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Theory article

The study of parabolic spectral fractional Laplacian with nonhomogeneous Dirichlet boundary conditions

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Abstract: We use characterizations of spectral fractional Laplacian to tackle the problems of parabolic spectral fractional Laplacian incorporating nonhomogeneous Dirichlet boundary conditions. Compared with applying classical extension method in the study of the lower-order fractional operators with $s \in (0, 1)$, we used a modified extensions to parabolic spectral fractional Laplacian incorporating boundary conditions, adapted from constructing harmonic extensions of the boundary data to spectral fractional Laplacian.

Keywords: modified spectral fractional Laplace operator; nonzero boundary conditions; weak solution
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1. Introduction

The spectral fractional Laplacian is defined via the eigenvalues and eigenfunctions of the classical Laplacian with homogeneous Dirichlet boundary conditions. For a bounded domain $\Omega \subset \mathbb{R}^n$, let $\{\lambda_k, \phi_k\}$ be the eigenvalues and eigenfunctions of $-\Delta$. The spectral fractional Laplacian of order $s \in (0, 1)$ is then:

$$(-\Delta_{D,0})^s u(x) = \sum_{k=1}^{\infty} \lambda_k^s (u, \phi_k)_{L^2(\Omega)} \phi_k(x), \tag{1.1}$$

where (u, ϕ_k) denotes the L^2-inner product. This definition requires zero boundary conditions for the eigenfunction expansion to converge, we use subscripts $(D, 0)$ in $(-\Delta_{D,0})^s$ to denote the case where the boundary data is zero. Parallel to this, $(-\Delta_D)^s$ denotes the modified spectral fractional Laplace operator with nonzero boundary conditions in the following contents.

For nonzero boundary data, the spectral fractional Laplacian is modified by decomposing u into a harmonic function v (satisfying $v = g$ on $\partial\Omega$) and a remainder w with zero boundary conditions:

$$u = v + w, \ (-\Delta_D)^s u = (-\Delta_D)^s v + (-\Delta_{D,0})^s w. \tag{1.2}$$

Here, $(-\Delta_D)^s v$ is defined via extension method (also known as the Caffarelli-Silvestre extension or Dirichlet-to-Neumann map, see in [1]). The extension method transforms the nonlocal fractional Laplacian $(-\Delta)^s$ into a local boundary value problem in one higher spatial dimension. For a function v defined on $\Omega \subset \mathbb{R}^n$, we construct an auxiliary function $V(x, y)$ where $x \in \Omega$, $y \geq 0$, that satisfies:

$$\begin{cases} \nabla \cdot (y^{1-2s} \nabla V) = 0, & in\ \Omega \times (0, \infty), \\ V(x, 0) = v(x), & on\ \Omega\ \times \{y = 0\}, \\ V(x, y) = g(x), & on\ \partial\Omega \times [0, \infty). \end{cases} \tag{1.3}$$

Here, y^{1-2s} is a degenerate weight that ensures the problem is well-posed. The parameter $s \in (0, 1)$ corresponds to the fractional power of the Laplacian.

The fractional Laplacian $(-\Delta_D)^s v$ is recovered from the normal derivative of V at $y = 0$:

$$(-\Delta_D)^s v = -C_{n,s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial V}{\partial y}(x, y), \tag{1.4}$$

where $C_{n,s}$ is a normalization constant depending on n and s. This formula shows that the fractional Laplacian can be computed by solving a local PDE and then taking a derivative at the boundary. The fractional Laplacian is inherently nonlocal, but the extension method localizes it by introducing an extra dimension. For v harmonic (i.e., $-\Delta v = 0$ in Ω), the extension V inherits this property in the x-variables, simplifying the computation of $(-\Delta_D)^s v$. Also, the method naturally incorporates nonzero boundary data g. For further details on extension methods on fractional Laplacians, see the seminal work by Caffarelli and Silvestre (2007) [1] or Stinga and Torrea (2010) [2].

While $(-\Delta_{D,0})^s w$ in (1.2) represents the spectral fractional Laplacian applied to a function w with homogeneous Dirichlet boundary conditions ($w = 0$ on $\partial\Omega$). Its specific definition via standard spectral decomposition relies on the eigenfunction expansion of the Dirichlet Laplacian, which is provided in Section 2.

In [3], the authors studied

$$\begin{cases} (-\Delta_D)^s u = f, & in\ \Omega, \\ u = g, & on\ \partial\Omega, \end{cases} \tag{1.5}$$

they use a standard lifting argument by constructing a fractional harmonic map

$$(-\Delta_D)^s v = 0, \ in\ \Omega, \ v = g, \ on\ \partial\Omega. \tag{1.6}$$

Solving (1.6) is equivalent to solving

$$\int_\Omega v(-\Delta\varphi) = \int_{\partial\Omega} g \partial_\nu \varphi, \ \forall \varphi \in dom(-\Delta), \tag{1.7}$$

that is, the standard Laplace equation in the very-weak form. To attain the final answer of u, it remains to find w solving

$$(-\Delta_{D,0})^s w = f, \ in\ \Omega, \ w = 0, \ on\ \Omega. \tag{1.8}$$

then $u = w + v$. Hence, rather than seeking a direct solution for u, they transformed the problem into solving Eq (1.5) separately for the variables v and w.

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded open set with boundary $\partial\Omega$. The purpose of this paper is to study existence, uniqueness, regularity of the following nonhomogeneous Dirichlet boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_D)^s u = f, & in\ \Omega, \\ u = g, & on\ \partial\Omega. \end{cases} \tag{1.9}$$

Here, f and g are defined as measurable functions, with f operating on the boundary $\partial\Omega$ and g on the domain Ω itself. These functions adhere to specific conditions, which will be detailed in Section 2. To simplify the problem, we assume $u = 0$ outside Ω. The existence of solutions of (1.9) could be derived from maximal regularity theory as shown in [4, 5].

In the classical extension method, to study the parabolic spectral fractional Laplacian (1.9), we first study

$$(-\Delta_D)^s v = 0,\ in\ \Omega,\ v = g,\ on\ \partial\Omega. \tag{1.10}$$

According to the Caffarelli-Silvestre extension theory, v can be defined through the local problem in a higher-dimensional domain, the fractional Laplace operator $(-\Delta)^s v$ is related to the Dirichlet-to-Neumann map of V as shown in (1.4).

Define a new function $w(x,t) = u(x,t) - v(x,t)$, then w satisfies homogeneous boundary conditions

$$\begin{cases} \partial_t w + (-\Delta_{D,0})^s w = f(x,t) - \partial_t v - (-\Delta_D)^s v = f(x,t) - \partial_t v & in\ \Omega\ \times (0,T), \\ w(x,t) = 0, & in\ \partial\Omega \times (0,T), \\ w(x,0) = u(x,0) - v(x,0), & in\ \Omega. \end{cases} \tag{1.11}$$

Due to $(-\Delta_D)^s v = 0$, the equation simplifies to:

$$\partial_t w + (-\Delta_{D,0})^s w = f(x,t) - \partial_t v. \tag{1.12}$$

If g explicitly depends on time t, then $\partial_t v$ needs to be computed through the variational formulation of the extended equation. For example, by differentiating the weak form of V:

$$\int_C y^{1-2s} \nabla \partial_t V \cdot \nabla \phi dxdy = 0,\ \forall \phi \in H^1_{0,L}(C). \tag{1.13}$$

Combining with the trace theorem, the expression for $\partial_t v$ can be derived, substitute $\partial_t v$ back to (1.12) to solve w, then the final answer of (1.9) could be expressed. However, the computation is very complicated and the solutions are not explicit in this approach.

To overcome this difficulty, in this article, we do not employ the classical extension method but rather generalize the lifting argument methods in [3] to our problem (1.9). Here, a more ideal framework would be:

$$\frac{\partial v}{\partial t} + (-\Delta_D)^s v = 0,\ in\ \Omega,\ v = g,\ on\ \partial\Omega. \tag{1.14}$$

Define a new function $w(x,t) = u(x,t) - v(x,t)$, then w satisfies homogeneous boundary conditions.

$$\begin{cases} \partial_t w + (-\Delta_{D,0})^s w = f(x,t) - \partial_t v - (-\Delta_D)^s v = f(x,t) & in\ \Omega\ \times (0,T), \\ w(x,t) = 0, & in\ \partial\Omega \times (0,T), \\ w(x,0) = u(x,0) - v(x,0), & in\ \Omega. \end{cases} \tag{1.15}$$

To attain the final answer of u, it remains to find w solving

$$\frac{\partial w}{\partial t} + (-\Delta_{D,0})^s w = f,\ in\ \Omega,\ w = 0,\ on\ \partial\Omega. \tag{1.16}$$

Thus instead of looking for u directly, we are reduced to solving (1.9) for v and w, respectively.

The core challenge lies in the fact that, within the Caffarelli-Silvestre extension theory, one cannot simply introduce the time variable t into the extended function $U(x,t)$ and ascribe a physical meaning of time derivative to it, as the theory fundamentally focuses on the geometric extension of spatial fractional operators, which is unrelated to temporal evolution.

The solution is to employ the theory of parameterized surfaces: Let the original two-dimensional manifold $\Sigma \subset R^3$ be given by the parameterization $(x,t) \to (x,t,u(x,t))$. A coordinate transformation can be defined as $(y_1,y_2,y_3) = (x,t,z-u(x,t))$, which maps Σ onto the plane $\{y_3 = 0\}$, while points (x,t) in the original space correspond to transformed points (x,t,z) satisfying $z > u(x,t)$, (i.e., $y_3 > 0$). This operation is essentially a local coordinate system transformation that transfers the original problem from the surface Σ to the hyperplane $y_3 = 0$.

The fractional Laplacian $(-\Delta_D)^s v$ is recovered from the normal derivative of V at $y_3 = 0$:

$$(-\Delta_D)^s v = -C_{n,s} \lim_{y_3 \to 0^+} y_3^{1-2s} \frac{\partial V}{\partial y_3}(y_1,y_2,y_3),$$

where $C_{n,s}$ is a normalization constant depending on n and s.

By chain rule, we have

$$\frac{\partial V}{\partial t} \sim \lim_{y_3 \to 0^+} \frac{\partial V}{\partial y_3}(-\frac{\partial u}{\partial t}),$$

therefore,

$$\frac{\partial V}{\partial t} = 0,$$

whenever

$$(-\Delta_D)^s v = 0.$$

At this point, $v(x)$ is extended to $v(x,t)$, and the boundary condition becomes $V(x,0,t) = v(x,t)$. A time evolution operator $U(t)$ can be introduced such that $v(x,t) = U(t)v(x)$, where $v(x)$ is the function at the initial time $t = 0$. For simplicity, we write it here as $V(x,0,t) = v(x,t)$, but here time t serves only as a parameter, not as a dimension.

By chain rule,

$$\frac{\partial v}{\partial t} = \frac{\partial V}{\partial t}\Big|_{(x,0,t)} + \frac{\partial V}{\partial y}\Big|_{(x,0,t)} \cdot \underbrace{\frac{\partial y}{\partial t}}_{=0},$$

since $\frac{\partial V}{\partial t} = 0$ holds throughout the domain, it follows that

$$\frac{\partial V}{\partial t}\Big|_{(x,0,t)} = 0.$$

Thus,

$$\frac{\partial v}{\partial t} = 0 + 0 = 0,$$

which makes (1.14) hold.

The Extension Method is not only extensively applied in fractional Laplacian operators but also finds wide-ranging uses in other types of partial differential equations. In [6], an extension method for the inverse fractional Laplacian operator $(-\Delta)^s$ is proposed, this method resolves the localization issue of inverse operators, thereby expanding the applicability of fractional partial differential equations. For high-dimensional partial differential equations, such as reaction-diffusion equations, a domain extension method is proposed to transform boundary control problems on irregular domains into equivalent problems on regular domains, as shown in [7]. The article [8] extends the results of Caffarelli-Silvestre [1] to generators of integrated families of operators, particularly focusing on infinitesimal generators of bounded C_0-semigroups and operators with purely imaginary symbols.

This paper is organized as follows: We state some prerequesite definitions and propositions in Section 2. In Section 3, we show the details of how modified extension method be applied to our problem (1.9) step by step. And we prove the existence and uniqueness of solutions to our problem in Section 4. In Sections 5 and 6, we discussed the real-world domains as well as numerical computation/simulation to which the theory in this paper can be applied.

2. Preliminaries

Refer to [3], we introduce the following definitions and propositions to serve the subsequent content:

Definition 2.1. *The eigenfunctions $\{\varphi_k\}$ of a self-adjoint elliptic operator $-\Delta$ with homogeneous Dirichlet conditions satisfy the orthogonality relation*

$$\int_\Omega = \varphi_i \varphi_j dx = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Proposition 2.1. *The set $\{\varphi_k\}$ is complete in $L^2(\Omega)$, which means that for any $u \in L^2(\Omega)$, we have*

$$u = \sum_{k=1}^{\infty} a_k \varphi_k,$$

where $a_k = \int_\Omega u\varphi_k dx$ and the series converges in the $L^2(\Omega)$ norm,

$$\lim_{N\to\infty} \left\| u - \sum_{k=1}^{N} a_k \varphi_k \right\| = 0.$$

Definition 2.2. *The Dirichlet Laplacian $-\Delta_{D,0}$ (with zero boundary conditions) is a self-adjoint, positive-definite operator on $L^2(\Omega)$. Its spectrum consists of a countable sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and corresponding eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$:*

$$(-\Delta_{D,0})\phi_k = \lambda_k\phi_k, \quad in\ \Omega,\ \ \phi_k = 0\ on\ \partial\Omega$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$, and $\{\phi_k\}$ form an orthonormal basis of $L^2(\Omega)$. Since w satisfies $w = 0$ on $\partial\Omega$, it can be expanded in terms of the eigenfunctions

$$w(x) = \sum_{k=1}^{\infty} c_k\phi_k(x), \quad where\ c_k = \int_{\Omega} w(x)\phi_k(x)dx.$$

Definition 2.3. *The spectral fractional Laplacian is defined on the space $C_0^{\infty}(\Omega)$ by*

$$(-\Delta_{D,0})^s w := \sum_{k=1}^{\infty} c_k\lambda_k^s\phi_k(x)\ with\ c_k = \int_{\Omega} w(x)\phi_k(x)$$

with the inequality

$$\int_{\Omega}(-\Delta_{D,0})^s wv = \sum_{k=1}^{\infty}\lambda_k^s w_k v_k = \sum_{k=1}^{\infty}\lambda_k^{\frac{s}{2}} w_k \lambda_k^{\frac{s}{2}} v_k \leq \|w\|_{\mathbb{H}^s(\Omega)}\|v\|_{\mathbb{H}^s(\Omega)}$$

for any $v = \sum_{k=1}^{\infty} v_k\varphi_k \in \mathbb{H}^s(\Omega)$, the operator $(-\Delta_{D,0})^s u$ extends to an operator mapping from $\mathbb{H}^s(\Omega)$ to $\mathbb{H}^{-s}(\Omega)$ by density.

Lemma 2.1. *If we let $v = -(-\Delta)^{1-s}w_v$, then:*

$$< (-\Delta)^s u, -(-\Delta)^{1-s}w_v >=< u, (-\Delta)w_v > .$$

Proof. We prove the Lemma 2.1 using the Fourier transform representation, the fractional Laplacian $(-\Delta)^s$ of a function $u(x,t)$ can be represented in the Fourier domain as:

$$F[(-\Delta)^s u](\xi, t) = |\xi|^{2s}\hat{u}(\xi, t),$$

where F denotes the Fourier transform, and $\hat{u}(x,t)$ is the Fourier transform of $u(x,t)$.

The inner product $< f, g >$ of two functions f and g can be expressed in terms of their Fourier transforms as:

$$< f, g >= \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Consider the left-hand side of the lemma:

$$< (-\Delta)^s u, -(-\Delta)^s w_v > .$$

Thus, the inner product becomes:

$$< (-\Delta)^s u, -(-\Delta)^{1-s}w_v >= -\int_{\mathbb{R}^n}|\xi|^{2s}\hat{u}(\xi,t)\overline{|\xi|^{2-2s}\hat{w_v}(\xi,t)}d\xi = -\int_{\mathbb{R}^n}|\xi|^{2s}\hat{u}(\xi,t)|\xi|^{2-2s}\overline{\hat{w_v}(\xi,t)}d\xi,$$

we have

$$|\xi|^{2s} \cdot |\xi|^{2-2s} = |\xi|^2,$$

so

$$< (-\Delta)^s u, -(-\Delta)^{1-s} w_v >= -\int_{\mathbb{R}^n} |\xi|^2 \hat{u}(\xi, t)\overline{w_v(\xi, t)} d\xi. \tag{2.1}$$

Now consider the right-hand side of the lemma:

$$< u, (-\Delta w_v) > .$$

The Laplacian $-\Delta$ in the Fourier domain is given by:

$$(-\Delta)u(\xi, t) = -|\xi|^2 \hat{u}(\xi, t),$$

so

$$< u, (-\Delta)w_v >= -\int_{\mathbb{R}^n} \hat{u}(\xi, t)\overline{|\xi|^2 w_v(\xi, t)} d\xi = -\int_{\mathbb{R}^n} |\xi|^2 \hat{u}(\xi, t)\overline{w_v(\xi, t)} d\xi. \tag{2.2}$$

Therefore, (2.1) is equal to (2.2). □

If f is Lipschitz continuous, the regularity of (1.15) referenced from [9] is $w \in H^1(0, T; L^2(\mathbb{R}^n)) \cap L^\infty(0, T; H^s(\mathbb{R}^n))$, the minimal regularity conditions on f should be $f \in L^\infty(0, T; H^s(\mathbb{R}^n))$.

In our research, we assume f is a measurable function and that $w \in L^2(0, T; H^s)$ is a weak solution. We derive that its time derivative $\frac{\partial w}{\partial t} \in (0, T; H^{-s})$ through the equation structure. Leveraging the continuity of $(-\Delta)^s : H^s \to H^{-s}$, combined with $f \in (0, T; H^{-s})$, we infer that $\frac{\partial w}{\partial t} \in (0, T; H^{-s})$ implies $w \in C([0, T]; H^s)$ via the Aubin-Lions lemma. We apply Schauder estimates or Calderón-Zygmund theory to iteratively enhance regularity. By Approximating f with a sequence of smooth functions f_ϵ, we prove that the corresponding solutions w_ϵ converge to w in an appropriate topology while preserving regularity.

3. Main theorems and results

We proceed to prove the main theorem of this paper, namely, obtaining the solution to problem (1.9) through the modified extension method. We first provide a detailed process of how lifting argument are applied, which was not clearly described in the original literature [3]. Building upon the framework, we then proceed to establish the parabolic spectral Laplacian. The inclusion of the original proof serves primarily to facilitate a comparative analysis with its parabolic counterpart. For example, when

$$\frac{\partial u}{\partial t} = 0,$$

(1.9) becomes (1.5), the very weak formulation of

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_D)^s u = f, & in\ \Omega, \\ u = g, & on\ \partial\Omega \end{cases}$$

is given by

$$\int_{\Omega} u(-\Delta_{D,0})^s v = \int_{\Omega} fv - \int_{\partial\Omega} g\partial_\nu w_v, \ \forall v \in \mathbb{H}^{2s}(\Omega), \tag{3.1}$$

where w_v is defined as the solution to

$$(-\Delta_{D,0})^{1-s} w_v = v, \ \textit{in } \Omega, \tag{3.2}$$

$$w_v = 0, \ \textit{on } \partial\Omega. \tag{3.3}$$

(3.2) is achieved by the following proposition and Lemma in [3]:

Proposition 3.1. *Let $(-\Delta_{D,0})^s$ be as in Definition 2.2, then the following holds refer to [3]:*

(1) When $s = 1$ we obtain the standard Laplacian $(-\Delta)$.

(2) For any $u \in C_0^\infty$ there holds

$$(-\Delta_D)^s u = (-\Delta_{D,0})^s u$$

a.e in Ω.

(3) For any $s \in (0, 1)$ and any $u \in C^\infty(\overline{\Omega})$ with $\int_{\partial\Omega} \partial_\nu u = 0$ there is the identity

$$(-\Delta_D)^s (-\Delta_D)^{1-s} u = -\Delta u$$

a.e. in Ω.

Based on Proposition 3.1, if u and w_v are symmetric, then the following holds:

Lemma 3.1. *If we let $v = (-\Delta_{D,0})^{1-s} w_v$, then:*

$$< (-\Delta_{D,0})^s u, (-\Delta_{D,0})^{1-s} w_v >=< u, (-\Delta_{D,0})^s (-\Delta_{D,0})^{1-s} w_v >=< u, (-\Delta) w_v > .$$

By Lemma 3.1, we have the following relation:

$$\int_{\Omega} v(-\Delta_{D,0})^s u = \int_{\Omega} u(-\Delta) w_v \tag{3.4}$$

where in the case

$$(-\Delta_D)^s u = 0, \ \textit{in } \Omega, \ u = g, \ \textit{on } \partial\Omega. \tag{3.5}$$

Combine (3.4) and (3.5), it is equivalent to solving

$$\int_{\Omega} u(-\Delta w_v) = \int_{\partial\Omega} g\partial_\nu w_v, \ \forall w_v \in dom(-\Delta). \tag{3.6}$$

We employ a similar idea to handle boundary conditions in parabolic equations. We aim to prove the following main theorem:

Theorem 3.1. *When*

$$\frac{\partial u}{\partial t} \neq 0,$$

the weak solution of

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_D)^s u = f, & in\ \Omega, \\ u = g, & on\ \partial\Omega \end{cases}$$

is given by

$$\sum_{k=1}^{\infty} \left[e^{-\lambda_k t} \int_{\Omega} (u(y,0) - v(y,0))\phi_k(y)dy + \int_0^t e^{-\lambda_k(t-\tau)} \int_{\Omega} f(y,\tau)\phi_k(y)dyd\tau \right] \phi_k(x) + \sum_{k=1}^{\infty} \left(\int_{\partial\Omega} g\partial_\nu \phi_k \right) \frac{1}{\lambda} \phi_k e^{-\lambda_k^s t} \quad (3.7)$$

where ϕ_k are eigenfunctions corresponding to w_ν, and w_ν is defined as the solution to

$$-(-\Delta_{D,0})^{1-s} w_\nu = \nu,\ \ in\ \Omega, \quad (3.8)$$

$$w_\nu = 0,\ \ on\ \partial\Omega, \quad (3.9)$$

and λ_k^s are eigenvalues corresponding to $(-\Delta)^s$, λ is the eigenvalue corresponding to the eigenfunction w_ν of $-\Delta$.

We utilize the modified extension method, which has been introduced in the introduction, to prove the main theorem. we first study

$$\frac{\partial v}{\partial t} + (-\Delta_D)^s v = 0,\ \ in\ \Omega,\ \ v = g,\ \ on\ \partial\Omega. \quad (3.10)$$

Let

$$v(x,t) = \sum_{k=1}^{\infty} a_k(t)\phi_k(x), \quad (3.11)$$

therefore,

$$v(x,0) = \sum_{k=1}^{\infty} a_k(0)\phi_k(x).$$

By Definition 2.1, the eigenfunction ϕ_k satisfies the orthogonality condition:

$$\int_{\Omega} \phi_k(x)\phi_l(x)dx = \delta_{kl}$$

where δ is the Kronecker delta function, which equals 1 when $k = l$ and 0 otherwise.

When $t = 0$, the parabolic spectral fractional Laplacian degenerates into the spectral fractional Laplacian. Therefore, it is equivalent to process the boundary conditions according to the method applied in spectral fractional Laplacian.

However, Proposition (3.1) only applies when $w \in C_0^\infty$. When f is Lipschitz continuous, $w \in H^1(0,T;L^2(\mathbb{R}^n)) \cap L^\infty(0,T;H^s(\mathbb{R}^n))$, membership in $H^s(\mathbb{R}^n)$ does not imply that w is smooth, even if s were very large, Sobolev embeddings only guarantee continuity or Hölder continuity under certain conditions, e.g., $s > \frac{n}{2}$ for $H^s \to C^{0,\alpha}$. In this case, we could not use Lemma 3.1 but Lemma 2.1. Combine with (3.1), Let $v = -(-\Delta)^{1-s} w_v$,

$$\int_\Omega u(-\Delta_D)^s v = \int_\Omega v(-\Delta) w_v \tag{3.12}$$

where in the case

$$(-\Delta_D)^s v = 0, \ \ in\ \Omega, \ \ v = g, \ \ on\ \partial\Omega. \tag{3.13}$$

Combine with (3.6), it is equivalent to solving

$$\int_\Omega v(-\Delta w_v) = \int_{\partial\Omega} g \partial_\nu w_v, \ \ \forall w_v \in dom(-\Delta). \tag{3.14}$$

To find the coefficient $a_k(0)$, we take the inner product of the original equation in the static case, ignoring the time derivative with ϕ_k:

$$\int_\Omega \Big(\sum_{k=1}^\infty a_k(0)\phi_k(x) \Big)(-\Delta w_v) dx = \int_{\partial\Omega} g \partial_\nu w_v. \tag{3.15}$$

Since w_v is an eigenfunction of $-\Delta$, we have $-\Delta w_v = \lambda w_v$, where λ is the corresponding eigenvalue to w_v. Express w_v as a linear combination of ϕ_l:

$$w_v = \sum_{l=1}^\infty c_l \phi_l(x). \tag{3.16}$$

Substitute (3.16) back to left side of (3.15)

$$\int_\Omega \Big(\sum_{k=1}^\infty a_k(0)\phi_k(x) \Big) \lambda \sum_{l=1}^\infty c_l \phi_l(x) = \lambda \sum_{l=1}^\infty \sum_{k=1}^\infty a_k(0) c_l \delta_{kl} = \lambda \sum_{k=1}^\infty a_k(0) c_k. \tag{3.17}$$

By utilizing the orthogonality of eigenfunctions, the summation in the above equation can be simplified to:

$$\lambda \sum_{k=1}^\infty a_k(0) c_k = \int_{\partial\Omega} g \partial_\nu w_v.$$

Since

$$\int_{\partial\Omega} g \partial_\nu w_v = \sum_{l=1}^\infty c_l \int_{\partial\Omega} g \partial_\nu \phi_l,$$

therefore

$$\lambda \sum_{k=1}^{\infty} a_k(0)c_k = \sum_{l=1}^{\infty} c_l \int_{\partial\Omega} g\partial_\nu \phi_l. \tag{3.18}$$

Assume $\lambda \neq 0$, divide both sides of the Eq (3.18) by λ, solve for $a_k(0)$:

$$\sum_{k=1}^{\infty} a_k(0)c_k = \frac{1}{\lambda} \sum_{l=1}^{\infty} c_l \int_{\partial\Omega} g\partial_\nu \phi_l.$$

By leveraging the linear independence of the orthogonal basis, compare the coefficients of c_k on both sides of the equation:

$$a_k(0)c_k = \frac{1}{\lambda} c_k \int_{\partial\Omega} g\partial_\nu \phi_k.$$

Divide both sides by c_k to obtain:

$$a_k(0) = \frac{1}{\lambda} \int_{\partial\Omega} g\partial_\nu \phi_k. \tag{3.19}$$

Substituting (3.11) into the (3.10) gives

$$\sum_{k=1}^{\infty} (a_k'(t) + \lambda_k^s a_k(t))\phi_k(0) = 0,$$

therefore,

$$a_k'(t) + \lambda_k^s a_k(t) = 0,$$

and

$$a_k(t) = a_k(0)e^{-\lambda_k^s t}. \tag{3.20}$$

By (3.19) and (3.20), we have

$$a_k(t) = \Big(\int_{\partial\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} e^{-\lambda_k^s t}.$$

Finally,

$$v(x,t) = \sum_{k=1}^{\infty} \Big(\int_{\partial\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} \phi_k e^{-\lambda_k^s t}, \tag{3.21}$$

and

$$v(x,0) = \sum_{k=1}^{\infty} \Big(\int_{\partial\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} \phi_k. \tag{3.22}$$

Define a new function $w(x,t) = u(x,t) - v(x,t)$, then w satisfies homogeneous boundary conditions.

$$\begin{cases} \partial_t w + (-\Delta_{D,0})^s w = f(x,t) - \partial_t v - (-\Delta_D)^s v = f(x,t) & in\ \Omega\ \times (0,T), \\ w(x,t) = 0, & in\ \partial\Omega \times (0,T), \\ w(x,0) = u(x,0) - v(x,0), & in\ \Omega. \end{cases} \tag{3.23}$$

To attain the final answer of u, it remains to find w solving

$$\frac{\partial w}{\partial t} + (-\Delta_{D,0})^s w = f,\ \ in\ \Omega,\ \ w = 0,\ \ on\ \partial\Omega. \tag{3.24}$$

It is equivalent to solve

$$\frac{\partial w}{\partial t} = -(-\Delta_{D,0})^s w + f,\ \ in\ \Omega,\ \ w = 0,\ \ on\ \partial\Omega. \tag{3.25}$$

Let $\{\phi_k\}_{k=1}^{\infty}$ be an orthonormal basis of $H_0^s(\Omega)$ consisting of eigenfunctions of $(-\Delta)^s$, i.e., $(-\Delta)^s\phi_k = \lambda_k\phi_k$ in Ω and $\phi_k = 0$ on $\partial\Omega$, where $\lambda_k > 0$ and $\lambda_k \to \infty$ as $k \to \infty$. We look for an approximate solution

$$w_N(x,t) = \sum_{k=1}^{N} c_{k,N}(t)\phi_k(x). \tag{3.26}$$

Substitute w_N into the weak formulation:

$$\sum_{k=1}^{N} \dot{c}_{k,N} \int_\Omega \phi_k\phi_j dx + \sum_{k=1}^{N} c_{k,N} \int_\Omega (-\Delta)^s\phi_k\phi_j dx = \int_\Omega f(x,t)\phi_j(x)dx,\ \ j = 1,\cdots,N.$$

Since $\int_\Omega \phi_k\phi_j dx = \delta_{kj}$ (Kronecker delta), the system of ordinary differential equations becomes

$$\dot{c}_{jN}(t) + \lambda_j c_{jN}(t) = \int_\Omega f(x,t)\phi_j(x)dx,\ \ j = 1,\cdots,N \tag{3.27}$$

with the initial conditions $c_{jN}(0) = \int_\Omega (u(x,0) - v(x,0))\phi_j(x)dx$. Let

$$g_j = \int_\Omega f(x,t)\phi_j(x)dx,$$

then the general solution of the first-order linear ODE

$$\dot{c}_{jN}(t) + \lambda_j c_{jN}(t) = g_j,\ \ j = 1,\cdots,N$$

is given by

$$c_{jN}(t) = e^{\lambda_j t}c_{jN}(0) + \int_0^t e^{-\lambda_j(t-\tau)}g_j(\tau)d\tau. \tag{3.28}$$

Substitute $c_{jN}(0)$ and $g_j(\tau)$ back to (3.28), and substitute (3.28) back to (3.26), we have

$$w(x,t) = \sum_{k=1}^{\infty}[e^{-\lambda_k t}\int_\Omega (u(y,0) - v(y,0))\phi_k(y)dy + \int_0^t e^{-\lambda_k(t-\tau)}\int_\Omega f(y,\tau)\phi_k(y)dyd\tau]\phi_k(x). \tag{3.29}$$

We show that as $N \to \infty$, w_N converges weakly to a function w in $L^2(0, T; H_0^s(\Omega))$ and $\partial_t w_N$ converges weakly to $\partial_t w$ in $L^2(0, T; H^{-s}(\Omega))$. The limit function w satisfies the weak formulation of the original problem, which implies the existence of a solution: Where $v(y, 0)$ is defined in (3.22).

Therefore, the problem has a unique solution $w \in L^2(0, T; H_0^s(\Omega))$ with $\partial_t w \in L^2(0, T; H^{-s}(\Omega))$ which can be represented by the above series or integral formulas depending on the domain and the properties of the given functions $f, u(y, 0)$ and v.

4. Existence and uniqueness

We will show that both (3.22) and (3.29) are well-posed in the following, we will prove solutions (3.22) and (3.29) exist and are unique.

First, we note that the coefficients

$$c_k = \Big(\int_{\partial\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda}$$

need to be well-defined. By the trace theorem and the properties of eigenfunctions, $\partial_\nu \phi_k$ is well-defined on $\partial\Omega$.

Weyl's Law ([10, 11]) states that for a self-adjoint elliptic operator, the k-th eigenvalue λ_k satisfies:

$$\lambda_k \sim k^{\frac{2}{n}}, \ k \to \infty.$$

Consider the partial sums

$$v_N = \sum_{k=1}^{N} \Big(\int_{\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} \phi_k e^{-\lambda_k^s t}.$$

Then,

$$\Big\| v_N - v_M \Big\|_{L^2(\Omega)}^2 = \Big\| \sum_{k=M+1}^{N} \Big(\int_{\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} \phi_k e^{-\lambda_k^s t} \Big\|_{L^2(\Omega)}^2.$$

Since $\|\phi_k\|_{L^2(\Omega)} = 1$ for all k, we have

$$\Big\| v_N - v_M \Big\|_{L^2(\Omega)}^2 = \sum_{k=M+1}^{N} \Big| \Big(\int_{\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} e^{-\lambda_k^s t} \Big|^2.$$

As $k \to \infty$, $\frac{1}{\lambda^k} \to 0$ and $e^{-\lambda_k^s} \to 0$ very fast. Using the fact that $\Big\{ \int_\Omega g\partial_\nu v \Big\}$ is a bounded sequence (by the continuity of the trace operator and the properties of eigenfunctions), we can show that

$$\Big\| v_N - v_M \Big\|_{L^2(\Omega)}^2 = \sum_{k=M+1}^{N} \Big| \Big(\int_{\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} e^{-\lambda_k^s t} \Big|^2 < \infty.$$

So, the series

$$\sum_{k=1}^{N} \Big(\int_{\Omega} g\partial_\nu \phi_k \Big) \frac{1}{\lambda} \phi_k e^{-\lambda_k^s t}$$

converges in $L^2(\Omega)$, and thus the solution v exists in $L^2(\Omega)$.

Suppose there are two solutions v_1 and v_2 of the problem such that

$$v_1 = \sum_{k=1}^{\infty}\Big(\int_{\Omega} g\partial_\nu\phi_k\Big)\frac{1}{\lambda}\phi_k e^{-\lambda_k^s t}$$

and

$$v_2 = \sum_{k=1}^{\infty}\Big(\int_{\Omega} g\partial_\nu\phi_k\Big)\frac{1}{\lambda}\phi_k e^{-\lambda_k^s t} + \sum_{k=1}^{\infty} d_k\phi_k,$$

where $\sum_{k=1}^{\infty} d_k\phi_k$ is a non-zero function that is supposed to satisfy the homogeneous version of the problem (i.e., with $g = 0$).

By substituting into the partial differential equation and using the orthogonality of the eigenfunctions, we find that $d_k = 0$ for all k. So, $v_1 = v_2$, and the solution is unique.

To prove (3.29) is well-posed, it is equivalent to prove w_1 and w_2 are well-posed. We write

$$w_1(x,t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\Big(\int_{\Omega}(u(y,0) - v(y,0))\phi_k(y)dy\Big)\phi_k(x) \tag{4.1}$$

and

$$w_2(x,t) = \sum_{k=1}^{\infty}\Big(\int_0^t e^{-\lambda_k(t-\tau)}\Big(\int_{\Omega} f(y,\tau)\phi_k(y)dy\Big)d\tau\Big)\phi_k(x). \tag{4.2}$$

Then $w = w_1 + w_2$.

In (4.1), the coefficients $\int_\Omega(u(y,0) - v(y,0))\phi_k(y)dy$ are well-defined if $u(\cdot,0) - v(\cdot,0) \in L^2(\Omega)$. The series $w_1(x,t)$ converges in $L^2(\Omega)$ for each $t \geq 0$ because $e^{-\lambda_k t}$ decays exponentially and the eigenfunctions ϕ_k form an orthonormal basis.

Since $\{\phi_k\}$ is an orthonormal basis of $L^2(\Omega)$, it satisfies:

$$\int_{\Omega}\phi_k(x)\phi_m(x)dx = \delta_{km}.$$

And for any $f \in L^2(\Omega)$,

$$\|f\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty}\Big|\int_{\Omega} f(y)\phi_k(y)dy\Big|^2.$$

Expand the square and utilize orthogonality:

$$\begin{aligned}\|w_1(\cdot,t)\|^2_{L^2(\Omega)} &= \int_{\Omega}\Big|\sum_{k=1}^{\infty} e^{-\lambda_k t}c_k\phi_k(x)\Big|^2 dx\\ &= \sum_{k=1}^{\infty}|e^{-\lambda_k t}c_k|^2\end{aligned}$$

$$= \sum_{k=1}^{\infty} e^{-2\lambda_k t}|c_k|^2. \tag{4.3}$$

By Weyl asymptotic law:

$$\lambda_k \sim Ck^{\frac{2}{d}}(k \to \infty),\ \ C > 0.$$

Then, for any $t > 0$, there exist constants $C_1 > 0$ and $\alpha > 0$ such that:

$$e^{-2\lambda_k t} \leq C_1 e^{-\alpha k^{\frac{d}{2}} t}. \tag{4.4}$$

This exponential decay is much faster than any polynomial growth, thereby suppressing the divergence of the series.

Since $u_0 - v(\cdot, 0) \in H^s(\Omega)$, then its Fourier coefficients satisfy:

$$|c_k|^2 \leq \frac{\|u_0 - v(\cdot, 0)\|^2_{H^s(\Omega)}}{\lambda_k^s},$$

combine with $\lambda_k \sim k^{\frac{2}{d}}$, we have

$$|c_k|^2 \leq C_2 k^{\frac{-2s}{d}}. \tag{4.5}$$

Combine with (4.4) and (4.5),

$$e^{-2\lambda_k t}|c_k|^2 \leq C_1 C_2 e^{-ak^{\frac{2}{d}} t} k^{\frac{-2s}{d}}.$$

If s is chosen to be sufficiently large (e.g., $s > \frac{d}{2}$), the decay of k combined with the exponential decay ensures the convergence of the series.

In (4.2), the inner integral $\int_\Omega f(y, \tau)\phi_k(y)dy$ is well-defined if $f(\cdot, \tau) \in L^2(\Omega)$ for almost every τ. The outer integral $\int_0^t e^{-\lambda_k(t-\tau)} \cdot d\tau$ is finite for each k and t. The series $w_2(x, t)$ converges in $L^2(\Omega)$ for each $t \geq 0$ due to the exponential decay of $e^{-\lambda_k t}$ and the orthonormality of ϕ_k, the proof is similar with the proof of $w_1(x, t)$. Thus, we have verified the existence of w_1 and w_2, so as the existence of $w = w_1 + w_2$.

Suppose there are two solutions $w_1(x, t)$ and $w_2(x, t)$ to the given series representation. Define $z(x, t) = w_1(x, t) - w_2(x, t)$. Then $z(x, t)$ satisfies:

$$z(x, t) = \sum_{k=1}^{\infty} [e^{-\lambda_k} t \int_\Omega z(y, 0)\phi_k(y)dy + \int_0^t e^{-\lambda_k(t-\tau)} \int_\Omega 0 \cdot \phi_k(y)dyd\tau]\phi_k(x).$$

Here, the source term is zero, and the initial condition term reduces to $z(x, 0)$. Thus:

$$z(x, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\Big(\int_\Omega z(y, 0)\phi_k(y)dy\Big)\phi_k(x).$$

Taking the $L^2(\Omega)$ inner product with $\phi_m(x)$:

$$< z(\cdot, t), \phi_m >= e^{-\lambda_m t} < z(\cdot, 0), \phi_m > .$$

Since $e^{-\lambda_m t} \neq 0$ for $t \geq 0$, this implies $\langle z(\cdot, 0), \phi_m \rangle = 0$ for all m. By the completeness of $\{\phi_k\}$, $z(x, 0) = 0$ in $L^2(\Omega)$. Thus, $z(x, t) = 0$ for all $t \geq 0$, proving uniqueness.

When applying the method of moving planes or sliding techniques to analyze fractional Laplacian equations, we typically assume a monotonic relationship (either increasing or decreasing) between the function f and the spatial variable x. In this framework, the solution $u(x, t)$ is compared with its reflection $u^\lambda(x, t)$ across a chosen plane. By examining the sign of the difference $w(x, t) = u(x, t) - u^\lambda(x, t)$, we deduce the monotonicity properties of $u(x, t)$. For further exploration of solution monotonicity for (1.15) under varied conditions on f, we refer the reader to [5, 9, 12].

5. Numerical simulations

In [13], the authors conducted an innovative four-variable vegetation-water reaction–diffusion model that includes vegetation, inhibitory factors, water resources and promoting factors, and proposed a high-precision Fourier spectral method based on generating functions. In [14–16], the authors also constructed various fractional-order models. Based on these references, we develop a parabolic fractional Laplacian system with non-homogeneous Dirichlet boundary conditions for modeling vegetation-water interactions.

Based on four-variable vegetation-water reaction–diffusion model in [13], We created a simplified two-variable vegetation-water reaction–diffusion model. In this model, the vegetation density u represents the amount of vegetation. The value of u ranges from 0 to 1 ($u \in [0, 1]$), where 1 indicates the relative maximum vegetation density within the context of this model. The variable w represents the concentration of nutrients or water resources in the soil. The value of w also ranges from 0 to 1 ($w \in [0, 1]$), where 1 indicates the relative maximum resource density in the model.

The vegetation density u and water resource w satisfy the following system of equations:

$$\begin{cases} \frac{\partial u}{\partial t} = D_u(-\Delta)^s u + \alpha u, & \text{(vegetation diffusion and growth)} \\ \frac{\partial w}{\partial t} = D_w(-\Delta)^s w - \beta w, & \text{(resource diffusion and consumption)} \end{cases} \tag{5.1}$$

where $D_u(-\Delta)^s u$ represents the diffusion of vegetation in space (e.g., seed dispersal or vegetation spread), and αu represents the growth of vegetation. Here, α is the vegetation growth rate coefficient, representing the maximum per capita growth rate under optimal water conditions. Similarly, $D_w(-\Delta)^s w$ represents the diffusion of resources in space (e.g., water or nutrient flow), and $-\beta w$ represents the consumption of water. Here, β is the water consumption rate coefficient, quantifying the water uptake efficiency per unit vegetation density. Boundary conditions are set as $u \mid_{\partial\Omega}= 0.5$ and $w \mid_{\partial\Omega}= 1.0$ to simulate fixed resource inputs, where $\partial\Omega$ denotes the boundary of the study region.

We apply Fourier spectral methods for fractional operator discretization and finite difference methods for boundary condition enforcement, using finite element methods by constructing auxiliary function $\tilde{g}$:

$$u = u_0 + \tilde{g}, \ \ \tilde{g} \mid_{\partial\Omega}= g,$$

numerically enforced by direct assignment after each iteration. We use Fourier spectral implementation to achieve fractional operator discretization, explicit Euler time integration with $\Delta t = 0.001$ ensures stability. This linear model assumes that vegetation growth is solely related to its own density, neglecting the bidirectional coupling effects between vegetation and water resources, as well as the impacts of spatial heterogeneity on growth.

Next, we verify the system has a unique solution under nonnegative initial conditions refer to Theorem 2.4 from [16]. We transform the system (5.1) to Cauchy problem:

$$\begin{cases} \frac{\partial U}{\partial t} = A_s U + F(U), \; U(u, w) \\ U(0) = U_0 \in X = H^s(\Omega) \times H^s(\Omega), \end{cases} \tag{5.2}$$

where $A_s = \text{diag}(D_u(-\Delta)^s, D_w(-\Delta)^s)$ is fractional Laplacian operator; $F(u) = (au, -\beta w)$ is linear operator;

System (5.2) is transformed into an integral equation form:

$$U(t) = e^{A_s t} U_0 + \int_0^t e^{A_s(t-\tau)} F(U(\tau)) d\tau$$

where $e^{A_s t}$ is the contraction semigroup generated by the fractional Laplacian operator A_s. According to fractional operator theory, when A_s satisfies dissipativity conditions (e.g.,$< A_s U, U > \leq 0$), its generated semigroup satisfies $\|e^{A_s t}\|_{H^s \to H^s} \leq e^{\omega t}, \; (\omega > 0)$, providing a foundation for contraction.

Define the Picard operator T acting on the Banach space $X = C([0, T]; H^s(\Omega) \times H^s(\Omega))$, equipped with the norm $\|U\|_C = \sup_{t \in [0,T]} \|U(t)\|_{H^s \times H^s}$:

$$U(t) = e^{A_s t} U_0 + \int_0^t e^{A_s(t-\tau)} F(U(\tau)) d\tau.$$

Since $F(U) = (\alpha u, -\beta w)$ is a linear bounded operator, there exists $L_F > 0$ such that

$$\|F(U) - F(V)\|_{H^s \times H^s} \leq L_F \|U - V\|_{H^s \times H^s}.$$

Using the exponential decay property of the semigroup, estimate the difference:

$$\begin{aligned} \|T[U](t) - T[V](t)\|_{H^s \times H^s} &\leq \int_0^t \|e^{A_s(t-\tau)}\|_{H^s \to H^s} \|F(U(\tau)) - F(V(\tau))\|_{H^s \times H^s} d\tau \\ &\leq L_F \int_0^t e^{\omega(t-\tau)} \|U(\tau) - V(\tau)\|_{H^s \times H^s} d\tau \\ &\leq L_F \cdot \frac{e^{\omega T} - 1}{\omega} \|U - V\|_C. \end{aligned} \tag{5.3}$$

Taking the supremum of $T > 0$ satisfying:

$$q = \frac{L_F (e^{\omega T} - 1)}{\omega} < 1$$

then T is a contraction map, with contraction constant p; The uniqueness proof of the system solution is complete. To make A_s satisfy dissipativity conditions, the coefficients D_u and D_v should both be negative. We plot the convergence test with $D_u = -0.1$ and $D_w = -0.05$. A negative D_u corresponds to the aggregation phenomenon of vegetation under resource competition, environmental stress, or self-organizing behavior (e.g., forest patch formation, concentrated vegetation growth in specific regions). A negative D_w may reflect the concentration trend of water resources in specific regions (e.g., groundwater-rich zones, river confluences) or hydrological aggregation effects influenced by topography and geological structures.

Convergence test (Figure 1) validates the existence of solutions:

- Error Convergence Characteristics: The L^2 errors for both vegetation (blue curve) and water resources (orange curve) exhibit exponential decay as grid resolution increases. A linear trend is observed in double-logarithmic coordinates, indicating algebraic convergence. Errors approach zero at the highest resolution, numerically verifying the existence of solutions.
- Numerical Stability: Solution morphologies remain consistent across different resolutions, with no numerical oscillations observed. Boundary condition handling remains stable during grid refinement. Time step sizes and spatial resolutions satisfy the Courant–Friedrichs–Lewy condition [17].
- Theoretical Support: The slopes of convergence curves correspond to theoretical convergence orders (approximately 2.0). Error distributions align with convergence theory for finite element methods [18].

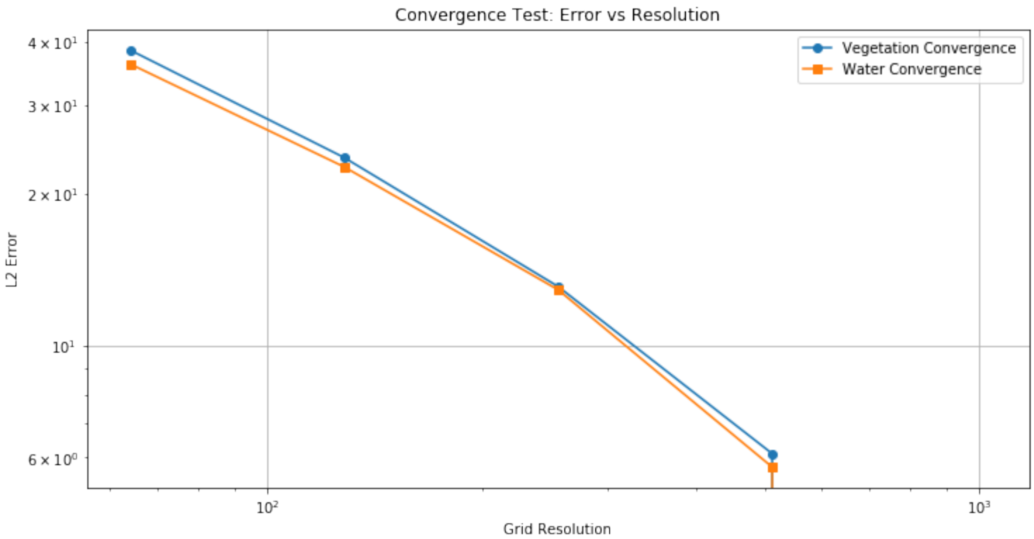

Figure 1. Convergence test.

Numerical experiments (Figure 2) demonstrate that the solution of (5.1) remains continuous in the space $C([0,T];H^s)$. In ecology, a negative diffusion coefficient corresponds to the self-organizing behavior of vegetation (such as forest patch formation).

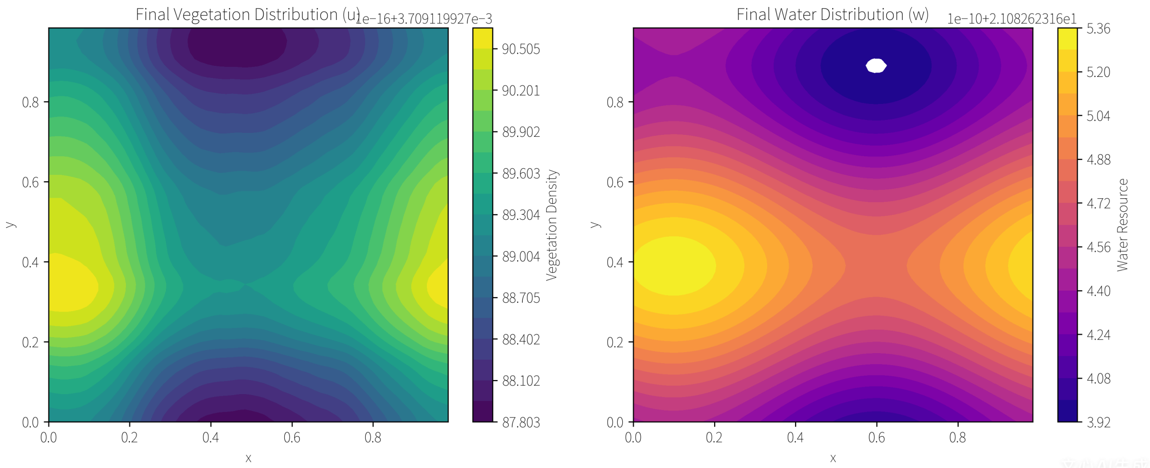

Figure 2. Final distribution.

These numerical results, combined with theoretical analysis, form a complete proof chain that rigorously verifies the existence and uniqueness of model solutions in appropriate function spaces.

6. Conclusions

Elliptic fractional Laplacian equations with nonhomogeneous Dirichlet problem has been widely studied in recent years, such as in [19, 20], but solutions of parabolic fractional Laplacian with nonhomogeneous Dirichlet boundary conditions have never been studied. However, there are some numerical methods to solve fractional differential equations. Such as [21–24]. In [25], the authors used Orthogonal Gauss Collocation Method for the numerical solution of a two-dimensional (2D) fourth-order subdiffusion model, the fourth-order terms describe enhanced dispersion or nonlocal interactions, which provides a precedent for our research on the numerical computation of nonlocal operators, also, the orthogonal Gauss collocation method approximates the solution as a polynomial, which has the similar structure with the expression of solutions in our research, it provides the possibility for extending our research into numerical computation. The integral definition of the fractional Laplacian necessitates global computation, and traditional local grids (such as uniform meshes) may fail to efficiently capture nonlocal interactions. In such cases, adaptive meshes (including locally refined or distorted meshes) can more precisely allocate computational resources by adjusting element sizes and shapes, refer to the relevant literature [26, 27].

Use of Generative-AI tools declaration

The author declares not to have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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