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Research article

On stability and convergence of a novel iterative method for fixed point problems

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Abstract: Regarded as a cornerstone of mathematics, the fixed-point (fp) explores invariant outcomes under defined operators, thus offering powerful tools for problems that arise in mathematics, physics, engineering, computer science, and economics. This paper presents a novel iterative method to approximate the fps of non-expansive maps in Banach spaces (BSs). We investigate the stability of the proposed method and provide its convergence analysis. A numerical example further illustrates its performance in comparison to existing iterations. Consequently, we theoretically and numerically prove that our new iterative algorithm converges faster than some leading iterative algorithms in the literature for non-expansive maps. Hence, our results generalize and improve several well-known results in the existing literature.

Keywords: non-expansive map; iterative methods; fixed point problems; weak stability; convergence rate
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1. Introduction

Let $\mathcal{B}$ be a Banach space (BS). A map $U : \mathcal{B} \to \mathcal{B}$ is said to be a Lipschitzian map with the constant $\eta > 0$ if for all $x, y \in \mathcal{B}$, we have the following:

$$\|Ux - Uy\| \leq \eta \cdot \|x - y\|.$$

If $\eta \in [0, 1)$, then U is called a contraction map, and if $\eta = 1$, then U is called a non-expansive map. Banach [1] established the first fundamental fixed-point (fp) theorem in 1922, and proved that every contraction map on a complete metric space has a unique fp, which can be approximated via the Picard iteration. After Banach's fp theorem, it took four decades to prove the first fp theorem for non-expansive maps. In 1965, Browder [2, 3], Göhde [4], and Kirk [5] independently established fp theorems for non-expansive maps in BSs. Browder and Göhde used a uniformly convex Banach space (UCBS) to prove the fp theorem, while Kirk used a BS with a normal structure.

The existence theorem guarantees the presence of an fp but does not provide a constructive method to find it. It is well known that the Picard iteration fails to converge in the case of non-expansive maps. To overcome this difficulty, it was shown that the Mann iteration [6] converges to a fp of a non-expansive map. Motivated by the desire to achieve both favorable behaviors of iterative processes and a faster convergence, various iterative schemes have been proposed in the literature. Some of these include Ishikawa [7], Noor [8], Agarwal et al. [9], Abbas and Nazir [10], Thakur et al. [11], M^* [12], M [13], K [14], and K^* [15] iterations, among others. In 2019, Piri et al. [16] introduced a faster iteration process to approximate the fps of generalized α-nonexpansive maps in BSs. In 2020, Garodia and Uddin [17] proposed a new iteration process to solve the split feasibility problem. In recent years, some researchers have extended iterative methods beyond BSs to more general frameworks such as hyperbolic spaces. Şahin and Öztürk [18] studied the KF-iteration process in hyperbolic spaces and obtained new fp results that demonstrated the convergence and stability of the method. Subsequently, Şahin and Kalkan [19] modified the AA-iterative algorithm into hyperbolic spaces and applied it to the solution of integral equations on time scales. These investigations emphasize the growing interest in iterative methods within hyperbolic space settings and highlight the need to develop more efficient iterative processes with enhanced convergence properties.

Iterative methods are fundamental tools with wide-ranging applications in mathematics, science, and engineering. They play an important role in numerical simulations [20], image processing [21], training artificial intelligence and machine learning algorithms [22], solving nonlinear equations [23], analyzing integral equations [24], optimizing control systems [25], modeling renewable energy devices [26], and improving robotic control [27]. In 2024, Saif et al. [28] and Almarri et al. [29] used fp iterations to solve boundary value problems. These studies collectively show that iterative approaches are essential for both theoretical investigations and practical problem-solving across diverse fields.

Motivated by the aforementioned papers, we introduce a new iterative method defined as follows:

$$\begin{cases} u_0 \in H, \\ w_k = U((1-a_k)u_k + a_k U u_k), \\ v_k = U((1-b_k)U u_k + b_k U w_k), \\ u_{k+1} = U((1-c_k)v_k + c_k U v_k), \quad \forall k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where H is a non-empty, closed, and convex subset of a BS $\mathcal{B}$, and $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ are real sequences in $[0, 1]$. It is worth mentioning that this iterative method is reduced to the iteration introduced by Garodia and Uddin [17] when $c_k = 0$ for all $k \geq 0$.

In this paper, we study the stability and convergence of the iterative method defined by (1.1). Furthermore, we investigate the convergence behavior of the proposed iterative scheme. Additionally, a numerical example is provided to compare the performance of this iterative method with other iterative schemes. Our results show that the new iteration process (1.1) generalizes the Garodia and Uddin iterative algorithm and provides faster convergence to the fp than the Garodia and Uddin iteration and some other iterations in the literature.

2. Preliminaries

In this section, we include essential definitions and results that will serve as foundational tools for the primary discussions in our main section.

Definition 2.1. *[30, Definition 1] A BS $\mathcal{B}$ is formally defined as uniformly convex if, for each $\epsilon \in (0, 2]$, one can find a corresponding $\delta(\epsilon) > 0$ such that the following holds:*
whenever $x, y \in \mathcal{B}$ satisfy $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geq \epsilon$, it must be true that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

Example 2.2. *[30] (i) Euclidean spaces of all dimensions, Hilbert spaces, and hyper-Hilbert spaces are all uniformly convex.*
(ii) For $p > 1$, the well-known spaces L_p and l_p are uniformly convex.

Lemma 2.3. *[31] A BS $\mathcal{B}$ is uniformly convex if and only if for each number $R > 0$, there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(s) = 0 \Leftrightarrow s = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\psi(\|x - y\|),$$

for all $\lambda \in [0, 1]$, and all $x, y \in \mathcal{B}$ such that $\|x\| \leq R$ and $\|y\| \leq R$.

Lemma 2.4. *[32, Lemma 1.3] Let $\mathcal{B}$ be a UCBS, and $\{t_k\}$ be a sequence in $[e, f]$ for some $e, f \in (0, 1)$ and all $k \geq 0$. If the sequences $\{u_k\}$ and $\{v_k\}$ in $\mathcal{B}$ satisfy $\limsup_{k\to\infty} \|u_k\| \leq R$, $\limsup_{k\to\infty} \|v_k\| \leq R$, and $\lim_{k\to\infty} \|t_k u_k + (1 - t_k)v_k\| = R$ for some $R \geq 0$, then $\lim_{k\to\infty} \|u_k - v_k\| = 0$.*

Definition 2.5. *[33, p. 136] A BS $\mathcal{B}$ is defined to satisfy Opial's condition if, for any sequence $\{u_k\}$ in $\mathcal{B}$ that weakly converges to an element $x \in \mathcal{B}$ (denoted as $u_k \rightharpoonup x$), then the following inequality holds:*

$$\limsup_{k\to\infty} \|u_k - x\| < \limsup_{k\to\infty} \|u_k - y\|,$$

for all $y \in \mathcal{B}$ such that $y \neq x$.

Example 2.6. *[33] (i) Every Hilbert space satisfies the Opial condition.*
(ii) Let $\mathcal{B} = \mathbb{R} \times l_2$ with the norm $\|(a, y)\| = \max\{|a|, \|y\|_2\}$, where $\|y\|_2 = (\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}$. Then, $\mathcal{B}$ does not satisfy the Opial condition, even though $\mathbb{R}$ and l_2 satisfy it.

Definition 2.7. *[33, Definition 5.2.8] A map $U : H \to \mathcal{B}$ is said to be demiclosed at $y \in \mathcal{B}$ if, for every sequence $\{u_k\}$ in H, then the conditions $u_k \rightharpoonup x \in H$ and $Uu_k \to y$ together imply that $Ux = y$.*

Lemma 2.8. *[34, Theorem 4.1] Let H be a non-empty closed convex subset of a BS $\mathcal{B}$ with Opial's condition, and U be a non-expansive self-map on H. If a sequence $\{u_k\}$ weakly converges to a point z, and the condition $\lim_{k\to\infty} \|u_k - Uu_k\| = 0$ holds, then $Uz = z$, that is, the operator $I - U$ is demiclosed at zero, where I is the identity map on $\mathcal{B}$.*

Let H be a non-empty closed convex subset of a BS $\mathcal{B}$, and consider a bounded sequence $\{u_k\} \subset H$. For $x \in H$, define the following:

$$r(x, \{u_k\}) = \limsup_{k\to\infty} \|u_k - x\|.$$

The asymptotic radius of $\{u_k\}$ with respect to the set H is given by the following:

$$r(H, \{u_k\}) = \inf\{r(x, \{u_k\}) : x \in H\}.$$

Correspondingly, the asymptotic center of the sequence relative to H, denoted by $A(H, \{u_k\})$, is the set of points in H where this asymptotic radius is attained:

$$A(H, \{u_k\}) = \{x \in H : r(x, \{u_k\}) = r(H, \{u_k\})\}.$$

In a UCBS, $A(H, \{u_k\})$ is a singleton, that is, it contains exactly one element (see [33, Theorem 3.1.6]).

The following result will be the key to deducing the convergence rates for the iterative method (1.1).

Lemma 2.9. *[35, Lemma 3.2] Let $\{\beta_k\}$ and $\{\gamma_k\}$ be sequences of positive numbers. Assume the following conditions hold:*

(i) $\{\beta_k\}$ is non-summable (i.e., $\sum_{k=0}^{\infty} \beta_k = \infty$);
(ii) $\{\gamma_k\}$ is decreasing and;
(iii) $\sum_{i=0}^{\infty} \beta_i \gamma_i$ converges (i.e., $\sum_{i=0}^{\infty} \beta_i \gamma_i < \infty$).

Then, we have the following:

$$\gamma_k = o\Big(1/\sum_{i=0}^{k} \beta_i\Big),$$

where the notation o means that $s_k = o(1/t_k)$ iff $\lim_{k\to\infty} s_k t_k = 0$ for positive sequences $\{s_k\}$ and $\{t_k\}$.

3. Stability result

An iterative process is said to be numerically stable if slight disturbances, which arise from approximations and rounding errors, only lead to minor deviations in the approximate value of the fp computed by the method. In 1988, Harder and Hicks [36] gave the following formal definition of stability and proved the stability results of the Picard, Mann, and Kirk iteration methods under various contractive conditions.

Definition 3.1. *Let U be a self-map on a BS $\mathcal{B}$, and $\{u_k\}$ be an iterative sequence produced by the map U such that*

$$\begin{cases} u_0 \in \mathcal{B}, \\ u_{k+1} = \Upsilon(U, u_k), \end{cases} \tag{3.1}$$

where u_0 is a given starting point, and Υ is a function that defines the iteration. Assume that $\{u_k\}$ strongly converges to $\xi^* \in F_U$, where F_U denotes the set of all fps of U. For any sequence $\{x_k\} \subset \mathcal{B}$, if

$$\lim_{k\to\infty} \|x_{k+1} - \Upsilon(U, x_k)\| = 0 \text{ implies that } \lim_{k\to\infty} x_k = \xi^*,$$

then $\{u_k\}$ is said to be stable with respect to U.

Definition 3.2. *[37] Consider two sequences, $\{u_k\}$ and $\{x_k\}$, in a BS $\mathcal{B}$. The sequences are said to be equivalent if $\lim_{k\to\infty} \|u_k - x_k\| = 0$.*

In 2010, Timiş [38] provided the following definition of weak stability.

Definition 3.3. *Let $\mathcal{B}$ be a BS, U be a self map on $\mathcal{B}$, and $\{u_k\} \subset \mathcal{B}$ be the iterative sequence (3.1) which strongly converges to $\xi^* \in F_U$. For any equivalent sequence $\{x_k\} \subset \mathcal{B}$ of $\{u_k\}$, if*

$$\lim_{k\to\infty} \|x_{k+1} - \Upsilon(U, x_k)\| = 0 \textit{ implies that } \lim_{k\to\infty} x_k = \xi^*,$$

then the iterative sequence $\{u_k\}$ is said to be weak w^2-stable with respect to U.

Theorem 3.4. *Let H be a non-empty, closed, and convex subset of a BS $\mathcal{B}$, $U : H \to H$ be a non-expansive map with $F_U \neq \emptyset$, and $\{u_k\}$ be the iterative sequence (1.1) which strongly converges to $\xi^* \in F_U$. Then, the iteration process (1.1) is weak w^2-stable with respect to U.*

Proof. Let $\{x_k\}$ be the equivalent sequence of $\{u_k\}$. Set

$$\varepsilon_k = \|x_{k+1} - \Upsilon(U, x_k)\|.$$

Suppose that $lim_{k\to\infty}\varepsilon_k = 0$. Since U is non-expansive, we obtain

$$\begin{aligned}
\|u_{k+1} - \Upsilon(U, x_k)\| &= \|U((1-c_k)v_k + c_kUv_k) - U((1-c_k)y_k + c_kUy_k)\| \\
&\le \|((1-c_k)v_k + c_kUv_k) - ((1-c_k)y_k + c_kUy_k)\| \\
&\le (1-c_k)\|v_k - y_k\| + c_k\|Uv_k - Uy_k\| \\
&\le (1-c_k)\|v_k - y_k\| + c_k\|v_k - y_k\| \\
&= \|v_k - y_k\|,
\end{aligned}$$

and

$$\begin{aligned}
\|v_k - y_k\| &= \|U((1-b_k)Uu_k + b_kUw_k) - U((1-b_k)Ux_k + b_kUz_k)\| \\
&\le \|((1-b_k)Uu_k + b_kUw_k) - ((1-b_k)Ux_k + b_kUz_k)\| \\
&\le (1-b_k)\|Uu_k - Ux_k\| + b_k\|Uw_k - Uz_k\| \\
&\le (1-b_k)\|u_k - x_k\| + b_k\|w_k - z_k\| \\
&= (1-b_k)\|u_k - x_k\| + b_k\|U((1-a_k)u_k + a_kUu_k) - U((1-a_k)x_k + a_kUx_k)\| \\
&\le (1-b_k)\|u_k - x_k\| + b_k\{(1-a_k)\|u_k - x_k\| + a_k\|u_k - x_k\|\} \\
&= (1-b_k)\|u_k - x_k\| + b_k\|u_k - x_k\| \\
&= \|u_k - x_k\|.
\end{aligned}$$

Hence, we obtain the following:

$$\|u_{k+1} - \Upsilon(U, x_k)\| \le \|u_k - x_k\|.$$

Then,

$$\begin{aligned}
\|x_{k+1} - \xi^*\| &\le \|x_{k+1} - \Upsilon(U, x_k)\| + \|\Upsilon(U, x_k) - u_{k+1}\| + \|u_{k+1} - \xi^*\| \\
&\le \varepsilon_k + \|u_k - x_k\| + \|u_{k+1} - \xi^*\|.
\end{aligned}$$

From the hypothesis of the theorem, we know that $lim_{k\to\infty}\|u_k - \xi^*\| = 0$. Moreover, we have $\lim_{k\to\infty}\|u_k - x_k\| = 0$ because $\{u_k\}$ and $\{x_k\}$ are equivalent sequences. Now, taking the limit as $k \to \infty$ on both sides of the above inequality and then using the assumption $lim_{k\to\infty}\varepsilon_k = 0$, we obtain the following:

$$\lim_{k\to\infty} \|x_{k+1} - \xi^*\| = 0.$$

Thus, $\{u_k\}$ is weak w^2-stable with respect to U. □

Example 3.5. *Let $\mathcal{B} = \mathbb{R}$, $H = [-0.5, 0.5]$, and $U : H \to H$ be a map defined by $Ux = x^2$. Clearly, we can see that U is a non-expansive map. Let us take $a_k = b_k = c_k = \frac{1}{2}$ in* (1.1)*. To show* (1.1) *is weak w^2−stable, let us take the sequence $x_k = \frac{1}{k}$. Clearly, we can verify that $\{x_k\}$ is equivalent to $\{u_k\}$. We can see that*

$$\lim_{k\to\infty} \|x_{k+1} - \Upsilon(U, x_k)\| = 0 \textit{ implies that } \lim_{k\to\infty} x_k = 0.$$

Therefore, the iteration (1.1) *is weak w^2-stable with respect to U.*

4. Convergence theorems

First, we prove two preliminary results as follows.

Theorem 4.1. *Let U be a non-expansive map defined on a non-empty, closed, and convex subset H of a BS $\mathcal{B}$ with $F_U \neq \emptyset$. If $\{u_k\}$ is the iterative sequence defined by (1.1), then $\lim_{k\to\infty} \|u_k - \xi^*\|$ exists for all $\xi^* \in F_U$.*

Proof. Let $\xi^* \in F_U$. Then, using (1.1), we obtain the following:

$$\begin{aligned} \|w_k - \xi^*\| &= \|U((1 - a_k)u_k + a_k U u_k) - \xi^*\| \\ &\leq \|(1 - a_k)u_k + a_k U u_k - \xi^*\| \\ &\leq (1 - a_k)\|u_k - \xi^*\| + a_k\|U u_k - \xi^*\| \\ &\leq (1 - a_k)\|u_k - \xi^*\| + a_k\|u_k - \xi^*\| \\ &= \|u_k - \xi^*\|. \end{aligned} \tag{4.1}$$

Furthermore,

$$\begin{aligned} \|v_k - \xi^*\| &= \|U((1 - b_k)U u_k + b_k U w_k) - \xi^*\| \\ &\leq \|(1 - b_k)U u_k + b_k U w_k - \xi^*\| \\ &\leq (1 - b_k)\|U u_k - \xi^*\| + b_k\|U w_k - \xi^*\| \\ &\leq (1 - b_k)\|u_k - \xi^*\| + b_k\|w_k - \xi^*\| \\ &\leq (1 - b_k)\|u_k - \xi^*\| + b_k\|u_k - \xi^*\| \\ &= \|u_k - \xi^*\|. \end{aligned} \tag{4.2}$$

Now,

$$\begin{aligned} \|u_{k+1} - \xi^*\| &= \|U((1 - c_k)v_k + c_k U v_k) - \xi^*\| \\ &\leq \|(1 - c_k)v_k + c_k U v_k - \xi^*\| \\ &\leq (1 - c_k)\|v_k - \xi^*\| + c_k\|U v_k - \xi^*\| \\ &\leq (1 - c_k)\|v_k - \xi^*\| + c_k\|v_k - \xi^*\| \\ &= \|v_k - \xi^*\| \\ &\leq \|u_k - \xi^*\|. \end{aligned} \tag{4.3}$$

From (4.3), we can see that $\{\|u_k - \xi^*\|\}$ is a non-increasing and bounded below sequence. Hence, $\lim_{k\to\infty} \|u_k - \xi^*\|$ exists. □

Theorem 4.2. *Let U be a non-expansive map defined on a non-empty, closed, and convex subset H of a UCBS $\mathcal{B}$, and $\{u_k\}$ be the iterative sequence defined by (1.1) with the real sequence $\{a_k\} \subset [e, f]$ for some $e, f \in (0, 1)$ and all $k \geq 0$. Then, $F_U \neq \emptyset$ iff $\{u_k\}$ is bounded and $\lim_{k\to\infty} \|u_k - Uu_k\| = 0$.*

Proof. Let $\xi^* \in F_U$. Then, by Theorem 4.1, $\lim_{k\to\infty} \|u_k - \xi^*\|$ exists. Let

$$\lim_{k\to\infty} \|u_k - \xi^*\| = R.$$

Then

$$\limsup_{k\to\infty} \|Uu_k - \xi^*\| \leq R.$$

Using (4.1) and (4.2), we have

$$\limsup_{k\to\infty} \|w_k - \xi^*\| \leq R \tag{4.4}$$

and

$$\limsup_{k\to\infty} \|v_k - \xi^*\| \leq R, \tag{4.5}$$

respectively. Since

$$\|u_{k+1} - \xi^*\| \leq \|v_k - \xi^*\|,$$

we get

$$R \leq \liminf_{k\to\infty} \|v_k - \xi^*\|. \tag{4.6}$$

From (4.5) and (4.6), we obtain the following:

$$\lim_{k\to\infty} \|v_k - \xi^*\| = R.$$

Now

$$\|v_k - \xi^*\| \leq (1 - b_k)\|u_k - \xi^*\| + b_k\|w_k - \xi^*\|;$$

thus,

$$\|v_k - \xi^*\| - \|u_k - \xi^*\| \leq b_k\{\|w_k - \xi^*\| - \|u_k - \xi^*\|\} \leq \|w_k - \xi^*\| - \|u_k - \xi^*\|.$$

Therefore,

$$\|v_k - \xi^*\| \leq \|w_k - \xi^*\|.$$

Hence, we obtain the following:

$$R \leq \liminf_{k\to\infty} \|w_k - \xi^*\|. \tag{4.7}$$

Using (4.4) and (4.7), we obtain the following:

$$\lim_{k\to\infty} \|w_k - \xi^*\| = R.$$

Since

$$\|w_k - \xi^*\| \leq \|(1 - a_k)u_k + a_k Uu_k - \xi^*\| \leq \|u_k - \xi^*\|,$$

which gives

$$\lim_{k\to\infty} \|(1-a_k)u_k + a_k U u_k - \xi^*\| = R,$$

then, by Lemma 2.4, we obtain the following:

$$\lim_{k\to\infty} \|u_k - U u_k\| = 0.$$

Conversely, suppose that $\{u_k\}$ is bounded and $\lim_{k\to\infty} \|u_k - U u_k\| = 0$. Let $\xi^* \in A(H, \{u_k\})$. Then, we have the following:

$$\begin{aligned} r(U\xi^*, \{u_k\}) &= \limsup_{k\to\infty} \|u_k - U\xi^*\| \\ &\leq \limsup_{k\to\infty} \|u_k - U u_k\| + \limsup_{k\to\infty} \|U u_k - U\xi^*\| \\ &\leq \limsup_{k\to\infty} \|u_k - \xi^*\| = r(\xi^*, \{u_k\}). \end{aligned}$$

This implies that $U\xi^* \in A(H, \{u_k\})$. Since $\mathcal{B}$ is uniformly convex, then $A(H, \{u_k\})$ will be singleton. Therefore, we obtain $U\xi^* = \xi^*$, that is, $F_U \neq \emptyset$. □

Now, we are ready for the weak and strong convergence theorems of our iteration method in a BS.

Theorem 4.3. *Let $\mathcal{B}$, H, U, and $\{u_k\}$ be as in Theorem 4.2 and $F_U \neq \emptyset$. If $\mathcal{B}$ satisfies Opial's condition, then $\{u_k\}$ weakly converges to a fp of U.*

Proof. Let $\xi^* \in F_U$. It follows from Theorem 4.1 that $\lim_{k\to\infty} \|u_k - \xi^*\|$ exists. Consider the sequence $\{u_k\}$ defined by (1.1), which is assumed to weakly converge to both u and v. By Theorem 4.2, we have $\lim_{k\to\infty} \|u_k - U u_k\| = 0$. Furthermore, utilizing Lemma 2.8, it is established that the operator $I - U$ is demiclosed at zero. These results collectively imply that $u, v \in F_U$.

Next, we proceed to demonstrate the uniqueness of this weak limit. Since $u, v \in F_U$, the limits $\lim_{k\to\infty} \|u_k - u\|$ and $\lim_{k\to\infty} \|u_k - v\|$ must exist. For contradiction, assume that $u \neq v$. Then, applying Opial's condition, we arrive at the following contradiction:

$$\begin{aligned} \lim_{k\to\infty} \|u_k - u\| &= \lim_{j\to\infty} \|u_{k_j} - u\| \\ &< \lim_{j\to\infty} \|u_{k_j} - v\| \\ &= \lim_{k\to\infty} \|u_k - v\| \\ &= \lim_{l\to\infty} \|u_{k_l} - v\| \\ &< \lim_{l\to\infty} \|u_{k_l} - u\| \\ &= \lim_{k\to\infty} \|u_k - u\|. \end{aligned}$$

This logical inconsistency implies that our initial assumption $u \neq v$ must be false. Therefore, we conclude that $u = v$. This proves that the sequence $\{u_k\}$ weakly converges to a unique fp of U. □

Theorem 4.4. *Let $\mathcal{B}$, H, U, and $\{u_k\}$ satisfy the hypotheses of Theorem 4.3. Then, $\{u_k\}$ strongly converges to a point of F_U iff* $\liminf_{k\to\infty} d(u_k, F_U) = 0$, *where* $d(x, F_U) = \inf\{\|x - \xi^*\| : \xi^* \in F_U\}$.

Proof. The proof is straightforward and can be directly derived from Theorem 4.4 of Thakur et al. [39]. □

Next, we give the final strong convergence result using condition (A) defined in [40].

Theorem 4.5. *Let the assumptions of Theorem 4.3 hold. If U satisfies the condition (A), that is, there exists a non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(s) = 0 \Leftrightarrow s = 0$ such that $\|x - Ux\| \geq \psi(d(x, F_U))$ for all $x \in H$, then $\{u_k\}$ strongly converges to a point in F_U.*

Proof. The proof follows the lines of Theorem 8 of Abbas and Nazir [10]. □

5. Convergence rate

In this section, we focus on analyzing the convergence rate of $\{u_k\}$. Throughout, we use the quantity $\|(I - U)u_k\|$ as a measure of the convergence rate since $\|(I - U)x\| = 0$ iff $Ux = x$, and the property $\lim_{k\to\infty} \|(I - U)u_k\| = 0$ always holds when $F_U \neq \emptyset$. Using Lemma 2.9, we show that $\|(I - U)u_k\|$ converges to zero at a rate of $o(\frac{1}{\sqrt{\beta_k}})$ when $\sum_{k=0}^{\infty} \beta_k = \infty$ (i.e., $\lim_{k\to\infty} \sqrt{\beta_k}\|(I - U)u_k\| = 0$).

Theorem 5.1. *Let H be a non-empty, closed, and convex subset of a UCBS $\mathcal{B}$, and $U : H \to H$ be a non-expansive map with $F_U \neq \emptyset$. Let $\{u_k\}$ be defined by the iteration process (1.1), where $u_0 \in H$, $\beta_k = \sum_{j=0}^{k} a_j b_j c_j (1 - a_j)$ for $k \in \mathbb{N}$, and $\sum_{k=0}^{\infty} \beta_k = \infty$. Then, the convergence rate estimate*

$$\|(I - U)u_k\| = o\left(\frac{1}{\sqrt{\beta_k}}\right)$$

holds (i.e., $\lim_{k\to\infty} \sqrt{\beta_k}\|(I - U)u_k\| = 0$).

Proof. Let $\xi^* \in F_U$. Then, by Lemma 2.3, we obtain the following:

$$\begin{aligned}
\|u_{k+1} - \xi^*\|^2 &= \|U((1 - c_k)v_k + c_k U v_k) - \xi^*\|^2 \\
&\leq \|(1 - c_k)v_k + c_k U v_k - \xi^*\|^2 \\
&\leq (1 - c_k)\|v_k - \xi^*\|^2 + c_k\|U v_k - \xi^*\|^2 - c_k(1 - c_k)\|v_k - U v_k\|^2 \\
&\leq (1 - c_k)\|v_k - \xi^*\|^2 + c_k\|U v_k - \xi^*\|^2 \\
&\leq (1 - c_k)\|u_k - \xi^*\|^2 + c_k\|v_k - \xi^*\|^2 \\
&= (1 - c_k)\|u_k - \xi^*\|^2 + c_k\|U((1 - b_k)U u_k + b_k U w_k) - \xi^*\|^2 \\
&\leq (1 - c_k)\|u_k - \xi^*\|^2 + c_k\|(1 - b_k)U u_k + b_k U w_k - \xi^*\|^2 \\
&\leq (1 - c_k)\|u_k - \xi^*\|^2 + c_k[(1 - b_k)\|U u_k - \xi^*\|^2 + b_k\|U w_k - \xi^*\|^2 - b_k(1 - b_k)\|U u_k - U w_k\|^2] \\
&\leq (1 - c_k)\|u_k - \xi^*\|^2 + c_k(1 - b_k)\|u_k - \xi^*\|^2 + c_k b_k\|w_k - \xi^*\|^2 \\
&= (1 - c_k b_k)\|u_k - \xi^*\|^2 + c_k b_k\|U((1 - a_k)u_k + a_k U u_k) - \xi^*\|^2 \\
&\leq (1 - c_k b_k)\|u_k - \xi^*\|^2 + c_k b_k\|(1 - a_k)u_k + a_k U u_k - \xi^*\|^2 \\
&\leq (1 - c_k b_k)\|u_k - \xi^*\|^2 + c_k b_k[(1 - a_k)\|u_k - \xi^*\|^2 + a_k\|U u_k - \xi^*\|^2 - a_k(1 - a_k)\|u_k - U u_k\|^2] \\
&\leq (1 - c_k b_k)\|u_k - \xi^*\|^2 + c_k b_k\|u_k - \xi^*\|^2 - a_k c_k b_k(1 - a_k)\|u_k - U u_k\|^2 \\
&= \|u_k - \xi^*\|^2 - a_k b_k c_k(1 - a_k)\|u_k - U u_k\|^2;
\end{aligned}$$

thus,

$$a_kb_kc_k(1-a_k)\|u_k - Uu_k\|^2 \le \|u_k - \xi^*\|^2 - \|u_{k+1} - \xi^*\|^2.$$

This gives $\lim_{k\to\infty} \sqrt{a_kb_kc_k(1-a_k)}\|u_k - Uu_k\| = 0$. Now, by taking the summation from 0 to l, we obtain the following:

$$\sum_{k=0}^{l} a_kb_kc_k(1-a_k)\|u_k - Uu_k\|^2 \le \|u_0 - \xi^*\|^2.$$

Taking the limit as $l \to \infty$ on the above inequality, we see that

$$\sum_{k=0}^{\infty} a_kb_kc_k(1-a_k)\|u_k - Uu_k\|^2 < \infty.$$

Additionally, we know $\sum_{k=0}^{\infty}\beta_k = \infty$ from the hypothesis. Since the assumptions of Lemma 2.9 hold with $\beta_k = a_kb_kc_k(1-a_k)$ and $\gamma_k = \|(I-U)u_k\|^2$, then we obtain the following:

$$\|(I-U)u_k\|^2 = o\Big(\frac{1}{\beta_k}\Big).$$

Hence, we can conclude that

$$\|(I-U)u_k\| = o\Big(\frac{1}{\sqrt{\beta_k}}\Big).$$

□

Remark 5.2. *Since $\beta_k = a_kb_kc_k(1-a_k)$ in our iteration while $\beta_k = a_kb_k(1-a_k)$ in the Garodia and Uddin [17] iteration, the presence of the extra parameter c_k concludes a faster rate of convergence for our scheme.*

6. A numerical example

Example 6.1. *Let $\mathcal{B} = \mathbb{R}$, $H = [1, 400]$, and $U : H \to H$ be a map defined by $Ux = \sqrt[3]{311x^2 + 51x + 100}$. First, we will show that U is a non-expansive map. Observe that the function $f(x) = \sqrt[3]{311x^2 + 51x + 100} - x$, $\forall x \in [1, 400]$ has the derivative*

$$f'(x) = \frac{1}{3}\Big(\frac{1}{(311x^2 + 51x + 100)^{2/3}}\Big)(622x + 51) - 1,$$

for all $x \in [1, 400]$. Since $x \ge 1$, we have $\frac{1}{3}\Big(\frac{1}{(311x^2+51x+100)^{2/3}}\Big)(622x + 51) \le 1$; hence,

$$f'(x) \le 0,$$

for all $x \in [1, 400]$. This means that f is a decreasing function on $[1, 400]$. Let $x, y \in [1, 400]$ with $x \le y$, which implies that

$$f(y) \le f(x);$$

thus, we have the following:

$$\begin{aligned} \sqrt[3]{311y^2+51y+100}-y &\leq \sqrt[3]{311x^2+51x+100}-x, \\ \sqrt[3]{311y^2+51y+100}-\sqrt[3]{311x^2+51x+100} &\leq y-x, \\ |\sqrt[3]{311x^2+51x+100}-\sqrt[3]{311y^2+51y+100}| &\leq |x-y|. \end{aligned}$$

Hence, we obtain the following:

$$|Ux - Uy| \leq |x - y|.$$

Therefore, U is a non-expansive map.

Let $a_k = \frac{1}{\sqrt{3k}}$, $b_k = \frac{1}{k^2}$, and $c_k = \frac{1}{k^2+1}$ for all $k \in \mathbb{N}$; then, we obtain Tables 1 and 2.

Table 1. Comparison of the convergence rates of the new iteration with other iterative methods for the initial point $u_0 = 1$.

Iter.	Agarwal et al.	Abbas and Nazir	M-iteration	Thakur et al.	Piri et al.	Garodia and Uddin	New iteration
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	19.8094	34.7808	49.7522	49.7522	91.7536	137.9112	160.1425
13	302.2403	308.6609	311.1364	311.0949	311.1491	311.1646	311.1647
14	305.1899	309.5856	311.1529	311.1338	311.1583	311.1648	311.1649
15	307.1708	310.1671	311.1599	311.1511	311.1621	311.1649	311.1649
20	310.6372	311.0621	311.1649	311.1647	311.1649	311.1649	311.1649
22	310.9304	311.1231	311.1649	311.1649	311.1649	311.1649	311.1649
37	311.1644	311.1649	311.1649	311.1649	311.1649	311.1649	311.1649
42	311.1649	311.1649	311.1649	311.1649	311.1649	311.1649	311.1649

Table 2. Numbers of iterations required to obtain the fp 311.1649 for different initial points.

u_0	Agarwal et al.	Abbas and Nazir	M-iteration	Thakur et al.	Piri et al.	Garodia and Uddin	New iteration
1	42	37	20	22	20	15	14
150	40	34	19	21	18	14	13
250	37	32	18	19	17	13	12
311	22	18	11	12	10	8	7

Additionally, we obtain the influence of the parameters for various iterative methods with the initial point $u_0 = 1$ in Table 3.

Table 3. Numbers of iterations required to obtain the fp 311.1649 for different parameters.

Parameters	Agarwal et al.	Abbas and Nazir	M-iteration	Thakur et al.	Piri et al.	Garodia and Uddin	New iteration
$a_k = \frac{1}{\sqrt{3k}}$, $b_k = \frac{1}{k^2}$, $c_k = \frac{1}{k^2+1}$	42	37	20	22	20	15	14
$a_k = 0.1$, $b_k = 0.1$, $c_k = 0.1$	22	20	12	12	11	8	7
$a_k = 0.1$, $b_k = 0.5$, $c_k = 0.9$	21	11	11	11	9	7	5
$a_k = 0.9$, $b_k = 0.9$, $c_k = 0.9$	13	9	9	9	7	6	5

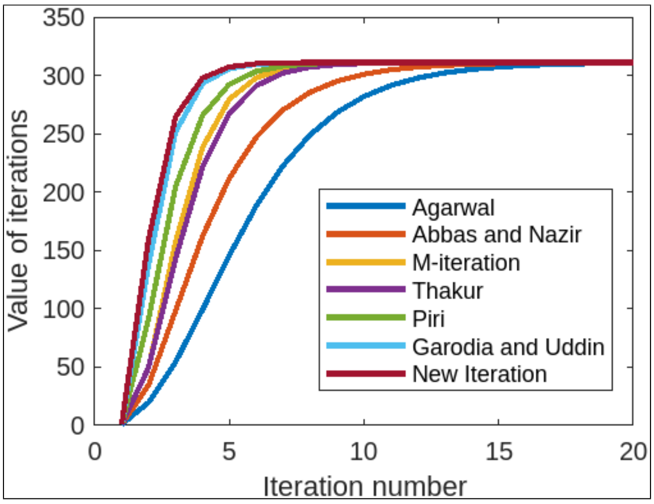

Figure 1. Graphical comparison of the iterative methods corresponding to Table 1.

Table 1 and Figure 1 show that the iterative method (1.1) converges faster than other iteration processes to the fp in $F_U = \{311.1649\}$. *Additionally, Tables 2 and 3 show that this iteration has a better convergence rate than other iterative schemes for non-expansive maps.*

7. Conclusions

As a significant domain of mathematical research, the fp theory provides broad utility in disciplines such as physics, engineering, economics, and computer science. This theory establishes a rigorous framework through which one can investigate both the existence and qualitative behavior of solutions to complex equations and nonlinear systems. Motivated by these broad applications, this paper introduced a novel iteration method in BSs and established its weak w^2-stability for non-expansive maps. Furthermore, we analyzed our proposed iteration and demonstrated its convergence (both weak and strong) under various conditions. Our convergence rate analysis revealed that this new method outperforms existing schemes. Finally, a numerical example (using the MATLAB software) was provided to visually confirm our findings, thereby illustrating the scheme's faster convergence to an fp compared to previously established iterations.

Future research can be directed towards several extensions of the present work. One natural line is to investigate the proposed iteration method for broader classes of non-expansive maps in BSs; see [41–44] for detailed discussions of these classes. In addition, given the ubiquity of nonlinear phenomena in nature, it is crucial to develop efficient techniques that can effectively handle nonlinear problems. Once fixed-point approximation results are established in BSs, extending them to more general frameworks becomes both natural and valuable. In this regard, analyzing the convergence properties of the proposed iteration process in Kohlenbach hyperbolic spaces [45], which generalize BSs and provide a suitable framework for real-world applications, appears to be a particularly promising direction for further research.

Author contributions

Gaurav Aggarwal: Writing–original draft; Aynur Şahin: Writing–review and editing, validation; Izhar Uddin: Supervision; Sabiya Khatoon: Writing–original draft. All authors have read and agreed to the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools to create this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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