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Research article

C*-algebra-valued perturbed modular metric spaces and existence results of fourth-order boundary value problems

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Abstract: The primary objective of perturbed metric spaces was to set route for the advancement of problems involving fixed point findings for a distraught structure, where errors inevitably affected the measurement of distance between two points. This objective developed into a broadly applicable context when the range set of the distance function was a C^*-algebra. Inspired by this, the notion of C^*-algebra-valued perturbed modular metric space was introduced in this paper. Thereafter, some fixed point theorems in the new space were studied, using paired contractive mappings. In order to demonstrate the novelty of the concepts presented herein and to generalize some significant related discoveries in the literature, contrasting examples were given. Under this comparative illustration, numerical and graphical approaches were adopted to study the rate of convergence of Picard-type and paired contractive operators. As an application, new conditions for the existence and uniqueness of a solution to fourth-order boundary value problems were obtained.

Keywords: C^*-algebra; perturbed metric; fixed point; integral equation
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1. Introduction

Fixed point (FP) theorems, specifically the Banach's fixed point theorem (BFPT), has been an intriguing topic of research in the literature owing to its extensive range of utility in mathematical sciences. There are numerous extensions of the BFPT in the literature. One of the ways of presenting this generalization is by introducing new contractions. Along this lane, [7] developed a novel contraction, known as paired contraction (PCt), and established some FP results in metric spaces (M.S). [8] gave an extension of the idea of [7] by proving some FP results in M.S, using a paired Chatterjea-type contraction mapping. Another strategy for enhancing the current FP outcomes is by incorporating a novel setting. In this direction, Chistyakov [5] started up the concept of modular M.S as an enlargement of an M.S which substitutes standard metric with a modular function, enabling a more flexible, nonlinear, or structured measurement of distance. In [6], certain significant FP results in such space were explored. In numerous practical scenarios, such as physics, computer science, and optimization, distance may not function optimally. To widen the scope of metric space theory by including imperfect, uncertain, or generalized concepts of distance, Jleli and Samet [12] initiated the framework of perturbed M.S and obtained a generalization of BFPT in such spaces. Using the key discoveries of [12], Dobritoiu [10] proposed a novel existence and uniqueness result for the solution of a nonlinear Fredholm integral equation.

The study of novel spaces and their characteristics has been a captivating field of study within the realm of mathematics. Along this path, Ma et al. [13] defined C^*-algebra-valued (C^*-alg-v) M.S by substituting the set of nonnegative real numbers with a unital C^*-algebra. They also derived new FP theorems for contractive and nonexpansive mappings on such spaces. Following [13], several improvements of the idea of C^*-alg-v M.S have been presented in different directions. Consequently, Shateri [18] launched the idea of C^*-alg-v modular space and proved some FP theorems in such spaces. Later, Moeine [14] developed the framework of C^*-alg-v modular M.S as a generalization of a modular M.S due to [5] and proved some common FP theorems for generalized contractive-type mappings on such spaces. Recently, Ahmad et al. [1] initiated the idea of C^*-alg-v perturbed M.S and established some FP results with an application. For related developments, see [2, 3, 17].

The existing literature indicates that insufficient investigation was conducted on the concept of C^*-alg-v perturbed modular M.S. Moreover, it is revealing from the literature that, the idea of PCt on C^*-alg-v perturbed modular M.S is probably not yet considered. It is also interesting to note that the concept of PCt, proposing the idea of three distinct points that are pair-wise disjoint is a significant contribution to the metric FP theory, particularly the Banach contraction principle. In this work, we initiate a hybrid concept of PCt on C^*-alg-v perturbed modular M.S and discuss some FP results in this frame. Comparative examples are created to illustrate that the acquired results improve some notable ones in the existing research. The proposed finding is shown to be applicable by providing new criteria for examining the existence and uniqueness of a solution to an integral equation. It is noteworthy that the study of modular M.S has enormous utility in functional analysis, particularly in the research of Musielak-Orlicz spaces, Lebesgue spaces and other function spaces where the conventional metric or norm might not be adequate.

The fundamental component of a modular metric space is the modular function, which substitutes a more flexible, functional measure for the usual distance. It governs the space's geometry, defines convergence, induces the topology, and forms the basis for more complex analytical techniques like operator theory and fixed-point theorems. Because of this, modular spaces are effective in situations when conventional metrics are either excessively restrictive or inadequate. A distinguishing and applicable axiom of the modular metrics is the condition involving the nondecreasing property of the parameter λ. Furthermore, a number of well-known applications of C^*-algebra and perturbed M.S currently exist. Thus, a hybrid of these ideas, as put forth in this paper, is an interesting area of research.

This manuscript is organized into four sections: An overview of pertinent literature and introduction are provided in Section 1. Section 2 compiles some essential preliminaries required in this work. The main findings and a few corollaries acquired from our results are discussed in Section 3. Lastly, an application that offers new existence and uniqueness criteria of an integral equation solution is shown in Section 4.

2. Preliminaries

All through this paper, $\mathbb{W}$ refers to a unital C^*-algebra and $\mathbb{W}_h = \{\eta \in \mathbb{W} : \eta = \eta^*\}$. We consider $\eta \in \mathbb{W}$ to be a positive element, denoted by $\eta \succeq \theta$, with $\eta \in \mathbb{W}_h$ and $\sigma(\eta) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(\eta)$ is the spectrum of η. A partial ordering $\preceq$ on $\mathbb{W}_h$ by employing the positive elements can be defined as follows: $\eta \preceq \gamma$ if and only if $\gamma - \eta \succeq \theta$, where θ denotes the zero element in $\mathbb{W}$. Henceforth, $\mathfrak{S}$ represents a nonvoid set, $\mathbb{W}_+$ denotes the set $\{\eta \in \mathbb{W} : \eta \succeq \theta\}$ and $|\eta| = (\eta\eta^*)^{\frac{1}{2}}$.

Ma et al. [13] started-up the framework of C^*-alg-v M.S as follows.

Definition 2.1. [13] Suppose $\rho : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{W}$ is a mapping satisfying the following:

(1) $\theta \preceq \rho(\varsigma, \mu)$ and $\rho(\varsigma, \mu) = \theta \Leftrightarrow \varsigma = \mu$ for all $\varsigma, \mu \in \mathfrak{S}$;
(2) $\rho(\varsigma, \mu) = \rho(\mu, \varsigma)$ for all $\varsigma, \mu \in \mathfrak{S}$;
(3) $\rho(\varsigma, \mu) \preceq \rho(\varsigma, \varpi) + \rho(\varpi, \mu)$ for all $\varsigma, \mu, \varpi \in \mathfrak{S}$.

Then, ρ is termed as a C^*-alg-v metric on $\mathfrak{S}$ and $(\mathfrak{S}, \mathbb{W}, \rho)$ is termed as a C^*-alg-v M.S.

Definition 2.2. [13] Suppose that $(\mathfrak{S}, \mathbb{W}, \rho)$ is a C^*-alg-v M.S. We call a mapping $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ a C^*-alg-v contractive mapping on $\mathfrak{S}$, if there exists an $\mathfrak{A} \in \mathbb{W}$ with $\|\mathfrak{A}\| < 1$ such that

$$\rho(\Gamma\varsigma, \Gamma\mu) \preceq \mathfrak{A}^*\rho(\varsigma, \mu)\mathfrak{A}, \ \forall\, \varsigma, \mu \in \mathfrak{S}. \tag{2.1}$$

Theorem 2.3. *[13] Let $(\mathfrak{S}, \mathbb{W}, d)$ be a complete C^*-alg-v M.S and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping on $\mathfrak{S}$ satisfying* (2.1). *Then, Γ has a unique FP in $\mathfrak{S}$.*

Definition 2.4. [9] A point $\varsigma \in \mathfrak{S}$ is called a periodic point of period n if $\Gamma^n(\varsigma) = \varsigma$. The smallest positive integer n for which $\Gamma^n(\varsigma) = \varsigma$ is referred to as the prime period of ς.

The following is the main result of [7]:

Definition 2.5. [7] In the context of an M.S $(\mathfrak{S}, d)$ with a cardinality of at least three, denoted as $|\mathfrak{S}| \geq 3$, a mapping $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ is said to have a PCt on $\mathfrak{S}$ if there exists a real number $\tau \in [0, 1)$ such that the following inequality holds:

$$d(\Gamma\varsigma, \Gamma\mu) + d(\Gamma\mu, \Gamma\varpi) \leq \tau(d(\varsigma, \mu) + d(\mu, \varpi)), \qquad \forall\, \varsigma, \mu, \varpi \in \mathfrak{S}. \tag{2.2}$$

Theorem 2.6. *[7] Let $(\mathfrak{S}, d)$ be a complete M.S with cardinality of at least three and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping with the PCt property on $\mathfrak{S}$. Then, Γ has an FP if, and only if, T does not have periodic points of prime period two. Moreover, the number of FPs is at most two.*

In what follows, we recall the idea of perturbed M.S introduced by Jleli and Samet [12].

Definition 2.7. [12] Let $\mathcal{D},\mathcal{P}:\mathfrak{S}\times\mathfrak{S}\longrightarrow[0,\infty)$ be two given mappings. Then, $\mathcal{D}$ is said to be a perturbed metric on $\mathfrak{S}$ with respect to $\mathcal{P}$, if $\mathcal{D}-\mathcal{P}:\mathfrak{S}\times\mathfrak{S}\longrightarrow[0,\infty)$ is a metric on $\mathfrak{S}$, that is:

i) $\theta\preceq(\mathcal{D}-\mathcal{P})(\varsigma,\mu)$ and $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)=\theta\Leftrightarrow\ \varsigma=\mu,\ \forall\ \varsigma,\mu\in\mathfrak{S}$;
ii) $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)=(\mathcal{D}-\mathcal{P})(\mu,\varsigma)$;
iii) $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)\preceq(\mathcal{D}-\mathcal{P})(\varsigma,\varpi)+(\mathcal{D}-\mathcal{P})(\varpi,\mu),\ \forall\ \varsigma,\mu,\varpi\in\mathfrak{S}$.

$\mathcal{P}$ is said to be a perturbed mapping, $\rho=\mathcal{D}-\mathcal{P}$ is an exact metric, and the triplet $(\mathfrak{S},\mathcal{D},\mathcal{P})$ is called a perturbed M.S.

Theorem 2.8. *[12] Let $(\mathfrak{S},\mathcal{D},\mathcal{P})$ be a complete perturbed M.S and $\Gamma:\mathfrak{S}\longrightarrow\mathfrak{S}$ be a given mapping. Assume that Γ is a perturbed continuous mapping and satisfies the Banach contractive-type operator, that is, there exists $\tau\in[0,1)$ such that*

$$\mathcal{D}(\Gamma\varsigma,\Gamma\mu)\preceq\tau\mathcal{D}(\varsigma,\mu),\qquad\forall\ \varsigma,\mu\in\mathfrak{S}.$$

Then, Γ admits one, and only one, FP.

Ahmad et al. [1] extended the main idea of [12] to the context of C^*-algebra as follows.

Definition 2.9. [1] Let $\mathcal{D},\mathcal{P}:\mathfrak{S}\times\mathfrak{S}\longrightarrow\mathbb{W}$ be the two mapping. Then, $\mathcal{D}$ is referred to as a C^*-alg-v perturbed metric on $\mathfrak{S}$ with respect to $\mathcal{P}$, if $\mathcal{D}-\mathcal{P}:\mathfrak{S}\times\mathfrak{S}\longrightarrow\mathbb{W}$ is a C^*-alg-v metric on $\mathfrak{S}$, that is,

i) $\theta\preceq(\mathcal{D}-\mathcal{P})(\varsigma,\mu)$, and $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)=\theta\Leftrightarrow\ \varsigma=\mu,\ \forall\ \varsigma,\mu\in\mathfrak{S}$;
ii) $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)=(\mathcal{D}-\mathcal{P})(\mu,\varsigma)$;
iii) $(\mathcal{D}-\mathcal{P})(\varsigma,\mu)\preceq(\mathcal{D}-\mathcal{P})(\varsigma,\varpi)+(\mathcal{D}-\mathcal{P})(\varpi,\mu),\ \forall\ \varsigma,\mu,\varpi\in\mathfrak{S}$.

$\mathcal{P}$ is called a C^*-alg-v perturbed mapping, $\rho=\mathcal{D}-\mathcal{P}$ is an exact C^*-alg-v metric and the quadruple $(\mathfrak{S},\mathcal{D},\mathcal{P},\mathbb{W})$ is termed a C^*-alg-v perturbed M.S.

Definition 2.10. [1] Let $(\mathfrak{S},\mathbb{W},\mathcal{D},\mathcal{P})$ be a C^*-alg-v perturbed M.S, $\{\varsigma_n\}$ be a sequence in $\mathfrak{S}$, and $\Gamma:\mathfrak{S}\longrightarrow\mathfrak{S}$. Then,

i. $\{\varsigma_n\}$ is known as a C^*-alg-v perturbed covergent sequence in $(\mathfrak{S},\mathbb{W},\mathcal{D},\mathcal{P})$, if $\{\varsigma_n\}$ is a convergent sequence in the C^*-alg-v M.S $(\mathfrak{S},\mathbb{W},\rho)$.
ii. $\{\varsigma_n\}$ is known as a C^*-alg-v perturbed Cauchy sequence in $(\mathfrak{S},\mathbb{W},\mathcal{D},\mathcal{P})$, if $\{\varsigma_n\}$ is a Cauchy sequence in the C^*-alg-v M.S $(\mathfrak{S},\mathbb{W},\rho)$.
iii. $(\mathfrak{S},\mathbb{W},\mathcal{D},\mathcal{P})$ is a complete C^*-alg-v perturbed M.S, if $(\mathfrak{S},\mathbb{W},\rho)$ is a complete C^*-alg-v M.S.
iv. Γ is known as a C^*-alg-v perturbed continuous mapping, if Γ is continuous with respect to the exact C^*-alg-v metric ρ.

Theorem 2.11. *[1] Let $(\mathfrak{S},\mathbb{W},\mathcal{D},\mathcal{P})$ be a complete C^*-alg-v perturbed M.S and $\Gamma:\mathfrak{S}\longrightarrow\mathfrak{S}$ be a mapping. Suppose that Γ is a perturbed continuous mapping and Γ satisfies; $\forall\ \varsigma,\mu\in\mathfrak{S}$, there exists $\mathfrak{A},\mathfrak{B},\mathfrak{C}\in\mathbb{W}'_+$ with $\|\mathfrak{A}+\mathfrak{B}+\mathfrak{C}\|<1$, such that*

$$\mathcal{D}(\Gamma\varsigma,\Gamma\mu)\preceq\mathfrak{A}\mathcal{D}(\varsigma,\mu)+\mathfrak{B}\mathcal{D}(\varsigma,\Gamma\varsigma)+\mathfrak{C}\mathcal{D}(\mu,\Gamma\mu).\tag{2.3}$$

Then Γ has a unique FP in $\mathfrak{S}$.

Chistyakov [5] initiated the concept of modular M.S as an extension of M.S. In this line of progress, Moeini et al. [14] developed the frame of C^*-alg-v modular M.S as follows:

Definition 2.12. [14] A function $\sigma_\lambda : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{W}^+$ is known as a C^*-alg-v modular metric on $\mathfrak{S}$ if the subsequent axioms are met:

A1) $\sigma_\lambda(\varsigma, \mu) = \theta$ if, and only if, $\varsigma = \mu$ for all $\varsigma, \mu \in \mathfrak{S}$ and $\lambda > 0$;
A2) $\sigma_\lambda(\varsigma, \mu) = \sigma_\lambda(\mu, \varsigma)$ for all $\varsigma, \mu \in \mathfrak{S}$ and $\lambda > 0$;
A3) $\sigma_{\lambda+\mu}(\varsigma, \mu) \preceq \sigma_\lambda(\varsigma, \varpi) + \sigma_\mu(\varpi, \mu)$ for all $\varsigma, \mu, \varpi \in \mathfrak{S}$ and $\lambda, \mu > 0$.

3. Main results

In this section, we introduce the idea of C^*-alg-v perturbed modular M.S, as a generalization of C^*-alg-v perturbed M.S according to Ahmad et al. [1] and modular M.S due to [5] .

Definition 3.1. Let $\mathcal{D}_\lambda, \mathcal{P}_\lambda : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{W}$ be two mappings. Then, $\mathcal{D}_\lambda$ is said to be a C^*-alg-v perturbed modular metric on $\mathfrak{S}$ with respect to $\mathcal{P}_\lambda$, if $\mathcal{D}_\lambda - \mathcal{P}_\lambda : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{W}$ is a C^*-alg-v modular metric on $\mathfrak{S}$, i.e., given $\varsigma, \mu, \varpi \in \mathfrak{S}$

Di) $(\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \mu) = \theta$ for all $\lambda > 0 \;\Leftrightarrow\; \varsigma = \mu$;
Dii) $(\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \mu) = (\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\mu, \varsigma)$ for all $\lambda > 0$;
Diii) $(\mathcal{D}_{\lambda+\mu} - \mathcal{P}_{\lambda+\mu})(\varsigma, \mu) \preceq (\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \varpi) + (\mathcal{D}_\mu - \mathcal{P}_\mu)(\varpi, \mu)$ for all $\lambda, \mu > 0$.

$\mathcal{P}_\lambda$ is termed a C^*-alg-v perturbed modular mapping, $\rho_\lambda = \mathcal{D}_\lambda - \mathcal{P}_\lambda$ is an exact C^*-alg-v modular metric and the quadruple $(\mathfrak{S}, \mathcal{D}_\lambda, \mathcal{P}_\lambda, \mathbb{W})$ is termed a C^*-alg-v perturbed modular M.S.

Remark 1. A C^*-alg-v perturbed modular metric on $\mathfrak{S}$ is not necessarily a C^*-alg-v modular metric on $\mathfrak{S}$ as shown in the following examples.

Example 3.2. Let $\mathfrak{S} = \mathbb{R}$, $\mathbb{W} = \mathbb{M}_2(\mathbb{R})$. Define $\mathcal{D}_\lambda, \mathcal{P}_\lambda : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{W}$ by $\mathcal{D}_\lambda(\varsigma, \mu) = e^{-\lambda}\,\mathrm{diag}\left(\varsigma^2 - \mu^2 + \varsigma\mu, \tau(\varsigma^2 - \mu^2 + \varsigma\mu)\right)$ and $\mathcal{P}_\lambda(\varsigma, \mu) = e^{-\lambda}\,\mathrm{diag}\,(\varsigma\mu, \tau\varsigma\mu)$ for all $\lambda > 0$, $\tau \in \mathbb{R}^+$, and $\varsigma, \mu \in \mathbb{R}$. A natural ordering on $\mathbb{W}$ is given by $(\eta, \gamma) = (\kappa, \upsilon) \Leftrightarrow \eta = \kappa,\ \gamma = \upsilon$.

To show that $\mathcal{D}_\lambda$ is a C^*-alg-v perturbed modular metric on $\mathfrak{S}$, we need to show that $\mathcal{D}_\lambda - \mathcal{P}_\lambda$ is a C^*-alg-v modular metric on $\mathfrak{S}$.

Suppose $(\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \mu) = \theta$, then $e^{-\lambda}\,\mathrm{diag}\left(\varsigma^2 - \mu^2, \tau(\varsigma^2 - \mu^2)\right) = \theta$

$$\Rightarrow\ \left(\varsigma^2 - \mu^2, \tau(\varsigma^2 - \mu^2)\right) = 0 \ \Rightarrow\ \varsigma^2 - \mu^2 = 0 \ \Rightarrow\ \varsigma = \mu.$$

Conversely, if $\varsigma = \mu$, then $(\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \varsigma) = e^{-\lambda}\,\mathrm{diag}\left(\varsigma^2 - \varsigma^2, \tau(\varsigma^2 - \varsigma^2)\right)$

$$= e^{-\lambda}\,\mathrm{diag}(0, 0) = \theta, \text{ which gives } Di).$$

$$\begin{aligned}
(\mathcal{D}_{\lambda+\mu} - \mathcal{P}_{\lambda+\mu})(\varsigma, \mu) &= e^{-(\lambda+\mu)}\,\mathrm{diag}\left(\varsigma^2 + \varpi^2 - \varpi^2 - \mu^2, \tau(\varsigma^2 + \varpi^2 - \varpi^2 - \mu^2)\right) \\
&\preceq e^{-(\lambda+\mu)}\,\mathrm{diag}\left(|\varsigma^2 - \varpi^2| + |\varpi^2 - \mu^2|, \tau(|\varsigma^2 - \varpi^2| + |\varpi^2 - \mu^2|)\right) \\
&= e^{-(\lambda+\mu)}\,\mathrm{diag}\left(|\varsigma^2 - \varpi^2||, \tau(|\varsigma^2 - \varpi^2|)\right) + e^{-(\lambda+\mu)}\,\mathrm{diag}\left(|\varpi^2 - \mu^2|, \tau(|\varpi^2 - \mu^2|)\right) \\
&\preceq e^{-\lambda}\,\mathrm{diag}\left(|\varsigma^2 - \varpi^2|, \tau(|\varsigma^2 - \varpi^2|)\right) + e^{-\mu}\,\mathrm{diag}\left(|\varpi^2 - \mu^2|, \tau(|\varpi^2 - \mu^2|)\right) \\
&= (\mathcal{D}_\lambda - \mathcal{P}_\lambda)(\varsigma, \varpi) + (\mathcal{D}_\mu - \mathcal{P}_\mu)(\varpi, \mu),
\end{aligned}$$

which gives Diii). Hence, $\mathcal{D}_\lambda - \mathcal{P}_\lambda$ is a C^*-alg-v modular metric on $\mathfrak{S}$ and $\mathcal{D}_\lambda$ is a C^*-alg-v perturbed modular metric on $\mathfrak{S}$.

We can see that $\mathcal{D}_\lambda$ is not a C^*-alg-v modular metric on $\mathfrak{S}$, since

$$\begin{aligned}\mathcal{D}_\lambda(\varsigma,\varsigma) &= e^{-\lambda}\operatorname{diag}\left(\varsigma^2 - \varsigma^2 + \varsigma^2, \tau(\varsigma^2 - \varsigma^2 + \varsigma^2)\right)\\ &= e^{-\lambda}\operatorname{diag}\left(\varsigma^2, \tau\varsigma^2\right) \neq \theta.\end{aligned}$$

Example 3.3. Let $\mathfrak{S} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^2$. Define $\mathcal{D}_\lambda : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{W}$ by $\mathcal{D}_\lambda(\varsigma,\mu) = \left(\frac{|\varsigma-\mu|}{\lambda} + \frac{\varsigma^2+\mu^2}{\lambda}, 0\right)$ for all $\lambda > 0$, $\varsigma,\mu\in\mathbb{R}$. Then, $\mathcal{D}_\lambda$ is a C^*-alg-v perturbed modular metric on $\mathfrak{S}$ with respect to the C^*-alg-v perturbed modular mapping $\mathcal{P}_\lambda : \mathfrak{S}\times\mathfrak{S}\longrightarrow\mathbb{W}$ given by $\mathcal{P}_\lambda(\varsigma,\mu) = \left(\frac{\varsigma^2+\mu^2}{\lambda}, 0\right)$ for all $\lambda > 0$, $\varsigma,\mu\in\mathbb{R}$.

In this case, the exact C^*-alg-v modular metric is the mapping $\rho_\lambda = \mathcal{D}_\lambda - \mathcal{P}_\lambda : \mathfrak{S}\times\mathfrak{S}\longrightarrow\mathbb{W}$ defined by

$$\rho_\lambda(\varsigma,\mu) = \left(\frac{|\varsigma-\mu|}{\lambda}, 0\right) \ \forall\, \lambda > 0, \ \ \varsigma,\mu\in\mathbb{R}.$$

We see that $\mathcal{D}_\lambda$ is not a C^*-alg-v modular metric on $\mathfrak{S}$ since $\mathcal{D}_\lambda(\varsigma,\varsigma) = \left(\frac{|\varsigma-\varsigma|}{\lambda} + \frac{\varsigma^2+\varsigma^2}{\lambda}, 0\right) = \left(\frac{2\varsigma^2}{\lambda}, 0\right) \neq \theta$.

Example 3.4. Let $\mathfrak{S} = L^\infty(\Omega)$ and $\mathcal{H} = L^2(\Omega)$, where Ω is a Lebesgue measurable set, and $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators in $\mathcal{H}$. It is well-known that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. Define $\mathcal{D}_\lambda : \mathfrak{S}\times\mathfrak{S}\longrightarrow\mathcal{B}(\mathcal{H})$ by

$$\mathcal{D}_\lambda(\varsigma,\mu) = \pi_{\frac{1}{\lambda}|\varsigma-\mu+\varsigma\mu|}, \quad \forall\,\lambda > 0, \ \varsigma,\mu\in\mathfrak{S},$$

where $\pi_h : \mathcal{H}\longrightarrow\mathcal{H}$ is a multiplication operator defined by $\pi_h(\varpi) = h.\varpi$, $\varpi\in\mathcal{H}$.
Then, $\mathcal{D}_\lambda$ is a C^*-alg-v perturbed modular metric with respect to the C^*-alg-v perturbed mapping $\mathcal{P}_\lambda : \mathfrak{S}\times\mathfrak{S}\longrightarrow\mathcal{B}(\mathcal{H})$ defined by

$$P(\varsigma,\mu) = \pi_{\frac{1}{\lambda}|\varsigma\mu|}, \quad \forall\,\lambda > 0, \ \varsigma,\mu\in\mathfrak{S}.$$

The exact C^*-alg-v modular metric on $\mathfrak{S}$ is given by $\rho_\lambda(\varsigma,\mu) = \pi_{\frac{1}{\lambda}|\varsigma-\mu|}$. In consistency with Definition 3.1, $(\mathfrak{S},\mathbb{W},\mathcal{D}_\lambda,\mathcal{P}_\lambda)$ it is a complete C^*-alg-v perturbed modular M.S, since $(\mathfrak{S},\mathbb{W},\rho_\lambda)$ is complete (see [14]).

Definition 3.5. Let $(\mathfrak{S},\mathbb{W},\mathcal{D}_\lambda,\mathcal{P}_\lambda)$ be a C^*-alg-v perturbed modular M.S with cardinality of at least three. A mapping $\Gamma : \mathfrak{S}\longrightarrow\mathfrak{S}$ is called a PCt on $\mathfrak{S}$ if there exists $\mathfrak{A}\in\mathbb{W}_+$ with $\|\mathfrak{A}\| < 1$, such that $\forall\,\lambda > 0$, $\varsigma,\mu,\varpi\in\mathfrak{S}$,

$$\mathcal{D}_\lambda(\Gamma\varsigma,\Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu,\Gamma\varpi) \preceq \mathfrak{A}^*\left(\mathcal{D}_\lambda(\varsigma,\mu) + \mathcal{D}_\lambda(\mu,\varpi)\right)\mathfrak{A}. \tag{3.1}$$

Theorem 3.6. *Let $(\mathfrak{S},\mathbb{W},\mathcal{D}_\lambda,\mathcal{P}_\lambda)$ be a C^*-alg-v perturbed modular M.S with cardinality of at least three. Consider a perturbed continuous mapping $\Gamma : \mathfrak{S}\longrightarrow\mathfrak{S}$ with PCt property on $\mathfrak{S}$. Then,*

i) Γ has an FP if, and only if, Γ does not have a periodic point of prime period two.
ii) The number of FP is at most two.

Proof. Let Γ be a mapping with no peiodic point of prime period two. Suppose $\varsigma_0 \in \mathfrak{S}$ is an initial point such that $\varsigma_1 = \Gamma\varsigma_0,\ \varsigma_2 = \Gamma\varsigma_1$, and so on, forming a sequence $\varsigma_0, \varsigma_1, \cdots$.

Assuming none of the points ς_i are FPs of Γ for every $i = 0, 1, \cdots$, we can see that all ς_i are distinct. Since ς_i is not an FP, then we have $\varsigma_i \neq \Gamma\varsigma_i = \varsigma_{i+1}$. Furthermore, since prime period two's periodic point fails to exist, we deduce that $\varsigma_{i+2} = \Gamma(\Gamma\varsigma_i) \neq \varsigma_i$. Also, if ς_{i+1} is not an FP, then $\varsigma_{i+1} \neq \Gamma\varsigma_{i+1} = \varsigma_{i+2}$. Consequently, $\varsigma_i,\ \varsigma_{i+1}$, and ς_{i+2} are all distinct from each other. Now,

$$\begin{aligned} \mathcal{D}_\lambda(\varsigma_n, \varsigma_{n+1}) + \mathcal{D}_\lambda(\varsigma_{n+1}, \varsigma_{n+2}) &= \mathcal{D}_\lambda(\Gamma\varsigma_{n-1}, \Gamma\varsigma_n) + \mathcal{D}_\lambda(\Gamma\varsigma_n, \Gamma\varsigma_{n+1}) \ \ \forall\ \lambda > 0 \\ &\preceq \mathfrak{A}^* \left(\mathcal{D}_\lambda(\varsigma_{n-1}, \varsigma_n) + \mathcal{D}_\lambda(\varsigma_n, \varsigma_{n+1})\right) \mathfrak{A} \\ &\preceq (\mathfrak{A}^*)^2 \left(\mathcal{D}_\lambda(\varsigma_{n-2}, \varsigma_{n-1}) + \mathcal{D}_\lambda(\varsigma_{n-1}, \varsigma_n)\right) \mathfrak{A}^2 \\ &\preceq \cdots \preceq (\mathfrak{A}^*)^n \left(\mathcal{D}_\lambda(\varsigma_{n-2}, \varsigma_{n-1}) + \mathcal{D}_\lambda(\varsigma_{n-1}, \varsigma_n)\right) \mathfrak{A}^n. \end{aligned}$$

This reveals that,

$$\mathcal{D}_\lambda(\varsigma_n, \varsigma_{n+1}) + \mathcal{D}_\lambda(\varsigma_{n+1}, \varsigma_{n+2}) \preceq (A^*)^n \left(\mathcal{D}_\lambda(\varsigma_{n-2}, \varsigma_{n-1}) + \mathcal{D}_\lambda(\varsigma_{n-1}, \varsigma_n)\right) \mathfrak{A}^n \ \forall\ \lambda > 0. \tag{3.2}$$

Denoting $R_0 = \mathcal{D}_\lambda(\varsigma_0, \varsigma_1) + \mathcal{D}_\lambda(\varsigma_1, \varsigma_2)$, $R_1 = \mathcal{D}_\lambda(\varsigma_1, \varsigma_2) + \mathcal{D}_\lambda(\varsigma_2, \varsigma_3), \cdots, R_n = \mathcal{D}_\lambda(\varsigma_n, \varsigma_{n+1}) + \mathcal{D}_\lambda(\varsigma_{n+1}, \varsigma_{n+2})\ \forall\ \lambda > 0,\ n \in \mathbb{N}$ and recalling that $\rho_\lambda = \mathcal{D}_\lambda - \mathcal{P}_\lambda$ is the exact C^*-alg-v modular metric on $\mathfrak{S}$, from 3.2, we have

$$\rho_\lambda(\varsigma_n, \varsigma_{n+1}) + \rho_\lambda(\varsigma_{n+1}, \varsigma_{n+2}) + \mathcal{P}_\lambda(\varsigma_n, \varsigma_{n+1}) + \mathcal{P}_\lambda(\varsigma_{n+1}, \varsigma_{n+2}) \preceq (A^*)^n R_0 A^n \ \ \forall\ \lambda > 0, \text{ and } n \in \mathbb{N}.$$

Since $\rho_\lambda \preceq \rho_\lambda + \mathcal{P}_\lambda$, then

$$\rho_\lambda(\varsigma_n, \varsigma_{n+1}) + \rho_\lambda(\varsigma_{n+1}, \varsigma_{n+2}) \preceq (\mathfrak{A}^*)^n R_0 \mathfrak{A}^n \ \ \forall\ \lambda > 0, \text{ and } n \in \mathbb{N}.$$

Now, for $n, p > 1$, we have

$$\begin{aligned} \rho_\lambda(\varsigma_n, \varsigma_{n+p}) &\preceq \rho_{\frac{\lambda}{n}}(\varsigma_n, \varsigma_{n+1}) + \rho_{\frac{\lambda}{n}}(\varsigma_{n+1}, \varsigma_{n+2}) + \cdots + \rho_{\frac{\lambda}{n}}(\varsigma_{n+p-1}, \varsigma_{n+p}) \\ &\preceq (\mathfrak{A}^*)^n R_0 \mathfrak{A}^n + (\mathfrak{A}^*)^{n+1} R_0 \mathfrak{A}^{n+1} + \cdots + (\mathfrak{A}^*)^{n+p-1} R_0 \mathfrak{A}^{n+p-1} \\ &= \sum_{k=n}^{n+p-1} (\mathfrak{A}^*)^k R_0 \mathfrak{A}^k = \sum_{k=n}^{n+p-1} (\mathfrak{A}^*)^k R_0^{\frac{1}{2}} R_0^{\frac{1}{2}} \mathfrak{A}^k \\ &\preceq \sum_{k=n}^{n+p-1} \left(R_0^{\frac{1}{2}} \mathfrak{A}^k\right)^* \left(R_0^{\frac{1}{2}} \mathfrak{A}^k\right) \\ &= \sum_{k=n}^{n+p-1} \left|R_0^{\frac{1}{2}} \mathfrak{A}^k\right|^2 \preceq \left\| \sum_{k=n}^{n+p-1} \left|R_0^{\frac{1}{2}} \mathfrak{A}^k\right|^2 \right\| \\ &\preceq \|R_0^{\frac{1}{2}}\|^2 \sum_{k=n}^{n+p-1} \left\|\mathfrak{A}^k\right\|^2 I \\ &= \|R_0^{\frac{1}{2}}\|^2 \frac{\|\mathfrak{A}\|^n}{1 - \|\mathfrak{A}\|} I \ \ \to\ \theta \text{ as } n \to \infty. \end{aligned}$$

This proves that in the C^*-alg-v modular M.S $(\mathfrak{S}, \mathbb{W}, \rho_\lambda)$, the sequence $\{\varsigma_n\}$ is ρ_λ-Cauchy and so, it is perturbed Cauchy with respect to the C^*-alg-v perturbed modular M.S $(\mathfrak{S}, \mathbb{W}, \mathcal{D}_\lambda, \mathcal{P}_\lambda)$. The completeness of this space prompts us to deduce that there is $\varsigma \in \mathfrak{S}$ such that

$$\lim_{n\to\infty} \rho_\lambda(\varsigma_n, \varsigma) = \theta.$$

We then show that ς is an FP of Γ. Since Γ is perturbed continuous, then

$$\lim_{n\to\infty} \rho_\lambda(\Gamma\varsigma_n, \Gamma\varsigma) = \theta, \text{ that is,}$$

$\lim_{n\to\infty} \rho_\lambda(\varsigma_{n+1}, \Gamma\varsigma) = \theta$, but ρ_λ is a C^*-alg-v modular metric. Then, $\varsigma = \Gamma\varsigma$.

Suppose that there exist three pairwise distinct FPs, denoted as ς, μ, ϖ. This implies that $\Gamma\varsigma = \varsigma$, $\Gamma\mu = \mu$, and $\Gamma\varpi = \varpi$, however, this contradicts the PCt. On the other hand if we assume that Γ has an FP u and let u be a periodic point of prime period two, where $\Gamma\mu = \Gamma(\Gamma u) = u$, then

$$\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma u) + \mathcal{D}_\lambda(\Gamma u, \Gamma\mu) = \mathcal{D}_\lambda(\varsigma, u) + \mathcal{D}_\lambda(u, \mu), \quad \forall\, \lambda > 0,$$

which is a contradiction. □

Example 3.7. Let $\mathcal{D}_\lambda$ be a C^*-alg-v perturbed modular metric as in Example 3.3 and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping defined by $\Gamma\varsigma = \frac{\varsigma}{2}$. Then, we see that $\Gamma(1) = \frac{1}{2}$ and $\Gamma(\Gamma 1) = 2$, that is, Γ does not have a periodic point of prime period two. By PCt (2.2),

$$\begin{aligned}
\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi) &= \mathcal{D}_\lambda(\frac{\varsigma}{2}, \frac{\mu}{2}) + \mathcal{D}_\lambda\left(\frac{\mu}{2}, \frac{\varpi}{2}\right) \\
&= \left(\frac{|\frac{\varsigma}{2} - \frac{\mu}{2}|}{\lambda} + \frac{\frac{\varsigma^2}{4} + \frac{\mu^2}{4}}{\lambda}, 0\right) + \left(\frac{|\frac{\mu}{2} - \frac{\varpi}{2}|}{\lambda} + \frac{\frac{\mu^2}{4} + \frac{\varpi^2}{4}}{\lambda}, 0\right) \\
&= \left(\frac{1}{2}\frac{|\varsigma - \mu|}{\lambda} + \frac{1}{4}\left(\frac{\varsigma^2 + \mu^2}{\lambda}\right), 0\right) + \left(\frac{1}{2}\frac{|y - z|}{\lambda} + \frac{1}{4}\left(\frac{\mu^2 + \varpi^2}{\lambda}\right), 0\right) \\
&\preceq \left(\frac{1}{2}\left(\frac{|\varsigma - \mu|}{\lambda} + \frac{\varsigma^2 + \mu^2}{\lambda}\right), 0\right) + \left(\frac{1}{2}(\frac{|y - z|}{\lambda} + \frac{\mu^2 + \varpi^2}{\lambda}), 0\right) \\
&= \frac{1}{2}\left(\mathcal{D}_\lambda(\varsigma, \mu) + \mathcal{D}_\lambda(\mu, \varpi)\right) \quad \forall\, \lambda > 0.
\end{aligned}$$

This implies that, $\|\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi)\| \leq \frac{1}{2}\, \|\mathcal{D}_\lambda(\varsigma, \mu) + \mathcal{D}_\lambda(\mu, \varpi)\|$, that is, $\|\mathfrak{A}\| = \frac{1}{2} < 1$.

By Theorem 3.6, Γ possesses at most two FPs. To find the FP of Γ, notice that $\varsigma = \Gamma\varsigma \;\Rightarrow\; \varsigma = \frac{\varsigma}{2}$, which implies that $\varsigma = 0$ is the FP. However, the main theorem of Ma et.al [13] fails in this case, since $\mathcal{D}_\lambda$ is a C^*-alg-v perturbed modular metric.

We further demonstrate that the sequence $\{\Gamma\varsigma_n\}$ indeed converges to the FP of Γ. For this, let $\varsigma_n = \Gamma\varsigma_{n-1}$, $n \geq 0$ and choose $\varsigma_0 = 0.1$. Then, we record the iterations of (ς_n) in the following table.

We see from Table 1 and Figure 1 that the sequence (ς_n) converges to the FP $\varsigma = 0$.

Table 1. Iterative sequence.

n	Sequence ς_n	$\Gamma\varsigma = \frac{\varsigma}{2}$
0	0.1	–
1	0.05	$\varsigma_1 = \Gamma\varsigma_0$
2	0.025	$\varsigma_2 = \Gamma\varsigma_1$
3	0.0125	$\varsigma_3 = \Gamma\varsigma_2$
4	0.00625	$\varsigma_4 = \Gamma\varsigma_3$
5	0.003125	$\varsigma_5 = \Gamma\varsigma_4$
6	0.001563	$\varsigma_6 = \Gamma\varsigma_5$
7	0.000781	$\varsigma_7 = \Gamma\varsigma_6$
8	0.000391	$\varsigma_8 = \Gamma\varsigma_7$
9	0.000196	$\varsigma_9 = \Gamma\varsigma_8$

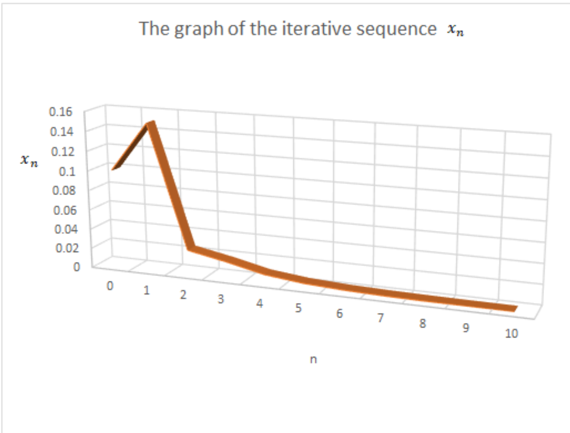

Figure 1. Graph of the iterative sequence ς_n.

The subsequent result is a version of the primary finding of Ma et al. [13].

Corollary 1. *Let* $(\mathfrak{S}, \mathbb{W}, \mathcal{D}_\lambda, \mathcal{P}_\lambda)$ *be a complete* C^**-alg-v perturbed modular M.S with a contractive mapping* $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ *satisfying;*

$$\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) \preceq \mathfrak{A}^* \mathcal{D}_\lambda(\varsigma, \mu)\mathfrak{A}, \tag{3.3}$$

for all $\lambda > 0$, $\varsigma, \mu \in \mathfrak{S}$, $\mathfrak{A} \in \mathbb{W}_+$ *with* $\|\mathfrak{A}\| < 1$. *Then,* Γ *has at most two FPs.*

Proof. For the case where the cardinality of $\mathfrak{S}$ is 1 or 2, the proof is straightforward. However, if $|\mathfrak{S}| \geq 3$, then we have the following argument: Suppose there exists an element $\varsigma \in \mathfrak{S}$ such that $\Gamma(\Gamma\varsigma) = \varsigma$, where we have

$$\mathcal{D}_\lambda(\Gamma\varsigma, \varsigma) = \mathcal{D}_\lambda(\Gamma\varsigma, \Gamma(\Gamma\varsigma)),$$

that is,

$$\|\mathcal{D}_\lambda(\Gamma\varsigma, \varsigma)\| = \|\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma^2\varsigma)\|. \tag{3.4}$$

However, by (3.3) with $\mu = \Gamma\varsigma$, $\|\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma^2\varsigma)\| < \|\mathcal{D}_\lambda(\varsigma, \Gamma\varsigma)\|$, which contradicts (3.4).

Thus, Γ cannot have a periodic points of prime period two. Otherwise, the result will be equivalent to the contraction mapping principle. Now, let $\varsigma, \mu, \varpi \in \mathfrak{S}$ be pairwise distinct, and by applying (3.3), we have

$$\begin{aligned} \mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) &\preceq \mathfrak{A}^*\mathcal{D}_\lambda(\varsigma, \mu)\mathfrak{A}, \text{ and} \\ \mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi) &\preceq \mathfrak{A}^*\mathcal{D}_\lambda(\mu, \varpi)\mathfrak{A} \ \forall \lambda > 0. \end{aligned}$$

So, $\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi) \preceq \mathfrak{A}^* (\mathcal{D}_\lambda(\varsigma, \mu) + \mathcal{D}_\lambda(\mu, \varpi)) \mathfrak{A}$.

This shows that Γ is a paired contractive mapping on $\mathfrak{S}$. Therefore, by Theorem 3.6, Γ possesses at most two FPs. □

Corollary 2. *Let $(\mathfrak{S}, \mathbb{W}, \rho_\lambda)$ be a complete C^*-alg-v modular M.S and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a mapping satisfying the following: for all $\varsigma, \mu \in \mathfrak{S}$, there exists $\mathfrak{A} \in \mathbb{W}_+$ with $\|\mathfrak{A}\| < 1$ such that*

$$\rho_\lambda(\Gamma\varsigma, \Gamma\mu) \preceq \mathfrak{A}^*\rho_\lambda(\varsigma, \mu)\mathfrak{A}, \text{ for all } \lambda > 0. \tag{3.5}$$

Then, Γ has an FP.

Proof. Taking $\mathcal{P}_\lambda = \theta$, then $(\mathfrak{S}, \mathbb{W}, \mathcal{D}_\lambda, \mathcal{P}_\lambda) = (X, \mathbb{W}, \rho_\lambda)$. Hence,

$$\begin{aligned} \rho_\lambda(\Gamma\varsigma, \Gamma\mu) + \rho_\lambda(\Gamma\mu, \Gamma\varpi) &\preceq \mathfrak{A}^*\rho_\lambda(\varsigma, \mu)\mathfrak{A} + \mathfrak{A}^*\rho_\lambda(\mu, \varpi)\mathfrak{A} \\ &= \mathfrak{A}^* (\rho_\lambda(\varsigma, \mu) + \rho_\lambda(\mu, \varpi)) \mathfrak{A} \ \forall \lambda > 0. \end{aligned}$$

Therefore, by Theorem 3.6, Γ has at most two FPs.

For uniqueness, suppose ς and μ are FP of Γ, that is,, $\Gamma\varsigma = \varsigma$ and $\Gamma\mu = \mu$. Then,

$$\begin{aligned} \rho_\lambda(\varsigma, \mu) &\preceq \rho_\lambda(\varsigma, \Gamma\varsigma_n) + \rho_\lambda(\Gamma\varsigma_n, \mu) \\ &= \rho_\lambda(\Gamma\varsigma, \Gamma\varsigma_n) + \rho_\lambda(\Gamma\varsigma_n, \Gamma\mu) \\ &\preceq \mathfrak{A}^* (\rho_\lambda(\varsigma, \varsigma_n) + \rho_\lambda(\varsigma_n, \mu)) \mathfrak{A}, \ \forall \lambda > 0. \end{aligned}$$

As $n \to \infty$ and considering $\|\mathfrak{A}\| < 1$, we have $\rho_\lambda(\varsigma, \mu) = \theta$, which implies $\varsigma = \mu$. □

Corollary 3. *[1] Let $(\mathfrak{S}, \mathbb{W}, \mathcal{D}, \mathcal{P})$ be a complete C^*-alg-v perturbed M.S and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a pertubed continuous mapping satisfying the following: For all $\varsigma, \mu \in \mathfrak{S}$, there exists $\mathfrak{A} \in \mathbb{W}_+$ with $\|\mathfrak{A}\| < 1$ such that*

$$\mathcal{D}(\Gamma\varsigma, \Gamma\mu) \preceq \mathfrak{A}^*\mathcal{D}(\varsigma, \mu)\mathfrak{A}. \tag{3.6}$$

Then, Γ has at most two FP.

Proof. Put $\lambda = 1$ in Corollary 1. □

Corollary 4. *See [12] Let $(\mathfrak{S}, \mathcal{D}, \mathcal{P})$ be a complete perturbed M.S and $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a perturbed continuous mapping satisfying the following: For all $\varsigma, \mu \in \mathfrak{S}$, there exists $\tau \in (0, 1)$ such that*

$$\mathcal{D}(\Gamma\varsigma, \Gamma\mu) \preceq \tau\mathcal{D}(\varsigma, \mu).$$

Then, Γ has an FP in $\mathfrak{S}$.

Proof. Let $\mathbb{W} = \mathbb{R}$ in Corollary 3. □

In the following example, we demonstrate that the paired contractive operator has a higher rate of convergence compared to the Banach contractive operator in perturbed M.S.

Example 3.8. Let $\{\varsigma_n\}_{n\geq 0} = \{\frac{1}{2^n}\}$ be a sequence in $\mathfrak{S} = [0,1]$ and $\mathcal{D} : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{R}$, given by $\mathcal{D}(\varsigma,\mu) = |\varsigma-\mu|+\varsigma\mu$, be a perturbed metric on $\mathfrak{S}$ with respect to the perturbed mapping $\mathcal{P} : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{R}$, given by $\mathcal{P}(\varsigma,\mu) = \varsigma\mu$ for all $\varsigma,\mu \in \mathfrak{S}$. Suppose $\Gamma : \mathfrak{S} \longrightarrow \mathfrak{S}$ is a mapping for each $n = 0, 1, 2, 3, ...$ defined by

$$\Gamma\varsigma_n = \begin{cases} \frac{\varsigma_n}{2}, & \text{if } \varsigma_n \in [0, \frac{1}{2}), \\ \\ \frac{3}{4} - \frac{\varsigma_n}{2}, & \text{if } \varsigma_n \in [\frac{1}{2}, 1]. \end{cases}$$

Assume $\{\varsigma_n, \mu_n, \varpi_n\} = \{\frac{1}{2^n}, \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}\} \subset \mathfrak{S}$ for $n = 0, 1, 2, 3, ...$

Case I. For $\varsigma_n, \mu_n \in [\frac{1}{2}, 1]$, $\varpi_n \in [0, \frac{1}{2})$, we require $n = 0$, so that $\{\varsigma, \mu, \varpi\} = \{1, \frac{1}{2}, \frac{1}{4}\}$.

Then,

$$\begin{aligned} \mathcal{D}(\Gamma 1, \Gamma\frac{1}{2}) + \mathcal{D}(\Gamma\frac{1}{2}, \Gamma\frac{1}{4}) &= \mathcal{D}(\frac{3}{4} - \frac{1}{2}, \frac{3}{4} - \frac{1}{4}) + \mathcal{D}(\frac{3}{4} - \frac{1}{4}, \frac{1}{8}) \\ &= \left|\frac{3}{4} - \frac{1}{2} - \frac{2}{4}\right| + (\frac{6}{16} - \frac{2}{8}) + \left|\frac{3}{4} - \frac{1}{4} - \frac{1}{8}\right| + \frac{3}{32} - \frac{1}{32} \\ &= \frac{3}{8} + \frac{7}{16} = \frac{13}{16}, \end{aligned}$$

$$\begin{aligned} \mathcal{D}(1, \frac{1}{2}) + \mathcal{D}(\frac{1}{2}, \frac{1}{4}) &= \left|1 - \frac{1}{2}\right| + \frac{1}{2} + \left|\frac{1}{2} - \frac{1}{4}\right| + \frac{1}{18} \\ &= 1 + \frac{3}{8} = \frac{11}{8}. \end{aligned}$$

By PCt, we require $\tau \in [0, 1)$ such that $\frac{13}{16} \leq \tau\frac{11}{8}$. That is, $\frac{13}{22} \leq \tau$, which implies $\tau \in [0.591, 1)$.

Case II. For $\varsigma_n \in [\frac{1}{2}, 1]$, $\mu_n, \varpi_n \in [0, \frac{1}{2})$, we require $n = 1$, so that $\{\varsigma, \mu, \varpi\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$. Then,

$$\begin{aligned} \mathcal{D}(\Gamma\frac{1}{2}, T\frac{1}{4}) + \mathcal{D}(\Gamma\frac{1}{4}, T\frac{1}{8}) &= \mathcal{D}(\frac{3}{4} - \frac{1}{4}, \frac{1}{8}) + \mathcal{D}(\frac{1}{8}, \frac{1}{16}) \\ &= \left|\frac{3}{4} - \frac{1}{4} - \frac{1}{8}\right| + (\frac{3}{32}) - \frac{1}{32}) + \left|\frac{1}{8} - \frac{1}{16}\right| + \frac{1}{128} \\ &= \frac{7}{16} + \frac{9}{128} = \frac{65}{128}, \end{aligned}$$

$$\begin{aligned} \mathcal{D}(\frac{1}{2}, \frac{1}{4}) + \mathcal{D}(\frac{1}{4}, \frac{1}{8}) &= \left|\frac{1}{2} - \frac{1}{4}\right| + \frac{1}{8} + \left|\frac{1}{4} - \frac{1}{8}\right| + \frac{1}{32} \\ &= \frac{3}{8} + \frac{5}{32} = \frac{17}{32}. \end{aligned}$$

By PCt, $\frac{65}{128} \leq \tau\frac{17}{32}$, which implies that $\frac{65}{68} \leq \tau$, that is, $\tau \in [0.956, 1)$. Consequently, all the conditions of Theorem 3.6 are fulfilled.

However, Γ is not a Banach-type operator in the sense of [12]. To see this, take $\varsigma = \frac{1}{2}$ and $\mu = \frac{1}{4}$. Then, $\mathcal{D}(\Gamma\frac{1}{2}, \Gamma\frac{1}{4}) = \frac{7}{16} > \mathcal{D}(\frac{1}{2}, \frac{1}{4}) = \frac{3}{8}$.

Table 2 gives a comprehensive analysis, demonstrating that the rate of convergence of the PC operator is faster compared to that of the Banach contractive operator. In order to achieve precision to four decimal places, we start with the initial value $\varsigma_0 = 1$. The Banach contractive operator is defined as $\Gamma\varsigma = \frac{\varsigma}{2}$ and the PC operator is given as $\Gamma\varpi = \frac{1}{2^{n+2}}$ for $n \in \mathbb{N}$.

We see from Table 2 and Figure 2 that the PC operator converges faster than the Banach contractive operator.

Table 2. Comparison between Banach contractive operator and PC operators.

n	Sequence ς_n	Banach contractive operator	PC operator
0	–	1	1
1	$\varsigma_1 = \Gamma\varsigma_0$	0.5	0.0625
2	$\varsigma_2 = \Gamma\varsigma_1$	0.25	0.0313
3	$\varsigma_3 = \Gamma\varsigma_2$	0.125	0.0156
4	$\varsigma_4 = \Gamma\varsigma_3$	0.0625	0.0078
5	$\varsigma_5 = \Gamma\varsigma_4$	0.0313	0.0039
6	$\varsigma_6 = \Gamma\varsigma_5$	0.0156	0.0020
7	$\varsigma_7 = \Gamma\varsigma_6$	0.0078	0.0010
8	$\varsigma_8 = \Gamma\varsigma_7$	0.0039	0.0005
9	$\varsigma_9 = \Gamma\varsigma_8$	0.0020	0.0003
10	$\varsigma_{10} = \Gamma\varsigma_9$	0.0010	0.0002
11	$\varsigma_{11} = \Gamma\varsigma_{10}$	0.0005	0.0001
12	$\varsigma_{12} = \Gamma\varsigma_{11}$	0.0003	0.0000
13	$\varsigma_{13} = \Gamma\varsigma_{12}$	0.0002	0.0000
14	$\varsigma_{14} = \Gamma\varsigma_{13}$	0.0001	0.0000
15	$\varsigma_{15} = \Gamma\varsigma_{14}$	0.0000	0.0000

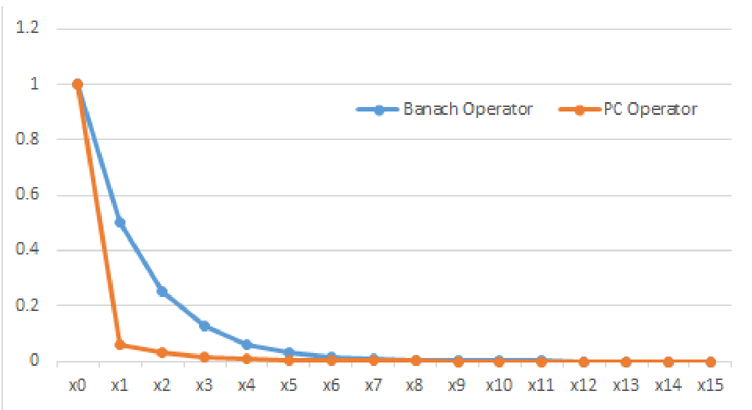

Figure 2. Graph of Banach contractive and PC operators.

4. Applications to fourth-order boundary value problems

The mathematical basis for demonstrating the existence, uniqueness, or positivity of solutions to fourth-order boundary value problems, even in the absence of explicit formulas is provided by fixed point theory. In this direction, Younis et al. [19] established some novel convergence results to the Helmholtz problem with mixed boundary conditions and demonstrated the existence and uniqueness of the solution by applying graph-controlled contractions. Similarly, Pasha et al. [16] showcased the utility of some common fixed point theorems using commuting mapping in the solution of boundary value problem via C^*-alg-v bipolar metric space. In this section, we employ one of the major findings of this manuscript to analyze a boundary value problem in a C^*-alg-v perturbed modular metric setting. In this regard, consider the fourth-order boundary value problem (BVP):

$$\begin{cases} \varsigma^{iv}(\ell) = \Re\left(\ell, \varsigma(\ell), \varsigma'(\ell), \varsigma''(\ell), \varsigma'''(\ell)\right), & \ell \in \Omega, \\ \varsigma(\alpha) = \varsigma'(\alpha) = \varsigma''(\beta) = \varsigma'''(\beta) = \alpha, \end{cases} \tag{4.1}$$

where $\Re : \Omega \times \mathbb{R}^4 \longrightarrow \mathbb{R}$, Ω is a Lebesgue measurable set. Suppose $\mathfrak{S} = L^\infty(\Omega)$, $\mathcal{H} = L^2(\Omega)$, and $\mathbb{W} = B(\mathcal{H})$. Define $\mathcal{D}_\lambda : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{W}$ by $\mathcal{D}_\lambda(\varsigma, \mu) = \pi_{\frac{1}{\lambda}|\varsigma-\mu+\varsigma\mu|}$ $\forall\, \lambda > 0,\ \varsigma, \mu \in \mathfrak{S}$ and $\mathcal{P}_\lambda : \mathfrak{S}\times\mathfrak{S} \longrightarrow \mathbb{W}$ by $\mathcal{P}_\lambda(\varsigma, \mu) = \pi_{\frac{1}{\lambda}|\varsigma\mu|}$, $\ \forall\, \lambda > 0,\ \varsigma, \mu \in \mathfrak{S}$. Then, the quadruple $(\mathfrak{S}, \mathbb{W}, \mathcal{D}_\lambda, \mathcal{P}_\lambda)$ is a complete C^*-alg-v perturbed modular M.S (see Example 3.4).

The BVP (4.1) can be written in integral form as

$$\varsigma(\ell) = \int_\Omega G(\ell, \wp)\Re(\wp, \varsigma(\wp), \varsigma'(\wp)d\wp), \tag{4.2}$$

where $G(\ell, \wp) : \Omega\times\Omega \longrightarrow \mathbb{R}$ is the Green's function associated to (4.1). It is interesting to note that the Green's function represents solutions using integral formulae, is linear and symmetric for self-adjoint problems, fulfills continuity, and is suited to specific boundary conditions. These properties are useful particulary in the existence theory of integro-differential equations.

We study conditions for the existence of a unique solution to the integral Eq (4.2) under the following hypotheses:

Theorem 4.1. *Let $\Re : \Omega \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ be a continuous nondecreasing function. Suppose that the following conditions are satisfied:*

i) $\sup_{\ell\in\Omega} \int_\Omega G(\ell, \wp)d\wp \leq \tau$, *for* $\tau \in (0, 1)$, $\ell, \wp \in \Omega$*;*
ii) $|\Re(\ell, \wp, \varsigma'(\wp)) - \Re(\ell, \wp, \mu'(\wp))| \leq \frac{\tau}{2}|\varsigma(\wp) - \mu(\wp)|$ *for* $\ell, \wp \in \Omega$*;*
iii) $|\Re(\ell, \wp, \varsigma'(\wp))\Re(\ell, \wp, \mu'(\wp))| \leq \frac{\tau}{2}|(\varsigma(\wp)\mu(\wp))|$.

Then, the BVP (4.1) *has at most two solutions in* $L^2(\Omega)$.

Proof. Let $\Gamma : L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$ be given by

$$\Gamma\varsigma(\ell) = \int_\Omega G(\ell, \wp)\Re(\ell, \wp, \varsigma'(\wp))d\wp, \qquad \ell \in \Omega.$$

Now, for any $h \in \mathcal{H}$, using the property of norm, we have $\wp$

$$\|\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi)\| \leq \|\mathcal{D}_\lambda(\Gamma\varsigma, \Gamma\mu)\| + \|\mathcal{D}_\lambda(\Gamma\mu, \Gamma\varpi)\|. \tag{4.3}$$

From (4.3), $\|\mathcal{D}_\lambda(\Gamma\varsigma,\Gamma\mu)\| = \sup\left\langle \pi_{\frac{1}{\lambda}|\Gamma\varsigma-\Gamma\mu+\Gamma\varsigma\Gamma\mu|}h, h\right\rangle$

$$
\begin{aligned}
&= \sup_{\|h\|=1}\int_\Omega\Big(\Big|\int_\Omega \frac{1}{\lambda}G(\ell,\wp)(\Re(\wp,\varsigma(\wp),\varsigma'(\wp)) - \Re(\wp,\mu(\wp),\mu'(\wp)) + \\
&\quad \Re(\wp,\varsigma(\wp),\varsigma'(\wp)\Re(\wp,\mu(\wp),\mu'(\wp))d\wp\Big|\Big)h(\ell)\overline{h(\ell)}d\ell \\
&\leq \sup_{\|h\|=1}\int_\Omega\Big(\frac{1}{\lambda}\int_\Omega |G((\ell,\wp)|\Big(|\Re(\wp,\varsigma(\wp),\varsigma'(\wp)) - \Re(\wp,\mu(\wp),\mu'(\wp))| + \\
&\quad |\Re(\wp,\varsigma(\wp),\varsigma'(\wp)\Re(\wp,\mu(\wp),\mu'(\wp)|\Big)d\wp\Big)h(\ell)\overline{h(\ell)}d\ell \\
&\leq \sup_{\|h\|=1}\int_\Omega\Big(\frac{1}{\lambda}\int_\Omega |G(\ell,\wp)|\left(\frac{\tau}{2}|\varsigma(\wp)-\mu(\wp)| + \frac{\tau}{2}|\varsigma(\wp)\mu(\wp)|\right)d\wp\Big)h(\ell)\overline{h(\ell)}d\ell \\
&\leq \sup_{\|h\|=1}\int_\Omega\Big(\frac{1}{\lambda}\int_\Omega \frac{\tau}{2}|G(\ell,\wp)|\sup_{\ell\in\Omega}\left(|\varsigma(\ell)-\mu(\ell)| + |\varsigma(\ell)\mu(\ell)|\right)d\wp\Big)h(\ell)\overline{h(\ell)}d\ell \\
&= \sup_{\|h\|=1}\int_\Omega\Big(\frac{1}{\lambda}\int_\Omega \frac{\tau}{2}\sup_{\ell\in\Omega}\left(|\varsigma(\ell)-\mu(\ell)| + |\varsigma(\ell)\mu(\ell)|\right)\int_\Omega |G(\ell,\wp)|d\wp\Big)h(\ell)\overline{h(\ell)}d\ell \\
&\leq \left(\frac{\tau^2}{2\lambda}\sup_{\ell\in\Omega}(|\varsigma(\ell)-\mu(\ell)| + |\varsigma(\ell).\mu(\ell)|)\right)\sup_{\|h\|=1}\int_\Omega |h(\ell)|^2 d\ell \\
&\leq \frac{\tau^2}{2\lambda}\|\varsigma-\mu+\varsigma\mu)\|_\infty \\
&= \xi\|\mathcal{D}_\lambda(\varsigma,\mu)\|, \quad \text{where } \xi = \frac{\tau^2}{2} \ \text{ for } \tau\in(0,1).
\end{aligned}
$$

This implies that

$$\|\mathcal{D}_\lambda(\Gamma\varsigma,\Gamma\mu)\| \leq \xi\|\mathcal{D}_\lambda(\varsigma,\mu)\|, \quad \forall\ \lambda > 0. \tag{4.4}$$

Similarly,

$$\|\mathcal{D}_\lambda(\Gamma\mu,\Gamma\varpi)\| \leq \xi\|\mathcal{D}_\lambda(\mu,\varpi)\|, \quad \forall\ \lambda > 0. \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.3), we obtain

$$\|\mathcal{D}_\lambda(\Gamma\varsigma,\Gamma\mu) + \mathcal{D}_\lambda(\Gamma\mu,\Gamma\varpi)\| \leq \xi\left(\|\mathcal{D}_\lambda(\varsigma,\mu)\| + \|\mathcal{D}_\lambda(\mu,\varpi)\|\right), \quad \forall\ \lambda > 0.$$

Hence, it follows from Theorem 3.6 that (4.1) has at most two solutions in $L^\infty(\Omega)$. □

5. Conclusions

The significance of a perturbed M.S arises from the necessity to understand how minor alterations (perturbations) to an original metric influence the geometry, topology, and analysis within that space, as many issues in mathematics, physics, computer science, and applied sciences are predicated on such modifications. Taking this factor into consideration, this manuscript initiated the context of C^*-alg-v perturbed modular M.S, and some FP results were studied. The primary difference between the existing spaces (see [1, 12]) and this new one is the significance of function space and perturbation in the context of C^*-algebra. Examples were generated to illustrate the applicability and efficacy of the proposed results. With the aid of numerical and graphical techniques, the rate of convergence of paired contractive operator was shown to be faster than that of the Banach contractive operator, using one of the presented examples. As an application, one of the obtained theorems was applied to investigate the conditions for the existence and uniqueness of solution of a BVP transformed into an integral equation. Further research can be done in improving the main idea of this work. For example, the ideas of b-M.S, G-M.S, fuzzy M.S, and so on may be examined in the direction of this research. In particular, since every set is completely characterized by its indicator function, the idea of this paper can be extended to fuzzy fixed point results and some hybrid fuzzy models such as: Intuitionistic fuzzy set, L-Fuzzy set, soft set, and fuzzy-soft set, to mention a few.

Author contributions

Mohammed Shehu Shagari: Conceptualization, investigation, methodology, writing-original draft; Zulaihatu Tijjani Ahmad: Conceptualization, methodology; Faryad Ali: Investigation, writing-review and editing; Ghada Ali Basendwah: Formal analysis, investigation, writing-review and editing; Mohammed A. Al-Kadhi: Formal analysis, methodology; Akbar Azam: Formal analysis, investigation, writing-review and editing. All authors have read and agreed to the submitted version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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