{\eta}(s)\,\mathrm{d}s \leq\ & -\int_{\top_2}^{\top}F(\top,s)\omega'(s)\,\mathrm{d}s + \int_{\top_2}^{\top}\frac{F(\top,s)\rho'(s)\omega(s)}{\rho(s)}\mathrm{d}s \\ & -\int_{\top_2}^{\top}\frac{F(\top,s)\beta_1(b(s),\top_1)b'(s)}{\rho(s)}\omega^2(\top)\,\mathrm{d}s \\ & -\frac{R_0}{\iota_0}\left(\int_{\top_2}^{\top}F(\top,s)\upsilon'(s)\,\mathrm{d}s + \int_{\top_2}^{\top}\frac{F(\top,s)\rho'(s)\upsilon(s)}{\rho(s)}\mathrm{d}s\right) \\ & -\frac{R_0}{\iota_0}\int_{\top_2}^{\top}\frac{F(\top,s)\beta_1(b(s),\top_1)b'(s)}{\rho(s)}\upsilon^2(s)\,\mathrm{d}s.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_{\top_2}^{\top}\rho(s)F(\top,s)\widetilde{\eta}(s)\,\mathrm{d}s \leq\ & F(\top,\top_2)\omega(\top_2) \\ & -\int_{\top_2}^{\top}\left(-\frac{\partial}{\partial(s)}F(\top,s) - \frac{\rho'(s)}{\rho(s)}F(\top,s)\right)\omega(s)\,\mathrm{d}s \\ & -\int_{\top_2}^{\top}F(\top,s)\frac{\beta_1(b(s),\top_1)b'(s)}{\rho(s)}\omega^2(s)\,\mathrm{d}s + \frac{R_0}{\iota_0}F(\top,\top_2)\upsilon(\top_2) \\ & -\frac{R_0}{\iota_0}\int_{\top_2}^{\top}\left[-\frac{\partial}{\partial(s)}F(\top,s) - \frac{1}{\rho(s)}\rho'(s)F(\top,s)\right]\upsilon(s)\,\mathrm{d}s \\ & -\int_{\top_2}^{\top}\frac{F(\top,s)\beta_1(b(s),\top_1)b'(s)}{\rho(s)}\upsilon^2(s)\,\mathrm{d}s,\end{aligned}$$

that is,

$$\begin{aligned}& \int_{\top_2}^{\top}F(\top,s)\rho(s)\widetilde{\eta}(s)\,\mathrm{d}s \\ \leq\ & F(\top,\top_2)\omega(\top_2) + \frac{R_0}{\iota_0}F(\top,\top_2)\upsilon(\top_2) \\ & +\int_{\top_2}^{\top}\left[\frac{h(\top,s)(F(\top,s))^{\frac{1}{2}}}{\rho(s)}\omega(s) - \frac{F(\top,s)\beta_1(b(s),\top_1)b'(\top)}{\rho(s)}\omega^2(s)\right]\mathrm{d}s \\ & +\frac{R_0}{\iota_0}\int_{\top_2}^{\top}\left[\frac{h(\top,s)(F(\top,s))^{\frac{1}{2}}}{\rho(s)}\upsilon(s) - F(\top,s)\frac{\beta_1(b(s),\top_1)b'(\top)}{\rho(s)}\upsilon^2(s)\right]\mathrm{d}s.\end{aligned} \tag{3.42}$$

Using (3.42) and (3.19), we find

$$F^{-1}(\top,\top_2)\int_{\top_2}^{\top}\left[F(\top,s)\rho(s)\widetilde{\eta}(s)-\frac{\iota_0+R_0}{4\iota_0}\frac{(h(\top,s))^2(F(\top,s))^2}{(\rho(s)\beta_1(b(s),\top_1)b'(s))}\right]\mathrm{d}s$$
$$\leq \omega(\top_2)+\frac{R_0\upsilon(\top_2)}{\iota_0},$$

and this contradicts (3.40). The proof is complete. □

Theorem 3.6. *Assume that (3.3) is satisfied, $b'(\top)>0$, and $b(\top)\leq\iota(\top)$. If*

$$\frac{h(\top,s)(F(\top,s))^{1/2}}{\rho(s)}=-\frac{\partial}{\partial s}F(\top,s)-\frac{\rho'(s)F(\top,s)}{\rho(s)},\quad(\top,s)\in D_0 \tag{3.43}$$

and

$$\limsup_{\top\to\infty}F^{-1}(\top,\top_2)\int_{\top_1}^{\top}(F(\top,s)\rho(s)\widetilde{\eta}(s)-G_2(\top,s))\,\mathrm{d}s=\infty, \tag{3.44}$$

for $\top_2>\top_1$, then (1.1) has property F.

Proof. Substituting (3.6) and (3.17) in Theorem 3.1, the derivation process is similar to the proof of Theorem 3.5. The proof is complete. □

Theorem 3.7. *Let (3.3) and (3.39) be satisfied, and $b(\top)\geq\iota(\top)$. If*

$$\limsup_{\top\to\infty}F^{-1}(\top,\top_2)\int_{\top_2}^{\top}(\rho(s)F(\top,s)\widetilde{\eta}(s)-G_3(\top,s)\,\mathrm{d}s)=\infty, \tag{3.45}$$

for $\top_2>\top_1$, then (1.1) has property F.

Proof. From (3.29) in Theorem 3.2, similar to the proof of that of Theorem 3.5. The proof is complete. □

Theorem 3.8. *Let (3.3) and (3.43) be satisfied and $b(\top)\geq\iota(\top)$. If*

$$\limsup_{\top\to\infty}F^{-1}(\top,\top_2)\int_{\top_2}^{\top}(F(\top,s)\rho(s)\widetilde{\eta}(s)-G_4(\top,s))\,\mathrm{d}s=\infty, \tag{3.46}$$

for $\top_2>\top_1$, then any solution x of (1.1) is oscillatory or satisfies (1.5).

Proof. From Theorem, similar to the proof of that of Theorem 3.5. The proof is complete. □

Remark 3.2. *From Theorems 3.5–3.8, we can obtain some oscillation criteria for (1.1) with different choices of ρ and F.*

In the following theorem, we extend our results to include the equation (1.8).

Theorem 3.9. *Suppose that (1.9) is satisfied, $b(\top)\leq\iota(\top)$, and $b'(\top)>0$. If*

$$\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left[\frac{\rho(s)\widetilde{\eta}(s)}{2^{\mu-1}}-\frac{\left(\iota_0+R_0^{\mu}\right)}{(\mu+1)^{\mu+1}\iota_0 b'(s)\rho(s)\beta_1(b(s),\top_1)}((\rho'(s))_+)^{\mu+1}\right]\mathrm{d}s=\infty,$$

then (1.1) has property F.

4. Examples

Example 4.1. *Consider the equation*

$$\left(\top^{-3}\left(\left[x(\top)+R_0x(0.5\top)\right]''\right)\right)'+\left(\top^{-6}\lambda\right)x(0.5\top)=0, \tag{4.1}$$

where $\lambda>0$, $R=R_0>0$. *Here,* $\varrho(\top)=\top^{-3}, b(\top)=\iota(\top)=0.5\top$, $\iota_0=0.5$, *and* $\eta(\top)=\lambda\top^{-6}$.

That is, we have

$$\widetilde{\eta}(\top)=\min\{\eta(\top),\eta(\iota(\top))\}=\lambda\top^{-6},$$

and

$$\beta_1(\top,\top_1):=\int_{\top_1}^{\top}s^3\mathrm{d}s\geq\top^4.$$

Thus, (3.3) holds. Putting $\rho(\top)=\top^5$ *and according to Theorem 3.1, we find*

$$\begin{aligned}&\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left[\rho(s)\widetilde{\eta}(s)-\frac{(\iota_0+R_0)}{4\iota_0b'(s)\rho(s)\beta_1(b(s),\top_1)}(\rho'(s))^2\right]\mathrm{d}s\\ \geq\ &\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left(\lambda s^{-1}-\frac{1}{4s}2^2\left(\frac{1}{2}+R_0\right)2^4.5^2\right)\mathrm{d}s=\infty,\end{aligned}$$

which implies that

$$\left(\lambda-2^3.5^2(2R_0+1)\right)\limsup_{\top\to\infty}\int_{\top_2}^{\top}s^{-1}\mathrm{d}s=\infty.$$

Thus, we conclude that any solution of (4.1) is oscillatory or satisfies (1.5) if

$$\lambda>2^3.5^2(2R_0+1). \tag{4.2}$$

Example 4.2. *Consider the equation*

$$\left(\top^{-4}\left[x(\top)+R_0x(1/6\top)\right]''\right)'+\lambda\top^{-6}x(1.5\top)=0, \tag{4.3}$$

where $\lambda>0$ *and* $R_0>0$. *Here,* $\varrho(\top)=\top, b(\top)=1.5\top, \iota(\top)=1/6\top$, *and* $\eta(\top)=\lambda\top^{-6}$. *It is easy to see that* $\iota'(\top)=\iota_0=1/6$,

$$\widetilde{\eta}(\top)=\min\{\eta(\top),\eta(\iota(\top))\}=\lambda\top^{-7},$$

and

$$\beta_1(\top,\top_1)=\int_{\top_1}^{\top}s^4\mathrm{d}s\geq\frac{1}{5}\top^5.$$

Putting $\rho(\top)=\top^6$ *and according to Theorem 3.2, we have*

$$\begin{aligned}&\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left[\rho(s)\widetilde{\eta}(s)-\frac{(\iota_0+\rho_0)\left((\rho'(s))_+\right)^2}{4\iota_0\iota'(s)\rho(s)\beta_1(\iota(s),\top_1)}\right]\mathrm{d}s\\ \geq\ &\limsup_{\top\to\infty}\int_{\top_2}^{\top}\left(s^6\lambda s^{-7}-\frac{(\iota_0+\rho_0)}{4\left(\frac{1}{5}\right)\left(\frac{1}{6^7}\right)}\frac{6^2s^{10}}{s^6s^5}\right)=\infty,\end{aligned}$$

which implies that

$$\int_{\top_2}^{\top}\left[\lambda-\frac{5.6^8\left(1+6\rho_0\right)}{4}\right]\limsup_{\top\to\infty}\int_{\top_2}^{\top}s^{-1}\mathrm{d}s=\infty.$$

Hence, if

$$\lambda>\frac{5.6^8\left(1+6\rho_0\right)}{4},$$

then any solution of (4.3) is oscillatory or satisfies (1.5).

Example 4.3. *Consider the equation*

$$\left(\top\left(\left[x(\top)+2x\left(\frac{\top}{2}\right)\right]''\right)^3\right)'+\left(\frac{\lambda}{\top^6}\right)x^3\left(\frac{\top}{2}\right)=0, \tag{4.4}$$

where $\lambda>0$, and $\mu=3$. Here, $\varrho(\top)=\top, b(\top)=\iota(\top)=0.5\top$, $\iota_0=0.5$, and $\eta(\top)=\lambda\top^{-6}$. It is not difficult to see that (1.9) holds. By choosing $\rho(\top)=\top^5$, and using Theorem 3.9, we conclude that any solution of (4.4) is oscillatory or satisfies (1.5) if

$$\lambda>5312.5. \tag{4.5}$$

Remark 4.1. *Figure 1 shows that the condition (4.5) ensures the appearance of oscillatory solutions of (4.4) by imposing a threshold on the parameter λ, which controls the system's behavior. For values of λ above this threshold, the system's dynamics become conducive to oscillations. As λ increases, the amplitude of the oscillations grows. In other words, larger values of λ result in more pronounced oscillations, potentially altering both their frequency and amplitude and making the system more responsive to changes in the initial conditions.*

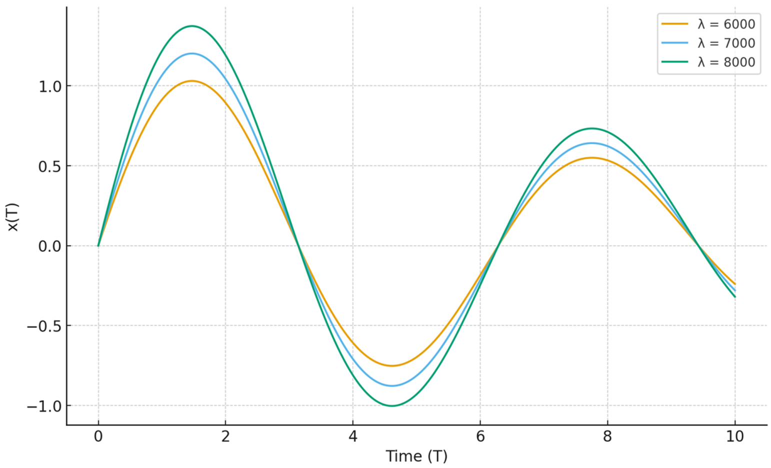

Figure 1. Numerical simulation of some solutions to Eq (4.4).

Example 4.4. *Consider the equation*

$$\left(\top\left(\left[x(\top)+R_0x(0.5\top)\right]''\right)^3\right)'+\left(\top^{-6}\lambda\right)x^3(0.5\top)=0, \tag{4.6}$$

where $\lambda > 0$, $R(\top) = R_0 > 0$, *and* $\mu = 3$. *Here,* $\varrho(\top) = \top, b(\top) = \iota(\top) = 0.5\top$, $\iota_0 = 0.5$, *and* $\eta(\top) = \lambda\top^{-6}$. *It is easy to see that*

$$\beta_1(\top, \top_1) := \int_{\top_1}^{\top} s^{-1/3} \mathrm{d}s \geq \top^{2/3}.$$

Thus, (1.9) holds. Set $\rho(\top) = \top^5$, *by Theorem 3.9, we conclude that any solution of (4.6) is oscillatory or satisfies (1.5) if*

$$\text{Condition } 1 : \lambda > \frac{2^5.\left(5^4\right)\left(2R_0^3 + 1\right)}{4^3}. \tag{4.7}$$

By Theorem 1.1, we see that (1.9) holds and

$$\liminf_{\top \to \infty} \frac{\top^2 \lambda}{2^6} \int_{\top}^{\infty} \frac{1}{s^3} \mathrm{d}s > \frac{(6)^3}{(4)^4 (1 - R_0)^3},$$

that is

$$\text{Condition } 2 : \lambda > \frac{2^7 (6)^3 (1 - R_0)^{-3}}{4^4}. \tag{4.8}$$

Based on the comparison presented in Table 1, we conclude that our results (Condition 1) significantly outperform the previous work (Condition 2) in [27, Corollary 1]. This improvement is particularly evident in regions where R_0 *exceeds* 0.312. *Furthermore, we observe that the magnitude of this enhancement increases as* R_0 *approaches higher values (Figure 2).*

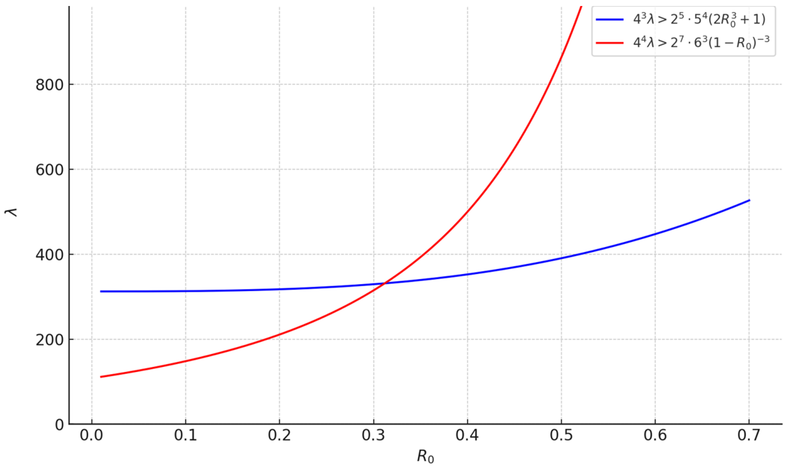

Figure 2. Behavior of the oscillation criteria across varying R_0 values.

Table 1. Numerical comparison of λ values required for oscillation.

	Special case of R_0					
	$R_0 = 0.05$	$R_0 = 0.1$	$R_0 = 0.312$	$R_0 = 0.4$	$R_0 = 0.5$	$R_0 = 0.51$
Condition 1 ($\lambda >$ ..)	312.58	313.13	331.49	352.50	390.63	395.41
Condition 2 ($\lambda >$ ..)	126.06	148.15	332.31	500.00	864.00	918.00

5. Conclusions

This study investigates the oscillatory behavior of solutions to Eq (1.1), building upon and refining previous work by relaxing the conditions on the functions associated with the equation. Specifically, we replace the commonly used conditions $0 \leq R(\top) < 1$, $-1 < R(\top) < 0$, and $-1 < R(\top) < 1$ studied in [22, 23, 30, 35] with a more general and flexible constraint, $0 \leq R(\top) \leq R_0 < \infty$. The proof incorporates Riccati assumptions in various forms, which facilitate the derivation of results that are applicable to a wide range of models. Additionally, we explore the Philos-type oscillatory behavior of Eq (1.1) through Theorems 3.5–3.8.

This study contributes to advancing the current body of knowledge by establishing clear and precise criteria for evaluating the nature of oscillatory solutions, thereby providing a solid foundation for future research. The application of the proposed methodology to higher-order differential equations, in particular, represents a fertile area for further exploration. Investigating oscillatory behavior within the framework of higher-order equations clearly enhances the understanding of the underlying mathematical structure of these systems and allows for the discovery of new patterns. Among the promising directions for future research, one could consider removing the constraint in (1.4) and expanding the scope of the methods presented here. This approach could also be applied to analyze more generalized equations by replacing the corresponding function $Q(\top)$ with

$$Q(\top) = x(\top) + \int_c^d R(\top, s)\, x(\iota(\top, s))\, \mathrm{d}s.$$

Author contributions

Conceptualization, A.A.-J., F.A. and B.Q.; methodology, A.A.-J., B.Q. and F.A.; validation, A.A.-J., B.Q. and F.A.; investigation, A.A.-J., B.Q.; resources, B.Q. and A.A.-J.; data curation, A.A.-J., B.Q.; writing—original draft preparation, A.A.-J., I.I., F.A. and B.Q.; writing—review and editing, B.Q., A.A.-J., I.I., D.C. and F.A.; visualization, A.A.-J., F.A., I.I. and B.Q.; supervision, A.A.-J., D.C., F.A. and B.Q.; project administration, B.Q. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Funding

This work was supported by Researchers Supporting Project number (PNURSP2025R406), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

Dimplekumar Chalishajar is the Guest Editor of special issue "Mathematical Control Theory, Analysis, and Applications" for AIMS Mathematics. Dimplekumar Chalishajar was not involved in the editorial review and the decision to publish this article.

All authors declare no conflicts of interest in this paper.

References

1. A. Bacciotti, L. Rosier, *Liapunov functions and stability in control theory*, 2Eds, Berlin, Heidelberg: Springer, 2005. https://doi.org/10.1007/b139028
2. H. Tsukamoto, S. J. Chung, J. J. E. Slotine, Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview, *Annu. Rev. Control*, **52** (2021), 135–169. https://doi.org/10.1016/j.arcontrol.2021.10.001
3. I. Jadlovská, G. E. Chatzarakis, J. Džurina, S. R. Grace, On sharp oscillation criteria for general third-order delay differential equations, *Mathematics*, **9** (2021), 1675. https://doi.org/10.3390/math9141675
4. J. R. Graef, I. Jadlovská, E. Tunç, Sharp asymptotic results for third-order linear delay differential equations, *J. Appl. Anal. Comput.*, **11** (2021), 2459–2472. https://doi.org/10.11948/20200417
5. S. R. Grace, Oscillation criteria for third order nonlinear delay differential equations with damping, *Opuscula Math.*, **35** (2015), 485–497. https://doi.org/10.7494/OpMath.2015.35.4.485
6. B. Baculíková, J. Džurina, Oscillation of the third order Euler differential equation with delay, *Math. Bohem.*, **139** (2014), 649–655. https://doi.org/10.21136/MB.2014.144141
7. J. K. Hale, *Theory of functional differential equations*, 2Eds, New York: Springer, 1977. https://doi.org/10.1007/978-1-4612-9892-2
8. R. C. Dorf, R. H. Bishop, *Modern control systems*, 13 Eds, Pearson Education, 2016.
9. W. Weaver, S. P. Timoshenko, D. H. Young, *Vibration problems in engineering*, Wiley, 1991.
10. K. Ogata, *Modern control engineering*, 5Eds, Prentice Hall, 2010.
11. B. Batiha, N. Alshammari, F. Aldosari, F. Masood, O. Bazighifan, Asymptotic and oscillatory properties for even-order nonlinear neutral differential equations with damping term, *Symmetry*, **17** (2025), 87. https://doi.org/10.3390/sym17010087
12. M. Aldiaiji, B. Qaraad, L. F. Iambor, E. M. Elabbasy, On the asymptotic behavior of class of third-order neutral differential equations with symmetrical and advanced argument, *Symmetry*, **15** (2023), 1165. https://doi.org/10.3390/sym15061165
13. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for difference and functional differential equations*, Springer Dordrecht, 2000. https://doi.org/10.1007/978-94-015-9401-1

14. A. Al-Jaser, C. Cesarano, B. Qaraad, L. F. Iambor, Second-order damped differential equations with superlinear neutral term: New criteria for oscillation, *Axioms*, **13** (2024), 234. https://doi.org/10.3390/axioms13040234

15. K. Gopalsamy, *Stability and oscillations in delay differential equations of population dynamics*, Springer Dordrecht, 1992. https://doi.org/10.1007/978-94-015-7920-9

16. J. Grawitter, R. van Buel, C. Schaaf, H. Stark, Dissipative systems with nonlocal delayed feedback control, *New J. Phys.*, **20** (2018), 113010. https://doi.org/10.1088/1367-2630/aae998

17. B. Baculíková, J. Džurina, Oscillation of third-order nonlinear differential equations, *Appl. Math. Lett.*, **24** (2011), 466–470. https://doi.org/10.1016/j.aml.2010.10.043

18. S. R. Grace, R. P. Agarwal, R. Pavani, E. Thandapani, On the oscillation of certain third-order nonlinear functional differential equations, *Appl. Math. Comput.*, **202** (2008), 102–112. https://doi.org/10.1016/j.amc.2008.01.025

19. S. H. Saker, J. Džurina, On the oscillation of certain class of third-order nonlinear delay differential equations, *Math. Bohem.*, **135** (2010), 225–237. https://doi.org/10.21136/MB.2010.140700

20. B. Batiha, N. Alshammari, F. Aldosari, F. Masood, O. Bazighifan, Nonlinear neutral delay differential equations: Novel criteria for oscillation and asymptotic behavior, *Mathematics*, **13** (2025), 147. https://doi.org/10.3390/math13010147

21. F. Masood, S. Aljawi, O. Bazighifan, Novel iterative criteria for oscillatory behavior in nonlinear neutral differential equations, *AIMS Mathematics*, **10** (2025), 6981–7000. https://doi.org/10.3934/math.2025319

22. T. Candan, Oscillation criteria and asymptotic properties of solutions of third-order nonlinear neutral differential equations, *Math. Method. Appl. Sci.*, **38** (2015), 1379–1392. https://doi.org/10.1002/mma.3153

23. T. Candan, Asymptotic properties of solutions of third-order nonlinear neutral dynamic equations, *Adv. Differ. Equ.*, **2014** (2014), 35. https://doi.org/10.1186/1687-1847-2014-35

24. O. Özdemir, Ş. Kaya, Comparison theorems on the oscillation of third-order functional differential equations with mixed deviating arguments in neutral term, *Differ. Equ. Appl.*, **14** (2022), 17–30. https://doi.org/10.7153/dea-2022-14-02

25. A. Al Themairi, B. Qaraad, O. Bazighifan, K. Nonlaopon, New conditions for testing the oscillation of third-order differential equations with distributed arguments, *Symmetry*, **14** (2022), 2416. https://doi.org/10.3390/sym14112416

26. S. R. Grace, New criteria on oscillatory behavior of third order half-linear functional differential equations, *Mediterr. J. Math.*, **20** (2023), 180. https://doi.org/10.1007/s00009-023-02342-0

27. B. Baculíková, J. Džurina, Oscillation of third-order neutral differential equations, *Math. Comput. Model.*, **52** (2010), 215–226. https://doi.org/10.1016/j.mcm.2010.02.011

28. Y. Wang, F. Meng, J. Gu, Oscillation criteria of third-order neutral differential equations with damping and distributed deviating arguments, *Adv. Differ. Equ.*, **2021** (2021), 515. https://doi.org/10.1186/s13662-021-03661-w

29. Y. Tian, Y. Cai, Y. Fu, T. Li, Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, *Adv. Differ. Equ.*, **2015** (2015), 267.

30. B. Karpuz, Ö. Öcalan, S. Öztürk, Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations, *Glasgow Math. J.*, **52** (2010), 107–114. https://doi.org/10.1017/S0017089509990188

31. J. Džurina, S. R. Grace, I. Jadlovská, On nonexistence of Kneser solutions of third-order neutral delay differential equations, *Appl. Math. Lett.*, **88** (2019), 193–200. https://doi.org/10.1016/j.aml.2018.08.016

32. S. Y. Zhang, Q. R. Wang, Oscillation of second-order nonlinear neutral dynamic equations on times cales, *Appl. Math. Comput.*, **216** (2010), 2837–2848. https://doi.org/10.1016/j.amc.2010.03.134

33. C. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math.*, **36** (1981), 168–178. https://doi.org/10.1007/BF01223686

34. Y. Kitamura, T. Kusano, Oscillation of first-order nonlinear differential equations with deviating arguments, *Proc. Amer. Math. Soc.*, **78** (1980), 64–68.

35. Z. Han, T. Li, C. Zhang, S. Sun, An oscillation criteria for third order neutral delay differential equations, *J. Appl. Anal.*, **16** (2010), 295–303.

AIMS Press