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Research article

Recovering of dual time-varying source terms in a system of coupled time-fractional diffusion equations

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Abstract: The present study is devoted to investigating the inverse problem of simultaneously reconstructing two source terms that depend solely on time in a system of coupled fractional reaction–diffusion equations. Such coupled systems are fundamental for modelling multispecies anomalous diffusion processes, where the evolution of each state variable is governed by unknown time-varying sources. The inverse problem is tackled using supplementary measurements of the state variables over the spatial domain. The coupling between the two equations presents a distinct complexity, making the simultaneous reconstruction problem notably challenging and important. Results on the unique solvability, are established by employing the Rothe method, whereby the problem is first discretized in time and then stability results for the semi-discrete approximations are derived. These estimates, together with compactness arguments, are then used to rigorously prove the convergence of Rothe approximations to a unique weak solution. Finally, the proposed method's effectiveness and stability are further validated through a series of numerical simulations.

Keywords: inverse source problem; coupled fractional system; temporal source identification; Rothe method; numerical reconstruction
Mathematics Subject Classification: 35R30, 35R11, 35K57, 65M32

1. Introduction

Diffusion and transport phenomena involving multiple interacting species have attracted considerable attention and arise naturally in a wide range of applications, serving as fundamental tools for modelling complex phenomena in physics, biology, and the social sciences. They provide a mathematical framework for describing the simultaneous evolution of several interacting processes, where the dynamics of each component are influenced not only by its own state but also by the behaviour of other variables in the system. Such models appear, for example, in fluid dynamics [1], biology [2, 3], chemical reactions [4, 5], physics, and material sciences [6]. In particular, coupled parabolic systems are often employed to capture diffusion and reaction mechanisms between multiple species or physical quantities. The interaction terms describe how different components exchange mass, energy, or momentum, while diffusion operators account for spatial spreading. Fractional time derivatives endow diffusion models with the capacity to represent memory and anomalous diffusion, leading to a more physically realistic portrayal of complex phenomena in heterogeneous media (for or an overview on this topic, see [7]). Coupled fractional diffusion systems naturally arise in various applications. For examples, in biology, interactions between multiple species, such as predator–prey or competitive populations, can be modelled using coupled fractional diffusion–reaction equations, where memory effects influence the dynamics. In materials science, the diffusion of multiple interacting chemical species or ions in heterogeneous media often requires coupled fractional models to capture nonlocal transport behavior. Other examples include coupled transport in porous media, heat and mass transfer in multi–phase materials, and anomalous diffusion processes in complex systems (see e.g., [8, 9]).

Recent research, several studies [10, 11], have reported that anomalous diffusion models can provide a more accurate description of the experimental observations. For instance, anomalous diffusion frequently arises in materials exhibiting memory effects, such as in the viscoelastic substances, as well as kinetics of particles moving in quenched random force fields and in polymer physics (e.g., [12, 13]). In these contexts, the presence of structural heterogeneities, trapping mechanisms, and long-range correlations in particle motion often give rise to deviations from classical Gaussian diffusion, thereby necessitating fractional-order models for a more accurate description.

Owing to the rich mathematical framework provided by fractional-derivative models, whether governed by a single scalar equation or by a coupled system of equations, for capturing complex phenomena, their study has attracted increasing attention in recent years. This growing interest is evidenced by the extensive body of research devoted to the subject, ranging from theoretical investigations to numerical analyses.

The focus of this paper is on addressing an initial–boundary value problem governed by a system of two coupled time-fractional diffusion equations and, in particular, formulating and studying a corresponding inverse problem of simultaneously recovering two time-dependent source terms.

Let $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) be a bounded open set with a sufficiently smooth boundary $\partial\Omega$, and where $T < \infty$ stands for a fixed final time. In this paper, we are interested in the following problem for two coupled time-fractional parabolic equations:

$$\begin{cases} \partial_t^\gamma u - \Delta u + a(x,t)u + b(x,t)v = \mathcal{F}(x,t), & (x,t) \in \Omega \times (0,T), \\ \partial_t^\gamma v - \Delta v + c(x,t)v + d(x,t)u = \mathcal{G}(x,t), & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.1}$$

supplemented with the homogeneous Neumann boundary conditions and initial data

$$\begin{cases} \nabla u(x,t) \cdot \vec{n} = 0, \ \nabla v(x,t) \cdot \vec{n} = 0, \ (x,t) \in \Gamma \times (0,T), \\ u(x,0) = \tilde{u}_0(x), \ v(x,0) = \tilde{v}_0(x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

We assume throughout this paper that the coefficients $a, b, c, d : \Omega \times (0, T) \to \mathbb{R}$ are uniformly positive and bounded, with essentially bounded time derivatives. More precisely, there are positive constants for which the following conditions hold:

$$\begin{aligned} &(H_1) \quad a, b, c, d \in W^{1,\infty}(0, T; L^\infty(\Omega)), \\ &(H_2) \quad a_0 \le a(x,t) \le a_1, \;\; b_0 \le b(x,t) \le b_1, \;\; c_0 \le c(x,t) \le c_1, \;\; d_0 \le d(x,t) \le d_1, \\ &(H_3) \quad \min(a_0, c_0) \ge \frac{1}{2}(b_1 + d_1). \end{aligned} \tag{1.3}$$

Here, ∂_t^γ denotes the Caputo fractional derivative of order $\gamma \in (0, 1)$, defined as follows:

$$\partial_t^\gamma z(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma}\, \partial_s z(s)\, ds, \quad t > 0.$$

Furthermore, the Caputo derivative introduced above has a representation as a convolution, namely

$$\partial_t^\gamma z(t) = (\omega^\gamma * \partial_t z)(t), \quad t > 0,$$

where the kernel is given by $\omega^\gamma(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$ for all $t > 0$, and the symbol $*$ denotes the standard convolution product. It is worth noting that the kernel $\omega^\gamma \in L^1(0, T)$ is non-negative for $t > 0$ and exhibits a singularity at $t = 0$. Moreover, it satisfies (see [14, Corollary 2.2])

$$\partial_t \omega^\gamma(t) \le 0, \quad \partial_{tt} \omega^\gamma(t) \ge 0, \quad \text{for all } t > 0.$$

In what follows, we assume that the source terms $\mathcal{F}(x,t)$ and $\mathcal{G}(x,t)$ admit the following representation:

$$\mathcal{F}(x,t) = p(t) f(x) + F(x,t), \quad \mathcal{G}(x,t) = q(t) g(x) + G(x,t), \quad (x,t) \in \Omega \times [0, T].$$

In the system (1.1), the functions u and v represent interacting species or state variables, while the parameters $a, b, c,$ and d capture the reaction couplings between them. The source terms $\mathcal{F}(x,t)$ and $\mathcal{G}(x,t)$ can be interpreted as external inputs or forcing terms. When both $\mathcal{F}(x,t)$ and $\mathcal{G}(x,t)$ are fully known, the task reduces to determining the solutions u and v, which corresponds to the direct problem. In contrast, when some components of these functions are unknown and need to be identified, one must rely on the available measurements of the system, which leads to the inverse problem studied in this paper. We emphasize that, in the present work, the unknown source terms considered are $p(t)$ and $q(t)$, while $f(x), g(x), F(x,t)$, and $G(x,t)$ are assumed to be known. This situation gives rise to an inverse problem (for a comprehensive overview of the theory of inverse problems, see [15]).

The present study is concerned with the dual identification problem of two time-varying source terms in the system (1.1) of the unknowns data u and v from the following time-dependent measurements:

$$\int_\Omega u(x,t) dx = \theta(t), \quad \int_\Omega v(x,t) dx = \vartheta(t), \quad t \in [0, T], \tag{1.4}$$

where $\theta(t)$ and $\vartheta(t)$ denote a priori measured data, which may be affected by noise. The integral measurement represents the spatial average of the state variable over the domain Ω. This formulation is motivated by practical experimental setups where pointwise measurements may be unavailable or unreliable.

Inverse problems associated with partial differential equations, which aim at the identification or reconstruction of unknown parameters, and stand as a cornerstone in practical various scientific and engineering applications, including medical imaging for the recovery of internal structures [16], source identification in environmental studies [17], characterization of material properties in engineering [18], and exploration problems such as locating mineral deposits and assessing oil and gas reservoirs [19].

Over the years, both theoretical and computational studies of this class of inverse problems have attracted considerable attention, especially regarding the precise identification and reconstruction of the source terms in models with fractional features. Several numerical and regularization techniques have been proposed for addressing inverse problems of this type. The Tikhonov regularization methods is a widely recognized and extensively used method for addressing ill-posed problems. Until recently, its use has been extended to address a diverse set of inverse problems for time-fractional diffusion equations [20–22]. Additional studies have employed optimization-based methods to tackle this kind of problem [23, 24]. Other approaches, including iterative and boundary-value methods and quasi-reversibility methods, have been proposed to effectively address the inverse problem [25–27].

Along a different line of research, the Rothe method has been applied in the context of time-fractional evolution equations to address inverse problems of identifying time-dependent parameters. In the pioneering work [28], Slodička and Šišková rigorously developed the theoretical framework underlying the application of the Rothe method to inverse problems for time-fractional parabolic equations. They focused on an identification problem of a time-dependent source term in a semilinear time-fractional parabolic model, establishing fundamental results on unique resolvability of the inverse problem. Another significant contribution [29] was made by Hendy and Van Bockstal. This work applies the Rothe method to the problem of recovering a unique time-dependent source term in a time-fractional diffusion equation, a challenge compounded by the presence of nonsmooth solutions. Furthermore, in [30], the authors addressed the source identification problem in a time-fractional degenerate diffusion model using the Rothe method, establishing the existence and uniqueness of solutions and developing a stable numerical scheme for reconstructing the missing source terms.

A review of the relevant literature shows that most existing studies have concentrated on the identification of unknown parameters or sources in single-equation fractional models. Although the analysis of such problems is already very challenging, the extension to coupled systems is far from straightforward. The presence of interaction terms introduces additional analytical and numerical difficulties that preclude the direct use of established approaches. In the setting of time-dependent source identification, the situation becomes even more intricate: The inverse problem is essentially very ill-posed, necessitating the adaptation and refinement of existing techniques to enable the decomposition of the joint effect of $p(t)$ and $q(t)$. The coupling also implies that each source influences both state variables, rendering the measurements for u and v interdependent. This interdependence raises critical issues of identifiability and stability that simply do not arise in single-source reconstruction problems.

The present work contributes to the understanding of inverse problems for coupled fractional systems by addressing a problem that, to the best of our knowledge, has not been previously studied. Against this background, the contributions of the paper are twofold. First, we establish a rigorous theoretical framework for the inverse problem. By reformulating it in a coupled weak form and applying the Rothe method, we prove the existence and uniqueness of a weak solution. This analysis relies on deriving nontrivial a priori estimates for the semi-discrete approximations and employing compactness arguments to pass to the limit. Second, we design a stable and computationally efficient numerical scheme within the Rothe framework, which naturally yields a time-step algorithm. At each step, the unknown source terms are recovered explicitly from the delayed state solutions and the measurement data, after which the system for the new states is solved. This direct, noniterative approach is consistent with the theoretical analysis and is highly efficient, as it avoids the computational overhead of the iterative optimization methods commonly employed in inverse problems. Finally, numerical experiments confirm the accuracy and robustness of the proposed scheme, demonstrating its ability to reconstruct the source terms with high fidelity, even in the presence of significant measurement noise.

The following sections comprise the remainder of this work. In the next section, we present some preliminaries and reformulate the inverse problem into a suitable weak formulation. In Section 3, we introduce the semi-discretization in time via the Rothe method, detailing the construction of the semi-discrete scheme and establishing the unique solvability and essential a priori estimates for the discrete approximations. Section 4 presents the theoretical findings, where our main results on existence and uniqueness are established by passing to the limit in the discrete problem and employing stability estimates together with compactness arguments. Section 5 addresses the numerical reconstruction, illustrating the practical implementation of the method, and presenting a series of numerical experiments that validate the efficacy and robustness of our approach. Finally, the paper concludes with a summary and discussion of future research directions.

2. Reformulation of the inverse problem

The starting point for studying the inverse problem is to reformulate it appropriately. For simplicity, and without loss of generality, we restrict ourselves to the case $F = G = 0$. As a first step, we multiply the two equations of the problem (1.1) by the test functions $\phi, \psi \in H^1(\Omega)$, integrate with respect to x over Ω, and subsequently use Green formula, which yields the following system:

$$\begin{cases} \langle (\omega^\gamma * \partial_t u)(t), \varphi \rangle_{H^1(\Omega)} + L_1(t)(u, v; \varphi) = p(t)(f, \varphi), \\ \langle (\omega^\gamma * \partial_t v)(t), \psi \rangle_{H^1(\Omega)} + L_2(t)(v, u; \psi) = q(t)(g, \psi), \end{cases} \tag{2.1}$$

where

$$\begin{cases} L_1(t)(u(t), v(t); \varphi) := (\nabla u(t), \nabla \varphi) + (a(t)u(t) + b(t)v(t), \varphi), \\ L_2(t)(v(t), u(t); \psi) := (\nabla v(t), \nabla \psi) + (c(t)v(t) + d(t)u(t), \psi). \end{cases}$$

The next step is to express the unknown source terms p and q in terms of u, v and the available measurements data θ, ϑ. By taking $\varphi = 1$ and $\psi = 1$ as test functions in (2.1) and employing the integral measurements (1.4), we obtain

$$\begin{cases} (\omega^\gamma * \theta')(t) + \int_\Omega (au + bv)dx = p(t) \int_\Omega f dx, \\ (\omega^\gamma * \vartheta')(t) + \int_\Omega (cv + du)dx = q(t) \int_\Omega g dx. \end{cases}$$

Assuming that $\int_\Omega f\,dx \neq 0$ and $\int_\Omega g\,dx \neq 0$, we can immediately obtain

$$p(t) = \frac{(\omega^\gamma * \theta')(t) + \int_\Omega (au+bv)dx}{\int_\Omega f\,dx}, \quad q(t) = \frac{(\omega^\gamma * \vartheta')(t) + \int_\Omega (cv+du)dx}{\int_\Omega g\,dx}. \tag{2.2}$$

Accordingly, the inverse problem can be interpreted in the following sense: Find a quadruple $\mathcal{S} = (u, v, p, q)$

$$u, v \in L^2(0,T;H^1(\Omega)), \quad p, q \in L^2(0,T), \quad \text{with } \partial_t^\gamma u, \partial_t^\gamma v \in L^2(0,T;L^2(\Omega)),$$

such that (u, v) satisfies for almost all $t \in (0, T)$ and for all $\varphi, \psi \in H^1(\Omega)$, the system (2.1), with $p(t)$ and $q(t)$ are determined by the equations of (2.2).

3. Time-discretization

The Rothe method relies basically on semi-discretization with in the time variable. For doing so, we partition the interval $[0, T]$ into n uniform subintervals $[t_{i-1}, t_i]$ of length $\tau = \frac{T}{n}$, where $t_i = i\tau$ for $i = 1, \ldots, n$. For a function z, we use z_i to denote the approximation of $z(t_i)$. The same convention is adopted for other functions as well. The temporal derivative at t_i is discretized using the explicit Euler scheme, that is,

$$\partial_t z(t_i) \approx \delta z_i := \frac{z_i - z_{i-1}}{\tau}.$$

Similarly, we define

$$(W * z)(t_i) \approx (W * z)_i := \sum_{k=1}^{i} W_{i+1-k} z_k \tau.$$

In view of the definition above, we have the following identity:

$$\delta(W * z)_i = \frac{(W * z)_i - (W * z)_{i-1}}{\tau} = W_1 z_1 + \sum_{k=1}^{i-1} \delta W_{i+1-k} z_k \tau, \quad i \geq 1, \tag{3.1}$$

as $(W * z)_0 := 0$. Furthermore, (3.1) implies

$$\delta(W * z)_i = W_i z_0 + \sum_{k=1}^{i} \delta z_k W_{i+1-k} \tau = W_i z_0 + (W * \delta z)_i, \quad i \geq 1. \tag{3.2}$$

In terms of the consideration above, the weak problem in (2.1), (2.2) is approximated by a sequence of boundary value problems governed by a system of coupled elliptic equations, which consists of finding $u_i, v_i \in H^1(\Omega)$, and $(p_i, q_i) \in \mathbb{R}^2$ for all $i = 1, \cdots, n$ satisfying the following for all $\varphi, \psi \in H^1(\Omega)$:

$$\begin{cases} \langle (\omega^\gamma * \delta u)_i, \varphi \rangle + L_1^i(u_i, v_{i-1}; \varphi) = p_i(f, \varphi), \\ \langle (\omega^\gamma * \delta v)_i, \psi \rangle + L_2^i(v_i, u_{i-1}; \psi) = q_i(g, \psi), \\ u_0 = \tilde{u}_0, \quad v_0 = \tilde{v}_0, \end{cases} \tag{3.3}$$

and

$$p_i = \frac{(\omega^\gamma * \theta')_i + \displaystyle\int_\Omega (a_i u_{i-1} + b_i v_{i-1})\, dx}{\displaystyle\int_\Omega f\, dx}, \quad q_i = \frac{(\omega^\gamma * \vartheta')_i + \displaystyle\int_\Omega (c_i v_{i-1} + d_i u_{i-1})\, dx}{\displaystyle\int_\Omega g\, dx}. \tag{3.4}$$

We note that the fourth equations in (3.3) and (3.4) are linear and decoupled. Hence, for each $1 \le i \le n$, the pair (p_i, q_i) is computed from (3.4) using (u_{i-1}, v_{i-1}), after which (u_i, v_i) is obtained from (3.3). We pass now to proving the existence of a unique solution to (3.3) and (3.4).

Theorem 3.1. *Assume that $\tilde{u}_0, \tilde{v}_0 \in L^2(\Omega)$, and $f, g \in L^2(\Omega)$ with*

$$\left|\int_\Omega f(x)dx\right| \ge C_f > 0, \quad \left|\int_\Omega g(x)dx\right| \ge C_g > 0.$$

Moreover, assume that the assumptions (1.3) *hold. Then, for all $i = 1, \cdots, n$ there is a unique quadruple $\mathcal{S}_i = (u_i, v_i, p_i, q_i)$ such that*

$$u_i, v_i \in H^1(\Omega), \quad \text{and } p_i, q_i \in \mathbb{R},$$

solving the discrete problems (3.3) *and* (3.4).

Proof. Let us assume $\mathcal{U}_i = (u_i, v_i), \mathcal{V} = (\varphi, \psi)$ and define the product space $\mathbb{H}(\Omega) := H^1(\Omega) \times H^1(\Omega)$ equipped with the norm

$$\|(\varphi, \psi)\|_{\mathbb{H}(\Omega)} := \left(\|\varphi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}, \quad \forall(\varphi, \psi) \in \mathbb{H}(\Omega).$$

Now, we first rewrite the problem (3.3) in the following form

$$\mathcal{A}_i(\mathcal{U}, \mathcal{V}) = \mathcal{F}_i(\mathcal{V}), \tag{3.5}$$

where $\mathcal{A}_i$ is bilinear on $\mathbb{H}(\Omega)$ given explicitly as

$$\mathcal{A}_i(\mathcal{U}_i, \mathcal{V}) := \omega^\gamma(\tau)\left((u_i, \varphi) + (v_i, \psi)\right) + (\nabla u_i, \nabla\varphi) + (\nabla v_i, \nabla\psi) + (a_i u_i, \varphi) + (c_i v_i, \psi)$$

and $\mathcal{F}_i : \mathbb{H}(\Omega) \to \mathbb{R}$ is a linear form that collects all the contributions from the previous time steps $i-1$, together with the source terms at the current step, $p_i(f, \varphi)$ and $q_i(g, \psi)$. It is given explicitly by

$$\begin{aligned}\mathcal{F}_i(\mathcal{V}) := {} & p_i(f, \varphi) + q_i(g, \psi) + \omega^\gamma(\tau)\big((u_{i-1}, \varphi) + (v_{i-1}, \psi)\big) \\ & - (\omega^\gamma * \delta u)_{i-1}\tau - (\omega^\gamma * \delta v)_{i-1}\tau + (b v_{i-1}, \varphi) + (d u_{i-1}, \psi).\end{aligned}$$

For $i = 1$, by knowing $u_0 = \tilde{u}_0$ and $v_0 = \tilde{v}_0$, (3.4) determines the unique (p_1, q_1). Conversely, it can be readily verified that $\mathcal{A}_1$ is continuous and coercive on $\mathbb{H}(\Omega)$ under the assumptions of (1.3), and $\mathcal{F}_1$ is a bounded linear form on $\mathbb{H}(\Omega)$. Hence, the existence and uniqueness of a solution (u_1, v_1) for (3.5) follows directly from the Lax–Milgram lemma. Furthermore, for $i \ge 2$, one can prove, by induction, the existence of a unique solution $(u_i, v_i) \in \mathbb{H}(\Omega)$ to the problem (3.5) by assuming that we have solved up to the step $i - 1$, that the right-hand side $\mathcal{F}_i$ is known, and that the Lax–Milgram lemma applies again due to the coercivity of $\mathcal{A}_i$. □

The following lemmas constitute key components in the establishment of the necessary a priori estimates. It should be emphasized that the proof follows analogous arguments to those used in [28], which we extend to fit our framework.

Lemma 3.1. *Let $(z_i)_{i\in\mathbb{N}}$ be a sequence of $L^2(\Omega)$, and let $(W_i)_{i\in\mathbb{N}}$ be a real sequences. Assume that $(W_i)_{i\in\mathbb{N}}$ is non-negative, bounded, and decreasing. It then holds that*

$$2(\delta(W * z)_i, z_i) \;\geq\; \delta(W * \|z\|^2)_i \;+\; W_i\|z_i\|^2,$$

where

$$(W * \|z\|^2)_j = \sum_{i=1}^{j} W_{j+1-i}\,\|z_i\|^2\tau.$$

Proof. Let us write $\mathcal{J} := 2(\delta(W * z_i), z_i) - \delta(W * \|z\|^2)_i - W_i\|z_i\|^2$. Now, by expanding the backward discrete differences, we obtain

$$\mathcal{J} = 2\left(\sum_{k=1}^{i} W_{i+1-k}(z_k, z_i) - \sum_{k=1}^{i-1} W_{i-k}(z_k, z_i)\right) - \left(\sum_{k=1}^{i} W_{i+1-k}\|z_k\|^2 - \sum_{k=1}^{i-1} W_{i-k}\|z_k\|^2\right) - W_i\|z_i\|^2.$$

Performing some arrangement of the terms, we obtain

$$\mathcal{J} = \sum_{k=1}^{i-1}(W_{i+1-k} - W_{i-k})\Big(2(z_k, z_i) - \|z_k\|^2\Big) + (W_1 - W_i)\|z_i\|^2.$$

Notice that

$$\sum_{k=1}^{i-1}(W_{i+1-k} - W_{i-k}) = W_i - W_1.$$

Consequently, we obtain

$$\begin{aligned}\mathcal{J} &= \sum_{k=1}^{i-1}(W_{i+1-k} - W_{i-k})\Big(2(z_k, z_i) - \|z_k\|^2 - \|z_i\|^2\Big) \\ &= \sum_{k=1}^{i-1}(W_{i+1-k} - W_{i-k})(-\|z_k - z_i\|^2).\end{aligned} \tag{3.6}$$

The sequence $(W_i)_{i\in\mathbb{N}}$ is decreasing, so we have $W_{i+1-k} - W_{i-k} \leq 0$ for each $1 \leq k \leq i-1$. Hence, all the terms in the sum $\mathcal{J}$ are positive, as they arise from the product of two negative quantities, which yields that $\mathcal{J} \geq 0$. From this last result, we deduce the desired result. □

Lemma 3.2. *Let the hypotheses of Lemma 3.1 be satisfied. It then holds that*

$$2\sum_{i=1}^{j}(\delta(W * z)_i\, z_i) \;\geq\; (W * \|z\|^2)_j \;+\; \sum_{i=1}^{j} W_i\|z\|_i^2, \quad 1 \leq i \leq j \leq n.$$

Proof. For each $i \geq 1$, we have according to Lemma 3.1 that

$$2(\delta(W * z)_i, z_i) \;\geq\; \delta(W * \|z\|^2)_i + W_i \|z_i\|^2.$$

Summing over $i = 1, \ldots, j$ yields

$$2\sum_{i=1}^{j} (\delta(W * z)_i, z_i)\tau \;\geq\; \sum_{i=1}^{j} \delta(W * \|z\|^2)_i \tau + \sum_{i=1}^{j} W_i \|z_i\|^2 \tau.$$

In view of the convention $(W * \|u\|^2)_0$, we can verify that

$$\sum_{i=1}^{j} \delta(W * \|z\|^2)_i \tau = (W * \|z\|^2)_j - (W * \|z\|^2)_0 = (W * \|z\|^2)_j.$$

Consequently, we obtain

$$2\sum_{i=1}^{j} (\delta(W * z)_i, z_i)\tau \;\geq\; (W * \|z\|^2)_j + \sum_{i=1}^{j} W_i \|z_i\|^2 \tau,$$

which is the desired result. □

Lemma 3.3. *For a function $m \in C([0, T])$, suppose that a positive constant $C_\gamma > 0$ and a real number $\tilde{\gamma}$ with $0 < \gamma \leq \tilde{\gamma} \leq 1$, exist such that*

$$|m(t)| \leq C_\gamma \, t^{\tilde{\gamma}-1}, \quad \textit{for all } 0 < t \leq T. \tag{3.7}$$

The following discrete estimate then holds

$$\left|(\omega^\gamma * m)_i\right| \;\leq\; C(\gamma, \tilde{\gamma}, T), \quad i = 1, \ldots, n. \tag{3.8}$$

Proof. Fix $i \geq 1$. For $s \in [t_{k-1}, t_k]$ with $1 \leq k \leq i$, we have

$$t_{i-k} = t_i - t_k \;\leq\; t_i - s \;\leq\; t_i - t_{k-1} = t_{i+1-k}.$$

Since $t \mapsto \omega^\gamma(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$ is decreasing on $(0, T]$, it follows that

$$\omega^\gamma_{i+1-k} \;\leq\; \omega^\gamma(t_i - s) \;\leq\; \omega^\gamma_{i-k}.$$

Integrating over $[t_{k-1}, t_k]$ gives

$$\tau\, \omega^\gamma_{i+1-k} \;\leq\; \int_{t_{k-1}}^{t_k} \omega^\gamma(t_i - s)\, ds.$$

Therefore, using the assumption (3.7), we obtain

$$\begin{aligned} \left|(\omega^\gamma * m)_i\right| &= \left| \sum_{k=1}^{i} \omega^\gamma_{i+1-k}\, m(t_k)\, \tau \right| \\ &\leq C_\gamma \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} \omega^\gamma(t_i - s)\, ds \right) t_k^{\tilde{\gamma}-1}. \end{aligned}$$

Keeping in mind that $\tilde{\gamma} - 1 < 0$, the function $s \mapsto s^{\tilde{\gamma}-1}$ is decreasing. Hence, $t_k^{\tilde{\gamma}-1} \leq s^{\tilde{\gamma}-1}$ for all $s \in [t_{k-1}, t_k]$. Therefore, it follows that

$$\left|(\omega^\gamma * m)_i\right| \leq C_\gamma \int_0^{t_i} \omega^\gamma(t_i - s)\, s^{\tilde{\gamma}-1}\, ds.$$

Recalling $\omega^\gamma(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$, we obtain

$$\begin{aligned}\left|(\omega^\gamma * m)_i\right| &\leq \frac{C_\gamma}{\Gamma(1-\gamma)} \int_0^{t_i} (t_i - s)^{-\gamma}\, s^{\tilde{\gamma}-1}\, ds \\ &\leq t_i^{\tilde{\gamma}-\gamma} \frac{\Gamma(\tilde{\gamma})\Gamma(1-\gamma)}{\Gamma(\tilde{\gamma}+1-\gamma)} \\ &\leq C_\gamma \frac{\Gamma(\tilde{\gamma})}{\Gamma(\tilde{\gamma}+1-\gamma)} T^{\tilde{\gamma}-\gamma},\end{aligned}$$

which proves (3.8). □

Lemma 3.4. *Let the assumptions of Theorem 3.1 be satisfied. Suppose that* $\tilde{u}_0, \tilde{v}_0 \in H^1(\Omega)$*,* $f, g \in L^2(\Omega)$ *and the assumptions of* (1.3) *are fulfilled. Moreover, assume that* $\theta, \vartheta \in C^2([0,T])$ *satisfying for some constant* $C > 0$*,*

$$|\theta'(t)| + |\vartheta'(t)| \leq C t^{\tilde{\gamma}-1}, \quad 0 < t \leq T, \tag{3.9}$$

with fixed $\tilde{\gamma} \in (\gamma, 1)$*. Then* $C > 0$ *and* τ_0 *exist such that for any* $0 < \tau < \tau_0$*, the following estimates hold*

$$\max_{1\leq j\leq n} \left(\omega^\gamma * \|u\|^2\right)_j + \sum_{i=1}^n \omega^\gamma \|u_i\|^2 \tau + \sum_{i=1}^n \|u_i\|^2_{H^1(\Omega)} \tau \leq C, \tag{3.10}$$

$$\max_{1\leq j\leq n} \left(\omega^\gamma * \|v\|^2\right)_j + \sum_{i=1}^n \omega^\gamma \|v_i\|^2 \tau + \sum_{i=1}^n \|v_i\|^2_{H^1(\Omega)} \tau \leq C. \tag{3.11}$$

Proof. Starting from (3.4), we have

$$|p_i| \leq \frac{|(\omega_\gamma * \theta')_i| + \displaystyle\int_\Omega |a_i u_{i-1}|\, dx + \int_\Omega |b_i v_{i-1}|\, dx}{\left|\displaystyle\int_\Omega f\, dx\right|}.$$

By Lemma 3.3, the first term on the numerator admits an upper bound. For the other terms, applying the Cauchy–Schwartz inequality yields factors involving a_i and b_i, which are uniformly bounded, since $a_i, b_i \in L^\infty(\Omega)$ according to (H_1) in (1.3). Hence, we obtain

$$|p_i| \leq C\left(1 + \|u_{i-1}\| + \|v_{i-1}\|\right). \tag{3.12}$$

Similarly, one can obtain

$$|q_i| \leq C\left(1 + \|u_{i-1}\| + \|v_{i-1}\|\right). \tag{3.13}$$

Now, if we set $\varphi = u_i\tau$ and $\psi = v_i\tau$ in (3.3) and sum up for $i = 1, \cdots, j$ with $1 \leq j \leq n$, we obtain

$$\sum_{i=1}^j ((w^\gamma * \delta u)_i, u_i)\,\tau + \sum_{i=1}^j \|\nabla u_i\|^2 \tau + \sum_{i=1}^j (a_i u_i, u_i)\tau = -\sum_{i=1}^j (b_i v_{i-1}, u_i)\tau + \sum_{i=1}^j p_i (f, u_i)\tau, \tag{3.14}$$

and

$$\sum_{i=1}^{j}((w^{\gamma}*\delta v)_i,v_i)\,\tau+\sum_{i=1}^{j}\|\nabla v_i\|^2\tau+\sum_{i=1}^{j}(c_iv_i,v_i)\tau=-\sum_{i=1}^{j}(d_iu_{i-1},v_i)\tau+\sum_{i=1}^{j}q_i(g,v_i)\tau. \tag{3.15}$$

For the first term on the left-hand side of the identity (3.14), we invoke Lemma 3.2, which yields the following lower bound:

$$\begin{aligned}\sum_{i=1}^{j}((w^{\gamma}*\delta u)_i,u_i)\,\tau &\geq \sum_{i=1}^{j}(\delta(w^{\gamma}*u)_i,u_i)\,\tau-\sum_{i=1}^{j}w_i^{\gamma}(u_0,u_i)\tau\\ &\geq \left(w^{\gamma}*\|u\|^2\right)_j+\frac{1}{2}\sum_{i=1}^{j}w_i^{\gamma}\|u_i\|^2\tau-\sum_{i=1}^{j}w_i^{\gamma}\|u_i\|\|\tilde{u}_0\|\tau \\ &\geq \frac{1}{2}\left(w^{\gamma}*\|u\|^2\right)_j+\left(\frac{1}{2}-\varepsilon\right)\sum_{i=1}^{j}w_i^{\gamma}\|u_i\|^2\tau-C_{\varepsilon}.\end{aligned} \tag{3.16}$$

Making use of the assumption (H_2) in (1.3), we obtain

$$\sum_{i=1}^{j}(au_i,u_i)\tau\geq a_0\sum_{i=1}^{j}\|u_i\|^2\tau.$$

On the other hand, the right hand-side of (3.14) can be estimated using the Cauchy–Schwartz and ε-Young inequalities, namely

$$\begin{aligned}-\sum_{i=1}^{j}(v_{i-1},u_i)\tau+\sum_{i=1}^{j}p_i(f_i,u_i)\tau &\leq \sum_{i=1}^{j}\|v_{i-1}\|\|u_i\|\tau+\sum_{i=1}^{j}|p_i|\|f_i\|\|u_i\|\tau\\ &\leq \frac{1}{2\varepsilon}\sum_{i=1}^{j}\|v_{i-1}\|^2\tau+\varepsilon\sum_{i=1}^{j}\|u_i\|^2\tau+\frac{1}{2\varepsilon}\sum_{i=1}^{j}|p_i|^2\|f\|^2\tau.\end{aligned}$$

In view of the estimate (3.12), it follows that

$$\begin{aligned}-\sum_{i=1}^{j}(v_{i-1},u_i)\tau+\sum_{i=1}^{j}p_i(f_i,u_i)\tau &\leq C_{\varepsilon}\sum_{i=1}^{j}\left(1+\|u_{i-1}\|+\|v_{i-1}\|\right)^2\tau+\varepsilon\sum_{i=1}^{j}\|u_i\|^2\tau\\ &\leq C_{\varepsilon}\left(1+\sum_{i=1}^{j-1}\|u_i\|^2\tau+\sum_{i=1}^{j-1}\|v_i\|^2\tau\right)+\varepsilon\sum_{i=1}^{j}\|u_i\|^2\tau.\end{aligned}$$

Collecting all estimates leads to the following bound:

$$\left(w^{\gamma}*\|u\|^2\right)_j+(1-\varepsilon)\sum_{i=1}^{j}w_i^{\gamma}\|u_i\|^2\tau+\sum_{i=1}^{j}\|\nabla u_i\|^2\tau\,(a_0-\varepsilon)\sum_{i=1}^{j}\|u_i\|^2\tau\leq C_{\varepsilon}\left(1+\sum_{i=1}^{j-1}\|u_i\|^2\tau+\sum_{i=1}^{j-1}\|v_i\|^2\tau\right).$$

Now, by choosing a sufficiently small $\varepsilon>0$, it results in

$$\left(w^{\gamma}*\|u\|^2\right)_j+\sum_{i=1}^{j}w_i^{\gamma}\|u_i\|^2\tau+\sum_{i=1}^{j}\|u_i\|^2_{H^1(\Omega)}\tau\leq C\left(1+\sum_{i=1}^{j-1}\|u_i\|^2\tau+\sum_{i=1}^{j-1}\|v_i\|^2\tau\right). \tag{3.17}$$

Estimating, similar to the above, with both sides of (3.13), following the same line, one can obtain

$$\left(w^\gamma * \|v\|^2\right)_j + \sum_{i=1}^{j} w_i^\gamma \|v_i\|^2\tau + \sum_{i=1}^{j} \|v_i\|^2_{H^1(\Omega)}\tau \le C\left(1 + \sum_{i=1}^{j-1} \|u_i\|^2\tau + \sum_{i=1}^{j-1} \|v_i\|^2\tau\right). \tag{3.18}$$

By summing (3.17) and (3.18) , we obtain the following discrete inequality:

$$S_j \le C\left(1 + \sum_{i=1}^{j-1} S_i\right), \quad 1 \le i < j \le n,$$

where S_j collects together all the terms that we intend to estimate, that is

$$S_j = \left(w^\gamma * \|u\|^2\right)_j + \left(w^\gamma * \|v\|^2\right)_j + \sum_{i=1}^{j} w_i^\gamma (\|u_i\|^2 + \|u_i\|^2)\tau + \sum_{i=1}^{j} \left(\|u_i\|^2_{H^1(\Omega)} + \|v_i\|^2_{H^1(\Omega)}\right)\tau.$$

Now, an application of the discrete Grönwall inequality leads to the fact that $S_j \le C$ for all $j = 1, \cdots, n$, which is exactly the desired result. □

Lemma 3.5. *Let the assumptions of Lemma 3.4 be satisfied. Then* $C > 0$ *and* τ_0 *exist such that for any* $0 < \tau < \tau_0$*, the following estimates hold*

$$\max_{1\le i\le n} (\|u_i\|^2_{H^1(\Omega)} + \|v_i\|^2_{H^1(\Omega)}) + \sum_{i=1}^{n} \left(\|u_i - u_{i-1}\|^2_{H^1(\Omega)} + \|v_i - v_{i-1}\|^2_{H^1(\Omega)}\right) \le C, \tag{3.19}$$

$$\max_{1\le i\le n} |p_i| + \max_{1\le i\le n} |q_i| \le C. \tag{3.20}$$

Proof. For each $i \ge 1$, we test the first equation in (3.3) with $\varphi = \delta u_i \tau$ and summing from $i = 1$ to j with $1 \le i < j \le n$, we obtain

$$\sum_{i=1}^{j} ((\omega^\gamma * \delta u)_i,\ \delta u_i)\,\tau + \sum_{i=1}^{j}(\nabla u_i, \nabla\delta u_i)\tau + \sum_{i=1}^{j}(a_i u_i, \delta u_i)\,\tau = -\sum_{i=1}^{j}(b_i v_{i-1}, \delta u_i)\,\tau + \sum_{i=1}^{j} p_i(f, \delta u_i)\,\tau. \tag{3.21}$$

The analogous computation for the second equation, tested with $\psi = \delta v_i \tau$ and summed, gives

$$\sum_{i=1}^{j} ((\omega^\gamma * \delta v)_i,\ \delta v_i)\,\tau + \sum_{i=1}^{j}(\nabla v_i, \nabla\delta v_i)\tau + \sum_{i=1}^{j}(c_i v_i, \delta v_i)\,\tau = -\sum_{i=1}^{j}(d_i u_{i-1}, \delta v_i)\,\tau + \sum_{i=1}^{j} q_i(g, \delta v_i)\,\tau. \tag{3.22}$$

According to [31, Eq 3.2] the positivity of the first term on the left-hand side is guaranteed. Furthermore we have

$$\sum_{i=1}^{j}(\nabla u_i, \nabla\delta u_i)\tau = \frac{1}{2}\left(\|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2\right).$$

On the other hand, one can readily verify that

$$2\sum_{i=1}^{j}(a_i u_i, \delta u_i)\,\tau = (a_j u_j, u_j) - (a_0 u_0, u_0) - \sum_{i=1}^{j}(\delta a_i u_{i-1}, u_{i-1})\,\tau + \sum_{i=1}^{j}(a_i u_i - u_{i-1}, u_i - u_{i-1}).$$

Therefore, by (1.3), a lower bound for the third term on the left-hand side of (3.21) is obtained as follows:

$$\sum_{i=1}^{j}(a_i u_i, \delta u_i)\,\tau \geq \frac{1}{2}\left(a_0\|u_j\|^2 - (a_1+\zeta)\|u_0\|^2 - \tilde{\zeta}\sum_{i=1}^{j-1}\|u_i\|^2\tau + a_0\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2\right),$$

where $\zeta > 0$ is a constant satisfying $\|\partial_t a\| \leq \zeta$, whose existence follows from the assumption (H_3).

Now, we turn to seeking for an upper bound for the right-hand side of (3.22). For doing so, we rely on the assumption (1.3), and apply the Cauchy–Schwartz and Young inequalities to obtain

$$\begin{aligned}
\sum_{i=1}^{j}(b_i v_{i-1}, \delta u_i)\,\tau &= \sum_{i=1}^{j}(b_i v_{i-1}, u_i - u_{i-1}) \\
&\leq b_1\sum_{i=1}^{j}\|v_{i-1}\|\|u_i - u_{i-1}\| \\
&\leq C_\varepsilon\sum_{i=1}^{j}\|v_{i-1}\|^2 + \varepsilon\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 \\
&\leq C_\varepsilon\left(1 + \sum_{i=1}^{j-1}\|v_i\|^2\right) + \varepsilon\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2.
\end{aligned}$$

Furthermore, we have

$$\sum_{i=1}^{j} p_i(f, \delta u_i)\tau \leq C_\varepsilon\sum_{i=1}^{j}|p_i|^2\tau + \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau \leq C_\varepsilon + \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau.$$

From this inequality, it holds that

$$-\sum_{i=1}^{j}(b_i v_{i-1}, \delta u_i)\,\tau + \sum_{i=1}^{j} p_i(f, \delta u_i)\,\tau \leq C_\varepsilon + C_\varepsilon\left(1 + \sum_{i=1}^{j-1}\|v_i\|^2\right) + \varepsilon\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 + \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau.$$

Combining the obtained results, we arrive at

$$\|u_j\|^2_{H^1(\Omega)} + (1-\varepsilon)\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2_{H^1(\Omega)} \leq C_\varepsilon\left(1 + \sum_{i=1}^{j-1}\|v_i\|^2\right) + \varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau. \tag{3.23}$$

Now, if we write $\varepsilon = \tilde{\varepsilon}\tau$, it then follows that

$$\|u_j\|^2_{H^1(\Omega)} + (1-\varepsilon-\tilde{\varepsilon})\sum_{i=1}^{j}\|u_i - u_{i-1}\|^2_{H^1(\Omega)} \leq C_\varepsilon\left(1 + \sum_{i=1}^{j-1}\|v_i\|^2\right).$$

Notice that, since $\tau < \tau_0 < 1$, it follows that $1 - \varepsilon - \tilde{\varepsilon} \geq 1 - 2\tilde{\varepsilon}$. Therefore, by choosing $\tilde{\varepsilon} > 0$ to be sufficiently small and independent of τ, we ensure that the coefficient remains positive, which yields

$$\|u_j\|^2_{H^1(\Omega)} + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2_{H^1(\Omega)} \leq C\left(1 + \sum_{i=1}^{j-1}\|v_i\|^2\right). \tag{3.24}$$

In a complete analogy with the previous estimate for u_j, testing the second equation with $\psi = \delta v_i \tau$, summing over $i = 1, \ldots, j$, and applying the same arguments yields the bound

$$\|v_j\|^2_{H^1(\Omega)} + \sum_{i=1}^{j} \|v_i - v_{i-1}\|^2_{H^1(\Omega)} \le C\left(1 + \sum_{i=1}^{j-1} \|u_i\|^2\right). \tag{3.25}$$

Combining (3.24) and (3.25), and applying a discrete Gronwall lemma, we finally deduce the uniform estimate

$$\|u_j\|^2_{H^1(\Omega)} + \|v_j\|^2_{H^1(\Omega)} + \sum_{i=1}^{j} \left(\|u_i - u_{i-1}\|^2_{H^1(\Omega)} + \|v_i - v_{i-1}\|^2_{H^1(\Omega)}\right) \le C,$$

which, together with (3.12) and (3.13), yields the uniform boundness of p_i and q_i, respectively. □

In the subsequent analysis, we require the following compatibility condition at the initial time, i.e., we assume that the weak formulation (2.1) is also fulfilled at $t = 0$. Consequently, one obtains the following identities:

$$\begin{cases} (\omega^\gamma * \partial_t u)(0) + (\nabla \tilde{u}_0, \nabla \varphi) + (a(0)\tilde{u}_0 + b(0)\tilde{v}_0, \varphi)\, dx = p_0(f, \varphi), \\ (\omega^\gamma * \partial_t v)(0) + (\nabla \tilde{v}_0, \nabla \psi) + (c(0)\tilde{v}_0 + d(0)\tilde{u}_0, \psi)\, dx = q_0(g, \psi). \end{cases} \tag{3.26}$$

Since $(\omega^\gamma * \partial_t u)(0) = (\omega^\gamma * \partial_t v)(0) = 0$, and by choosing the test functions $\varphi \equiv 1$ and $\psi \equiv 1$, the identities in (3.26) allow us to define the initial values p_0 and q_0 explicitly as follows:

$$p_0 = \frac{\displaystyle\int_\Omega (a_0\tilde{u}_0 + b_0\tilde{v}_0)\ dx}{\displaystyle\int_\Omega f\ dx}, \quad \text{and} \quad q_0 = \frac{\displaystyle\int_\Omega (c_0\tilde{v}_0 + d_0\tilde{u}_0)\ dx}{\displaystyle\int_\Omega g\ dx}. \tag{3.27}$$

Lemma 3.6. *Let the assumptions* (1.3) *and those of Lemma 3.4 be fulfilled. Assume in addition that* $\theta, \vartheta \in C^2([0, T])$. *If* (3.27) *is satisfied, then we have*

$$|\delta p_i| + |\delta q_i| \le C(1 + t_i^{-\gamma}), \qquad \forall i = 1, \ldots, n. \tag{3.28}$$

Furthermore, the following estimate holds:

$$\sum_{i=1}^{n} |\delta p_i|\tau + \sum_{i=1}^{n} |\delta q_i|\tau \ \le\ C, \qquad \forall n \in \mathbb{N}. \tag{3.29}$$

Proof. We begin with δp_1 and δq_1. Subtracting (3.27) from (3.4) at $i = 1$ and subsequently dividing by τ, we obtain

$$\delta p_1 = \frac{\omega^\gamma(\tau)\, \theta'(\tau) + \displaystyle\int_\Omega (\tilde{u}_0\, \delta a_1 + \tilde{v}_0\, \delta b_1)\, dx}{\displaystyle\int_\Omega f\, dx}, \quad \delta q_1 = \frac{\omega^\gamma(\tau)\, \vartheta'(\tau) + \displaystyle\int_\Omega (\tilde{v}_0\, \delta c_1 + \tilde{u}_0\, \delta d_1)\, dx}{\displaystyle\int_\Omega g\, dx}.$$

Since $\tilde{u}_0, \tilde{v}_0 \in L^2(\Omega)$ and $a, b \in W^{1,\infty}(0, T; L^\infty(\Omega))$, it follows that

$$|\delta p_1| \ \le\ C\, |\omega^\gamma(\tau)\, \theta'(\tau)| + C.\ \le\ C\tau^{-\gamma} + C.$$

Furthermore, similar arguments can be used to obtain $|\delta q_1`| \le C(1+\tau^{-\gamma})$. Now let us consider the case $i \ge 2$. Taking the difference of (3.4) and dividing by τ, we obtain

$$\delta p_i = \frac{\delta(\omega^\gamma * \theta')_i + \int_\Omega \delta(a_i u_{i-1})\,dx + \int_\Omega \delta(b_i v_{i-1})\,dx}{\int_\Omega f\,dx},$$

and

$$\delta q_i = \frac{\delta(\omega^\gamma * \theta')_i + \int_\Omega \delta(c_i v_{i-1})\,dx + \int_\Omega \delta(d_i u_{i-1})\,dx}{\int_\Omega g\,dx}.$$

Let us focus on δp_i. Using the discrete convolution identity (3.2), we deduce

$$|\delta(\omega^\gamma * \theta')_i| \le |\omega^\gamma(t_i)\,\theta'(0)| + \sum_{k=1}^{i} |\delta\theta'_k|\,\omega^\gamma(t_{i+1-k})\,\tau.$$

Due to the regularity $\theta \in C^2([0,T])$, we have $|\delta\theta'_k| \le C$. Therefore,

$$\sum_{k=1}^{i} |\delta\theta'_k|\,\omega^\gamma(t_{i+1-k})\,\tau \le C\sum_{m=1}^{i} t_m^{-\gamma}\tau = C\Big(t_1^{-\gamma}\tau + \sum_{m=2}^{i} t_m^{-\gamma}\tau\Big).$$

For $m \ge 2$, the monotonicity of the function $s \mapsto s^{-\gamma}$ yields

$$\sum_{m=2}^{i} t_m^{-\gamma}\tau \le \sum_{m=2}^{i}\int_{t_{m-1}}^{t_m} s^{-\gamma}\,ds \le \int_0^{t_i} s^{-\gamma}\,ds = \frac{t_i^{1-\gamma}}{1-\gamma} \le \frac{T^{1-\gamma}}{1-\gamma}. \tag{3.30}$$

Moreover, $t_1^{-\gamma}\tau = \tau^{1-\gamma} \le T^{1-\gamma}$. Hence

$$\sum_{k=1}^{i} |\delta\theta'_k|\,\omega^\gamma(t_{i+1-k})\,\tau \;\le\; C.$$

Next, under the boundedness assumption (1.3), we find

$$\begin{aligned}\Big|\int_\Omega \delta(a_i u_{i-1})\,dx\Big| &\le \Big|\int_\Omega u_{i-1}\,\delta a_i\,dx\Big| + \Big|\int_\Omega a_{i-1}\,\delta u_{i-1}\,dx\Big| \\ &\le \|\delta a_i\|_\infty \Big|\int_\Omega u_{i-1}\,dx\Big| + \|a_{i-1}\|_\infty \Big|\int_\Omega \delta u_{i-1}\,dx\Big| \\ &\le C(|\theta_{i-1}| + |\delta\theta_{i-1}|) \;\le\; C.\end{aligned}$$

An entirely similar argument shows that $\Big|\int_\Omega \delta(b_i v_i)\,dx\Big| \le C$. Gathering all the preceding estimates, we arrive at the following bound:

$$|\delta p_i| \;\le\; C(1 + t_i^{-\gamma}), \qquad \forall i \ge 1.$$

Summing this inequality for $i = 1, \ldots, j$ and applying (3.30), we obtain

$$\sum_{i=1}^{j} |\delta p_i| \tau \ \leq \ C \sum_{i=1}^{j} (1 + t_i^{-\gamma}) \tau \ \leq \ C.$$

In a similar manner, analogous estimates can be derived for q_i and δq_i, leading to bounds of the same type as those obtained for δp_i. □

Lemma 3.7. *Let the assumptions of Lemma 3.4 be satisfied. Then $C > 0$ and τ_0 exist such that for any $0 < \tau < \tau_0$, the following estimates hold*

$$\max_{1 \le j \le n} (\omega^\gamma * \|\delta u\|^2)_j + \sum_{i=1}^{n} \omega_i^\gamma \|\delta u_i\|^2 \tau + \sum_{i=1}^{n} \|\delta u_i\|^2_{H^1(\Omega)} \tau \le C, \tag{3.31}$$

$$\max_{1 \le j \le n} (\omega^\gamma * \|\delta v\|^2)_j + \sum_{i=1}^{n} \omega_i^\gamma \|\delta v_i\|^2 \tau + \sum_{i=1}^{n} \|\delta v_i\|^2_{H^1(\Omega)} \tau \le C. \tag{3.32}$$

Proof. Taking the difference of (2.2), we obtain

$$\begin{cases} (\delta(\omega^\gamma * \delta u)_i, \varphi) + (\delta L_1^i)(u_i, v_{i-1}; \varphi) = \delta p_i (f, \varphi), \\ (\delta(\omega^\gamma * \delta v)_i, \psi) + (\delta L_2^i)(v_i, u_{i-1}; \psi) = \delta q_i (g, \psi). \end{cases} \tag{3.33}$$

The difference can be defined for $i \geq 2$. In the case $i = 1$, however, it is obtained by subtracting Eq (3.26) from (3.3). In addition, since the two equations can be handled similarly, we restrict our attention to the first one and obtain analogous results for the second. First, notice that

$$\begin{aligned} (\delta L_1^i)(u_i, v_{i-1}; \varphi) &= (\nabla \delta u_i, \nabla \varphi) + (\delta(a_i u_i) + \delta(b_i v_{i-1}), \varphi) \\ &= (\nabla \delta u_i, \nabla \varphi) + (u_i \delta a_i + a_{i-1} \delta u_i, \varphi) + (v_{i-1} \delta b_i + b_i \delta v_{i-1}, \varphi). \end{aligned}$$

Consequently, the first equation of (3.33) can be rewritten as follows:

$$(\delta(\omega^\gamma * \delta u)_i, \varphi) + (\nabla \delta u_i, \nabla \varphi) + (u_i \delta a_i + a_{i-1} \delta u_i, \varphi) = \delta p_i (f, \varphi) - (v_{i-1} \delta b_i + b_i \delta v_{i-1}, \varphi).$$

Now, if we take $\varphi = \tau \delta u_i$ in the equation above, and sum up to j, we obtain

$$\begin{aligned} &\sum_{i=1}^{j} (\delta(\omega^\gamma * \delta u)_i, \delta u_i) \tau + \sum_{i=1}^{j} \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^{j} (a_{i-1} \delta u_i, \delta u_i) \tau \\ = &\sum_{i=1}^{j} \delta p_i (f, \delta u_i) \tau - \sum_{i=1}^{j} (v_{i-1} \delta b_i + b_i \delta v_{i-1}, \delta u_i) \tau - \sum_{i=1}^{j} (u_i \delta a_i, \delta u_i) \tau. \end{aligned}$$

For obtaining a lower bound for the left-hand side, we treat each term separately. For the first term, by means of the same argument used for (3.16), we obtain

$$\sum_{i=1}^{j} (\delta(\omega^\gamma * \delta u)_i, \ \delta u_i) \tau \geq \frac{1}{2} (\omega^\gamma * \|\delta u\|^2)_j + \frac{1}{4} \sum_{i=1}^{j} \omega_i^\gamma \|\delta u_i\|^2 \tau + \frac{\omega_1^\gamma}{4} \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

In view of (1.3), we can derive

$$\sum_{i=1}^{j}(a_{i-1}\delta u_i, \delta u_i)\tau \geq \zeta_0 \sum_{i=1}^{j} \|\delta u_i\|^2\tau.$$

Let us pass to the right hand-side. In a standard way, we can obtain using the Cauchy–Schwartz and Young inequalities that

$$\begin{aligned}\sum_{i=1}^{j} \delta p_i(f, \delta)\tau &\leq \sum_{i=1}^{j} |\delta p_i|\|f\|\|\delta u_i\|\tau \\ &\leq C_\varepsilon \sum_{i=1}^{j} |\delta p_i|\|f\|^2\tau + \sum_{i=1}^{j} |\delta p_i|\|\delta u_i\|^2\tau.\end{aligned}$$

Making use of (3.28) and (3.29), we obtain

$$\begin{aligned}\sum_{i=1}^{j} \delta p_i(f, \delta u_i)\tau &\leq C_\varepsilon \sum_{i=1}^{j} |\delta p_i|\|f\|^2\tau + \varepsilon \sum_{i=1}^{j} |\delta p_i|\|\delta u_i\|^2\tau \\ &\leq C_\varepsilon + \varepsilon \sum_{i=1}^{j} t_i^{-\gamma}\|\delta u_i\|^2\tau \leq C_\varepsilon + \varepsilon \sum_{i=1}^{j} \omega_i^\gamma\|\delta u_i\|^2\tau.\end{aligned}$$

Furthermore, we can find by means of the assumption of(1.3), together with the Cauchy–Schwartz and Young inequalities that

$$\begin{aligned}\sum_{i=1}^{j} (v_{i-1}\delta b_{i-1} + b_i\delta v_{i-1}, \delta u_i)\tau &\leq C \sum_{i=1}^{j} \|v_{i-1}\|\,\|\delta u_i\|\,\tau + C \sum_{i=1}^{j} \|\delta v_{i-1}\|\,\|\delta u_i\|\,\tau \\ &\leq C_\varepsilon \sum_{i=1}^{j} \|v_{i-1}\|^2\,\tau + C_\varepsilon \sum_{i=1}^{j} \|\delta v_{i-1}\|^2\,\tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau \\ &\leq C_\varepsilon \left(1 + \sum_{i=1}^{j} \|\delta v_{i-1}\|^2\,\tau\right) + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau.\end{aligned}$$

Using analogous arguments, we obtain

$$\sum_{i=1}^{j} (u_i\delta a_i, \delta u_i)\tau \leq C \sum_{i=1}^{j} \|u_i\|\|\delta u_i\|\tau \leq C_\varepsilon \sum_{i=1}^{j} \|u_i\|^2\tau + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\tau \leq C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2\tau.$$

Collecting all the findings above, we deduce

$$\begin{aligned}\left(\omega^\gamma * \|\delta u\|^2\right)_j + (1-\varepsilon) \sum_{i=1}^{j} \omega_i^\gamma\,\|\delta u_i\|^2\,\tau + (\omega_1^\gamma - \varepsilon) \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau \\ + \sum_{i=1}^{j} \|\nabla\delta u_i\|^2\tau \leq C_\varepsilon \left(1 + \sum_{i=1}^{j} \|\delta v_{i-1}\|^2\,\tau\right).\end{aligned}$$

With an $\varepsilon > 0$ that is sufficiently small, it follows that

$$(\omega^\gamma * \|\delta u\|^2)_j + \sum_{i=1}^{j} \omega_i^\gamma \|\delta u_i\|^2 \tau + \sum_{i=1}^{j} \|\delta u_i\|_{H^1(\Omega)}^2 \tau \le C\left(1 + \sum_{i=1}^{j} \|\delta v_{i-1}\|^2 \tau\right). \tag{3.34}$$

Proceeding in a similar manner, one can obtain an analogous estimate as follows

$$(\omega^\gamma * \|\delta v\|^2)_j + \sum_{i=1}^{j} \omega_i^\gamma \|\delta v_i\|^2 \tau + \sum_{i=1}^{j} \|\delta v_i\|_{H^1(\Omega)}^2 \tau \le C\left(1 + \sum_{i=1}^{j} \|\delta u_{i-1}\|^2 \tau\right). \tag{3.35}$$

Therefore, combining the coupled estimates (3.34) and (3.35), and applying discrete Gronwall provides the desired uniform bound. □

4. Existence and uniqueness

In this subsection, we establish the existence of a weak solution to the inverse problem. For this purpose, we first extend the discrete approximations u_i, v_i, p_i, and q_i obtained at each time step to the entire time frame $[0, T]$. This extension is accomplished by constructing Rothe functions, which interpolate the discrete solutions continuously in time. Specifically, we define the following piecewise linear and piecewise constant functions: $u_n, \overline{u}_n : [0, T] \mapsto H^1(\Omega)$,

$$u_n : t \mapsto \begin{cases} u_0 & : \ t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & : \ t \in (t_{i-1}, t_i], \ 1 \le i \le n, \end{cases}$$

$$\overline{u}_n : t \mapsto \begin{cases} u_0 & : \ t = 0 \\ u_i & : \ t \in (t_{i-1}, t_i], \ 1 \le i \le n. \end{cases}$$

Similarly, we define v_n and $\tilde{v}_n$. Furthermore, we define the Rothe functions analogously for the source term and measurement, namely, $p_n, q_n, \overline{p}_n, \overline{q}_n, \overline{\theta}_n$, and $\overline{\vartheta}_n$. Consequently, the discrete weak problems (3.3) and (3.4) can be reformulated over the entire time frame as follows:

$$\begin{cases} \langle \partial_t(\omega^\gamma * u_n)(t) - \overline{\omega}_n^\gamma(t) u_0, \varphi \rangle_{H^1} + \bar{L}_n^1(t)(\overline{u}_n, \overline{v}_n; \varphi) = \overline{p}_n(t)(f, \varphi), \\ \langle \partial_t(\omega^\gamma * v_n)(t) - \overline{\omega}_n^\gamma(t) v_0, \psi \rangle_{H^1} + \bar{L}_n^2(t)(\overline{v}_n, \overline{u}_n; \psi) = \overline{q}_n(t)(g, \psi), \end{cases} \tag{4.1}$$

and

$$\begin{cases} \overline{p}_n(t) = \dfrac{(\overline{\omega}_n^\gamma * \bar{\theta}_n')(t) + \displaystyle\int_\Omega \bar{a}_n(t)\widetilde{u}_n(t)\,dx + \int_\Omega \bar{b}_n(t)\widetilde{v}_n(t)\,dx)}{\displaystyle\int_\Omega f\,dx}, \\ \overline{q}_n(t) = \dfrac{(\overline{\omega}_n^\gamma * \bar{\vartheta}_n')(t) + \displaystyle\int_\Omega \bar{c}_n(t)\widetilde{v}_n(t)\,dx + \int_\Omega \bar{d}_n(t)\widetilde{u}_n(t)\,dx)}{\displaystyle\int_\Omega g\,dx}, \end{cases} \tag{4.2}$$

where $\widetilde{u}$ and $\widetilde{v}$ stands for the delayed Rothe function, defined, respectively, as

$$\widetilde{u}_n(t) := \begin{cases} u_0, & t \in [0, \tau], \\ \overline{u}_n(t - \tau), & t \in (t_{i-1}, t_i], \quad 2 \le i \le n, \end{cases}$$

and

$$\widetilde{v}_n(t) := \begin{cases} v_0, & t \in [0, \tau], \\ \bar{v}_n(t-\tau), & t \in (t_{i-1}, t_i], \quad 2 \le i \le n. \end{cases}$$

In view of Theorem 3.1, each of the problems (4.1) and (4.2) admits a unique weak solution for all $i = 1, \dots, n$. Consequently, the associated Rothe functions are well defined and uniquely determined by construction.

At this stage, we can formulate an existence result.

Theorem 4.1. *Suppose that the conditions of Lemma 3.4 hold. Moreover, assume that the regularity assumptions* (1.3) *hold, and the measurements data* $\theta, \vartheta \in C^2([0,T])$ *satisfy the condition* (3.9)*. Then a quadruple* (u, v, p, q) *solution to the problem* (2.1)-(2.2) *exists such that*

$$u, v \in C(\bar{I}; L^2(\Omega)) \cap L^\infty(I; H^1(\Omega)), \quad p, q \in L^2(I),$$

where

$$(\omega^\gamma * \partial_t u),\ (\omega^\gamma * \partial_t v) \in L^2(I; L^2(\Omega)).$$

Proof. In view of the estimates (3.20) and (3.32), we immediately deduce

$$\sup_{0\le t\le T} \|\bar{u}_n(t)\|^2_{H^1(\Omega)} + \int_0^T \|\partial_t u_n(t)\|^2_{H^1(\Omega)} dt \le C.$$

Hence, by virtue of compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, according to [32, Lemma 1.3.13], the existence of a function

$$u^\dagger \in C(\bar{I}; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)), \quad \text{where } \partial_t u^\dagger \in L^2(0,T; L^2(\Omega)),$$

together with a subsequence (u_{n_k}) of (u_n), such that

$$\begin{cases} u_{n_k} \to u^\dagger, & \text{in } C([0,T], L^2(\Omega)), \\ u_{n_k}(t) \rightharpoonup u^\dagger(t), & \text{in } H^1(\Omega),\ \forall t \in [0,T], \\ \bar{u}_{n_k}(t) \rightharpoonup u^\dagger(t), & \text{in } H^1(\Omega),\ \forall t \in [0,T], \\ \partial_t u_{n_k} \rightharpoonup \partial_t u^\dagger, & \text{in } L^2(0,T; L^2(\Omega)). \end{cases}$$

Moreover, from Lemma 3.5, we also have

$$\int_0^T \left(\|u_{n_k}(t) - \bar{u}_{n_k}(t)\|^2 + \|\bar{u}_{n_k}(t) - \widetilde{u}_{n_k}(t)\|^2 \right) dt \le \frac{C}{n_k} \to 0, \quad \text{as } k \to \infty. \tag{4.3}$$

Furthermore, the triangle inequality then gives $\|u_{n_k} - \widetilde{u}_{n_k}\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $k \to 0$. Since $u_{n_k} \to u^\dagger$ strongly in $L^2(0,T; L^2(\Omega))$ and the differences vanish in that norm, we conclude that $\bar{u}_{n_k}, \widetilde{u}_{n_k}$ also converge strongly to the same limit $u^\dagger$ in $L^2(0,T; L^2(\Omega))$.

The sequence $(\partial_t u_{n_k})_{k\in\mathbb{N}}$ is uniformly bounded in the reflexive space $L^2(0,T; H^1(\Omega))$, and hence we can extract a subsequence $(\partial_t u_{n_k})_{k\in\mathbb{N}}$ which converges to w strongly in $L^2(0,T; H^1(\Omega))$. Keeping in mind that the embedding $L^2(0,T; H^1(\Omega)) \hookrightarrow L^2(0,T; L^2(\Omega))$ is continuous, and we already know that

$$\partial_t u_{n_k} \rightharpoonup \partial_t u^\dagger \quad \text{in } L^2(0,T; L^2(\Omega)),$$

and thus, the uniqueness of weak limits implies $w = \partial_t u^\dagger$. From this, we deduce that $u^\dagger \in W^{1,2}(0,T;H^1(\Omega)) \subset C([0,T],H^1(\Omega))$. Now, if we let $\xi_1,\xi_2 \in [0,T]$ be fixed, we have

$$\|u^\dagger(\xi_1) - u^\dagger(\xi_2)\| \le \|u^\dagger(\xi_1) - u_{n_k}(\xi_2)\| + \|u_{n_k}(\xi_1) - u_{n_k}(\xi_2)\| + \|u_{n_k}(\xi_1) - u^\dagger(\xi_2)\|.$$

Hence, by relying on Lemma 3.5, one can easily verify that

$$\|u^\dagger(\xi_1) - u^\dagger(\xi_2)\| \le 2 \sup_{0\le\xi\le T} \|u^\dagger(\xi) - u_{n_k}(\xi)\| + C|\xi_1 - \xi_2|,$$

so if we let $k \to \infty$ and take the uniform convergence of $(u_{n_k})_{k\in\mathbb{N}}$ into account, leads to

$$\|u^\dagger(\xi_1) - u^\dagger(\xi_2)\| \le C|\xi_1 - \xi_2|, \quad \forall \xi_1,\xi_2 \in [0,T],$$

which, in turn implies that $u^\dagger \in \mathrm{Lip}(0,T;L^2(\Omega))$.

Along the same lines, and using the same arguments based on uniform estimates and convergence results, one deduces the existence of a subsequence (v_{n_k}) of $(v_n)_{n\in\mathbb{N}}$, and a limit function

$$v^\dagger \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)),$$

such that,

$$\begin{cases} v_{n_k} \to v^\dagger, & \text{in } C([0,T],L^2(\Omega)), \\ v_{n_k}(t) \rightharpoonup v^\dagger(t), & \text{in } H^1(\Omega),\ \forall t \in [0,T], \\ \overline{v}_{n_k}(t) \rightharpoonup v^\dagger(t), & \text{in } H^1(\Omega),\ \forall t \in [0,T], \\ \partial_t v_{n_k} \rightharpoonup \partial_t v^\dagger, & \text{in } L^2(0,T;L^2(\Omega)). \end{cases}$$

We also see that the subsequences $(v_{n_k})_{k\in\mathbb{N}}, (\bar{v}_{n_k})_{k\in\mathbb{N}}$, and $(\widetilde{v}_{n_k})_{k\in\mathbb{N}}$ have the same strong limit $v^\dagger$ in $L^2(0,T;L^2(\Omega))$. Furthermore, the limit function $v^\dagger$ has the following regularity:

$$v^\dagger \in C([0,T],H^1(\Omega)) \cap \mathrm{Lip}(0,T;L^2(\Omega)), \quad \text{with } \partial_t v^\dagger \in L^2(0,T;H^1(\Omega)).$$

By Lemma 3.5, the sequences $(\overline{p}_n)$ and $(\overline{q}_n)$ are bounded in the reflexive space $L^2(0,T)$; therefore, the functions $p,q \in L^2(0,T)$ and subsequences $(\overline{p}_{n_k})_{k\in\mathbb{N}}$, $(\overline{q}_{n_k})_{k\in\mathbb{N}}$ exist, such that

$$\overline{p}_{n_k} \rightharpoonup p^\dagger, \qquad \overline{q}_{n_k} \rightharpoonup q^\dagger \qquad \text{weakly in } L^2(0,T).$$

Hence, one can write the following for an arbitrary $h \in L^2(0,T)$:

$$\int_0^T \overline{p}_{n_k}(t)\,h(t)\,dt \longrightarrow \int_0^T p^\dagger(t)\,h(t)\,dt, \quad \text{and} \int_0^T \overline{q}_{n_k}(t)\,h(t)\,dt \longrightarrow \int_0^T q^\dagger(t)\,h(t)\,dt.$$

Since (f,φ) and (g,ψ) are independent of t, they can be factored out. By weak convergence in $L^2(0,T)$ and testing with the characteristic function $h = \chi_{(0,\xi)} \in L^2(0,T)$, it results in

$$\begin{cases} \displaystyle\int_0^\xi \overline{p}_{n_k}(t)(f,\varphi)\,dt \to \int_0^\xi p^\dagger(t)(f,\varphi)\,dt, \\ \displaystyle\int_0^\xi \overline{q}_{n_k}(t)(g,\psi)\,dt \to \int_0^\xi q^\dagger(t)(g,\psi)\,dt, \end{cases} \tag{4.4}$$

for every $\varphi, \psi \in H^1(\Omega)$ and every $\xi \in (0, T]$.

Now, we integrate each one of the identities in (4.1) over the interval $(0, \xi)$ with $\xi \in (0, T]$, keeping the test functions $\varphi, \psi \in H^1(\Omega)$ fixed. For u_{n_k}, we obtain

$$\langle(\omega^\gamma * u_{n_k})(\xi) - \int_0^\xi \langle\overline{\omega}^\gamma_{n_k}(t)\tilde{u}_0, \varphi\rangle_{H^1}\, dt + \int_0^\xi \overline{L}^1_{n_k}(t)(\overline{u}_{n_k}(t), \overline{v}_{n_k}(t); \varphi)\, dt = \int_0^\xi \overline{p}_{n_k}(t)\,(f, \varphi)\, dt. \tag{4.5}$$

Similarly, for v_n^k, we obtain

$$\langle(\omega^\gamma * v_{n_k})(\xi) - \int_0^\xi \langle\overline{\omega}^\gamma_{n_k}(t)\tilde{v}_0, \psi\rangle_{H^1}\, dt + \int_0^\xi \overline{L}^2_{n_k}(t)(\overline{v}_{n_k}(t), \overline{u}_{n_k}(t); \psi)\, dt = \int_0^\xi \overline{q}_{n_k}(t)\,(g, \psi)\, dt. \tag{4.6}$$

Notice that the mapping $w \mapsto (\omega^\gamma * w)(\xi)$ is continuous from $C([0, T]; L^2(\Omega))$ into $L^2(\Omega)$ for all $\xi \in (0, T]$. Consequently, since $u_{n_k} \to u^\dagger$ strongly in $C([0, T], L^2(\Omega))$ it follows that $(\omega^\gamma * u_{n_k})(\xi) \to (\omega^\gamma * u^\dagger)(\xi)$, which, in turn, implies by pairing with $\varphi \in H^1(\Omega)$ that

$$\langle(\omega^\gamma * u_{n_k})(\xi), \varphi\rangle_{H^1} \longrightarrow \langle(\omega^\gamma * u^\dagger)(\xi), \varphi\rangle_{H^1}.$$

Analogously, since $v_n^k \to v^\dagger$ strongly in $C([0, T]; L^2(\Omega))$, we also have the following for any $\psi \in H^1(\Omega)$

$$\langle(\omega^\gamma * v_{n_k})(\xi), \psi\rangle_{H^1} \longrightarrow \langle(\omega^\gamma * v^\dagger)(\xi), \psi\rangle_{H^1}.$$

We now pass to treating the initial data terms in both identities. Since $\overline{\omega}^\gamma_{n,k} \to \omega^\gamma$ in $L^1(0, T)$, and the scalar products $(u_0, \varphi)_{L^2}$, $(v_0, \psi)_{L^2}$ are fixed, dominated convergence yields

$$\int_0^\xi \langle\overline{\omega}^\gamma_{n,k}(t)\, u_0, \varphi\rangle_{H^1}\, dt \longrightarrow \int_0^\xi \langle\omega^\gamma(t)\, u_0, \varphi\rangle_{H^1}\, dt,$$

and

$$\int_0^\xi \langle\overline{\omega}^\gamma_{n,k}(t)\, v_0, \psi\rangle_{H^1}\, dt \longrightarrow \int_0^\xi \langle\omega^\gamma(t)\, v_0, \psi\rangle_{H^1}\, dt.$$

Passing to the limit in (4.5) and (4.6) gives the following for every $\xi \in (0, T]$ and $\varphi \in H^1(\Omega)$

$$\langle(\omega^\gamma * u^\dagger)(\xi), \varphi\rangle_{H^1} - \int_0^\xi \langle\omega^\gamma(t)\, u_0, \varphi\rangle_{H^1}\, dt + \int_0^\xi L^1(t)(u\dagger, v\dagger; \varphi)\, dt = \int_0^\xi p \dagger (t)\,(f, \varphi)\, dt. \tag{4.7}$$

Differentiating (4.7) with respect to ξ yields the continuous weak form

$$\langle\partial_t(\omega^\gamma * u^\dagger)(t) - \omega^\gamma(t)\tilde{u}_0, \varphi\rangle_{H^1} + L^1(t)(u^\dagger, v^\dagger; \varphi) = p^\dagger(t)\,(f, \varphi), \quad \text{for a.e. } t \in (0, T). \tag{4.8}$$

The same argument applied to (4.6) produces

$$\langle\partial_t(\omega^\gamma * v^\dagger)(t) - \omega^\gamma(t)\tilde{v}_0, \psi\rangle_{H^1} + L^2(t)(v^\dagger, u^\dagger; \psi) = q^\dagger(t)\,(g, \psi), \quad \text{for a.e. } t \in (0, T). \tag{4.9}$$

By the construction of the Rothe scheme, we have $u_{n_k}(0) = \tilde{u}_0$ and $\tilde{u}_{n_k}(0) = \tilde{u}_0$ for all $k \in \mathbb{N}$. Consequently, from the strong convergence of $(u_{n_k})_{k\in\mathbb{N}}$, it follows that

$$\|u(0) - \tilde{u}_0\|_{L^2(\Omega)} = \lim_{k\to\infty} \|u_{n_k}(0) - \tilde{u}_0\|_{L^2(\Omega)} = 0.$$

Therefore, the limit function $u^\dagger$ satisfies the initial condition, that is, $u^\dagger(0) = \tilde{u}_0$. Analogous reasoning also shows that $v^\dagger(0) = \tilde{v}_0$. Consequently, the functions set $(u^\dagger, v^\dagger, p^\dagger, q^\dagger)$ is obviously a solution to the problem in (2.1) and(2.2), since

$$\partial_t(\omega^\gamma * u^\dagger)(t) - \omega^\gamma(t)\tilde{u}_0 = (\omega^\gamma * \partial_t u^\dagger)(t) \in L^2(0, T; H^1(\Omega)^*),$$

and

$$\partial_t(\omega^\gamma * v^\dagger)(t) - \omega^\gamma(t)\tilde{v}_0 = (\omega^\gamma * \partial_t v^\dagger)(t) \in L^2(0, T; H^1(\Omega)^*),$$

as $u^\dagger$ and $v^\dagger$ turn out to be absolutely continuous.

Next, we turn to the limit passage in the measured problem in (4.2). For this purpose, we integrate the discrete measured equations over $(0, \xi) \subset (0, T)$ for n_k instead of n. We obtain

$$\begin{cases} \displaystyle\int_0^\xi \overline{p}_{n_k}(t)(f, 1)\, dt = \int_0^\xi (\overline{\omega}^\gamma_{n_k} * \overline{\theta}'_{n_k})(t)\, dt + \int_0^\xi (\overline{a}(t)\tilde{u}(t) + \overline{b}(t)\tilde{v}(t), 1)\, dt, \\ \displaystyle\int_0^\xi \overline{q}_{n_k}(t)(g, 1)\, dt = \int_0^\xi (\overline{\omega}^\gamma_{n_k} * \overline{\vartheta}'_{n_k})(t)\, dt + \int_0^\xi (\overline{c}(t)\tilde{v}(t) + \overline{d}(t)\tilde{u}(t), 1)\, dt. \end{cases} \tag{4.10}$$

By the standard stability property of convolution in $L^1(0, T)$, the convergences $\overline{\omega}^\gamma_{n_k} \to \omega^\gamma$, $\overline{\theta}'_{n_k} \to \theta'$, and $\overline{\vartheta}'_{n_k} \to \vartheta'$ in $L^1(0, T)$ imply that $\overline{\omega}^\gamma_{n_k} * \overline{\theta}'_{n_k} \to \omega^\gamma * \theta'$ and $\overline{\omega}^\gamma_{n_k} * \overline{\vartheta}'_{n_k} \to \omega^\gamma * \vartheta'$ in $L^1(0, T)$; therefore, it follows that

$$\begin{cases} \displaystyle\int_0^\xi (\overline{\omega}^\gamma_{n_k} * \overline{\theta}'_{n_k})(t)\, dt \to \int_0^\xi (\omega^\gamma * \theta')(t)\, dt, \\ \displaystyle\int_0^\xi (\overline{\omega}^\gamma_{n_k} * \overline{\vartheta}'_{n_k})(t)\, dt \to \int_0^\xi (\omega^\gamma * \vartheta')(t)\, dt. \end{cases} \tag{4.11}$$

The limits of the remaining terms on the right-hand sides follow from the convergence results obtained previously as a particular case with $\varphi = \psi = 1$. Consequently, passing to the limit as $k \to \infty$ and differentiating the resulting relations with respect to ξ yields, for a.e. $t \in (0, T)$

$$\begin{cases} \displaystyle\int_0^\xi p^\dagger(t)(f, 1)\, dt = \int_0^\xi (\omega^\gamma * \theta')(t)\, dt + \int_0^\xi (a(t)u^\dagger(t) + b(t)v^\dagger(t), 1)\, dt, \\ \displaystyle\int_0^\xi q^\dagger(t)(g, 1)\, dt = \int_0^\xi (\omega^\gamma * \vartheta')(t)\, dt + \int_0^\xi (c(t)v^\dagger(t) + d(t)u^\dagger(t), 1)\, dt. \end{cases}$$

Since the limit relations hold for all $\xi \in (0, T]$ and both sides define absolutely continuous functions of ξ, the differentiation is valid almost everywhere. We thus conclude that the limit quadruple$(u^\dagger, v^\dagger, p^\dagger, q^\dagger)$ solves the measured problem (2.2), which achieves the proof. □

We next turn to the proof of uniqueness of the solution.

Theorem 4.2. *Suppose that the conditions of Theorem 4.1 are fulfilled. Then the system* (2.1)-(2.2) *admits at most one solution of the quadruple* $\mathcal{S} = (u, v, p, q)$.

Proof. Suppose that $\{(u_i, v_i, p_i, q_i)\}_{i=1,2}$ be two solutions to the inverse problem, and set

$$U = u_1 - u_2,\ V = v_1 - v_2,\ P = p_1 - p_2, \text{ and } Q = q_1 - q_2.$$

The couple (U, V) then satisfies, for all $\varphi, \psi \in H^1(\Omega)$, the following system:

$$\begin{cases} \langle (\omega^\gamma * \partial_t U)(t), \varphi \rangle_{H^1} + L^1(t)(U, V; \varphi) = P(t)(f, \varphi), \\ \langle (\omega^\gamma * \partial_t V)(t), \psi \rangle_{H^1} + L^2(t)(V, U; \psi) = Q(t)(g, \psi), \end{cases} \tag{4.12}$$

where P and Q are given by

$$P(t) = \frac{\int_\Omega a(t)U(t)\,dx + \int_\Omega b(t)V(t)\,dx}{\int_\Omega f\,dx}, \quad Q(t) = \frac{\int_\Omega c(t)V(t)\,dx + \int_\Omega d(t)U(t)\,dx}{\int_\Omega g\,dx}. \tag{4.13}$$

Putting $\varphi = U(t)$ and $\psi = V(t)$ in (4.12) yields

$$\langle (\omega^\gamma * \partial_t U)(t), U(t) \rangle_{H^1} + L^1(t)(U, V; U) = P(t)(f, U(t)), \tag{4.14}$$

$$\langle (\omega^\gamma * \partial_t V)(t), V(t) \rangle_{H^1} + L^2(t)(V, U; V) = Q(t)(g, V(t)). \tag{4.15}$$

We now make the standard coercivity estimate for the bilinear forms $L^1(t)$ and $L^2(t)$. In view of the assumptions in (H_2) in (1.3), we have

$$\begin{aligned} L^1(t)(U, V; U) + L^2(t)(V, U; V) \geq \|\nabla U(t)\|^2_{L^2(\Omega)} + \|\nabla V(t)\|^2_{L^2(\Omega)} \\ + a_0\|U(t)\|^2_{L^2(\Omega)} + c_0\|V(t)\|^2_{L^2(\Omega)} + \int_\Omega (b(t) + d(t))\, U(t)V(t)\,dx. \end{aligned} \tag{4.16}$$

Use the elementary bound

$$\int_\Omega (b + d)\, UV\,dx \;\geq\; -\frac{b_1 + d_1}{2}\,(\|U\|^2 + \|V\|^2).$$

By the hypothesis $\min(a_0, c_0) \geq \dfrac{b_1 + d_1}{2}$, we get

$$a_0\|U(t)\|^2 + c_0\|V(t)\|^2 + \int_\Omega (b(t) + d(t))\, U(t)V(t)\,dx \;\geq\; 0.$$

Therefore, from (4.16), one deduces the simpler lower bound

$$L^1(t)(U, V; U) + L^2(t)(V, U; V) \;\geq\; \|\nabla U(t)\|^2 + \|\nabla V(t)\|^2. \tag{4.17}$$

Summing (4.14) and (4.15) and using (4.17) yields, for a.e. t

$$\begin{aligned} \langle (\omega^\gamma * \partial_t U)(t), U(t) \rangle_{H^1} + \langle (\omega^\gamma * \partial_t V)(t), V(t) \rangle_{H^1} \\ + \|\nabla U(t)\|^2 + \|\nabla V(t)\|^2 \;\leq\; P(t)(f, U(t)) + Q(t)(g, V(t)). \end{aligned} \tag{4.18}$$

Using the Cauchy and Young inequalities in the standard manner, one readily obtains an upper bound.

$$P(t)(f, U(t)) + Q(t)(g, V(t)) \leq C(\|U(t)\|^2 + \|V(t)\|^2).$$

Applying the inequality [33], leading to

$$((\omega^\gamma * \partial_t U)(t), U(t)) + ((\omega^\gamma * \partial_t V)(t), V(t)) \geq \frac{1}{2}\partial_t^\gamma(\|U(t)\|^2 + \|V(t)\|^2).$$

Collecting the preceding estimates, we conclude that

$$\partial_t^\gamma Y(t) \leq C\, Y(t), \quad \text{with } \; Y(t) := \|U(t)\|^2 + \|V(t)\|^2. \tag{4.19}$$

Consequently, according to the fractional Grönwall inequality, the fractional differential inequality (4.19) with $Y(0) = 0$ implies $Y(t) = 0$ for all $t \in [0, T]$. Equivalently, the functions U and V vanish almost everywhere in $\Omega \times (0, T)$. From the explicit formulas (4.13) and $U = V = 0$, we immediately obtain $P(t) = 0$ and $Q(t) = 0$ for a.e. t. Consequently, $p_1 = p_2$ and $q_1 = q_2$. □

5. Numerical reconstruction

In this section, we address the numerical reconstruction of the unknown source terms $p(t)$ and $q(t)$ in the problem (1.1). For clarity of exposition, we focus on the one-dimensional case, with the spatial domain taken as $\Omega = (0, 1)$ and a fixed final time $T = 1$.

The numerical solution of the inverse source problem is performed by implementing the algorithms described in the preceding sections. Although the scheme can be derived directly from the discrete formulations (3.3) and (3.4), we employ the L^1-finite difference scheme to approximate for the Caputo fractional derivative in order to improve accuracy. The spatial discretization is carried out using standard finite element methods. The unknown temporal source functions $p(t)$ and $q(t)$ are computed directly from the discrete measurement equations, which naturally incorporate the additional integral data into the system. This results in a consistent and unified numerical scheme for both the forward and inverse problems.

For the numerical tests, the noisy data are generated by introducing a random perturbation, namely

$$\begin{cases} \theta_\varepsilon(t) = \theta_{exact}(t) + \varepsilon\, \theta_{exact}(t) \cdot (2\,\mathrm{rand}(\mathrm{size}\theta(t))) - 1), \\ \vartheta_\varepsilon(t) = \vartheta_{exact}(t) + \varepsilon\, \vartheta_{exact}(t) \cdot (2\,\mathrm{rand}(\mathrm{size}\vartheta(t))) - 1), \end{cases}$$

for $t \in (0, T)$, where $\varepsilon > 0$ represents the percentage of noise, and rand$(\cdot)$ produces random numbers uniformly distributed in $[0, 1]$. It is worth noting that if one directly uses the noisy data θ_ε and ϑ_ε, the noise will badly affect fractional differentiation when approximating $\partial_t^\gamma\theta$ and $\partial_t^\gamma\vartheta$. Instead of using noisy data directly, we can approximate it with a smooth function that fits the noisy samples but suppresses random oscillations.

The idea to overcome this drawback is to replace the noisy data with a smooth approximating function. The regularized data are then used instead of the noisy data for computing fractional derivatives. Subsequently, the perturbed data are regularized using a nonlinear least-squares approach, yielding a function that effectively approximates the noisy measurements. A standard way is to use the nonlinear least-squares method to obtain regularized data of the form

$$\theta_{\varepsilon,reg}(t) = \sum_{k=0}^{m} \lambda_k t^k, \quad \text{and } \; \vartheta_{\varepsilon,reg}(t) = \sum_{k=0}^{m} \mu_k t^k.$$

In order to measure the accuracy of the numerical solution, we determine the relative errors:

$$\mathcal{E}(p) = \frac{\|p_N - p^\dagger\|_{L^2(0,T)}}{\|p^\dagger\|_{L^2(0,T)}}, \quad \text{and} \quad \mathcal{E}(q) = \frac{\|q_N - q^\dagger\|_{L^2(0,T)}}{\|q^\dagger\|_{L^2(0,T)}}, \tag{5.1}$$

where p_N and q_N denote the approximation of the exact source terms $p^\dagger$ and $q^\dagger$, respectively.

5.1. Numerical implementation

In this subsection, we provide a concise description of the implementation of the numerical inversion method, which is fundamentally based on the time-discretization scheme previously employed to establish the theoretical results.

Let $\{t_n\}_{0\le n\le N}$ constitute a uniform partition of the temporal domain $[0, T]$ with time step $\tau = T/N$. To accurately approximate the Caputo fractional derivative, we employ the standard L1 scheme. For a sufficiently smooth function $z(t)$, the discrete approximation of the Caputo derivative at time t_{n+1} is given by:

$$\partial_t^\alpha z(t_{n+1}) \approx D_\tau^\alpha z_{n+1} := \frac{1}{\Gamma(2-\alpha)\tau^\alpha}\left[z_{n+1} - \sum_{i=1}^{n} (b_{i-1}^\alpha - b_i^\alpha)\, z_{n+1-i} - b_n^\alpha z_0\right],$$

where the coefficients

$$b_i^\alpha = (i+1)^{1-\alpha} - i^{1-\alpha}, \qquad b_0^\alpha = 1,$$

form a strictly decreasing sequence. This discretization can be compactly written as

$$D_\tau^\alpha z_{n+1} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}(z_{n+1} + H_n^\alpha),$$

where H_n^α collects the history contributions of the fractional derivative.

We discretize the spatial domain $\Omega = [0, 1]$ into M uniform elements with the nodes $\xi_0 = 0 < \xi_1 < \cdots < \xi_M = 1$, where $h = 1/M$ denotes the mesh size. The finite element space $V_h \subset H^1(\Omega)$ is chosen as the space of continuous functions that are linear on each subinterval, that is,

$$V_h = \{v_h \in C^0(\bar{\Omega}) : v_h|_{[\xi_{i-1},\xi_i]} \in P_1,\ i = 1, \ldots, M\}.$$

The standard nodal basis functions $\{\phi_j\}_{j=0}^M$ are defined by the property $\phi_j(\xi_i) = \delta_{ij}$, providing the representation

$$z_h(x) = \sum_{j=0}^{M} z_h(\xi_j)\phi_j(x), \quad \forall v_h \in V_h.$$

The fully discrete scheme for the coupled system (2.1) seeks the approximations $u_h^n, v_h^n \in V_h$ satisfying the following for all $\phi_h, \psi_h \in V_h$:

$$\begin{cases} (D_\tau^\alpha u_h^{n+1}, \phi_h) + (\nabla u_h^{n+1}, \nabla\phi_h) + (a^{n+1}u_h^{n+1} + b^{n+1}v_h^n, \phi_h) &= p^{n+1}(f, \phi_h), \\ (D_\tau^\alpha v_h^{n+1}, \psi_h) + (\nabla v_h^{n+1}, \nabla\psi_h) + (c^{n+1}v_h^{n+1} + d^{n+1}u_h^n, \psi_h) &= q^{n+1}(g, \psi_h), \end{cases}$$

where the coefficients $a^{n+1}, b^{n+1}, c^{n+1}$, and d^{n+1} are evaluated at t_{n+1}. Substituting the L1-approximation and rearranging the terms yields the following coupled linear system:

$$\begin{cases} (u_h^{n+1}, \phi_h) + \alpha_0(\nabla u_h^{n+1}, \nabla\phi_h) + \alpha_0(a^{n+1}u_h^{n+1}, \phi_h) = \\ \qquad - \alpha_0(H_{u,n}^\alpha, \phi_h) - \alpha_0(b^{n+1}v_h^n, \phi_h) + \alpha_0 p^{n+1}(f, \phi_h), \\ (v_h^{n+1}, \psi_h) + \alpha_0(\nabla v_h^{n+1}, \nabla\psi_h) + \alpha_0(c^{n+1}v_h^{n+1}, \psi_h) = \\ \qquad - \alpha_0(H_{v,n}^\alpha, \psi_h) - \alpha_0(d^{n+1}u_h^n, \psi_h) + \alpha_0 q^{n+1}(g, \psi_h), \end{cases}$$

where $\alpha_0 = \Gamma(2-\alpha)\tau^\alpha$, and $H_{u,n}^\alpha$ and $H_{v,n}^\alpha$ represent the discrete history terms for u and v, respectively.

The unknown source terms p^{n+1} and q^{n+1} are reconstructed at each time step using the discrete measurement equations

$$p^{n+1} = \frac{D_\tau^\alpha\theta_{n+1} + (a^{n+1}u_h^n + b^{n+1}v_h^n, 1)}{(f, 1)}, \quad q^{n+1} = \frac{D_\tau^\alpha\vartheta_{n+1} + (c^{n+1}v_h^n + d^{n+1}u_h^n, 1)}{(g, 1)}, \tag{5.2}$$

where θ_{n+1} and ϑ_{n+1} are the discrete measurement data, and $(f, 1)$, $(g, 1)$ are assumed to be non-zero. Choosing $\phi_h = \psi_h = \phi_j$ for $j = 1, \ldots, M-1$, we obtain the block matrix system

$$\begin{bmatrix} M + \alpha_0(A + M_a) & \alpha_0 M_b \\ \alpha_0 M_d & M + \alpha_0(A + M_c) \end{bmatrix} \begin{bmatrix} U^{n+1} \\ V^{n+1} \end{bmatrix} = \begin{bmatrix} -\alpha_0 H_{u,n} - \alpha_0 M_b V^n + \alpha_0 p^{n+1} F \\ -\alpha_0 H_{v,n} - \alpha_0 M_d U^n + \alpha_0 q^{n+1} G \end{bmatrix},$$

where M and A are the mass and stiffness matrices; M_a, M_b, M_c, and M_d are coefficient-weighted mass matrices; U^{n+1} and V^{n+1} are the solution vectors; F and G are the source term vectors; and $H_{u,n}$ and $H_{v,n}$ are the history vectors.

This coupled system is solved sequentially at each time step, with the source terms updated using the most recent state approximations. The resulting numerical solution (u_h, v_h, p_h, q_h) provides a complete approximation to the solution of the inverse problem, with the option for further refinement through iterative correction steps to enhance consistency between the reconstructed sources and state variables.

5.2. Numerical tests

Example 5.1. *In this first example, we consider a problem with constant coefficients as follows:*

$$\begin{cases} \partial_t^\gamma u - \Delta u + au + bv = p(t)f(x) + F(x,t), & (x,t) \in (0,1) \times (0,T), \\ \partial_t^\gamma v - \Delta v + cv + du = q(t)g(x) + G(x,t), & (x,t) \in (0,1) \times (0,T), \end{cases} \tag{5.3}$$

supplemented with homogeneous Neumann boundary conditions and the initial data

$$u(x,0) = 1 - \cos(\pi x), \quad v(x,0) = \cos^2(\pi x),$$

with the source terms defined by

$$\begin{aligned} F(x,t) &= -\pi^2(t^2+1)\cos(\pi x) + (1-\cos(\pi x))(t^2+1) + (\cos^2(\pi x))(t^3+1), \\ G(x,t) &= 2\pi^2(t^3+1)\cos(2\pi x) + (\cos^2(\pi x))(t^3+1) + (1-\cos(\pi x))(t^2+1), \\ f(x) &= 1-\cos(\pi x), \quad g(x) = \cos^2(\pi x). \end{aligned}$$

We set the exact solutions

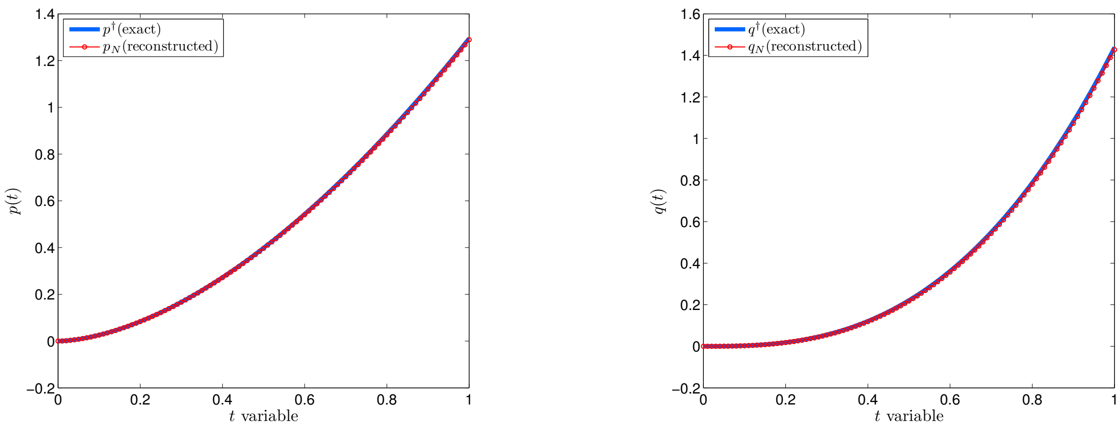
$$u(x,t) = (1-\cos(\pi x))(t^2+1), \quad v(x,t) = \cos^2(\pi x)(t^3+1),$$

and the unknown source amplitudes

$$p(t) = \frac{2}{\Gamma(3-\gamma)} t^{2-\gamma}, \quad q(t) = \frac{6}{\Gamma(4-\gamma)} t^{3-\gamma}.$$

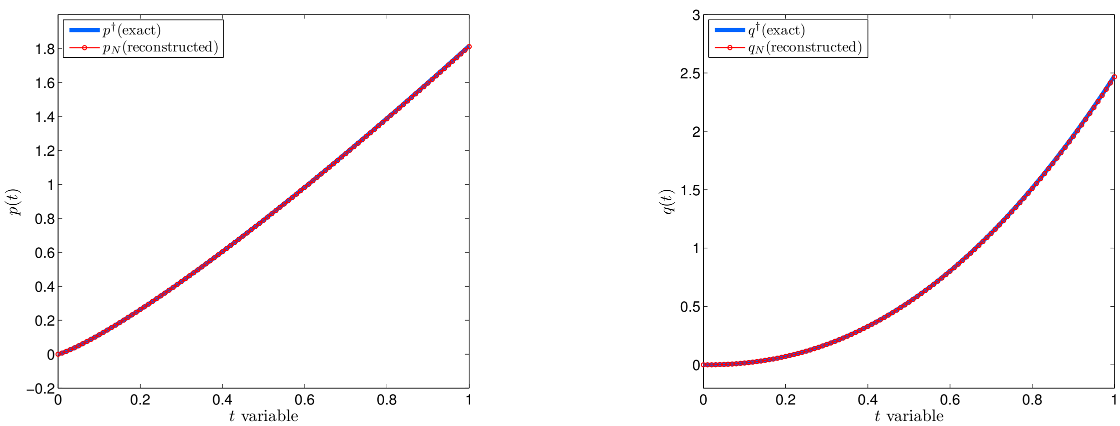
It can be readily verified that the functions $u(x,t)$, $v(x,t)$, $p(t)$, and $q(t)$ satisfy the system (5.3).

The numerical reconstructions of the source terms $p(t)$ and $q(t)$ in Example 5.1 with noise-free data are presented in Figures 1 and 2 for the fractional orders $\gamma = 0.3$ and $\gamma = 0.8$, respectively. As observed, the reconstructed profiles of $p(t)$ and $q(t)$ exhibit close agreement with the exact solutions in both cases. This observation is also supported by the quantitative results reported in Tables 1 and 2, which display the relative errors $\mathcal{E}(p)$ and $\mathcal{E}(q)$ for different values of the fractional order.



(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 1. Numerical reconstruction for Example 5.1 with zero noise ($\varepsilon = 0$) for $\gamma = 0.3$.



(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 2. Numerical reconstruction for Example 5.1 with zero noise ($\varepsilon = 0$) for $\gamma = 0.8$.

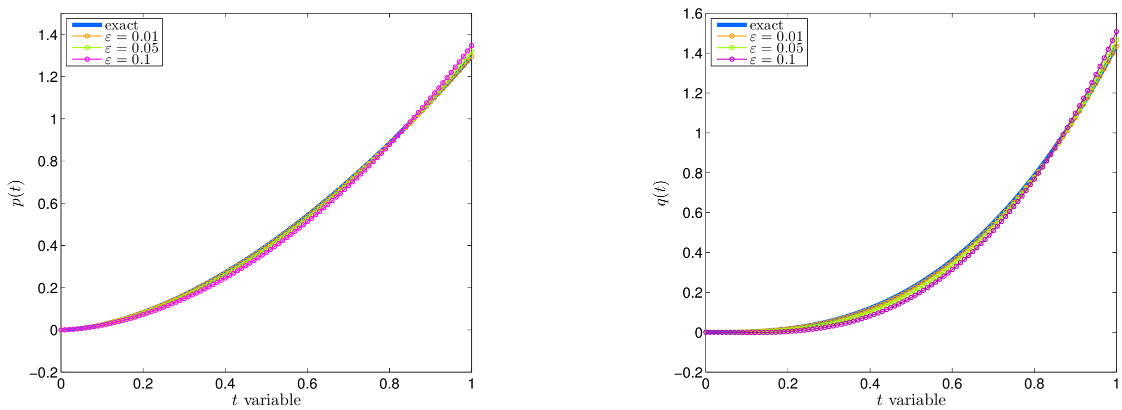
Table 1. Relative error $\mathcal{E}(p)$ of Example 5.1 for different fractional orders γ and noise levels ε.

γ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
0.3	0.0044	0.0066	0.0161	0.0282
0.5	0.0031	0.0052	0.0147	0.0279
0.8	0.0018	0.0047	0.0173	0.0332

Table 2. Relative error $\mathcal{E}(q)$ of Example 5.1 for different fractional orders γ and noise levels ε.

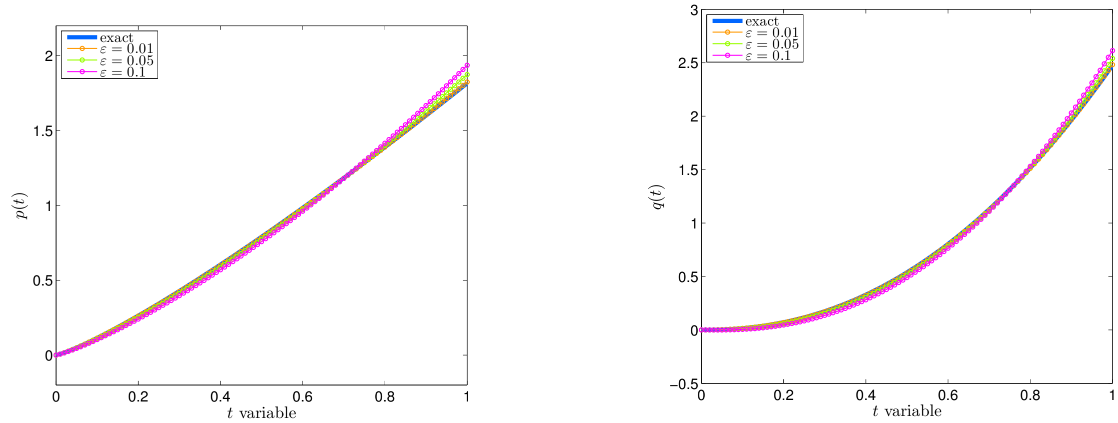
γ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
0.3	0.0096	0.0129	0.0276	0.0466
0.5	0.0061	0.0091	0.0227	0.0401
0.8	0.0033	0.0065	0.0204	0.0381

Figures 3 and 4 illustrate the numerical reconstruction of the source terms for the fractional orders $\gamma = 0.3$ and $\gamma = 0.8$ under different noise levels. Panels (a) and (b) in each figure show that the reconstructed profiles of $p(t)$ and $q(t)$ agree very well with the exact solutions, preserving accuracy and stability even at the relatively high noise level $\varepsilon = 0.1$.



(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 3. Numerical reconstruction for Example 5.1 for various noise levels ε and $\gamma = 0.3$.



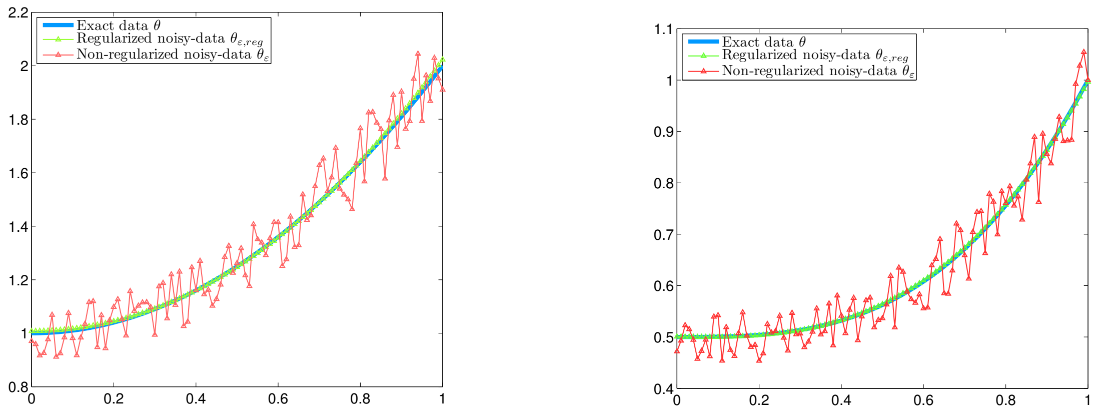
(a) Reconstruction of $p(t)$ **(b)** Reconstruction of $q(t)$

Figure 4. Numerical reconstruction for Example 5.1 for various noise levels ε and $\gamma = 0.8$.

Overall, the numerical reconstructions demonstrate high accuracy, with the recovered source terms closely matching the exact solutions and maintaining stability even under significant noise.

Figure 5 displays the perturbed measurements $\theta_\varepsilon(t)$ and $\varphi_\varepsilon(t)$ together with their regularized counterparts $\theta_{\varepsilon,\mathrm{reg}}(t)$ and $\varphi_{\varepsilon,\mathrm{reg}}(t)$. The noisy signals exhibit strong oscillations, while the regularized data provide smooth and differentiable profiles suitable for subsequent numerical computations.

These results confirms that the regularization step plays a crucial role in stabilizing the inversion process. Without it, the computation of the fractional derivatives of $\theta(t)$ and $\vartheta(t)$ would amplify noise and lead to unreliable reconstructions of the source terms.

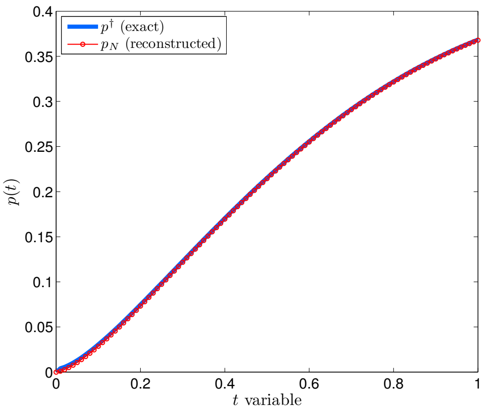


(a) Regularization of $\theta_\varepsilon(t)$ **(b)** Regularization of $\vartheta_\varepsilon(t)$

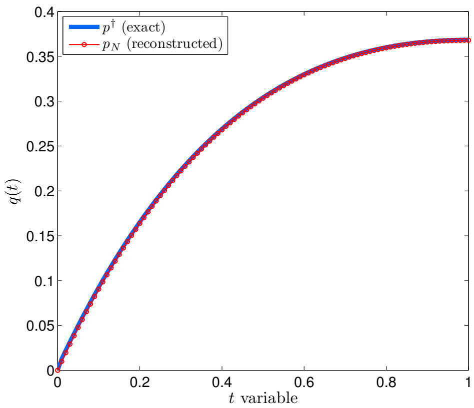
Figure 5. Regularization effect for noisy measurement data with $\varepsilon = 0.1$.

Example 5.2. *In this second example, we test a synthetic problem with the exact source terms $p(t) = t^{2-\gamma}e^{-t}$ and $q(t) = te^{-t}$, with the initial data $u_0(x) = \sin\left(\frac{\pi x}{2}\right)$ and $v_0(x) = \cos(\pi x)$. Here, the measurements data $\theta(t)$ and $\vartheta(t)$ are obtained by solving the forward problem.*

Figure 6 shows the numerical reconstruction of the source terms $p(t)$ and $q(t)$ for Example 5.2 *under noise-free conditions ($\varepsilon = 0$) with a fractional order $\gamma = 0.5$. The reconstructed profiles match the exact solutions very well.*



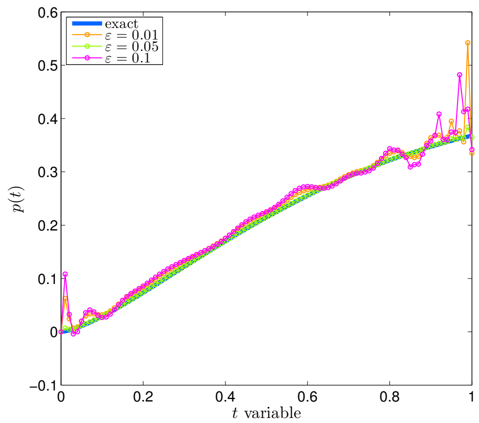
(a) Reconstruction of $p(t)$



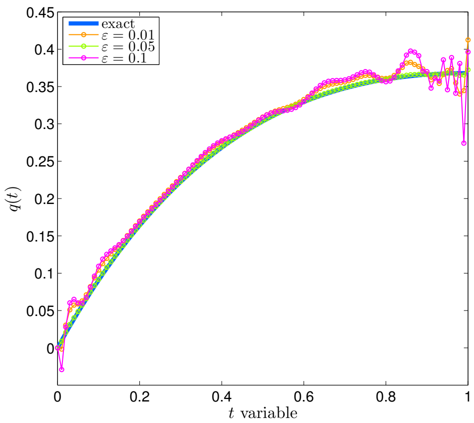
(b) Reconstruction of $q(t)$

Figure 6. Numerical reconstruction for Example 5.2 with zero noise ($\varepsilon = 0$) and $\gamma = 0.5$.

Figure 7 presents the numerical reconstruction of the source terms $p(t)$ and $q(t)$ in Example 5.2, under various noise levels $\varepsilon = 0.01, 0.05,$ and 0.1 and for the fractional order $\gamma = 0.5$. The reconstructed profiles demonstrate a remarkable ability to capture the exact solutions with high fidelity, even as the noise level increases to 10%. The method successfully recovers the underlying temporal source terms without significant deviation or instability.



(a) Reconstruction of $p(t)$



(b) Reconstruction of $q(t)$

Figure 7. Numerical reconstruction for Example 5.2 for various noise levels ε and $\gamma = 0.5$.

Tables 3 and 4 report the relative errors $\mathcal{E}(p)$ and $\mathcal{E}(q)$ for Example 5.2, under different fractional orders γ and noise levels ε. The results confirm the stability and accuracy of the proposed method: The errors remain small for all tested cases, even as the noise level increases to $\varepsilon = 0.1$. Moreover, it is observed that higher fractional orders (e.g., $\gamma = 0.8$) yield smaller reconstruction errors compared with lower orders. At present, we are unable to provide a theoretical explanation of this phenomenon, *as the conditional stability has not yet been established. Nevertheless, it is reasonable to ascribe this behaviour to the memory effects associated with fractional features.*

Table 3. Relative error $\mathcal{E}(p)$ of Example 5.2 for different fractional orders γ and noise levels ε.

γ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
0.3	0.0423	0.0765	0.0801	0.0930
0.5	0.0275	0.0557	0.0641	0.0852
0.8	0.0178	0.0472	0.0531	0.0701

Table 4. Relative error $\mathcal{E}(q)$ of Example 5.2 for different fractional orders γ and noise levels ε.

γ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
0.3	0.0376	0.0605	0.0718	0.0798
0.5	0.0224	0.0494	0.0563	0.0621
0.8	0.0145	0.0361	0.0442	0.0515

It is worth noting that the observed endpoint discrepancies are expected in inverse problems, which typically exhibit high sensitivity to noise near the temporal boundaries. While a complete theoretical justification for this specific behavior remains an open question, our numerical experiments suggest that it stems from the particular sensitivity of synthetic tests for time-dependent reconstruction to data perturbations. These results are consistent with the findings in prior studies and present a challenging question for future theoretical investigations.

6. Conclusions

In this work, we have investigated the inverse problem of simultaneously reconstructing two time-dependent source terms in a system of coupled time-fractional reaction–diffusion equations. By employing the Rothe method, we established the unique solvability of the problem under suitable assumptions on the data. The analysis was based on deriving stability estimates for the semi-discrete approximations and employing compactness arguments to rigorously pass to the limit. Within the same framework, we developed a direct and computationally efficient reconstruction procedure that avoids iterative optimization, thereby reducing the computational cost. Numerical experiments confirmed the accuracy and robustness of the proposed approach, showing that the source terms can be recovered with high fidelity even in the presence of significant measurement noise.

The findings of this paper contribute to the theoretical and numerical understanding of inverse problems in coupled fractional systems, a setting that introduces additional analytical and computational challenges compared with single-equation models. The consistent performance of the proposed method across a range of test cases highlights its potential as a reliable tool for the identification of time-dependent sources in multi-species anomalous diffusion processes.

As future work, it would be important on the theoretical side to relax the theoretical assumptions and to establish conditional stability estimates for the reconstruction problem. From the numerical perspective, incorporating more advanced regularization strategies could further improve stability under high noise levels. Another natural extension is to consider multi-term fractional problems, as well as systems involving the fractional Laplacian.

Author contributions

Maroua Nouar: Conceptualization, methodology, formal analysis, writing (original draft preparation); Maged Z. Youssef: Funding acquisition, methodology, formal analysis, validation, software, writing (review and editing); Hamed Ouled Sidi: Formal analysis, validation, software, writing (review and editing); Abdeldjalil Chattouh: Formal analysis, validation, software, project administration, supervision, writing (review and editing). All authors have read and agreed to the this version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author states that there is no conflict of interest regarding the publication of this paper.

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