Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 24730–24754.
DOI:10.3934/math.20251096
Received: 12 July 2025
Revised: 14 October 2025
Accepted: 17 October 2025
Published: 29 October 2025

Research article

Two classes of nearly optimal codebooks from generalized bent $\mathbb{Z}_4$-valued quadratic forms

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Abstract: Codebooks with small maximum cross-correlations are desirable in many fields, such as compressed sensing, direct spread code-division multiple-access (CDMA) systems, and space-time codes. The objective of this paper is the construction of codebooks. Based on the theory of $\mathbb{Z}_4$-valued quadratic forms, we propose two classes of generalized bent functions over $\mathbb{Z}_4$, and construct new families of codebooks from these functions. The codebooks obtained in this paper are nearly optimal with respect to the Welch bound, and could have a very small alphabet size, which is of importance in practical applications. Moreover, some Boolean bent functions are also derived.

Keywords: codebook; $\mathbb{Z}_4$-valued quadratic form; generalized bent function; Boolean bent function; nearly optimal
Mathematics Subject Classification: 11T23, 11T71, 94A60, 94B05

1. Introduction

Let $C = \{\mathbf{c}_0, \mathbf{c}_1, \cdots, \mathbf{c}_{N-1}\}$ be a set of N elements, where each $\mathbf{c}_i$, $0 \leq i \leq N-1$, is a unit norm $1 \times K$ complex vector. Such a set C is called an (N, K) codebook (also called signal set), and the vectors $\mathbf{c}_i$, $0 \leq i \leq N-1$ are called codewords of the codebook. The alphabet of the codebook is the set of all different complex values that the coordinates of all the codewords take. The alphabet size is the number of elements in the alphabet. In applications we prefer to employ codebooks with a small alphabet size [1]. As a performance measure of an (N, K) codebook C in practical applications, the maximum cross-correlation amplitude is defined as

$$I_{\max}(C) = \max_{0\leq i<j\leq N-1} \left|\mathbf{c}_i\mathbf{c}_j^H\right|,$$

where $\mathbf{c}_j^H$ denotes the conjugate transpose of the complex vector $\mathbf{c}_j$. Codebooks with small maximum cross-correlation amplitudes are preferred in many practical applications, such as unitary space-time modulations, multiple description coding over erasure channels, direct spread CDMA communications, and compressed sensing theory. For a given K, it is desirable to construct an (N, K) codebook with large N as well as small $I_{\max}(C)$ simultaneously. However, the following well-known bounds demonstrate a trade-off among these parameters.

Lemma 1. (1) (Welch Bound [2]) *For any* (N, K) *codebook* C *with* $N \geq K$,

$$I_{\max}(C) \geq I_W = \sqrt{\frac{N-K}{(N-1)K}}. \tag{1.1}$$

(2) (Levenshtein Bound [3]) For any real-valued (N, K) codebook C with $N > K(K+1)/2$,

$$I_{\max}(C) \geq I_L = \sqrt{\frac{3N-K^2-2K}{(N-K)(K+2)}}.$$

For any complex-valued (N, K) codebook C with $N > K^2$,

$$I_{\max}(C) \geq I_L = \sqrt{\frac{2N-K^2-K}{(N-K)(K+1)}}.$$

A codebook meeting the Welch bound with equality is called a maximum-Welch-bound-equality (MWBE) codebook [4]. The MWBE codebooks have applications in many fields, such as CDMA communications [5], space-time codes [6], compressed sensing [7], and combinatorial designs [8]. An MWBE codebook is an equiangular tight frame, and its construction is equivalent to line packing in Grassmannian spaces. When N is large, the Welch bound cannot be achieved. In that case, the Levenshtein bound turns out to be tighter than the Welch bound. However, it is very difficult to construct codebooks meeting the Welch bound or the Levenshtein bound (optimal codebooks), as pointed out by Sarwate [1]. Only a few optimal codebooks were provided in the literature (see [4,9–12] and references therein). Much research has been done instead to construct nearly optimal codebooks, i.e., the maximum cross-correlation amplitude $I_{\max}(C)$ is slightly higher than the Levenshtein bound or the Welch bound, but nearly achieves it for large enough K. That is to say,

$$\lim_{K\to\infty} \frac{I_{\max}(C)}{I_W} = 1 \quad \text{or} \quad \lim_{K\to\infty} \frac{I_{\max}(C)}{I_L} = 1.$$

There are many results on the constructions of nearly optimal codebooks. In 2006, Ding proposed several families of codebooks that nearly meet the Welch bound from almost difference sets [13]. Later, Ding and Feng extended this idea in [13] and presented a generic construction of complex codebooks using almost difference sets in arbitrary finite Abelian groups, from which several classes of codebooks nearly meeting the Welch bound were obtained [14]. In 2011, Zhou and Tang established a general connection between complex codebooks and relative difference sets, and used it to construct several classes of codebooks nearly meeting the Welch bound from some known relative difference sets [15]. Later, nearly optimal codebooks were proposed from difference sets and the product of Abelian groups [16], and from almost difference sets and partial difference sets [17]. In 2012, Zhang and Feng presented several general constructions of nearly optimal codebooks by using cyclotomic classes in finite fields, which generalized the codebooks derived from difference sets or almost difference sets in certain finite abelian groups [18]. Also in 2012, an (N, K) nearly optimal codebook was constructed from a $K \times N$ partial matrix with $K < N$, where each code vector is equivalent to a column of the matrix [19]. In 2014, Hong et al. proposed a class of nearly optimal partial Hadamard codebooks based on binary row selection sequences, which were generated by quadratic residue mapping of p-ary m-sequences [20]. In 2016, Tan et al. employed additive and multiplicative characters of finite fields to construct nearly optimal codebooks [21]. Similar methods were further utilized in subsequent works [22–26]. In 2018, Heng applied the generalized Jacobi sums and related character sums to obtain two classes of nearly optimal codebooks with respect to the Welch or Levenshtein bound, generalizing several known constructions [27]. Also in 2018, Luo and Cao introduced the hyper Eisenstein sum and used it to propose two codebook constructions [28]. Subsequently, based on p-ary linear codes and a hybrid character sum of a special kind of functions, they further proposed nearly optimal codebooks [29]. In 2023, Wang et al. introduced several classes of nearly optimal codebooks by selecting certain rows deterministically from circulant matrices, Fourier matrices, and Hadamard matrices, respectively [30]. In 2024, from Jacobi sums over Galois rings of arbitrary characteristics, Xu et al. produced a class of optimal codebooks [31].

In [9], Heng and Yue proposed a construction of codebooks from a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms, which satisfies the condition that the difference of arbitrary two distinct quadratic forms in the set is also generalized bent. It was pointed out in [9] that new generalized bent functions over $\mathbb{Z}_4$ satisfying the above condition are very difficult to construct in general. Later, Qi et al. [10] proposed a class of generalized bent $\mathbb{Z}_4$-valued quadratic forms of the form

$$Q(x) = \sum_{j=1}^{s} 2Q_j(a\gamma_j x) + \mathrm{Tr}_1^n(ax), \quad x \in \mathbb{T},$$

where

$$Q_j(x) = \sum_{i=1}^{\frac{t_j-1}{2}} \mathrm{Tr}_1^n(x^{1+2^{ik_j}}), \quad x \in \mathbb{T},$$

$\mathrm{Tr}_1^n(\cdot)$ is the trace function from the Galois ring $\mathbb{GR}(4, n)$ to $\mathbb{Z}_4$, $\mathbb{T}$ is the Teichmüller set of $\mathbb{GR}(4, n)$, n and s are positive integers, $1 \le k_1 < \cdots < k_{s+1} = n$, $k_j \mid k_{j+1}$, $t_j = \frac{n}{k_j}$ is odd for $1 \le j \le s$, and the coefficients a and γ_j $(1 \le j \le s)$ meet some restrictions. Based on the proposed functions, they introduced a set of generalized bent functions that satisfies the aforementioned condition. Then by employing the construction of Heng and Yue in [9], they derived a family of optimal codebooks achieving the Levenshtein bound from this set of generalized bent functions.

In this paper, inspired by the work of Heng and Yue [9] and Qi et al. [10], we investigate the $\mathbb{Z}_4$-valued quadratic forms with the form

$$Q(x) = \sum_{j=1}^{s} 2Q_j(x) + \mathrm{Tr}_1^n(cx), \quad x \in \mathbb{T}, \tag{1.2}$$

where

$$Q_j(x) = \sum_{i=1}^{\frac{n}{2k_j}-1} \mathrm{Tr}_1^n(a_{ji}x^{1+2^{ik_j}}) + \mathrm{Tr}_1^{\frac{n}{2}}(b_j x^{1+2^{\frac{n}{2}}}), \quad x \in \mathbb{T},$$

n and s are positive integers with $2|n$, $k_1, \cdots, k_s$ are distinct positive integers, each of which can divide $\frac{n}{2}$, and $a_{ji}, b_j, c \in \mathbb{T}$ for $1 \le j \le s$, $1 \le i \le \frac{n}{2k_j} - 1$. Here each $\frac{n}{k_j}$ is even for $1 \le j \le s$, which is different from the function of Qi et al. in [10]. Based on the theory of $\mathbb{Z}_4$-valued quadratic forms, we propose two classes of generalized bent $\mathbb{Z}_4$-valued quadratic forms of the form (1.2), by proving certain linear equations over the finite field have only the zero solution. Then we introduce two sets of generalized bent functions with the desirable properties. By the construction of Heng and Yue in [9], we obtain two families of codebooks from these sets. The results demonstrate that the obtained codebooks are nearly optimal with respect to the Welch bound, i.e., the ratio of the maximum cross-correlation amplitude of the codebooks to the Welch bound approaches 1. Moreover, new Boolean bent functions are derived from the proposed generalized bent functions. Compared with the codebooks in [9, 10], our codebooks have different parameters. Codebooks with a small alphabet size are of great importance in applications [1]. It is worth noting that our codebooks can have a smaller alphabet size than those in [9, 10] (see Remark 3 in Section 3 for a more detailed comparison).

The remainder of this paper is organized as follows. Section 2 introduces some notations and necessary background. Section 3 proposes two classes of generalized bent $\mathbb{Z}_4$-valued quadratic forms, and then generates two classes of nearly optimal codebooks with respect to the Welch bound from these quadratic forms. Section 4 concludes this study.

2. Preliminaries

In this section, we review some fundamental facts and results about Boolean bent functions, Galois rings, and $\mathbb{Z}_4$-valued quadratic forms.

2.1. Boolean bent functions

Let $\mathbb{F}_{2^n}$ be the finite field of 2^n elements, and $\mathbb{F}_{2^n}^*$ its multiplicative group. For two positive integers k and n with $k\,|\,n$, the *trace function* $\mathrm{tr}_k^n(x)$ from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^k}$ is defined to be

$$\mathrm{tr}_k^n(x) = x + x^{2^k} + x^{2^{2k}} + \cdots + x^{2^{(n/k-1)k}}.$$

When $k = 1$, $\mathrm{tr}_1^n(x)$ is called the *absolute trace function*. The trace function satisfies

(1) $\mathrm{tr}_k^n(x^{2^k}) \in \mathbb{F}_{2^k}$, $\mathrm{tr}_k^n(x^{2^k}) = \mathrm{tr}_k^n(x)$, where $x \in \mathbb{F}_{2^n}$;

(2) $\mathrm{tr}_k^n(x) = \mathrm{tr}_k^m(\mathrm{tr}_m^n(x))$, where n, k, m are positive integers with $k\,|\,m$ and $m\,|\,n$, and $x \in \mathbb{F}_{2^n}$;

(3) $\mathrm{tr}_k^n(ax + by) = a\mathrm{tr}_k^n(x) + b\mathrm{tr}_k^n(y)$, where $x, y \in \mathbb{F}_{2^n}$ and $a, b \in \mathbb{F}_{2^k}$.

Any function f: $\mathbb{F}_{2^n} \to \mathbb{F}_2$ is called a *Boolean function* in n variables. The Walsh transform of a Boolean function f on $\mathbb{F}_{2^n}$ is defined by

$$\widehat{f}(\lambda) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \mathrm{tr}_1^n(\lambda x)}, \quad \lambda \in \mathbb{F}_{2^n}.$$

Definition 1. [32] *A Boolean function f on $\mathbb{F}_{2^n}$ is bent if $|\widehat{f}(\lambda)| = 2^{n/2}$ for all $\lambda \in \mathbb{F}_{2^n}$.*

Bent functions are famous Boolean functions with the highest possible nonlinearity in an even number of variables. They were introduced by Rothaus in 1976 [32], and have been extensively studied [33, 34] due to their wide applications in several areas such as cryptography, coding theory, and sequence design. See [33, 34] for surveys on bent functions.

2.2. *Galois rings*

For a positive integer $n \geq 1$, let $\mathbb{Z}_4$ be the ring of integers modulo 4 and f a monic basic irreducible polynomial of degree n in $\mathbb{Z}_4[x]$. The ring $\mathbb{R} = \mathbb{Z}_4[x]/(f)$ is called the *Galois ring* of order 4^n with characteristic 4, which is a Galois extension of degree n over $\mathbb{Z}_4$ and denoted by $\mathbb{R} = \mathbb{GR}(4, n)$. Let ξ be a root of the polynomial f, then $\xi^{2^n-1} = 1$ and $\mathbb{R} \cong \mathbb{Z}_4[\xi]$.

Definition 2. [35] *The Teichmüller set* $\mathbb{T}$ *of the Galois ring* $\mathbb{R} = \mathbb{GR}(4, n)$ *is defined by*

$$\mathbb{T} = \{x \in \mathbb{R} \mid x^{2^n} = x\},$$

which can be expressed as $\mathbb{T} = \{0, 1, \xi, \cdots, \xi^{2^n-2}\}$.

The Teichmüller set in the Galois ring is named by analogy to its classical counterpart in p-adic integers, honoring mathematician Oswald Teichmüller, who first developed the concept of multiplicative representatives for residue fields. The functions we proposed in this paper are defined on the Teichmüller set of the Galois ring.

Let $\mathbb{T}$ be the Teichmüller set $\mathbb{T}$ of the Galois ring $\mathbb{R} = \mathbb{GR}(4, n)$. Let $\mu : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ denote the modulo-2 reduction. Naturally, the mapping μ induces a homomorphism from the Galois ring $\mathbb{R}$ to the finite field $\mathbb{F}_{2^n}$. Further, $\mu : \mathbb{T} \longrightarrow \mathbb{F}_{2^n}$ is an isomorphism. For simplicity, $\mu(z)$ is sometimes denoted by $\bar{z}$. Let $\mathbb{T}^*$ be the set of nonzero elements of $\mathbb{T}$, which is a multiplicative group. For two positive integers k and n with $k \mid n$, denote

$$\mathbb{T}_k = \{x \in \mathbb{T} \mid x^{2^k} = x\},$$

and

$$\mathbb{T}_k^* = \{x \in \mathbb{T}^* \mid x^{2^k} = x\},$$

respectively. Then for any $x \in \mathbb{T}_k$, we have $\mu(x) = \bar{x} \in \mathbb{F}_{2^k}$. For two different elements $x_1, x_2 \in \mathbb{T}$, we get $\bar{x}_1 \neq \bar{x}_2$. The addition operation in the Teichmüller set $\mathbb{T}$ is not closed. Specially, for any $x, y \in \mathbb{T}$, there exists a unique $t \in \mathbb{T}$ such that $t = x + y + 2\sqrt{xy}$. An operation $\oplus$ on $\mathbb{T}$ is defined as $x \oplus y = x + y + 2\sqrt{xy}$. Then $(\mathbb{T}, \oplus, \cdot) \cong (\mathbb{F}_{2^n}, +, \cdot)$.

For each $z \in \mathbb{R}$, it can be uniquely represented by $z = x + 2y$, where $x, y \in \mathbb{T}$. The trace function $\mathrm{Tr}_1^n : \mathbb{R} \longrightarrow \mathbb{Z}_4$ is defined by

$$\mathrm{Tr}_1^n(z) = \mathrm{Tr}_1^n(x + 2y) = \sum_{j=0}^{n-1} (x^{2^j} + 2y^{2^j}), \quad x, y \in \mathbb{T}.$$

Then $\mathrm{Tr}_1^n(az_1 + bz_2) = a\mathrm{Tr}_1^n(z_1) + b\mathrm{Tr}_1^n(z_2)$, where $a, b \in \mathbb{Z}_4$ and $z_1, z_2 \in \mathbb{R}$. Let $\mathrm{tr}_1^n(\cdot)$ denote the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$. The trace function over the Galois ring $\mathbb{R} = \mathbb{GR}(4, n)$ and that over the finite field $\mathbb{F}_{2^n}$ are related via the map μ as follows:

(1) $\overline{\mathrm{Tr}_1^n(z)} = \mathrm{tr}_1^n(\bar{z})$;

(2) $2\,\mathrm{Tr}_1^n(z) = 2\,\mathrm{tr}_1^n(\bar{z})$.

Lemma 2. [35] *The trace function over* $\mathbb{GR}(4, n)$ *has* 2*-adic expansion given by*

$$\mathrm{Tr}_1^n(x) = \mathrm{tr}_1^n(\bar{x}) + 2p(\bar{x}), \quad x \in \mathbb{T},$$

where $p(x)$ is defined by

$$p(x) = \begin{cases} \sum_{i=1}^{\frac{n}{2}-1} \mathrm{tr}_1^n(x^{1+2^i}) + \mathrm{tr}_1^{\frac{n}{2}}(x^{1+2^{\frac{n}{2}}}), & \text{if } n \text{ is even;} \\ \sum_{i=1}^{\frac{n-1}{2}} \mathrm{tr}_1^n(x^{1+2^i}), & \text{if } n \text{ is odd.} \end{cases} \tag{2.1}$$

For more information on Galois rings, the Teichmüller sets of Galois rings, and the trace function over Galois rings, the reader is referred to [35, 38].

2.3. $\mathbb{Z}_4$-valued quadratic forms

Definition 3. ([36]) *A symmetric bilinear form on* $\mathbb{T}$ *is a mapping* $B : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{Z}_2$ *which satisfies:*

(1) Symmetry: $B(x, y) = B(y, x)$;
(2) Bilinearity: $B(x \oplus y, z) = B(x, z) + B(y, z)$.

Definition 4. ([36]) *A* $\mathbb{Z}_4$*-valued quadratic form is a mapping* $Q : \mathbb{T} \longrightarrow \mathbb{Z}_4$ *with two properties:*

(1) $Q(0) = 0$;
(2) $Q(x \oplus y) = Q(x) + Q(y) + 2B(x, y)$, where B is a symmetric bilinear form defined as above.
The rank of a symmetric bilinear form B is defined as $\mathrm{rank}(B) = n - \dim_{\mathbb{Z}_2}(\mathrm{rad}(B))$, where

$$\mathrm{rad}(B) = \{x \in \mathbb{T} \mid B(x, y) = 0 \text{ for all } y \in \mathbb{T}\}.$$

The rank of a $\mathbb{Z}_4$-valued quadratic form Q is defined as $\mathrm{rank}(Q) = \mathrm{rank}(B)$, where B is the symmetric bilinear form associated with Q.

A $\mathbb{Z}_4$-valued quadratic form is a generalized Boolean function. For a $\mathbb{Z}_4$-valued quadratic form Q on $\mathbb{T}$, its Walsh transform is defined by

$$\widehat{Q}(\lambda) = \sum_{x \in \mathbb{T}} (\sqrt{-1})^{Q(x)+2\mathrm{Tr}_1^n(\lambda x)}, \quad \lambda \in \mathbb{T}.$$

Definition 5. [37] *A* $\mathbb{Z}_4$*-valued quadratic form* Q *on* $\mathbb{T}$ *is generalized bent if* $\left|\widehat{Q}(\lambda)\right| = 2^{n/2}$ *for all* $\lambda \in \mathbb{T}$.

According to Definition 5, if $Q(x)$ is generalized bent, then so is $Q(ax)$ for any $a \in \mathbb{T}^*$. Indeed, denote $G(x) = Q(ax)$, the Walsh transform of $G(x)$ at the point $\lambda \in \mathbb{T}$ is

$$\widehat{G}(\lambda) = \sum_{x \in \mathbb{T}} (\sqrt{-1})^{Q(ax)+2\mathrm{Tr}_1^n(\lambda x)} = \sum_{x \in \mathbb{T}} (\sqrt{-1})^{Q(ax)+2\mathrm{Tr}_1^n\left(\frac{\lambda}{a} \cdot ax\right)} = \sum_{y \in \mathbb{T}} (\sqrt{-1})^{Q(y)+2\mathrm{Tr}_1^n\left(\frac{\lambda}{a} \cdot y\right)} = \widehat{Q}\left(\frac{\lambda}{a}\right).$$

Thus, the generalized bentness of $Q(ax)$ follows.

The bentness of a $\mathbb{Z}_4$-valued quadratic form can be characterized in terms of its rank.

Lemma 3. [38] *A* $\mathbb{Z}_4$*-valued quadratic form* $Q(x)$ *is generalized bent if and only if it is of full rank, that is, if and only if* $\mathrm{rad}(B) = \{0\}$*, where* B *is the symmetric bilinear form associated to* Q.

The following lemma establishes a relationship between Boolean bent functions and generalized Boolean bent functions over $\mathbb{Z}_4$, which will be utilized in Section 4.

Lemma 4. ([39]) *Let $Q(x)$ be a generalized Boolean function over $\mathbb{Z}_4$ and $Q(x) = g(\bar{x}) + 2h(\bar{x})$ be its 2-adic expansion, where $x \in \mathbb{T}$, $g(\cdot)$ and $h(\cdot)$ are Boolean functions over $\mathbb{F}_{2^n}$. If n is even, then $Q(x)$ (with $x \in \mathbb{T}$) is bent if and only if both $h(x)$ and $g(x) + h(x)$ (with $x \in \mathbb{F}_{2^n}$) are bent.*

3. A construction of codebooks from a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms

We first recall a construction of codebooks from a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms, which was proposed in [9].

Let E_{2^n} denote the set formed by the standard basis of the 2^n-dimensional Hilbert space:

$$\begin{gathered}(1, 0, 0, ..., 0, 0),\\ (0, 1, 0, ..., 0, 0),\\ \vdots\\ (0, 0, 0, ..., 0, 1).\end{gathered}$$

Let $\mathbb{T}$ be the Teichmüller set of the Galois ring $\mathbb{GR}(4, n)$, and let $\xi_0, \xi_1, \cdots, \xi_{2^n-1}$ denote all the elements of $\mathbb{T}$. Define the set

$$S_0 = \left\{ \frac{1}{\sqrt{2^n}} \Big((\sqrt{-1})^{2\mathrm{Tr}_1^n(\lambda\xi_0)}, (\sqrt{-1})^{2\mathrm{Tr}_1^n(\lambda\xi_1)}, \cdots, (\sqrt{-1})^{2\mathrm{Tr}_1^n(\lambda\xi_{2^n-1})} \Big) \,\Big|\, \lambda \in \mathbb{T} \right\}.$$

Let $\mathcal{F}$ be a set of $\mathbb{Z}_4$-valued quadratic forms on $\mathbb{T}$, which satisfies the following two conditions:

1) Each quadratic form in $\mathcal{F}$ is generalized bent;
2) The difference of arbitrary two distinct quadratic forms in $\mathcal{F}$ is generalized bent.

For each $Q \in \mathcal{F}$, define the set

$$S_Q = \left\{ \frac{1}{\sqrt{2^n}} \Big((\sqrt{-1})^{Q(\xi_0)+2\mathrm{Tr}_1^n(\lambda\xi_0)}, (\sqrt{-1})^{Q(\xi_1)+2\mathrm{Tr}_1^n(\lambda\xi_1)}, \cdots, (\sqrt{-1})^{Q(\xi_{2^n-1})+2\mathrm{Tr}_1^n(\lambda\xi_{2^n-1})} \Big) \,\Big|\, \lambda \in \mathbb{T} \right\}.$$

Construct the following codebook from $\mathcal{F}$:

$$C_{\mathcal{F}} = \bigcup_{Q \in \mathcal{F}} S_Q \bigcup S_0 \bigcup E_{2^n}. \tag{3.1}$$

The parameters of the codebook $C_{\mathcal{F}}$ constructed from $\mathcal{F}$ are provided as follows. For completeness, we give its proof here.

Theorem 1. [9, Theorem 1] *Let $\mathcal{F}$ be a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms from $\mathbb{T}$ to $\mathbb{Z}_4$, such that the difference of arbitrary two distinct quadratic forms in $\mathcal{F}$ is generalized bent. Let $C_{\mathcal{F}}$ be the codebook constructed in* (3.1)*. Then, $C_{\mathcal{F}}$ is a $((|\mathcal{F}| + 2)2^n, 2^n)$ codebook with $I_{\max}(C_{\mathcal{F}}) = \frac{1}{\sqrt{2^n}}$ and alphabet size* 6*.*

Proof. It is clear that

$$|E_{2^n}| = |S_0| = |S_Q| = |\mathbb{T}| = 2^n,$$

the alphabet

$$A = \left\{0, 1, \frac{1}{\sqrt{2^n}}, \frac{-1}{\sqrt{2^n}}, \frac{\sqrt{-1}}{\sqrt{2^n}}, \frac{-\sqrt{-1}}{\sqrt{2^n}}\right\},$$

and each codeword is a unit norm complex vector of length 2^n over the alphabet A. Then, we get $N = (|\mathcal{F}| + 2)2^n$, $K = 2^n$, and the alphabet size is 6.

If $\mathbf{c}_1 \in E_{2^n}$ and $\mathbf{c}_2 \in \mathcal{S}_0$ or $\mathbf{c}_2 \in \mathcal{S}_Q$ for some $Q \in \mathcal{F}$, then $\left|\mathbf{c}_1\mathbf{c}_2^H\right| = \frac{1}{\sqrt{2^n}}$. For any $\mathbf{c}_1 \in \mathcal{S}_0$ and $\mathbf{c}_2 \in \mathcal{S}_Q$, we get

$$\left|\mathbf{c}_1\mathbf{c}_2^H\right| = \frac{1}{2^n}\left|\sum_{x\in\mathbb{T}}(\sqrt{-1})^{Q(x)+2\mathrm{T}_1^n\left((\lambda_1+\lambda_2)x\right)}\right| = \frac{1}{2^n} \times 2^{\frac{n}{2}} = \frac{1}{\sqrt{2^n}}$$

as Q is generalized bent, where $\lambda_1, \lambda_2 \in \mathbb{T}$. Let $Q_1, Q_2 \in \mathbb{T}$ be arbitrary two different quadratic forms. For any $\mathbf{c}_1 \in \mathcal{S}_{Q_1}$ and $\mathbf{c}_2 \in \mathcal{S}_{Q_2}$, we have

$$\left|\mathbf{c}_1\mathbf{c}_2^H\right| = \frac{1}{2^n}\left|\sum_{x\in\mathbb{T}}(\sqrt{-1})^{\left(Q_1(x)-Q_2(x)\right)+2\mathrm{Tr}_1^n\left((\lambda_1+\lambda_2)x\right)}\right| = \frac{1}{2^n} \times 2^{\frac{n}{2}} = \frac{1}{\sqrt{2^n}},$$

since $Q_1 - Q_2$ is generalized bent. For any two different vectors $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{S}_0$, we obtain

$$\left|\mathbf{c}_1\mathbf{c}_2^H\right| = \frac{1}{2^n}\left|\sum_{x\in\mathbb{T}}(\sqrt{-1})^{2\,\mathrm{Tr}_1^n\left((\lambda_1+\lambda_2)x\right)}\right| = \frac{1}{2^n}\left|\sum_{\bar{x}\in\mathbb{T}_{2^n}}(-1)^{\mathrm{tr}_1^n\left((\bar{\lambda}_1+\bar{\lambda}_2)\bar{x}\right)}\right| = 0,$$

due to the orthogonal property of additive characters over finite fields and the fact $\bar{\lambda}_1 \neq \bar{\lambda}_2$, where $\lambda_1, \lambda_2 \in \mathbb{T}$ are distinct. Similarly, we can prove that the inner product of any two different vectors in S_Q is also equal to 0. Thus,

$$I_{\max}(C) = \max_{0\le i<j\le N-1}\left|\mathbf{c}_i\mathbf{c}_j^H\right| = \frac{1}{\sqrt{2^n}}.$$

The proof is then completed. □

Codebooks with a small alphabet size are of importance in applications, according to Sarwate [1]. We further discuss the alphabet size of the codebooks defined by (3.1) in the following proposition. Its proof follows directly from the expressions of the coordinates of the codewords in the sets $\mathcal{S}_0$ and $\mathcal{S}_Q$.

Proposition 1. *Let $\mathcal{F}$ be the set of generalized bent $\mathbb{Z}_4$-valued quadratic forms in Theorem 1. If for each $x \in \mathbb{T}$ and each $Q \in \mathcal{F}$, $Q(x) = 0$ or 2, then the codebook $C_{\mathcal{F}}$ defined by* (3.1) *is real-valued, with alphabet $A = \left\{0, 1, \frac{1}{\sqrt{2^n}}, \frac{-1}{\sqrt{2^n}}\right\}$ and alphabet size* 4. *Otherwise, $C_{\mathcal{F}}$ is complex-valued, with alphabet $A = \left\{0, 1, \frac{1}{\sqrt{2^n}}, \frac{-1}{\sqrt{2^n}}, \frac{\sqrt{-1}}{\sqrt{2^n}}, \frac{-\sqrt{-1}}{\sqrt{2^n}}\right\}$ and alphabet size* 6.

4. Two classes of nearly optimal codebooks

In this section, we proposed two classes of nearly optimal codebooks by employing the construction of codebooks defined in (3.1). The key to constructing the codebook $C_{\mathcal{F}}$ in (3.1) is to obtain a set of $\mathbb{Z}_4$-valued quadratic forms $\mathcal{F}$, which satisfies two conditions above. It was pointed out in [9] that this set is very difficult to construct in general. In the following subsections, we study the $\mathbb{Z}_4$-valued quadratic forms with the form

$$Q(x)=\sum_{j=1}^{s}2Q_j(x)+\mathrm{Tr}_1^n(cx),\quad x\in\mathbb{T},\tag{4.1}$$

where $\mathbb{T}$ is the Teichmüller set of the Galois ring $\mathbb{GR}(4,n)$, s is a positive integer, $Q_j(x)$ $(1\le j\le s)$ are generalized Boolean functions from $\mathbb{T}$ to $\mathbb{Z}_4$, and $c\in\mathbb{T}$. For $Q_j(x)$, we consider two classes of specific functions, whose explicit expressions are given by (4.2) and (4.13), respectively. Then two classes of generalized bent functions of the form (4.1) are proposed. Based on these functions, we derive two families of the sets $\mathcal{F}$ that meet the aforementioned two conditions. Then two new classes of codebooks $C_{\mathcal{F}}$ in (3.1) are obtained from the sets $\mathcal{F}$. These codebooks are nearly optimal with respect to the Welch bound, and could have a very small alphabet size.

4.1. The first class of nearly optimal codebooks from generalized bent functions

Let n,s be positive integers with $4|n$, and $k_1,\cdots,k_s$ be distinct positive integers, each of which can divide $\frac{n}{4}$. For $1\le j\le s$, let $t_j=\frac{n}{4k_j}$, $a_j,b_j\in\mathbb{T}_{\frac{n}{4}}$, and define the generalized Boolean function $Q_j:\mathbb{T}\longrightarrow\mathbb{Z}_4$ as

$$Q_j(x)=\sum_{i=1}^{2t_j-1}\mathrm{Tr}_1^n(a_jx^{1+2^{ik_j}})+\mathrm{Tr}_1^{\frac{n}{2}}(b_jx^{1+2^{\frac{n}{2}}}).\tag{4.2}$$

Let $c\in\mathbb{T}_{\frac{n}{4}}$, then the generalized Boolean function $Q(x)$ in (4.1) can be written as

$$Q(x)=\sum_{j=1}^{s}\Big(\sum_{i=1}^{2t_j-1}2\mathrm{Tr}_1^n(a_jx^{1+2^{ik_j}})+2\mathrm{Tr}_1^{\frac{n}{2}}(b_jx^{1+2^{\frac{n}{2}}})\Big)+\mathrm{Tr}_1^n(cx),\quad x\in\mathbb{T}.\tag{4.3}$$

Proposition 2. *Let $Q(x)$ be the function defined by* (4.3) *and $B_Q(x,y)$ its symmetric bilinear form. Then $B_Q(x,y)$ satisfies $2B_Q(x,y)=2\mathrm{tr}_1^n\big(L_Q(\bar{x})\bar{y}\big)$, where*

$$L_Q(x)=\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}(\bar{a}_jx^{2^{ik_j}}+(\bar{a}_jx)^{2^{n-ik_j}})+\sum_{j=1}^{s}\bar{b}_jx^{2^{\frac{n}{2}}}+\bar{c}^2x.\tag{4.4}$$

Proof. Note that $x\oplus y=x+y+2\sqrt{xy}$ and $2\,\mathrm{Tr}_1^n(x)=2\,\mathrm{tr}_1^n(\bar{x})$. By Definition 4, the function $Q(x)$ in (4.3) is a $\mathbb{Z}_4$-valued quadratic form, and its corresponding symmetric bilinear form $B_Q(x,y)$ satisfies

$$\begin{aligned}2B_Q(x,y)&=Q(x\oplus y)-Q(x)-Q(y)\\&=\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}2\mathrm{Tr}_1^n\Big(a_j((x\oplus y)^{1+2^{ik_j}}-x^{1+2^{ik_j}}-y^{1+2^{ik_j}})\Big)+\\&\quad\sum_{j=1}^{s}2\mathrm{Tr}_1^{\frac{n}{2}}\Big(b_j((x\oplus y)^{1+2^{\frac{n}{2}}}-x^{1+2^{\frac{n}{2}}}-y^{1+2^{\frac{n}{2}}})\Big)+\mathrm{Tr}_1^n\Big(c((x\oplus y)-x-y)\Big)\\&=\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}2\mathrm{tr}_1^n\Big(\bar{a}_j((\bar{x}+\bar{y})^{1+2^{ik_j}}+\bar{x}^{1+2^{ik_j}}+\bar{y}^{1+2^{ik_j}})\Big)+\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{s} 2\mathrm{tr}_1^{\frac{n}{2}}\Big(\bar{b}_j((\bar{x}+\bar{y})^{1+2^{\frac{n}{2}}}+\bar{x}^{1+2^{\frac{n}{2}}}+\bar{y}^{1+2^{\frac{n}{2}}})\Big)+2\mathrm{Tr}_1^n(c\sqrt{xy})\\
=\ &\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1} 2\mathrm{tr}_1^n\Big(\bar{a}_j(\bar{x}^{2^{ik_j}}\bar{y}+\bar{x}\bar{y}^{2^{ik_j}})\Big)+\sum_{j=1}^{s} 2\mathrm{tr}_1^{\frac{n}{2}}\Big(\bar{b}_j(\bar{x}^{2^{\frac{n}{2}}}\bar{y}+\bar{x}\bar{y}^{2^{\frac{n}{2}}})\Big)+2\mathrm{tr}_1^n(\bar{c}\sqrt{\bar{x}\bar{y}})\\
=\ &\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1} 2\mathrm{tr}_1^n\Big(\bar{a}_j\bar{x}^{2^{ik_j}}\bar{y}+(\bar{a}_j\bar{x})^{2^{n-ik_j}}\bar{y}\Big)+\sum_{j=1}^{s} 2\mathrm{tr}_1^n\Big(\bar{b}_j\bar{x}^{2^{\frac{n}{2}}}\bar{y}\Big)+2\mathrm{tr}_1^n(\bar{c}^2\bar{x}\bar{y})\\
=\ &2\mathrm{tr}_1^n\Bigg(\Bigg(\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}\Big(\bar{a}_j\bar{x}^{2^{ik_j}}+(\bar{a}_j\bar{x})^{2^{n-ik_j}}\Big)+\sum_{j=1}^{s}\bar{b}_j\bar{x}^{2^{\frac{n}{2}}}+\bar{c}^2\bar{x}\Bigg)\bar{y}\Bigg)\\
=\ &2\mathrm{tr}_1^n\Big(L_Q(\bar{x})\bar{y}\Big),
\end{aligned}
$$

where the linearized polynomial $L_Q(x)$ associated with $Q(x)$ is defined by (4.4). This finishes the proof. □

In the sequel, the bentness of the function $Q(x)$ in (4.3) is characterized.

Lemma 5. *The function $Q(x)$ defined by* (4.3) *is generalized bent if and only if* $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$.

Proof. By Lemma 3 and Proposition 2, $Q(x)$ is generalized bent if and only if it has full rank, that is, if and only if the linear equation $L_Q(\bar{x})=0$ has only the zero solution in $\mathbb{T}$, where $L_Q(x)$ is given by (4.4). Recall that $\mathbb{T}$ is isomorphic to the finite field $\mathbb{F}_{2^n}$ under the mapping μ. Therefore, we can discuss the above equality over the finite field $\mathbb{F}_{2^n}$ instead of $\mathbb{T}$. Thus $Q(x)$ is generalized bent if and only if the equation $L_Q(x)=0$ has only the zero solution in $\mathbb{F}_{2^n}$. Notice that the equation $L_Q(x)=0$ can be reformulated as

$$
\sum_{j=1}^{s}\Big(\sum_{i=1}^{2t_j}\bar{a}_jx^{2^{ik_j}}+\sum_{i=1}^{2t_j}(\bar{a}_jx)^{2^{\frac{n}{2}+ik_j}}+(\bar{a}_j+\bar{b}_j)(x+x^{\frac{n}{2}})\Big)+\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)x=0. \tag{4.5}
$$

Thus, it suffices to prove that Eq (4.5) has only the zero solution in $\mathbb{F}_{2^n}$ if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$.

Suppose that (4.5) has only the zero solution in $\mathbb{F}_{2^n}$. Note that for $1\leq j\leq s$, $\bar{a}_j\in\mathbb{F}_{2^{\frac{n}{4}}}$. Then, it is easily checked that $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, otherwise each $x\in\mathbb{F}_{2^{\frac{n}{4}}}$ is a solution of (4.5) due to

$$
\sum_{i=1}^{2t_j}\bar{a}_jx^{2^{ik_j}}=\bar{a}_j\mathrm{tr}_{k_j}^{\frac{n}{2}}(x)=\bar{a}_j\mathrm{tr}_{k_j}^{\frac{n}{4}}\Big(x\mathrm{tr}_{\frac{n}{4}}^{\frac{n}{2}}(1)\Big)=0,
$$

and

$$
\sum_{i=1}^{2t_j}(\bar{a}_jx)^{2^{\frac{n}{2}+ik_j}}=\mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_jx)=\mathrm{tr}_{k_j}^{\frac{n}{4}}\Big(\bar{a}_jx\mathrm{tr}_{\frac{n}{4}}^{\frac{n}{2}}(1)\Big)=0,
$$

when $x\in\mathbb{F}_{2^{\frac{n}{4}}}$.

In the opposite direction, assume that $\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \neq 0$. Recall that $\bar{a}_j, \bar{b}_j \in \mathbb{F}_{2^{\frac{n}{4}}}$ for $1 \leq j \leq s$ and $\bar{c} \in \mathbb{F}_{2^{\frac{n}{4}}}$. By taking $2^{\frac{n}{2}}$-power on (4.5), we get

$$\sum_{j=1}^{s} \Big(\sum_{i=1}^{2t_j} \bar{a}_j x^{2^{\frac{n}{2}+ik_j}} + \sum_{i=1}^{2t_j} (\bar{a}_j x)^{2^{ik_j}} + (\bar{a}_j + \bar{b}_j)(x + x^{\frac{n}{2}}) \Big) + \Big(\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \Big) x^{2^{\frac{n}{2}}} = 0. \tag{4.6}$$

Adding (4.5) and (4.6) together gives

$$\sum_{j=1}^{s} \Big(\bar{a}_j \mathrm{tr}_{k_j}^{n}(x) + \mathrm{tr}_{k_j}^{n}(\bar{a}_j x) \Big) + \Big(\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \Big)(x + x^{2^{\frac{n}{2}}}) = 0. \tag{4.7}$$

Observe that for $1 \leq j \leq s$, $\mathrm{tr}_{k_j}^{n}(x), \mathrm{tr}_{k_j}^{n}(\bar{a}_j x) \in \mathbb{F}_{2^{\frac{n}{4}}}$ due to $k_j \mid \frac{n}{4}$. As $\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \neq 0$, it follows from (4.7) that the solutions of (4.5) satisfy $x + x^{2^{\frac{n}{2}}} \in \mathbb{F}_{2^{\frac{n}{4}}}$. Thus,

$$\mathrm{tr}_{k_j}^{n}(x) = \mathrm{tr}_{k_j}^{\frac{n}{4}}\Big((x + x^{2^{\frac{n}{2}}})\mathrm{tr}_{\frac{n}{4}}^{\frac{n}{2}}(1)\Big) = 0,$$

and

$$\mathrm{tr}_{k_j}^{n}(\bar{a}_j x) = \mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_j(x + x^{2^{\frac{n}{2}}})) = \mathrm{tr}_{k_j}^{\frac{n}{4}}\Big(\bar{a}_j(x + x^{2^{\frac{n}{2}}})\mathrm{tr}_{\frac{n}{4}}^{\frac{n}{2}}(1)\Big) = 0.$$

Then (4.7) is reduced to

$$\Big(\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \Big)(x + x^{2^{\frac{n}{2}}}) = 0,$$

which implies the solution of (4.5) must be in $\mathbb{F}_{2^{\frac{n}{2}}}$. So (4.5) can be written as

$$\sum_{j=1}^{s} \Big(\bar{a}_j \mathrm{tr}_{k_j}^{\frac{n}{2}}(x) + \mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_j x) \Big) + \Big(\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \Big) x = 0.$$

Since $\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \neq 0$, from the above equation we get that the solution of (4.5) necessarily belongs to $\mathbb{F}_{2^{\frac{n}{4}}}$, which leads to $\mathrm{tr}_{k_j}^{\frac{n}{2}}(x) = \mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_j x) = 0$. Therefore, (4.5) becomes $\Big(\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \Big) x = 0$, which means (4.5) has only the zero solution. Then the proof is finished. □

The generalized bent functions in Lemma 5 can be used to construct nearly optimal codebooks. By Lemma 5, $Q(x)$ is bent when $\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \neq 0$, and so is the function $Q(\alpha x)$, where $\alpha \in \mathbb{T}_{\frac{n}{4}}^{*}$. In order to construct codebooks, we consider the bentness of the difference $Q_{\alpha,\beta}(x)$ of $Q(\alpha x)$ and $Q(\beta x)$ for distinct $\alpha, \beta \in \mathbb{T}_{\frac{n}{4}}^{*}$.

For $\alpha, \beta \in \mathbb{T}_{\frac{n}{4}}^{*}$ with $\alpha \neq \beta$, let the generalized Boolean function $Q_{\alpha,\beta}(x)$ be defined as

$$Q_{\alpha,\beta}(x) = Q(\alpha x) - Q(\beta x), \tag{4.8}$$

where $Q(x)$ is defined by (4.3).

Proposition 3. *Let $Q_{\alpha,\beta}(x)$ be the function defined in* (4.8) *and $B_{Q_{\alpha,\beta}}(x,y)$ its symmetric bilinear form. Then $B_{Q_{\alpha,\beta}}(x,y)$ satisfies $2B_{Q_{\alpha,\beta}}(x,y) = 2\mathrm{tr}_1^n(L_{Q_{\alpha,\beta}}(\bar{x})\bar{y})$, where*

$$\begin{aligned} L_{Q_{\alpha,\beta}}(x) &= \sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}\Big(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})x^{2^{ik_j}}+(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})x)^{2^{n-ik_j}}\Big) \\ &\quad+\sum_{j=1}^{s}\bar{b}_j(\bar{\alpha}^2+\bar{\beta}^2)x^{2^{\frac{n}{2}}}+\bar{c}^2(\bar{\alpha}^2+\bar{\beta}^2)x. \end{aligned} \tag{4.9}$$

Proof. We have

$$\begin{aligned} Q_{\alpha,\beta}(x) &= \sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}2\mathrm{Tr}_1^n\Big(a_j(\alpha^{1+2^{ik_j}}-\beta^{1+2^{ik_j}})x^{1+2^{ik_j}}\Big) \\ &\quad+\sum_{j=1}^{s}2\mathrm{Tr}_1^{\frac{n}{2}}\Big(b_j(\alpha^2-\beta^2)x^{1+2^{\frac{n}{2}}}\Big)+\mathrm{Tr}_1^n(c(\alpha-\beta)x). \end{aligned}$$

The function $Q_{\alpha,\beta}(x)$ is a quaternary quadratic forms, and the symmetric bilinear form $B_{Q_{\alpha,\beta}}(x,y)$ of $Q_{\alpha,\beta}(x)$ is given by

$$\begin{aligned} 2B_{Q_{\alpha,\beta}}(x,y) &= Q_{\alpha,\beta}(x\oplus y)-Q_{\alpha,\beta}(x)-Q_{\alpha,\beta}(y) \\ &= \sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}2\mathrm{tr}_1^n\Big(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})(\bar{x}^{2^{ik_j}}\bar{y}+\bar{x}\bar{y}^{2^{ik_j}})\Big) \\ &\quad+\sum_{j=1}^{s}2\mathrm{tr}_1^{\frac{n}{2}}\Big(\bar{b}_j(\bar{\alpha}^2+\bar{\beta}^2)(\bar{x}^{2^{\frac{n}{2}}}\bar{y}+\bar{x}\bar{y}^{2^{\frac{n}{2}}})\Big)+2\mathrm{Tr}_1^n(c(\alpha-\beta)\sqrt{xy}) \\ &= \sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}2\mathrm{tr}_1^n\Big(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})\bar{x}^{2^{ik_j}}\bar{y}+(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})\bar{x})^{2^{n-ik_j}}\bar{y}\Big) \\ &\quad+\sum_{j=1}^{s}2\mathrm{tr}_1^n\Big(\bar{b}_j(\bar{\alpha}^2+\bar{\beta}^2)\bar{x}^{2^{\frac{n}{2}}}\bar{y}\Big)+2\mathrm{tr}_1^n(\bar{c}^2(\bar{\alpha}^2+\bar{\beta}^2)\bar{x}\bar{y}) \\ &= 2\mathrm{tr}_1^n\Bigg(\Big(\sum_{j=1}^{s}\sum_{i=1}^{2t_j-1}\Big(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})\bar{x}^{2^{ik_j}}+(\bar{a}_j(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})\bar{x})^{2^{n-ik_j}}\Big) \\ &\quad+\sum_{j=1}^{s}\bar{b}_j(\bar{\alpha}^2+\bar{\beta}^2)\bar{x}^{2^{\frac{n}{2}}}+\bar{c}^2(\bar{\alpha}^2+\bar{\beta}^2)\bar{x}\Big)\bar{y}\Bigg) \\ &= 2\mathrm{tr}_1^n(L_{Q_{\alpha,\beta}}(\bar{x})\bar{y}), \end{aligned}$$

where $L_{Q_{\alpha,\beta}}(x)$ is given by (4.9). □

Lemma 6. *The function $Q_{\alpha,\beta}(x)$ defined by* (4.8) *is generalized bent if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$.*

Proof. By Lemma 3 and Proposition 3, $Q_{\alpha,\beta}(x)$ is generalized bent if and only if the equation $L_{Q_{\alpha,\beta}}(\bar{x}) = 0$ has only the zero solution in $\mathbb{T}$, i.e., the equation $L_{Q_{\alpha,\beta}}(x) = 0$ has only the zero solution in $\mathbb{F}_{2^n}$, where

$L_{Q_{\alpha,\beta}}(x)$ is given by (4.9). Note that the equation $L_{Q_{\alpha,\beta}}(x) = 0$ can be equivalently rewritten as

$$\begin{aligned}&\sum_{j=1}^{s}\Big(\bar{\alpha}\Big(\sum_{i=1}^{2t_j}\bar{a}_j(\bar{\alpha}x)^{2^{ik_j}}+\sum_{i=1}^{2t_j}(\bar{a}_j\bar{\alpha}x)^{2^{\frac{n}{2}+ik_j}}\Big)+\bar{\beta}\Big(\sum_{i=1}^{2t_j}\bar{a}_j(\bar{\beta}x)^{2^{ik_j}}+\sum_{i=1}^{2t_j}(\bar{a}_j\bar{\beta}x)^{2^{\frac{n}{2}+ik_j}}\Big)\\&+(\bar{a}_j+\bar{b}_j)(\bar{\alpha}^2+\bar{\beta}^2)(x+x^{2^{\frac{n}{2}}})\Big)+\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)(\bar{\alpha}^2+\bar{\beta}^2)x=0.\end{aligned}\tag{4.10}$$

Then it remains to prove that Eq (4.10) has only the zero solution in $\mathbb{F}_{2^n}$ if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$.

First, we prove the sufficiency of the statement. Assume $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$. Recall that for $1\leq j\leq s$, $\bar{a}_j,\bar{b}_j\in\mathbb{F}_{2^{\frac{n}{4}}}$ and $\bar{c},\bar{\alpha},\bar{\beta}\in\mathbb{F}_{2^{\frac{n}{4}}}$. By taking $2^{\frac{n}{2}}$-power on both sides of (4.10), we have

$$\begin{aligned}&\sum_{j=1}^{s}\Big(\bar{\alpha}\Big(\sum_{i=1}^{2t_j}\bar{a}_j(\bar{\alpha}x)^{2^{\frac{n}{2}+ik_j}}+\sum_{i=1}^{2t_j}(\bar{a}_j\bar{\alpha}x)^{2^{ik_j}}\Big)+\bar{\beta}\Big(\sum_{i=1}^{2t_j}\bar{a}_j(\bar{\beta}x)^{2^{\frac{n}{2}+ik_j}}+\sum_{i=1}^{2t_j}(\bar{a}_j\bar{\beta}x)^{2^{ik_j}}\Big)\\&+(\bar{a}_j+\bar{b}_j)(\bar{\alpha}^2+\bar{\beta}^2)(x+x^{2^{\frac{n}{2}}})\Big)+\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)(\bar{\alpha}^2+\bar{\beta}^2)x^{2^{\frac{n}{2}}}=0.\end{aligned}\tag{4.11}$$

By adding (4.10) and (4.11) together, we get that

$$\begin{aligned}&\sum_{j=1}^{s}\Big(\bar{\alpha}\bar{a}_j\mathrm{tr}^n_{k_j}(\bar{\alpha}x)+\bar{\alpha}\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\alpha}x)+\bar{\beta}\bar{a}_j\mathrm{tr}^n_{k_j}(\bar{\beta}x)+\bar{\beta}\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\beta}x)\Big)\\&+\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)(\bar{\alpha}^2+\bar{\beta}^2)(x+x^{2^{\frac{n}{2}}})=0.\end{aligned}\tag{4.12}$$

Due to $k_j\mid\frac{n}{4}$, we have $\mathrm{tr}^n_{k_j}(\bar{\alpha}x),\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\alpha}x),\mathrm{tr}^n_{k_j}(\bar{\beta}x),\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\beta}x)\in\mathbb{F}_{2^{\frac{n}{4}}}$ for $1\leq j\leq s$. As $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$ and $\alpha\neq\beta$, Eq (4.12) implies that the solutions of (4.10) satisfy $x+x^{2^{\frac{n}{2}}}\in\mathbb{F}_{2^{\frac{n}{4}}}$, which yields

$$\mathrm{tr}^n_{k_j}(\bar{\alpha}x)=\mathrm{tr}^{\frac{n}{2}}_{k_j}\Big(\bar{\alpha}(x+x^{2^{\frac{n}{2}}})\Big)=\mathrm{tr}^{\frac{n}{4}}_{k_j}\Big(\bar{\alpha}(x+x^{2^{\frac{n}{2}}})\mathrm{tr}^{\frac{n}{2}}_{\frac{n}{4}}(1)\Big)=0,$$

and

$$\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\alpha}x)=\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{a}_j\bar{\alpha}(x+x^{2^{\frac{n}{2}}}))=\mathrm{tr}^{\frac{n}{4}}_{k_j}\Big(\bar{a}_j\bar{\alpha}(x+x^{2^{\frac{n}{2}}})\mathrm{tr}^{\frac{n}{2}}_{\frac{n}{4}}(1)\Big)=0.$$

Similarly, we get $\mathrm{tr}^n_{k_j}(\bar{\beta}x)=\mathrm{tr}^n_{k_j}(\bar{a}_j\bar{\beta}x)=0$. Accordingly, (4.12) becomes

$$\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)(\bar{\alpha}^2+\bar{\beta}^2)(x+x^{2^{\frac{n}{2}}})=0,$$

which means the solutions of (4.10) must lie in $\mathbb{F}_{2^{\frac{n}{2}}}$. Then (4.10) is equivalent to

$$\sum_{j=1}^{s}\Big(\bar{\alpha}\bar{a}_j\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{\alpha}x)+\bar{\alpha}\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{a}_j\bar{\alpha}x)+\bar{\beta}\bar{a}_j\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{\beta}x)+\bar{\beta}\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{a}_j\bar{\beta}x)\Big)+\Big(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\Big)(\bar{\alpha}^2+\bar{\beta}^2)x=0,$$

from which we get that the solutions of (4.10) necessarily belong to $\mathbb{F}_{2^{\frac{n}{4}}}$. Then

$$\mathrm{tr}^{\frac{n}{2}}_{k_j}(\bar{\alpha}x)=\mathrm{tr}^{\frac{n}{4}}_{k_j}\Big(\bar{\alpha}x\mathrm{tr}^{\frac{n}{2}}_{\frac{n}{4}}(1)\Big)=0.$$

Similarly, we obtain

$$\mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_j\bar{\alpha}x) = \mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{\beta}x) = \mathrm{tr}_{k_j}^{\frac{n}{2}}(\bar{a}_j\bar{\beta}x) = 0.$$

Consequently, (4.10) is further transformed into $\Big(\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2\Big)(\bar{\alpha}^2 + \bar{\beta}^2)x = 0$, which induces that $x = 0$ is the unique solution of (4.10).

For the necessity of the statement, if (4.10) has only the zero solution, then clearly $\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2 \neq 0$, otherwise each $x \in \mathbb{F}_{2^{\frac{n}{4}}}$ is a root of (4.10), which is a contradiction. This finishes the proof. □

Remark 1. *Let l be a positive integer. For $1 \leq i \leq l$, let $\alpha_i \in \mathbb{T}^*_{\frac{n}{4}}$, and $\sum_{i=1}^{l}\bar{\alpha}_i \neq 0$. By a similar analysis as in Lemma 6, we can obtain that $Q_{\alpha_1,\cdots,\alpha_l}(x) = \sum_{i=1}^{l}Q(\alpha_i x)$ is generalized bent if and only if $\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2 \neq 0$, where $Q(x)$ is defined in* (4.3).

Let $b_j = 0$ in (4.2) for $1 \leq j \leq s$. Then the following corollary is a straightforward consequence of Lemmas 5 and 6.

Corollary 1. *Let n, s be positive integers with $4|n$, and $k_1, \cdots, k_s$ be distinct positive integers, each of which divide $\frac{n}{4}$. For $1 \leq j \leq s$, let $t_j = \frac{n}{4k_j}$, $a_j \in \mathbb{T}_{\frac{n}{4}}$ and $c \in \mathbb{T}^*_{\frac{n}{4}}$. Then the function*

$$Q(x) = \sum_{j=1}^{s}\sum_{i=1}^{2t_j-1} 2\mathrm{Tr}_1^n(a_j x^{1+2^{ik_j}}) + \mathrm{Tr}_1^n(cx)$$

is generalized bent. Furthermore, for any $\alpha, \beta \in \mathbb{T}^*_{\frac{n}{4}}$ with $\alpha \neq \beta$, $Q(\alpha x) - Q(\beta x)$ is also generalized bent.

By Lemmas 5 and 6, we can immediately have the following result.

Theorem 2. *Let $Q(x)$ be the function defined by* (4.3) *with $\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2 \neq 0$. Define the set*

$$\mathcal{F} = \left\{Q(\alpha x)\,\middle|\,\alpha \in \mathbb{T}^*_{\frac{n}{4}}\right\}.$$

Then $\mathcal{F}$ is a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms. Moreover, the difference of any two distinct quadratic forms in $\mathcal{F}$ is also generalized bent.

With the above preparations, the following nearly optimal codebooks can be obtained.

Theorem 3. *Let $Q(x)$ be the function defined by* (4.3) *with $\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2 \neq 0$, and define the set $\mathcal{F} = \{Q(\alpha x) \mid \alpha \in \mathbb{T}^*_{\frac{n}{4}}\}$. Let $C_{\mathcal{F}}$ be the codebook constructed in* (3.1) *from the set $\mathcal{F}$. Then $C_{\mathcal{F}}$ is a $(2^{\frac{5n}{4}} + 2^n, 2^n)$ codebook with $I_{\max}(C_{\mathcal{F}}) = \frac{1}{\sqrt{2^n}}$ and alphabet size* 6*, which is nearly optimal with respect to the Welch bound. In particular, if $Q(x)$ satisfies $c = 0$ and $\sum_{j=1}^{s}\bar{b}_j \neq 0$, then the alphabet size of $C_{\mathcal{F}}$ reduces to* 4.

Proof. By Theorems 1 and 2, we derive that $C_{\mathcal{F}}$ is a codebook, $N = (|\mathcal{F}| + 2)2^n = 2^{\frac{5n}{4}} + 2^n$, $K = 2^n$, $I_{\max}(C_{\mathcal{F}}) = \frac{1}{\sqrt{2^n}}$, and the alphabet size of $C_{\mathcal{F}}$ is 6. Then by (1.1) in Lemma 1, we have

$$I_W = \sqrt{\frac{N-K}{(N-1)K}} = \sqrt{\frac{2^{\frac{n}{4}}}{2^{\frac{5n}{4}} + 2^n - 1}}.$$

Thus,

$$\lim_{K\to\infty} \frac{I_{\max}(C_{\mathcal{F}})}{I_W} = \lim_{n\to\infty} \sqrt{1 + \frac{1}{2^{\frac{n}{4}}} - \frac{1}{2^{\frac{5n}{4}}}} = 1,$$

which indicates that the codebook $C_{\mathcal{F}}$ nearly meets the Welch bound. If $Q(x)$ satisfies $c = 0$ and $\sum_{j=1}^{s} \bar{b}_j \neq 0$, then the alphabet size of $C_{\mathcal{F}}$ is 4 by Proposition 1. □

Example 1. *Let $n = 8$, $s = 2$, $k_1 = 1$, $k_2 = 2$, $t_1 = 2$, $t_2 = 1$. The function $Q(x)$ in (4.3) is given by $Q(x) = 2\mathrm{Tr}_1^8(a_1(x^3 + x^5 + x^9) + a_2x^5) + 2\mathrm{Tr}_1^4((b_1 + b_2)x^{17}) + \mathrm{Tr}_1^8(cx)$, where $a_1, a_2, b_1, b_2, c \in \mathbb{T}_2$ and $\bar{b}_1 + \bar{b}_2 + \bar{c}^2 \neq 0$. By Theorem 2, $\mathcal{F} = \{Q(\alpha x) \,|\, \alpha \in \mathbb{T}_2^*\}$ is a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms, which satisfies that the difference of arbitrary two distinct quadratic forms in the set is generalized bent. Then from Theorem 3, $C_{\mathcal{F}}$ in (3.1) is a (1280, 256) codebook with $I_{\max}(C_{\mathcal{F}}) = \frac{1}{16}$ and alphabet size 6. Further, if $c = 0$ and $b_1 \neq b_2$, then the alphabet size is 4. This example is also verified by a Magma program.*

To further demonstrate the asymptotic optimality of the codebooks from Theorem 3, we provide some explicit values of parameters of the codebooks in Table 1. From this table, we conclude that the maximum cross-correlation amplitude $I_{\max}(C_{\mathcal{F}})$ of these codebooks is very close to the Welch bound I_W as n increases. For example, when $n = 52$, it can be seen that $I_{\max}(C_{\mathcal{F}})/I_W \approx 1.00006$, which implies that the ratio $I_{\max}(C_{\mathcal{F}})/I_W$ is very close to 1. This confirms that the codebook $C_{\mathcal{F}}$ constructed by Theorem 3 is nearly optimal with respect to the Welch bound.

Table 1. The parameters of the (N, K) codebook $C_{\mathcal{F}}$ in Theorem 3.

n	N	K	$I_{\max}(C_{\mathcal{F}})$	I_W	$I_{\max}(C_{\mathcal{F}})/I_W$
4	48	16	0.25000	0.20628	1.21192
8	1280	256	0.06250	0.05592	1.11760
12	36864	4096	0.01563	0.01473	1.06065
16	1114112	65536	0.00391	0.00379	1.03078
20	34603008	1048576	0.00098	0.00096	1.01550
24	109051904	16777216	0.00024	0.00024	1.00778
28	34628173824	268435456	6.10352×10^{-5}	6.07981×10^{-5}	1.00390
32	1103806595072	4294967296	1.52588×10^{-5}	1.52291×10^{-5}	1.00195
36	35253091565568	68719476736	3.81470×10^{-6}	3.81098×10^{-6}	1.00098
40	1126999418470400	1099511627776	9.53674×10^{-7}	9.53209×10^{-7}	1.00049
44	36046389205008384	17592186044416	2.38419×10^{-7}	2.38360×10^{-7}	1.00024
48	1153202979583557632	281474976710656	5.96046×10^{-8}	5.95974×10^{-8}	1.00012
52	36897991747046473728	4503599627370496	1.49012×10^{-8}	1.49003×10^{-8}	1.00006

Furthermore, the generalized bent functions in Lemma 5 and Remark 1 can produce new Boolean bent functions.

From Lemma 2, the 2-adic expansion of the function $Q(x)$ in (4.3) is

$$Q(x) = g(\bar{x}) + 2h(\bar{x}),$$

where $g(\bar{x}) = \mathrm{tr}_1^n(\bar{c}\bar{x})$, and

$$h(\bar{x}) = \sum_{j=1}^{s}\Big(\sum_{i=1}^{2t_j-1} \mathrm{tr}_1^n(\bar{a}_j\bar{x}^{1+2^{ik_j}}) + \mathrm{tr}_1^{\frac{n}{2}}(\bar{b}_j\bar{x}^{1+2^{\frac{n}{2}}})\Big) + p(\bar{c}\bar{x})$$

are Boolean functions over $\mathbb{F}_{2^n}$, and $p(x)$ is defined by (2.1). By employing Lemma 4, we obtain the following Boolean bent functions.

Proposition 4. *Let n, s be positive integers with $4|n$, and $k_1, \cdots, k_s$ be distinct positive integers, each of which divides $\frac{n}{4}$. For $1 \le j \le s$, let $t_j = \frac{n}{4k_j}$ and $a_j, b_j, c \in \mathbb{F}_{2^{\frac{n}{4}}}$. Let the Boolean function $h(x)$ over $\mathbb{F}_{2^n}$ be defined by*

$$h(x) = \sum_{j=1}^{s}\Big(\sum_{i=1}^{2t_j-1} \mathrm{tr}_1^n(a_jx^{1+2^{ik_j}}) + \mathrm{tr}_1^{\frac{n}{2}}(b_jx^{1+2^{\frac{n}{2}}})\Big) + p(cx),$$

where $p(x)$ is defined by (2.1). Then $h(x)$ is bent if and only if $\sum_{j=1}^{s} b_j + c^2 \neq 0$.

Furthermore, let l be a positive integer. For $1 \le i \le l$, let $\alpha_i \in \mathbb{F}^*_{2^{\frac{n}{4}}}$ and $\sum_{i=1}^{l} \alpha_i \neq 0$. Then $\sum_{i=1}^{l} h(\alpha_i x)$ is bent if and only if $\sum_{j=1}^{s} b_j + c^2 \neq 0$.

4.2. The second class of nearly optimal codebooks from generalized bent functions

In the previous subsection, for $1 \le j \le s$, each function $Q_j(x)$ in (4.2) has the same coefficients in different trace terms from $\mathbb{T}$ to $\mathbb{Z}_4$. In this subsection, the coefficients of each trace term are not necessarily the same.

Let n, s be positive integers with $2|n$, and $k_1, \cdots, k_s$ be distinct positive integers, each of which divides $\frac{n}{2}$. For $1 \le j \le s$, denote $t_j = \frac{n}{2k_j}$ and let the generalized Boolean function $Q_j(x)$ from $\mathbb{T}$ to $\mathbb{Z}_4$ be defined by

$$Q_j(x) = \sum_{i=1}^{t_j-1} \mathrm{Tr}_1^n(a_{ji}x^{1+2^{ik_j}}) + \mathrm{Tr}_1^{\frac{n}{2}}(b_jx^{1+2^{\frac{n}{2}}}), \tag{4.13}$$

where $a_{ji} \in \mathbb{T}_{\frac{n}{2}}$, $a_{ji} = a_{j,t_j-i}^{2^{ik_j}}$ and $b_j \in \mathbb{T}_{\frac{n}{2}}$ for $1 \le j \le s$, $1 \le i \le t_j - 1$. Let $c \in \mathbb{T}_{\frac{n}{2}}$, then $Q(x)$ in (4.1) is given by

$$Q(x) = \sum_{j=1}^{s}\Big(\sum_{i=1}^{t_j-1} 2\mathrm{Tr}_1^n(a_{ji}x^{1+2^{ik_j}}) + 2\mathrm{Tr}_1^{\frac{n}{2}}(b_jx^{1+2^{\frac{n}{2}}})\Big) + \mathrm{Tr}_1^n(cx), \quad x \in \mathbb{T}. \tag{4.14}$$

The generalized Boolean function $Q(x)$ in (4.14) will be used to construct nearly optimal codebooks. To this end, we first characterize its bentness.

Lemma 7. *The function $Q(x)$ defined by* (4.14) *is generalized bent if and only if $\sum_{j=1}^{s} \bar{b}_j + \bar{c}^2 \neq 0$.*

Proof. The function $Q(x)$ is a $\mathbb{Z}_4$-valued quadratic form. By calculation, the symmetric bilinear form $B_Q(x,y)$ of $Q(x)$ is given by

$$2B_Q(x,y) = \mathrm{tr}_1^n\Big(L_Q(\bar{x})\bar{y}\Big),$$

where

$$L_Q(x) = \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}x^{2^{ik_j}} + (\bar{a}_{ji}x)^{2^{n-ik_j}}\Big) + \sum_{j=1}^{s}\bar{b}_j x^{2^{\frac{n}{2}}} + \bar{c}^2 x. \tag{4.15}$$

Then $Q(x)$ is generalized bent if and only if the equation $L_Q(\bar{x}) = 0$ has only the zero solution in $\mathbb{T}$, i.e., the equation $L_Q(x) = 0$ has only the zero solution in $\mathbb{F}_{2^n}$.

Observe that $a_{ji}^{2^{n-ik_j}} = a_{j,t_j-i}$ for $1 \le j \le s$, $1 \le i \le t_j - 1$. Then

$$\begin{aligned}
& \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}x^{2^{ik_j}} + (\bar{a}_{ji}x)^{2^{n-ik_j}}\Big) \\
= \; & \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}x^{2^{ik_j}} + \bar{a}_{j,t_j-i}x^{2^{n-ik_j}}\Big) \\
= \; & \sum_{j=1}^{s}\Big(\sum_{i=1}^{t_j-1}\bar{a}_{ji}x^{2^{ik_j}} + \sum_{i=1}^{t_j-1}\bar{a}_{j,t_j-i}x^{2^{\frac{n}{2}+(t_j-i)k_j}}\Big) \\
= \; & \sum_{j=1}^{s}\Big(\sum_{i=1}^{t_j-1}\bar{a}_{ji}x^{2^{ik_j}} + \sum_{l=1}^{t_j-1}\bar{a}_{jl}x^{2^{\frac{n}{2}+lk_j}}\Big) \\
= \; & \sum_{j=1}^{s}\Big(\sum_{i=1}^{t_j-1}\bar{a}_{ji}x^{2^{ik_j}} + \sum_{i=1}^{t_j-1}\bar{a}_{ji}x^{2^{\frac{n}{2}+ik_j}}\Big) \\
= \; & \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}\Big(x^{2^{ik_j}} + x^{2^{\frac{n}{2}+ik_j}}\Big)\Big).
\end{aligned}$$

Thus $L_Q(x)$ is further transformed into

$$L_Q(x) = \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}\Big(x^{2^{ik_j}} + x^{2^{\frac{n}{2}+ik_j}}\Big)\Big) + \sum_{j=1}^{s}\bar{b}_j(x + x^{2^{\frac{n}{2}}}) + \Big(\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2\Big)x. \tag{4.16}$$

Therefore, to finish the proof, we only need to prove that the equation

$$\sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}\Big(x^{2^{ik_j}} + x^{2^{\frac{n}{2}+ik_j}}\Big)\Big) + \sum_{j=1}^{s}\bar{b}_j(x + x^{2^{\frac{n}{2}}}) + \Big(\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2\Big)x = 0 \tag{4.17}$$

has only the zero solution in $\mathbb{F}_{2^n}$ if and only if $\sum_{j=1}^{s}\bar{b}_j + \bar{c}^2 \neq 0$.

Assume that $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$. Note that $\bar{a}_{ji},\bar{b}_j\in\mathbb{F}_{2^{\frac{n}{2}}}$ for $1\leq j\leq s$, $1\leq i\leq t_j-1$, and $\bar{c}\in\mathbb{F}_{2^{\frac{n}{2}}}$. It follows from (4.17) that the solution of (4.17) must belong to $\mathbb{F}_{2^{\frac{n}{2}}}$. As a result, (4.17) is equivalent to $\left(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\right)x=0$, which indicates that (4.17) has only the zero solution in $\mathbb{F}_{2^n}$.

On the other hand, if (4.17) has only the zero solution in $\mathbb{F}_{2^n}$, then it is obvious that $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, otherwise each $x\in\mathbb{F}_{2^{\frac{n}{2}}}$ is a solution of (4.17), a contradiction. Then the proof is completed. □

The following corollary is a straightforward consequence of Lemma 7.

Corollary 2. *Let n, s be positive integers with $2|n$, $k_1,\cdots,k_s$ be distinct positive integers, each of which can divide $\frac{n}{2}$, and $t_j=\frac{n}{2k_j}$ for $1\leq j\leq s$. Let $a_{ji}\in\mathbb{T}_{k_j}$, $a_{ji}=a_{j,t_j-i}$, $b_j\in\mathbb{T}_{\frac{n}{2}}$ for $1\leq j\leq s$, $1\leq i\leq t_j-1$, and $c\in\mathbb{T}_{\frac{n}{2}}$. If $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, then $Q(x)$ defined by* (4.14) *is generalized bent.*

From Lemma 7, $Q(x)$ is bent if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, and then $Q(\alpha x)$ is bent, where $\alpha\in\mathbb{T}^*_{\frac{n}{2}}$. For distinct $\alpha,\beta\in\mathbb{T}^*_{\frac{n}{2}}$, the following lemma gives the bentness of the difference $Q_{\alpha,\beta}(x)$ of $Q(\alpha x)$ and $Q(\beta x)$.

Lemma 8. *For $\alpha,\beta\in\mathbb{T}^*_{\frac{n}{2}}$ with $\alpha\neq\beta$, define the function from $\mathbb{T}$ to $\mathbb{Z}_4$ as $Q_{\alpha,\beta}(x)=Q(\alpha x)-Q(\beta x)$, where $Q(x)$ is defined in* (4.14). *Then $Q_{\alpha,\beta}(x)$ is generalized bent if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$.*

Proof. We have

$$\begin{aligned}Q_{\alpha,\beta}(x) &= \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}2\mathrm{Tr}_1^n\Big(a_{ji}(\alpha^{1+2^{ik_j}}-\beta^{1+2^{ik_j}})x^{1+2^{ik_j}}\Big)\\ &\quad+\sum_{j=1}^{s}2\mathrm{Tr}_1^{\frac{n}{2}}\Big(b_j(\alpha^2-\beta^2)x^{1+2^{\frac{n}{2}}}\Big)+\mathrm{Tr}_1^n(c(\alpha-\beta)x).\end{aligned}$$

The function $Q_{\alpha,\beta}(x)$ is a quaternary quadratic form. By calculation, its symmetric bilinear form $B_{Q_{\alpha,\beta}}(x,y)$ satisfies

$$2B_{Q_{\alpha,\beta}}(x,y)=2\mathrm{tr}_1^n\Big(L_{Q_{\alpha,\beta}}(\bar{x})\bar{y}\Big),$$

where

$$\begin{aligned}L_{Q_{\alpha,\beta}}(x) &= \sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})x^{2^{ik_j}}+\Big(\bar{a}_{ji}(\bar{\alpha}^{1+2^{ik_j}}+\bar{\beta}^{1+2^{ik_j}})x\Big)^{2^{n-ik_j}}\Big)\\ &\quad+\sum_{j=1}^{s}\bar{b}_j(\bar{\alpha}^2+\bar{\beta}^2)x^{2^{\frac{n}{2}}}+\bar{c}^2(\bar{\alpha}^2+\bar{\beta}^2)x.\end{aligned}$$

Then $Q_{\alpha,\beta}(x)$ is generalized bent if and only if the equation $L_{Q_{\alpha,\beta}}(x)=0$ has only the zero solution in $\mathbb{F}_{2^n}$. Note that $L_{Q_{\alpha,\beta}}(x)$ can be equivalently written as

$$L_{Q_{\alpha,\beta}}(x) = \bar{\alpha}\Big(\sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}(\bar{\alpha}x)^{2^{ik_j}}+(\bar{a}_{ji}\bar{\alpha}x)^{2^{n-ik_j}}\Big)+\sum_{j=1}^{s}\bar{b}_j(\bar{\alpha}x)^{2^{\frac{n}{2}}}+\bar{c}^2\bar{\alpha}x\Big)$$

$$+\bar{\beta}\Big(\sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}(\bar{\beta}x)^{2^{ik_j}}+(\bar{a}_{ji}\bar{\beta}x)^{2^{n-ik_j}}\Big)+\sum_{j=1}^{s}\bar{b}_j(\bar{\beta}x)^{2^{\frac{n}{2}}}+\bar{c}^2\bar{\beta}x\Big)$$
$$= \bar{\alpha}L_Q(\bar{\alpha}x)+\bar{\beta}L_Q(\bar{\beta}x),$$

where $L_Q(x)$ is defined by (4.15). Based on (4.16), the equation $L_{Q_{\alpha,\beta}}(x) = 0$ can be reformulated as

$$\begin{aligned}&\bar{\alpha}\sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}\Big((\bar{\alpha}x)^{2^{ik_j}}+(\bar{\alpha}x)^{2^{\frac{n}{2}+ik_j}}\Big)\Big)+\bar{\beta}\sum_{j=1}^{s}\sum_{i=1}^{t_j-1}\Big(\bar{a}_{ji}\Big((\bar{\beta}x)^{2^{ik_j}}+(\bar{\beta}x)^{2^{\frac{n}{2}+ik_j}}\Big)\Big)\\&+(\sum_{j=1}^{s}\bar{b}_j)(\bar{\alpha}^2+\bar{\beta}^2)(x+x^{2^{\frac{n}{2}}})+(\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2)(\bar{\alpha}^2+\bar{\beta}^2)x=0.\end{aligned} \tag{4.18}$$

Then, it is sufficient to prove that (4.18) has only the zero solution in $\mathbb{F}_{2^n}$ if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$. Notice that $\bar{\alpha}^2+\bar{\beta}^2\neq 0$. This statement can be similarly proved as in Lemma 7, and thus the proof is omitted here. □

Remark 2. *For any positive integer l, let $\alpha_i\in\mathbb{T}^*_{\frac{n}{2}}$ $(1\le i\le l)$ with $\sum_{i=1}^{l}\bar{\alpha}_i\neq 0$. Similarly as in Lemma 8, we derive that the generalized Boolean function $Q_{\alpha_1,\cdots,\alpha_l}(x)=\sum_{i=1}^{l}Q(\alpha_i x)$ is bent if and only if $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, where $Q(x)$ is defined in* (4.14).

By Lemmas 7 and 8, we obtain the following result.

Theorem 4. *Let $Q(x)$ be the function defined by* (4.14) *with $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$. Then*

$$\mathcal{F}=\Big\{Q(\alpha x)\,\Big|\,\alpha\in\mathbb{T}^*_{\frac{n}{2}}\Big\}$$

is a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms, which satisfies that the difference of arbitrary two distinct quadratic forms in the set is generalized bent.

Now, nearly optimal codebooks can be derived as follows.

Theorem 5. *Let $Q(x)$ be the generalized function defined by* (4.14) *with $\sum_{j=1}^{s}\bar{b}_j+\bar{c}^2\neq 0$, and define the set $\mathcal{F}=\{Q(\alpha x)\mid \alpha\in\mathbb{T}^*_{\frac{n}{2}}\}$. Let $C_{\mathcal{F}}$ be the codebook constructed in* (3.1) *from the set $\mathcal{F}$. Then $C_{\mathcal{F}}$ is a $(2^{\frac{3n}{2}}+2^n, 2^n)$ codebook nearly meeting the Welch bound, $I_{\max}(C_{\mathcal{F}})=\frac{1}{\sqrt{2^n}}$, and the alphabet size of $C_{\mathcal{F}}$ is* 6. *Further, if $Q(x)$ satisfies $c=0$ and $\sum_{j=1}^{s}\bar{b}_j\neq 0$, then the alphabet size of $C_{\mathcal{F}}$ reduces to* 4.

Proof. Together with Theorems 1 and 4, we derive that $C_{\mathcal{F}}$ is a codebook with $N=(|\mathcal{F}|+2)2^n=2^{\frac{3n}{2}}+2^n$, $K=2^n$ and $I_{\max}(C_{\mathcal{F}})=\frac{1}{\sqrt{2^n}}$. Moreover, the alphabet size of $C_{\mathcal{F}}$ is 6. Then by (1.1) in Lemma 1, we get

$$I_W=\sqrt{\frac{N-K}{(N-1)K}}=\sqrt{\frac{2^{\frac{n}{2}}}{2^{\frac{3n}{2}}+2^n-1}}.$$

Thus,

$$\lim_{K\to\infty}\frac{I_{\max}(C_{\mathcal{F}})}{I_W}=\lim_{n\to\infty}\sqrt{1+\frac{1}{2^{\frac{n}{2}}}-\frac{1}{2^{\frac{3n}{2}}}}=1,$$

which implies that $C_{\mathcal{F}}$ is a nearly optimal codebook with respect to the Welch bound. Given $Q(x)$ with $c=0$ and $\sum_{j=1}^{s}\bar{b}_j\neq 0$, the alphabet size of $C_{\mathcal{F}}$ is 4 from Proposition 1. □

In the sequel, we present two examples of the codebooks in Theorem 5.

Example 2. *Let $n=2$, and $\mathbb{T}$ be the Teichmüller set of $\mathbb{GR}(4,2)$. Then the function $Q(x)$ in Theorem 5 is $Q(x)=\mathrm{Tr}_1^2(x)$ or $Q(x)=2x^3$. Note that $\mathbb{T}_1^*=\{1\}$, then the set $\mathcal{F}=\{Q(x)\}$. Let $C_{\mathcal{F}}$ be the codebook constructed in* (3.1) *from the set $\mathcal{F}$.*

1) If we choose $Q(x)=\mathrm{Tr}_1^2(x)$, then by Theorem 5, $C_{\mathcal{F}}$ is a complex-valued codebook with alphabet size 6, and it is composed of the following 12 codewords of length 4:

$$\left(\frac{1}{2},-\frac{1}{2},-\frac{\sqrt{-1}}{2},-\frac{\sqrt{-1}}{2}\right),\left(\frac{1}{2},-\frac{1}{2},\frac{\sqrt{-1}}{2},\frac{\sqrt{-1}}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{\sqrt{-1}}{2},-\frac{\sqrt{-1}}{2}\right),\left(\frac{1}{2},\frac{1}{2},-\frac{\sqrt{-1}}{2},\frac{\sqrt{-1}}{2}\right),$$
$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),$$
$$(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1).$$

2) If we take $Q(x)=2x^3$, then by Theorem 5, $C_{\mathcal{F}}$ is a real-valued codebook with alphabet size 4, and it consists of the following 12 codewords of length 4:

$$\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right),$$
$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),$$
$$(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1).$$

For cases 1) and 2), the sets S_Q, S_0, and E_{2^n} are composed of the elements in the first, second, and third rows above, respectively.

Example 3. *Let $n=6$, $s=2$, $k_1=1$, $k_2=3$, $t_1=3$, $t_2=1$. Then $Q(x)$ in* (4.14) *can be written as $Q(x)=2\mathrm{Tr}_1^6(a_{11}x^3+a_{12}x^5))+2\mathrm{Tr}_1^3((b_1+b_2)x^9)+\mathrm{Tr}_1^6(cx)$, where $a_{11},a_{12},b_1,b_2,c\in\mathbb{T}_3$, $a_{11}=a_{12}^2$ and $\bar{b}_1+\bar{b}_2+\bar{c}^2\neq 0$. From Theorem 4, $\mathcal{F}=\{Q(\alpha x)\,|\,\alpha\in\mathbb{T}_3^*\}$ is a set of generalized bent $\mathbb{Z}_4$-valued quadratic forms, such that the difference of any two distinct quadratic forms in the set is generalized bent. Then by Theorem 5, $C_{\mathcal{F}}$ in* (3.1) *is a* $(576,64)$ *codebook with $I_{\max}(C_{\mathcal{F}})=\frac{1}{8}$ and alphabet size* 6. *Moreover, if $c=0$ and $b_1\neq b_2$, then the alphabet size is* 4. *This example is verified by a Magma program.*

In Table 2, we list the parameters of some specific codebooks generated by Theorem 5. The results demonstrate that $I_{\max}(C_{\mathcal{F}})$ is very close to I_W for large n. For instance, when $n=26$, the ratio $I_{\max}(C_{\mathcal{F}})/I_W\approx 1.00006$, which means that the ratio $I_{\max}(C_{\mathcal{F}})/I_W$ is much closer to 1. These numerical results further confirm the asymptotic optimality of the codebook, consistent with the statement in Theorem 5, and also guarantee the correctness of our main result.

The Boolean bent function below is an immediate consequence of Lemmas 2, 4, 7 and Remark 2.

Table 2. The parameters of the (N, K) codebook $C_{\mathcal{F}}$ in Theorem 5.

n	N	K	$I_{\max}(C_{\mathcal{F}})$	I_W	$I_{\max}(C_{\mathcal{F}})/I_W$
2	12	4	0.50000	0.42640	1.17260
4	80	16	0.25000	0.22502	1.11102
6	576	64	0.12500	0.11795	1.05974
8	4352	256	0.06250	0.06064	1.03066
10	33792	1024	0.03125	0.03077	1.01549
12	266240	4096	0.01563	0.01550	1.00778
14	2113536	16384	0.00781	0.00779	1.00390
16	16842752	65536	0.00391	0.00390	1.00195
18	134479872	262144	1.953125×10^{-3}	1.951220×10^{-3}	1.00098
20	1074790400	1048576	9.765625×10^{-4}	9.760860×10^{-4}	1.00049
22	8594128896	4194304	4.882813×10^{-4}	4.881621×10^{-4}	1.00024
24	68736253952	16777216	2.441406×10^{-4}	2.441108×10^{-4}	1.00012
26	549822922752	67108864	1.220703×10^{-4}	1.220629×10^{-4}	1.00006

Proposition 5. *Let n, s be positive integers with $2|n$, $k_1, \cdots, k_s$ be distinct positive integers, each of which can divide $\frac{n}{2}$, and $t_j = \frac{n}{2k_j}$ for $1 \leq j \leq s$. Let $a_{ji} \in \mathbb{F}_{2^{\frac{n}{2}}}$, $a_{ji} = a_{j,t_j-i}^{2^{ik_j}}$ and $b_j \in \mathbb{F}_{2^{\frac{n}{2}}}$ for $1 \leq j \leq s$, $1 \leq i \leq t_j - 1$, and $c \in \mathbb{F}_{2^{\frac{n}{2}}}$. Let the Boolean function $h(x)$ over $\mathbb{F}_{2^n}$ be defined by*

$$h(x) = \sum_{j=1}^{s} \Big(\sum_{i=1}^{t_j-1} \mathrm{tr}_1^n(a_{ji} x^{1+2^{ik_j}}) + \mathrm{tr}_1^{\frac{n}{2}}(b_j x^{1+2^{\frac{n}{2}}}) \Big) + p(cx),$$

where the Boolean function $p(x)$ is given by (2.1). Then $h(x)$ is bent if and only if $\sum_{j=1}^{s} b_j + c^2 \neq 0$.

Moreover, let l be a positive integer, $\alpha_i \in \mathbb{F}_{2^{\frac{n}{2}}}^*$ for $1 \leq i \leq l$, and $\sum_{i=1}^{l} \alpha_i \neq 0$. Then the Boolean function $\sum_{i=1}^{l} h(\alpha_i x)$ is bent if and only if $\sum_{j=1}^{s} b_j + c^2 \neq 0$.

To conclude this section, we remark that codebooks derived from generalized bent $\mathbb{Z}_4$-valued quadratic forms have been constructed in [9, 10] as well as in this paper. Below, we compare our codebooks with those of Heng and Yue [9] and Qi et al. [10].

Remark 3. *Both the codebooks in [10] and those proposed in this paper are derived by employing the construction in [9]. In [9], as well as in [10], a class of generalized bent functions was proposed, leading to a family of codebooks that achieve the Levenshtein bound. In this paper, we propose two classes of generalized bent functions and derive two families of codebooks that nearly achieve the Welch bound. The underlying generalized bent functions, as well as the parameters (including the parameter N and the alphabet size) of the codebooks obtained in this paper, are different from those in [9, 10]. Codebooks with a small alphabet size are highly desirable for practical applications [1]. It is worth noting that our codebooks can achieve a smaller alphabet size. More specifically, our codebooks could have alphabet size* 4*, compared to alphabet size* 6 *in [9, 10].*

5. Conclusions

In this paper, by proving certain linear equations over the finite field have only the zero solution, we propose two classes of generalized bent $\mathbb{Z}_4$-valued quadratic forms and introduce two sets of generalized bent functions satisfying certain conditions. Then two classes of codebooks are derived from these sets. The obtained codebooks are proven to be nearly optimal with respect to the Welch bound, and could have a very small alphabet size. As a product, new Boolean bent functions are obtained from the proposed generalized bent functions.

It would be possible to search for more sets of generalized bent functions with the desired properties leading to optimal or nearly optimal codebooks. Whether similar constructions of codebooks can be extended to other rings or beyond generalized bent functions deserves further study. We encourage interested readers to explore this direction.

Finally, we briefly discuss the potential applications of codebooks obtained in this paper. As pointed out by Sarwate, codebooks over a small alphabet are of importance in practical applications [1]. Hence, the proposed nearly optimal codebooks with alphabet size 6 or 4 are interesting. In code-division multiple-access (CDMA) communication systems, codebooks are used to distinguish signals from different users. To minimize multi-user interference, the maximum cross-correlation amplitude $I_{\max}$ of the codebook should be as small as possible. In synchronous direct-sequence CDMA (DS-CDMA) systems, where the number of users N exceeds the signal space dimension or the spreading factor K, optimal or nearly optimal codebooks are essential for interference suppression. The codebooks presented in this paper are nearly optimal and could have a very small alphabet size, making them well-suited for deployment in synchronous DS-CDMA systems. They are thus quite useful in third-generation (3G) and fourth-generation (4G) networks for increasing the subscriber capacity within the limited spectral resource [12].

Author contributions

Both authors of this article have been contributed equally. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Natural Science Foundation of Hubei Province of China (Grant No. 2021CFB275), the National Natural Science Foundation of China (Grant Nos. 12301667, 12201356), and the Natural Science Foundation of Shandong Province of China (Grant No. ZR2022QA048).

The authors would like to thank the editor and the anonymous referees for their helpful comments and suggestions.

Conflict of interest

The authors declare no conflicts of interest.

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