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Research article

Nonlinear three layer beam system: Well-posedness, asymptotic stability and numerical analysis

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Abstract: Our goal of this work is to study the well-posedness and the asymptotic behavior of solutions of a nonlinear triple beam system commonly known as the Rao-Nakra beam model. We investigated the effects of non-linear frictional and external forces to the beam model. The Garlekin approximation and the multiplier methods were techniques applied to achieve our goals. Numerical approximations were also performed with the finite difference method to validate our theoritical findings.

Keywords: well-posedness; polynomial and exponential stability; numerical approximation; nonlinear Rao-Nakra beam
Mathematics Subject Classification: 35D30, 35B35, 35L51,74J30, 74S20

1. Introduction

The Rao-Nakra beam model is an extension of the classical Euler-Bernoulli beam theory, which incorporates the effects of shear deformation and rotational inertia in the analysis of the beam. The three-layer sandwich beam model is used in aerospace and automotive structures. Due to their high strength-to-weight ratio, Rao-Nakra beam models are also used in shipbuilding. This model takes into account the motion of two outer face plates (assumed to be relatively rigid) and an inner core layer that is sandwiched between two compliant layers [1]. The Rao-Nakra model, captures the coupling between bending and shear forces and contributes to more accurate predictions of beam deflections, natural frequencies, and dynamic response. For three-layer or sandwich models with arbitrary boundary or constrained-layer, see [2, 3]. Different types of multilayer plates models can be found in [4, 5].

1.1. The Rao-Nakra model

Consider Figure 1, where h_j for $j = 1, 2, 3$ is the thickness of each layer in the beam, and $h = h_1 + h_2 + h_3$ is the thickness of the beam.

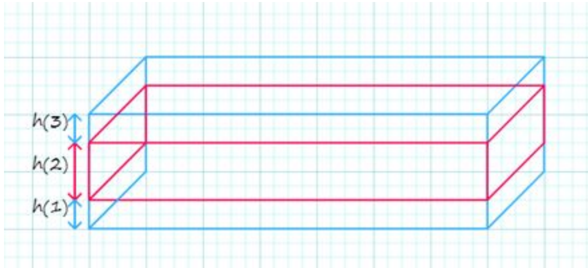


Figure 1. Triple layer beam.

Liu et al. [7] derived (1.1) as the generalized Rao-Nakra beam model after imposing the Kirchhoff hypothesis on the outer layer plates and assuming that the displacement through cross-sections are continuous in addition to being piecewise linear. The model according to Liu et al is as follows

$$\begin{cases} \rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - \delta = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + \delta = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho h c_{tt} + EI c_{xxxx} - G_1 h_1 k (c_x + y^1)_x - G_3 h_3 (c_x + y^3)_x - h_2 \delta_x = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_1 I_1 y^1_{tt} - E_1 I_1 y^1_{xx} - \dfrac{h_1}{2}\delta + G_1 h_1 (c_x + y^1) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_3 I_3 y^3_{tt} - E_3 I_3 y^3_{xx} - \dfrac{h_3}{2}\delta + G_3 h_3 (c_x + y^3) = 0, & (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{1.1}$$

where the bottom and top layers of the plate, longitudinal displacement, and shear angle are represented by a and b, y^1 and y^3, respectively. The transverse displacement of the beam is denoted by c, while the shear stress of the core layer is denoted by the symbol δ. Furthermore, $G_j, \rho_j > 0$ are the shear modulus and density for each layer $j = 1, 2, 3$. The parameters E_j and I_j represent Young's modulus and moments of inertia, respectively. Also, $EI = E_1 I_1 + E_2 I_2 + E_3 I_3$ and $\rho_h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$, (see $(1.1)_3$). Neglecting the rotary inertia in top and bottom layers of the beam, leads to $\rho_1 I_1 = \rho_3 I_3 = 0$ in $(1.1)_4$ and $(1.1)_5$. Furthermore, if we ignore the transverse shear, we obtain $c_x + y^1 = c_x + y^3 = 0$, which is the Euler-Bernoulli condition. Moreover, assuming the material in the core layer has a linear elastic property and that the relationship between the stress and strain is given by $\delta = 2G_2\varepsilon$, then

$$\varepsilon = \frac{1}{2h_2}(-a + b + \alpha c_x)\,, \text{and } \alpha = h_2 + \frac{h_1 + h_2}{2}$$

is the shear strain. Thus, we arrive at this particular Rao-Nakra beam model given by,

$$\begin{cases} \rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - k(-a + b + \alpha c_x) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + k(-a + b + \alpha c_x) = 0, & (x,t) \in (0,L) \times (0,\infty), \\ \rho h c_{tt} + EI c_{xxxx} - \alpha k (-a + b + \alpha c_x)_x = 0, & (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{1.2}$$

with $k = \dfrac{G_2}{h_2}$, $G_2 = \dfrac{E_2}{2(1+\nu)}$, and the poison ratio ν, is in the interval $(-1, \dfrac{1}{2})$.

1.2. Literature review

The Rao-Nakra beam model (1.2) is popular, and numerous researchers have investigated it with various kinds of damping on different layers. By coupling (1.2) with Maxwell-Cattaneo heat conduction, Raposo et al. [9] proved an exponential stability result. Mukiawa et al. [10] studied a Rao-Nakra beam model with a linear frictional force on $(1.2)_3$, Fourier law on $(1.2)_2$ to account for thermal heat, and viscoelastically damped on $(1.2)_1$. In addition to establishing the wellposedness of the problem, the authors obtained a general decay result. Mukiawa [25] investigated the stability of Rao-Nakra beam model through Gurtin-Pipkin's thermal law. Li et al. [14] investigated the system (1.2) and established unstable decay given that only one of the equations in (1.2) is damped. Furthermore, Li et al. [14] proved that when two of the three equations in (1.2) are damped, the system is polynomially stable. Last, when only one of the equations in (1.2) is damped, a polynomial decay result whose rate changes, and is dependent on the damping location and boundary conditions was demonstrated by Liu et al. [15]. Enyi [13], proved a general decay estimate for a triple layer beam model with time-varying internal damping and feedback. For the exact and boundary controllability of Rao-Nakra or multilayer beam models, see Hansen et al. [16–18] and Rajaram [19]. Cavalcanti et al. [20] studied the asymptotic behavior of the Rao-Nakra beam model with non-linear localised frictional and external forces. Fon more asymptotic analysis or long-time dynamics of the Rao-Nakra beam model, we refer the read to the articles in [21–23], and the references therein for more detailed results.

Some authors have considered a generalized Rao-Nakra beam model consisting of two viscous damping and proved the non-existence of exponential stability and existence of polynomial stability in some cases. Liu and Akil [12] considered the following Rao-Nakra beam model (1.3)

$$\begin{cases} \rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - \delta + e_1 a_t = 0, \\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + \delta + e_2 b_t = 0, \\ \rho h c_{tt} + EI c_{xxxx} - G_1 h_1 k (c_x + y^1)_x - G_3 h_3 (c_x + y^3)_x - h_2 \delta_x + e_3 c_t = 0, \\ \rho_1 I_1 y^1_{tt} - E_1 I_1 y^1_{xx} - \dfrac{h_1}{2}\delta + G_1 h_1 (c_x + y^1) + e_4 y^1_t = 0, \\ \rho_3 I_3 y^3_{tt} - E_3 I_3 y^3_{xx} - \dfrac{h_3}{2}\delta + G_3 h_3 (c_x + y^3) + e_5 y^3_t = 0. \end{cases} \tag{1.3}$$

Damping positions are organized into two categories, distinguished by the condition that both the upper and lower layers are either directly damped or not, with each of the categories having three subcategories. Different results consisting of necessary and sufficient conditions for strong stability and instability were proved, in addition to obtaining polynomial decay rate for these categories, as seen in the table below (see [12] for more details).

	Category **1**	Category **2**
Subcategory **1**	$e_1, e_4 > 0$ & $e_2 = e_3 = e_5 = 0$	$e_1, e_2 > 0$ & $e_3 = e_4 = e_5 = 0$
Subcategory **2**	$e_2, e_5 > 0$ & $e_1 = e_3 = e_4 = 0$	$e_1, e_3 > 0$ & $e_2 = e_4 = e_5 = 0$
Subcategory **3**	$e_1, e_5 > 0$ & $e_2 = e_3 = e_4 = 0$	$e_2, e_3 > 0$ & $e_1 = e_4 = e_5 = 0$
Stability type	Strong stability & instability	Strong stability & polynomial stability

Motivated by the results of Liu and Akil [12], for the first time, Ali et al. [8] investigated the well-posedness and stabilization of the generalized Rao-Nakra beam (1.4) when subjected to either local or global viscoelastic Kelvin-Voigt damping effects, by considering different boundary conditions, namely,

$$\begin{cases} \rho_1 h_1 a_{tt} - \left(E_1 h_1 a_x + D_1(x) a_{tx}\right)_x - \delta = 0, \\ \rho_1 I_1 y^1_{tt} - \left(E_1 I_1 y^1_x + D_2(x) y^1_{tx}\right)_x - \dfrac{h_1}{2}\delta + G_1 h_1 (c_x + y^1) = 0, \\ \rho h c_{tt} + (EI c_{xx} + D_3(x) c_{txx})_{xx} - G_1 h_1 k (c_x + y^1)_x - G_3 h_3 (c_x + y^3)_x - h_2 \delta_x = 0, \\ \rho_3 h_3 b_{tt} - \left(E_3 h_3 b_x + D_4(x) b_{tx}\right)_x + \delta = 0, \\ \rho_3 I_3 y^3_{tt} - \left(E_3 I_3 y^3_x + D_5(x) y^3_{tx}\right)_x - \dfrac{h_3}{2}\delta + G_3 h_3 (c_x + y^3) = 0, \end{cases} \tag{1.4}$$

in $(0, L) \times (0, T)$. To start, they showed that the relevant semigroup is analytic, contingent upon the presence of global Kelvin-Voigt damping on all five equations in system (1.4). Moreover, under the influence of only two local damping mechanisms affecting the shear angle displacements at the top and bottom layers (i.e., $D_2, D_5 > 0$ & $D_1 = D_3 = D_4 = 0$), they revealed that the energy of the system decreases polynomially. Quispe et al. [31] considered (1.1) and studied the stability of the system when the Kelvin-Voigt damping is allowed to act on the first and third equations. They showed global well-posedness and lack of exponential stability for the system; hence, they went ahead to show that the system has a polynomial stability at the decay rate of $t^{-\frac{1}{4}}$. In Al-Gharabli et al. [32], the authors considered a thermoplastic Rao–Nakra (sandwich beam) beam system having a nonlinear damping of variable exponent type on one of the equations, while the other two equations are damped by a thermal effect produced by the Coleman–Gurtin's thermal law. Utilizing the multiplier method, they derived overarching results regarding energy decay, with exponential decay serving as a specific example. Specifically, they considered the system

$$\begin{cases} \rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - \kappa(-a + b + \alpha c_x) + \delta_1 \theta_x = 0, \\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + \kappa(-a + b + \alpha c_x) - \delta_1 \theta + \delta_2 \vartheta_x = 0, \\ \rho h c_{tt} + EI c_{xxxx} - \alpha\kappa(-a + b + \alpha c_x)_x + \delta_3(t) \|c_t\|^{m(x)-2} c_t = 0, \\ \rho_4 \theta_t + (\beta_1 - 1)\theta_{xx} - \beta_1 \displaystyle\int_0^{+\infty} g_1(s) \theta_{xx}(x, t - s) ds + \delta_1 (a_{xt} + b_t) = 0, \\ \rho_5 \vartheta_t + (\beta_2 - 1)\theta_{xx} - \beta_2 \displaystyle\int_0^{+\infty} g_2(s) \vartheta_{xx}(x, t - s) ds + \delta_2 b_{xt} = 0, \end{cases} \tag{1.5}$$

in $(0, L) \times (0, T)$. There has been a considerable research on the linear Rao-Nakra beam system, considering linear and viscous damping positioned at different combinations of the five equations. Only few papers have considered the effect of nonlinear damping on the equations of the Rao-Nakra beam system. In this study, we investigate the role of nonlinear dampings on the stabilization of the Rao-Nakra beam system. The critical cases with $p = q = r = 2$ corresponds to the linear viscous damping, which has been extensively studied in the literature. Our focus is on the non-linear Rao-Nakra beam model, namely

$$\begin{cases}\rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - \kappa(-a + b + \alpha c_x) + \mu_1 |a_t|^{p-2} a_t = g_1(x,t), & \text{in } (0,L)\times(0,T),\\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + \kappa(-a + b + \alpha c_x) + \mu_2 |b_t|^{q-2} b_t = g_2(x,t), & \text{in } (0,L)\times(0,T),\\ \rho h c_{tt} + EI c_{xxxx} - \alpha\kappa(-a + b + \alpha c_x)_x + \mu_3 |c_t|^{r-2} c_t = g_3(x,t), & \text{in } (0,L)\times(0,T),\end{cases} \tag{1.6}$$

where $2 \le p, q, r < \infty$, $g_1, g_2, g_3 \in L^2((0,L)\times(0,T))$ are external forces and $\mu_1, \mu_2, \mu_3 > 0$ are damping coefficients. We endow system (1.6) with boundary conditions

$$\begin{cases} a(0,t) = b(0,t) = c(0,t) = c_{xx}(0,t) = 0, \ t \ge 0,\\ a(L,t) = b(L,t) = c(L,t) = c_{xx}(L,t) = 0, \ \ t \ge 0,\end{cases} \tag{1.7}$$

initial data

$$\begin{cases} a(x,0) = a_0(x), \ b(x,0) = b_0(x), \ c(x,0) = c_0(x), \quad x \in (0,L),\\ a_t(x,0) = a_1(x), \ b_t(x,0) = b_1(x), \ c_t(x,0) = c_1(x), \ x \in (0,L).\end{cases} \tag{1.8}$$

We study the contributions and influence of the exponents p, q, r on the stability dynamics of the nonlinear Rao-Nakra beam model in (1.6). We divide this into two broad cases: First is the case $(p,q,r) = (2,2,2)$ corresponding to the well studied linear Rao-Nakra beam with linear viscous damping and proved exponential decay of the system, and in the second case $(p,q,r) \neq (2,2,2)$ with different subcases. Indeed, we show that the energy functional corresponding to system (1.6) satisfies, for some $C, \lambda > 0$,

$$\Phi(t) \le \begin{cases} Ce^{-\lambda t}, & \text{if } (p,q,r) = (2,2,2),\\ C(1+t)^{-\frac{2}{\omega-2}}, & \text{if } (p,q,r) \neq (2,2,2),\\ & \text{where } \ \omega = \max\{p,q,r\}.\end{cases}$$

This article is ordered as follows: Section 2 contains some established facts that are fundamental to achieving our result. In Section 3, we establish the well-posedness of problem (1.6)–(1.8) while the proof of our main decay result is in Section 4. Finally, in Section 5, we provide numerical investigations to corroborate our analytical findings.

2. Preliminaries and functional setting

The following Lemmas are important for our results.

Lemma 2.1. *[26] Let $X \subset V \subset X^*$, with X a reflexive Banach space, V a Hilbert space, and X^* the dual of X such that the embeddings are continuous. If $z \in L^p((0,T), X)$ and $z_t \in L^q((0,T), X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$z \in C([0,T], V).$$

Lemma 2.2. *For any $y, z \in \mathbb{R}$ and $n \ge 2$, the inequality*

$$(|y|^{n-2}y - |z|^{n-2}z)(y - z) \ge 2^{2-n}|y - z|^n, \ \text{ holds.} \tag{2.1}$$

Lemma 2.3. *Komornik [27] Given that a function Φ defined from $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is non-increasing, and suppose there are two positive constants γ and λ such that*

$$\int_s^\infty \Phi^{1+\gamma}(t)dt \le \lambda\Phi(s), 0 \le s < \infty.$$

Then, for all $t \geq 0$, *we have that there exists two constants* $K, \omega > 0$ *such that*

$$\Phi(t) \leq \begin{cases} \dfrac{K}{(1+t)^{\frac{1}{\gamma}}}, & \text{if} \quad \gamma > 0, \\ Ke^{-\omega t}, & \text{if} \quad \gamma = 0. \end{cases} \tag{2.2}$$

We consider the space

$$H^2_{\otimes}(0,L) = H^2(0,L) \cap H^1_0(0,L),$$

and the inner product

$$\langle v, \tilde{v} \rangle_{H^2_{\otimes}(0,L)} = \int_0^L v_{xx} \tilde{v}_{xx} dx. \tag{2.3}$$

Remark 2.1. *[29,30] The norm* $\|.\|_{H^2_{\otimes}(0,L)}$ *and the usual* $H^2(0,L)$*-norm are equivalent. In fact,* $H^2_{\otimes}(0,L)$ *endowed with* $\langle \cdot, \cdot \rangle_{H^2_{\otimes}(0,L)}$ *is a complete inner product space. Also, for* $v \in H^2_{\otimes}(0,L)$*, the following embedding inequality holds*

$$\|v\|_{L^p(0,L)} \leq C_e \|v\|_{H^2_{\otimes}(0,L)} = C_e \|v_{xx}\|^2_{L^2(0,L)}, \tag{2.4}$$

where $C_e = C(L,p) > 0, 1 \leq p < +\infty$.

3. Existence and uniqueness

Going forward, we represent the usual norm on $L^2(0,L)$ by $\|.\|$. $C_p > 0$ stands for the Poincaré constant while $C > 0$, and $C_i > 0$ are generic constants. We then consider the weak form of system (1.6)–(1.8) formulated as follows:

Definition 3.1. For any $T > 0$, find

$$(a,b,c) : [0,T] \to H^1_0(0,L) \times H^1_0(0,L) \times H^2_{\otimes}(0,L),$$

such that

$$\begin{aligned} &(a(\cdot,0),\ b(\cdot,0),\ c(\cdot,0)) = (a_0,\ b_0,\ c_0), \\ &(a_t(\cdot,0),\ b_t(\cdot,0),\ c_t(\cdot,0)) = (a_1\ , b_1,\ c_1), \end{aligned} \tag{3.1}$$

for all $(\phi, \varphi, \psi) \in H^1_0(0,L) \times H^1_0(0,L) \times H^2_{\otimes}(0,L)$ satisfying

$$\begin{cases} \rho_1 h_1 \displaystyle\int_0^L a_{tt} \phi dx + E_1 h_1 \int_0^L a_x \phi_x dx - k \int_0^L (-a + b + \alpha c_x) \phi dx \\ \qquad\qquad\qquad\qquad + \mu_1 \displaystyle\int_0^L |a_t|^{p-2} a_t \phi dx = \int_0^L g_1 \phi dx, \\ \rho_3 h_3 \displaystyle\int_0^L b_{tt} \varphi dx + E_3 h_3 \int_0^L b_x \varphi_x dx + k \int_0^L (-a + b + \alpha c_x) \varphi dx \\ \qquad\qquad\qquad\qquad + \mu_2 \displaystyle\int_0^L |b_t|^{q-2} b_t \varphi dx = \int_0^L g_2 \varphi dx, \\ \rho h \displaystyle\int_0^L c_{tt} \psi dx + EI \int_0^L c_{xx} \psi_{xx} dx + \alpha k \int_0^L (-a + b + \alpha c_x) \psi_x dx \\ \qquad\qquad\qquad\qquad + \mu_3 \displaystyle\int_0^L |c_t|^{r-2} c_t \psi dx = \int_0^L g_3 \psi dx, \end{cases} \tag{3.2}$$

for almost every $t \in [0, T]$.

Next is the statement of the well-posedness theorem of our problem.

Theorem 3.1. *Let*

$$(a_0, b_0, c_0) \in H_0^1(0, L) \times H_0^1(0, L) \times H_\otimes^2(0, L),\ (a_1, b_1, c_1) \in L^2(0, L) \times L^2(0, L) \times L^2(0, L)$$

be given. Suppose $g_1, g_2, g_3 \in L^2((0, L) \times (0, T))$ *and* $2 \le p, q, r < \infty$. *Then, there exists a global weak solution* $(a, b, c) \in L^\infty\left([0, T]; H_0^1(0, L)\right) \times L^\infty\left([0, T]; H_0^1(0, L)\right) \times L^\infty\left([0, T]; H_\otimes^2(0, L)\right)$ *that is unique for the system* (1.6)–(1.8). *Furthermore, the solution satisfies the following*

$$\begin{aligned} a_t &\in L^\infty\left([0, T]; L^2(0, L)\right) \cap L^p\left((0, L) \times (0, T)\right), \\ b_t &\in L^\infty\left([0, T]; L^2(0, L)\right) \cap L^q\left((0, L) \times (0, T)\right), \\ c_t &\in L^\infty\left([0, T]; L^2(0, L)\right) \cap L^r\left((0, L) \times (0, T)\right). \end{aligned}$$

Proof. Existence: The spaces $H_0^1(0, L)$ and $H_\otimes^2(0, L)$ admit a set of orthonormal basis say $(\alpha_i)_{i\ge1}$, $(\beta_i)_{i\ge1}$, and $(\gamma_i)_{i\ge1}$, respectively. We then define the following finite dimensional spaces:

$$A_m := span\{\alpha_1, \alpha_2, ..., \alpha_m\},\ \ B_m := span\{\beta_1, \beta_2, ..., \beta_m\},\ C_m := span\{\gamma_1, \gamma_2, ..., \gamma_m\}$$

with dimensions, $m < \infty$, and denote

$$\begin{aligned} a_0^m &:= \sum_{i=1}^m \langle a_0, \alpha_i\rangle_{L^2(0,L)}\alpha_i \longrightarrow a_0 \in H_0^1(0, L),\ \ a_1^m := \sum_{i=1}^m \langle a_1, \alpha_i\rangle_{L^2(0,L)}\alpha_i \longrightarrow a_1 \in L^2(0, L), \\ b_0^m &:= \sum_{i=1}^m \langle b_0, \beta_i\rangle_{L^2(0,L)}\beta_i \longrightarrow b_0 \in H_0^1(0, L),\ \ b_1^m := \sum_{i=1}^m \langle b_1, \beta_i\rangle_{L^2(0,L)}\beta_i \longrightarrow b_1\ \in L^2(0, L), \\ c_0^m &:= \sum_{i=1}^m \langle c_0, \gamma_i\rangle_{L^2(0,L)}\gamma_i \longrightarrow c_0 \in H_\otimes^2(0, L),\ \ c_1^m := \sum_{i=1}^m \langle c_1, \gamma_i\rangle_{L^2(0,L)}\gamma_i \longrightarrow c_1 \in L^2(0, L). \end{aligned} \tag{3.3}$$

For each m, we need to determine solutions of the form

$$(a^m(x, t), b^m(x, t), c^m(x, t)) = \left(\sum_{i=1}^m u_i(t)\alpha_i(x), \sum_{i=1}^m v_i(t)\beta_i(x), \sum_{i=1}^m w_j(t)\gamma_i(x)\right)$$

satisfying the approximate problem below for $i = 1, 2, ..., m$,

$$
\begin{cases}
\rho_1 h_1 \dfrac{d}{dt}\displaystyle\int_0^L a_t^m \alpha_i dx + E_1 h_1 \int_0^L a_x^m \alpha_{ix} dx - k \int_0^L (-a^m + b^m + \alpha c_x^m)\alpha_i dx \\
\qquad\qquad\qquad\qquad\qquad + \mu_1 \displaystyle\int_0^L |a_t^m|^{p-2} a_t^m \alpha_i dx = \int_0^L g_1 \alpha_i dx, \\
\rho_3 h_3 \dfrac{d}{dt}\displaystyle\int_0^L b_t^m \beta_i dx + E_3 h_3 \int_0^L b_x^m \beta_{ix} dx + k \int_0^L (-a^m + b^m + \alpha c_x^m)\beta_i dx \\
\qquad\qquad\qquad\qquad\qquad + \mu_2 \displaystyle\int_0^L |b_t^m|^{q-2} b_t^m \beta_i dx = \int_0^L g_2 \beta_i dx, \\
\rho h \dfrac{d}{dt}\displaystyle\int_0^L c_t^m \gamma_i dx + EI \int_0^L c_{xx}^m \gamma_{ixx} dx + \alpha k \int_0^L (-a^m + b^m + \alpha c_x^m)\gamma_{ix} dx \\
\qquad\qquad\qquad\qquad\qquad \mu_3 \displaystyle\int_0^L |c_t^m|^{r-2} c_t^m \gamma_i dx = \int_0^L g_3 \gamma_i dx, \\
a^m(\cdot,0) = a_0^m,\ a_t^m(\cdot,0) = a_1^m,\ b^m(\cdot,0) = b_0^m,\ b_t^m(\cdot,0) = b_1^m, \\
c^m(\cdot,0) = c_0^m,\ c_t^m(\cdot,0) = c_1^m.
\end{cases}
\tag{3.4}
$$

Equation (3.4) yields system of ordinary differential equations with undetermined functions

$$(u_i(t), v_i(t), w_i(t)),\ \text{for } i = 1, 2, \cdots m.$$

For each $m \geq 1$, the classical theory of ODE guarantees the existence of a C^2- functions (a^m, b^m, c^m) on some finite interval $0 \leq t_m \leq T$, for the system (3.4). We now show that for each $m \geq 1$, t_m can be extended to T. First, by multiplying $(3.4)_1, (3.4)_2$, and $(3.4)_3$ by $u_i'(t), v_i'(t)$ and $w_i'(t)$ for $1 \leq i \leq m$, respectively, and adding the resulting equations lead to

$$
\begin{cases}
\dfrac{1}{2}\dfrac{d}{dt}\left[\rho_1 h_1 \|a_t^m\|^2 + E_1 h_1 \|a_x^m\|^2\right] - k \displaystyle\int_0^L (-a^m + b^m + \alpha c_x^m) a_t^m dx \\
\qquad\qquad\qquad\qquad\qquad + \mu_1 \displaystyle\int_0^L |a_t^m|^p dx = \int_0^L g_1 a_t^m dx, \\
\dfrac{1}{2}\dfrac{d}{dt}\left[\rho_3 h_3 \|b_t^m\|^2 + E_3 h_3 \|b_x^m\|^2\right] + k \displaystyle\int_0^L (-a^m + b^m + \alpha c_x^m) b_t^m dx \\
\qquad\qquad\qquad\qquad\qquad + \mu_2 \displaystyle\int_0^L |b_t^m|^q dx = \int_0^L g_2 b_t^m dx, \\
\dfrac{1}{2}\dfrac{d}{dt}\left[\rho h \|c_t^m\|^2 + EI \|c_{xx}^m\|^2\right] + \alpha k \displaystyle\int_0^L (-a^m + b^m + \alpha c_x^m) c_{xt}^m dx \\
\qquad\qquad\qquad\qquad\qquad + \mu_3 \displaystyle\int_0^L |c_t^m|^r dx = \int_0^L g_3 c_t^m dx.
\end{cases}
\tag{3.5}
$$

Adding the equations in (3.5) yields

$$
\begin{aligned}
\frac{d}{dt}\Phi^m(t) + \mu_1 \int_0^L |a_t^m|^p dx + \mu_2 \int_0^L |b_t^m|^q dx + \mu_3 \int_0^L |c_t^m|^r dx \\
= \int_0^L g_1 a_t^m dx + \int_0^L g_2 b_t^m dx + \int_0^L g_3 c_t^m dx,
\end{aligned}
\tag{3.6}
$$

where

$$\begin{aligned}\Phi^m(t) =&\frac{1}{2}\left[\rho_1 h_1\|a_t^m\|^2 + E_1 h_1\|a_x^m\|^2 + \rho_3 h_3\|b_t^m\|^2 + E_3 h_3\|b_x^m\|^2\right]\\ &+ \frac{1}{2}\left[\rho h\|c_t^m\|^2 + EI\|c_{xx}^m\|^2 + k\|(-a^m + b^m + \alpha c_x^m)\|^2\right].\end{aligned} \tag{3.7}$$

Integrating (3.6) over $(0, t)$, $0 \le t < t_m$, for each $m \ge 1$, we get

$$\begin{aligned}&\Phi^m(t) + \mu_1\int_0^t\int_0^L |a_t^m|^p dxds + \mu_2\int_0^t\int_0^L |b_t^m|^q dxds + \mu_3\int_0^t\int_0^L |c_t^m|^r dxds\\ &\qquad = \Phi^m(0) + \int_0^t\int_0^L g_1 a_t^m dxds + \int_0^t\int_0^L g_2 b_t^m dxds + \int_0^t\int_0^L g_3 c_t^m dxds.\end{aligned} \tag{3.8}$$

Furthermore, for each $0 < t < t_m$,

$$\begin{aligned}\Phi^m(t)+&\mu_1\int_0^t\int_0^L |a_t^m|^p dxds + \mu_2\int_0^t\int_0^L |b_t^m|^q dxds + \mu_3\int_0^t\int_0^L |c_t^m|^r dxds\\ \le&\Phi^m(0) + \epsilon_1\int_0^t\int_0^L |a_t^m|^2 dxds + \epsilon_2\int_0^t\int_0^L |b_t^m|^2 dxds + \epsilon_3\int_0^t\int_0^L |c_t^m|^2 dxds\\ &+ C_{\epsilon_1}\int_0^T\int_0^L |g_1|^2 dxdt + C_{\epsilon_2}\int_0^T\int_0^L |g_2|^2 dxdt + C_{\epsilon_3}\int_0^T\int_0^L |g_3|^2 dxdt,\end{aligned} \tag{3.9}$$

for some $\epsilon_i > 0, i \in \{1, 2, 3\}$, to be declared. In view of (3.3) and (3.7), we obtain

$$\begin{aligned}&\frac{\rho_1 h_1}{2}\|a_t^m\|^2 + \frac{E_1 h_1}{2}\|a_x^m\|^2 + \frac{\rho_3 h_3}{2}\|b_t^m\|^2 + \frac{E_3 h_3}{2}\|b_x^m\|^2 + \frac{\rho h}{2}\|c_t^m\|^2 + \frac{EI}{2}\|c_{xx}^m\|^2\\ &+ \frac{k}{2}\|(-a^m + b^m + \alpha c_x^m)\|^2 + \mu_1\int_0^t\int_0^L |a_t^m|^p dxds + \mu_2\int_0^t\int_0^L |b_t^m|^q dxds\\ &+ \mu_3\int_0^t\int_0^L |c_t^m|^r dxds\\ &\le C_{\epsilon_1,\epsilon_2,\epsilon_3} + \epsilon_1 T\sup_{0<t<t_m}\|a_t^m\|^2 + \epsilon_2 T\sup_{0<t<t_m}\|b_t^m\|^2 + \epsilon_3 T\sup_{0<t<t_m}\|c_t^m\|^2,\end{aligned} \tag{3.10}$$

with

$$C_{\epsilon_1,\epsilon_2,\epsilon_3} = \Phi(0) + C_{\epsilon_1}\int_0^T\int_0^L |g_1|^2 dxdt + C_{\epsilon_2}\int_0^T\int_0^L |g_2|^2 dxdt + C_{\epsilon_3}\int_0^T\int_0^L |g_3|^2 dxdt.$$

Equation (3.10) consists of positive terms, therefore taking supremum over t, $0 < t < t_m$, yields

$$\begin{aligned}&\frac{\rho_1 h_1}{2}\sup_{0<t<t_m}\|a_t^m\|^2 + \frac{E_1 h_1}{2}\sup_{0<t<t_m}\|a_x^m\|^2 + \frac{\rho_3 h_3}{2}\sup_{0<t<t_m}\|b_t^m\|^2 + \frac{E_3 h_3}{2}\sup_{0<t<t_m}\|b_x^m\|^2\\ &+ \frac{\rho h}{2}\sup_{0<t<t_m}\|c_t^m\|^2 + \frac{EI}{2}\sup_{0<t<t_m}\|c_{xx}^m\|^2 + \frac{k}{2}\sup_{0<t<t_m}\|(-a^m + b^m + \alpha c_x^m)\|^2\\ &+ \mu_1\int_0^T\int_0^L |a_t^m|^p dxdt + \mu_2\int_0^T\int_0^L |b_t^m|^q dxdt + \mu_3\int_0^T\int_0^L |c_t^m|^r dxdt\\ &\le 10C_{\epsilon_1,\epsilon_2,\epsilon_3} + 10\epsilon_1 T\sup_{0<t<t_m}\|a_t^m\|^2 + 10\epsilon_2 T\sup_{0<t<t_m}\|b_t^m\|^2 + 10\epsilon_3 T\sup_{0<t<t_m}\|c_t^m\|^2.\end{aligned} \tag{3.11}$$

Setting

$$\epsilon_1 = \frac{\rho_1 h_1}{40T},\ \epsilon_2 = \frac{\rho_3 h_3}{40T},\ \epsilon_3 = \frac{\rho h}{40T},$$

we get a constant C that does not depend on m which satisfies

$$\begin{aligned}
&\frac{\rho_1 h_1}{2}\sup_{0<t<t_m}\|a_t^m\|^2 + \frac{E_1 h_1}{2}\sup_{0<t<t_m}\|a_x^m\|^2 + \frac{\rho_3 h_3}{2}\sup_{0<t<t_m}\|b_t^m\|^2 + \frac{E_3 h_3}{2}\sup_{0<t<t_m}\|b_x^m\|^2 \\
&+ \frac{\rho h}{2}\sup_{0<t<t_m}\|c_t^m\|^2 + \frac{EI}{2}\sup_{0<t<t_m}\|c_{xx}^m\|^2 + \frac{k}{2}\sup_{0<t<t_m}\|(-a^m + b^m + \alpha c_x^m)\|^2 \\
&+ \mu_1 \int_0^T\int_0^L |a_t^m|^p dxdt + \mu_2 \int_0^T\int_0^L |b_t^m|^q dxdt + \mu_3 \int_0^T\int_0^L |c_t^m|^r dxdt \le C.
\end{aligned} \tag{3.12}$$

Therefore, the solution of (3.4) can be extended to the interval $[0, T]$. We can deduce from (3.12) that

$$\begin{aligned}
&\{a^m\} \text{ is bounded in } L^\infty((0,T); H_0^1(0,L)),\\
&\{a_t^m\} \text{ is bounded in } L^\infty((0,T); L^2(0,L)) \cap L^p((0,L)\times(0,T)),\\
&\{b^m\} \text{ is bounded in } L^\infty((0,T); H_0^1(0,L)),\\
&\{a_t^m\} \text{ is bounded in } L^\infty((0,T); L^2(0,L)) \cap L^q((0,L)\times(0,T)),\\
&\{c^m\} \text{ is bounded in } L^\infty((0,T); H_\otimes^2(0,L)),\\
&\{c_t^m\} \text{ is bounded in } L^\infty((0,T); L^2(0,L)) \cap L^r((0,L)\times(0,T)).
\end{aligned} \tag{3.13}$$

Thus, we can extract sub-sequences $\{(a^n, b^n, c^n)\}$ of $\{(a^m, b^m, c^m)\}$ such that

$$\begin{aligned}
&a^n \overset{*}{\rightharpoonup} a \in L^\infty((0,T); H_0^1(0,L)) \text{ and } a^n \rightharpoonup a \in L^2((0,T); H_0^1(0,L)),\\
&a_t^n \overset{*}{\rightharpoonup} a_t \in L^\infty((0,T); L^2(0,L)) \text{ and } a_t^n \rightharpoonup a_t \in L^2((0,T); L^2(0,L)),\\
&a_t^n \rightharpoonup a_t \in L^q((0,T); L^p(0,L)),\\
&b^n \overset{*}{\rightharpoonup} b \in L^\infty((0,T); H_0^1(0,L)) \text{ and } b^n \rightharpoonup b \in L^2((0,T); H_0^1(0,L)),\\
&b_t^n \overset{*}{\rightharpoonup} b_t \in L^\infty((0,T); L^2(0,L)) \text{ and } b_t^n \rightharpoonup b_t \in L^2((0,T); L^2(0,L)),\\
&b_t^n \rightharpoonup b_t \in L^q((0,T); L^q(0,L)),\\
&c^n \overset{*}{\rightharpoonup} c \in L^\infty((0,T); H_\otimes^2(0,L)) \text{ and } c^n \rightharpoonup c \in L^2((0,T); H_\otimes^2(0,L)),\\
&c_t^n \overset{*}{\rightharpoonup} c_t \in L^\infty((0,T); L^2(0,L)) \text{ and } c_t^n \rightharpoonup c_t \in L^2((0,T); L^2(0,L)),\\
&c_t^n \rightharpoonup c_t \in L^r((0,T); L^r(0,L)).
\end{aligned} \tag{3.14}$$

Since a_t^n, b_t^n, and c_t^n are respectively bounded sequences in

$$L^p\left((0,L)\times(0,T)\right)),\ L^q\left((0,L)\times(0,T)\right)),\ \text{and } L^r\left((0,L)\times(0,T)\right)),$$

we have

$$\begin{cases}
\displaystyle\int_0^T\int_0^L \left||a_t^n|^{p-2}a_t^n\right|^{\frac{p}{p-1}} dxdt = \int_0^T\int_0^L |a_t^n|^p dxdt \leqslant C,\\
\displaystyle\int_0^T\int_0^L \left||b_t^n|^{q-2}b_t^l\right|^{\frac{q}{q-1}} dxdt = \int_0^T\int_0^L |b_t^n|^q dxdt \leqslant C,\\
\displaystyle\int_0^T\int_0^L \left||c_t^n|^{r-2}c_t^n\right|^{\frac{r}{r-1}} dxdt = \int_0^T\int_0^L |c_t^n|^r dxdt \leqslant C.
\end{cases} \tag{3.15}$$

Consequently,

$$\begin{cases} |a_t^n|^{p-2}a_t^n \text{ is bounded in } L^{\frac{p}{p-1}}((0,L)\times(0,T)),\\ |b_t^n|^{q-2}b_t^n \text{ is bounded in } L^{\frac{q}{q-1}}((0,L)\times(0,T)),\\ |c_t^n|^{r-2}c_t^n \text{ is bounded in } L^{\frac{r}{r-1}}((0,L)\times(0,T)). \end{cases} \tag{3.16}$$

Therefore, there exist sub-sequences $(|a_t^l|^{p-2}a_t^l)_{l\geq 1}$, $(|b_t^l|^{q-2}b_t^l)_{l\geq 1}$ and $(|c_t^l|^{r-2}c_t^l)_{l\geq 1}$, such that

$$\begin{cases} |a_t^l|^{p-2}a_t^l \rightharpoonup \Upsilon \in L^{\frac{p}{p-1}}((0,L)\times(0,T)),\\ |b_t^l|^{q-2}b_t^l \rightharpoonup \Lambda \in L^{\frac{q}{q-1}}((0,L)\times(0,T)),\\ |c_t^l|^{r-2}c_t^l \rightharpoonup \Theta \in L^{\frac{r}{r-1}}((0,L)\times(0,T)), \end{cases} \tag{3.17}$$

where the functions Θ, Υ, and Λ are to be determined. Now, in view of (3.14), we obtain sub-sequences (a^l, b^l, c^l), such that

$$\begin{aligned} a^l &\rightharpoonup a \in L^2\left((0,T); H_0^1(0,L)\right),\\ a_t^l &\rightharpoonup a_t \in L^2\left((0,T); L^2(0,L)\right),\\ b^l &\rightharpoonup b \in L^2\left((0,T), H_0^1(0,L)\right),\\ b_t^l &\rightharpoonup b_t \in L^2\left((0,T); L^2(0,L)\right),\\ c^l &\rightharpoonup c \in L^2\left((0,T), H_\otimes^2(0,L)\right),\\ c_t^l &\rightharpoonup c_t \in L^2\left((0,T); L^2(0,L)\right). \end{aligned} \tag{3.18}$$

Since $H_\otimes^2(0,L)$ and $H_0^1(0,L)$ are compactly embedded in $L^2(0,L)$, the compacity lemma [26] guarantees the existence of sub-sequences (a^j, b^j, c^j) of (a^l, b^l, c^l), such that

$$\begin{aligned} a^j &\to a \in L^2\left((0,T); L^2(0,L)\right),\\ b^j &\to b \in L^2\left((0,T); L^2(0,L)\right),\\ c^j &\to c \in L^2\left((0,T); L^2(0,L)\right). \end{aligned} \tag{3.19}$$

Consequently,

$$\begin{aligned} a^j &\to a \in (0,L)\times(0,T) \ \text{ almost everywhere},\\ b^j &\to b \in (0,L)\times(0,T) \ \text{ almost everywhere},\\ c^j &\to c \in (0,L)\times(0,T) \ \text{ almost everywhere}. \end{aligned} \tag{3.20}$$

Considering (3.4), we replace (a^m, b^m, c^m) with (a^j, b^j, c^j) and then integrate over the open interval $0 < t < T$, we get

$$\begin{cases}\rho_1 h_1 \int_0^L a_t^j \alpha_i dx + E_1 h_1 \int_0^t \int_0^L a_x^j \alpha_{ix} dxds - k \int_0^t \int_0^L (-a^j + b^j + \alpha c_x^j)\alpha_i dxds \\ \qquad + \mu_1 \int_0^t \int_0^L |a_t^j|^{p-2} a_t^j \alpha_i dxds = \rho_1 h_1 \int_0^L a_1^j \alpha_i dx + \int_0^t \int_0^L g_1 \alpha_i dxds, \\ \rho_3 h_3 \int_0^L b_t^j \beta_i dx + E_3 h_3 \int_0^t \int_0^L b_x^j \beta_{ix} dxds + k \int_0^t \int_0^L (-a^j + b^j + \alpha c_x^j)\beta_i dxds \\ \qquad + \mu_2 \int_0^t \int_0^L |b_t^j|^{q-2} b_t^j \beta_i dxds = \rho_3 h_3 \int_0^L b_1^j \beta_i dx + \int_0^t \int_0^L g_2 \beta_i dxds, \\ \rho h \int_0^L c_t^j \gamma_i dx + EI \int_0^t \int_0^L c_{xx}^j \gamma_{ixx} dxds + \alpha k \int_0^t \int_0^L (-a^j + b^j + \alpha c_x^j)\gamma_{ix} dxds \\ \qquad + \mu_3 \int_0^t \int_0^L |c_t^j|^{r-2} c_t^j \gamma_i dxds = \rho h \int_0^L c_1^j \gamma_i dx + \int_0^t \int_0^L g_3 \gamma_i dxds \end{cases} \tag{3.21}$$

provided $i \leq j$.
For any $i \geq 1$, in view of (3.17)–(3.20) and letting $j \to \infty$, we obtain

$$\begin{cases}\rho_1 h_1 \int_0^L a_t \alpha_i dx + E_1 h_1 \int_0^t \int_0^L a_x \alpha_{ix} dxds - k \int_0^t \int_0^L (-a + b + \alpha c_x)\alpha_i dxds \\ \qquad + \mu_1 \int_0^t \int_0^L \Upsilon \alpha_i dxds = \rho_1 h_1 \int_0^L a_1 \alpha_i dx + \int_0^t \int_0^L g_1 \alpha_i dxds, \\ \rho_3 h_3 \int_0^L b_t \beta_i dx + E_3 h_3 \int_0^t \int_0^L b_x \beta_{ix} dxds + k \int_0^t \int_0^L (-a + b + \alpha c_x)\beta_i dxds \\ \qquad + \mu_2 \int_0^t \int_0^L \Lambda \beta_i dxds = \rho_3 h_3 \int_0^L b_1 \beta_i dx + \int_0^t \int_0^L g_2 \beta_i dxds, \\ \rho h \int_0^L c_t \gamma_i dx + EI \int_0^t \int_0^L c_{xx} \gamma_{ixx} dxds + \alpha k \int_0^t \int_0^L (-a + b + \alpha c_x)\gamma_{ix} dxds \\ \qquad + \mu_3 \int_0^t \int_0^L \Theta \gamma_i dxds = \rho h \int_0^L c_1 \gamma_i dx + \int_0^t \int_0^L g_3 \gamma_i dxds. \end{cases} \tag{3.22}$$

It follows that

$$\begin{cases}\rho_1 h_1 \int_0^L a_t \phi dx = - E_1 h_1 \int_0^t \int_0^L a_x \phi_x dxds + k \int_0^t \int_0^L (-a + b + \alpha c_x)\phi dxds \\ \qquad - \mu_1 \int_0^t \int_0^L \Upsilon \phi dxds + \rho_1 h_1 \int_0^L a_1 \phi dx + \int_0^t \int_0^L g_1 \phi dxds, \ \forall \phi \in H_0^1(0, L), \\ \rho_3 h_3 \int_0^L b_t \varphi dx = - E_3 h_3 \int_0^t \int_0^L v_x \varphi_x dxds - k \int_0^t \int_0^L (-a + b + \alpha c_x)\varphi dxds \\ \qquad - \mu_2 \int_0^t \int_0^L \Lambda \varphi dxds + \rho_3 h_3 \int_0^L b_1 \varphi dx + \int_0^t \int_0^L g_2 \varphi dxds, \ \forall \varphi \in H_0^1(0, L), \\ \rho h \int_0^L c_t \psi dx = - EI \int_0^t \int_0^L c_{xx} \psi_{xx} dxds - \alpha k \int_0^t \int_0^L (-a + b + \alpha c_x)\psi_x dxds \\ \qquad - \mu_3 \int_0^t \int_0^L \Theta \psi dxds + \rho h \int_0^L c_1 \psi dx + \int_0^t \int_0^L g_3 \psi dxds, \ \forall \psi \in H_{\otimes}^2(0, L). \end{cases} \tag{3.23}$$

Since the terms of on the right hand side of the three equations in (3.23) are differentiable, we deduce that provided $0 < t < T$, we have

$$\begin{cases} \rho_1 h_1 \int_0^L a_{tt}\phi dx + E_1 h_1 \int_0^L a_x \phi_x dx - k\int_0^L (-a + b + \alpha c_x)\phi dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_1 \int_0^L \Upsilon \phi dx = \int_0^L g_1 \phi dx, \\ \rho_3 h_3 \int_0^L b_{tt}\varphi dx + E_3 h_3 \int_0^L b_x \varphi_x dx + k\int_0^L (-a + b + \alpha c_x)\varphi dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_2 \int_0^L \Lambda \varphi dx = \int_0^L g_2 \varphi dx, \\ \rho h \int_0^L c_{tt}\psi dx + EI \int_0^L c_{xx}\psi_{xx} dx + \alpha k\int_0^L (-a + b + \alpha c_x)\psi_x dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_3 \int_0^L \Theta \psi dx = \int_0^L g_3 \psi dx, \end{cases} \tag{3.24}$$

for all $(\phi, \varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L) \times H_\otimes^2(0, L)$. Using the same steps as in [24, Eqs (3.19)–(3.30)] with modification to our case here, we obtain that

$$\Upsilon = |a_t|^{p-2}a_t\,, \quad \Lambda = |b_t|^{q-2}b_t \text{ and } \Theta = |c_t|^{r-2}c_t.$$

Therefore, for any $(\phi, \varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L) \times H_\otimes^2(0, L)$, system (3.24) becomes

$$\begin{cases} \rho_1 h_1 \int_0^L a_{tt}\phi dx + E_1 h_1 \int_0^L a_x \phi_x dx - k\int_0^L (-a + b + \alpha c_x)\phi dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_1 \int_0^L |a_t|^{p-2}a_t \phi dx = \int_0^L g_1 \phi dx, \\ \rho_3 h_3 \int_0^L b_{tt}\varphi dx + E_3 h_3 \int_0^L b_x \varphi_x dx + k\int_0^L (-a + b + \alpha c_x)\varphi dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_2 \int_0^L |b_t|^{q-2}b_t \varphi dx = \int_0^L g_2 \varphi dx, \\ \rho h \int_0^L c_{tt}\psi dx + EI \int_0^L c_{xx}\psi_{xx} dx + \alpha k\int_0^L (-a + b + \alpha c_x)\psi_x dx \\ \qquad\qquad\qquad\qquad\qquad + \mu_3 \int_0^L |c_t|^{r-2}c_t \psi dx = \int_0^L g_3 \psi dx. \end{cases} \tag{3.25}$$

Recalling (3.18) and in view of Lemma 2.1, we can extract up to a sub-sequence

$$\begin{aligned} &a^j \longrightarrow a \in C([0, T]; L^2(0, L)),\ b^j \longrightarrow b \in C([0, T]; L^2(0, L)), \\ &c^j \longrightarrow c \in C([0, T]; L^2(0, L)). \end{aligned}$$

It follows that, $a^j(x, 0), b^j(x, 0)$, and $c^j(x, 0)$ are well defined and

$$\begin{aligned} &a^j(x, 0) \longrightarrow a(x, 0) \in L^2(0, L),\quad b^j(x, 0) \longrightarrow b(x, 0) \in L^2(0, L), \\ &c^j(x, 0) \longrightarrow c(x, 0) \in L^2(0, L). \end{aligned} \tag{3.26}$$

In view of (3.3), we conclude the following convergences,

$$\begin{aligned}&a_0^j(x,0)=a_0^j(x)\longrightarrow a_0(x)\text{ in }H_0^1(0,L),\quad b_0^j(x,0)=b_0^j(x)\longrightarrow b_0(x)\text{ in }H_0^1(0,L),\\&c_0^j(x,0)=c_0^j(x)\longrightarrow c_0(x)\text{ in }H_\otimes^2(0,L).\end{aligned}\tag{3.27}$$

Therefore

$$a(x,0)=a_0(x),\ b(x,0)=b_0(x),\ c(x,0)=c_0(x),\ \text{in }(0,L).\tag{3.28}$$

Let $\eta,\kappa,\xi \in C_0^\infty(0,T)$, multiplying the equations in (3.4) by η,κ,ξ respectively, then replacing (a^m,b^m,c^m) by (a^j,b^j,c^j) in (3.4), and integrate over the open interval $(0,T)$. For all $i\le j$, we get

$$\begin{aligned}-\rho_1h_1\int_0^T\left(a_t^j,\alpha_i\right)\eta'(t)dt=&-E_1h_1\int_0^T\left(a_x^j,\alpha_{ix}\right)\eta(t)dt+k\int_0^T\left((-a^j+b^j+\alpha c_x^j),\alpha_i\right)\eta(t)dt\\&-\mu_1\int_0^T\left(|a_t^j|^{p-2}a_t^j,\alpha_i\right)\eta(t)dt+\int_0^T(g_1,\alpha_i)\,\eta(t)dt,\\-\rho_3h_3\int_0^T\left(b_t^j,\beta_i\right)\kappa'(t)dt=&-E_3h_3\int_0^T\left(b_x^j,\beta_{ix}\right)\kappa(t)dt-k\int_0^T\left((-a^j+b^j+\alpha c_x^j),\beta_i\right)\kappa(t)dt\\&-\mu_2\int_0^T\left(|b_t^j|^{q-2}b_t^j,\beta_i\right)\kappa(t)dt+\int_0^T(g_2,\beta_i)\,\kappa(t)dt,\\-\rho h\int_0^T\left(c_t^j,\gamma_i\right)\xi'(t)dt=&-EI\int_0^T\left(c_{xx}^j,\gamma_{ixx}\right)\xi(t)dt-\alpha k\int_0^T\left((-a^j+b^j+\alpha c_x^j),\gamma_{ix}\right)\xi(t)dt\\&-\mu_3\int_0^T\left(|c_t^j|^{r-2}c_t^j,\gamma_i\right)\xi(t)dt+\int_0^T(g_3,\gamma_i)\,\xi(t)dt.\end{aligned}$$

By letting $j\longrightarrow+\infty$, we get for any

$$\eta,\kappa,\xi\in C_0^\infty(0,T),$$

and for all

$$(\phi,\varphi,\psi)\in H_0^1(0,L)\times H_0^1(0,L)\times H_\otimes^2(0,L),$$

$$\begin{aligned}-\rho_1h_1\int_0^T(a_t,\phi)\,\eta'(t)dt=&-E_1h_1\int_0^T(a_x,\phi_x)\,\eta(t)dt+k\int_0^T((-a+b+\alpha c_x),\phi)\,\eta(t)dt\\&-\mu_1\int_0^T\left(|a_t|^{p-2}a_t,\phi\right)\eta(t)dt+\int_0^T(g_1,\phi)\,\eta(t)dt,\\-\rho_3h_3\int_0^T(b_t,\varphi)\,\kappa'(t)dt=&-E_3h_3\int_0^T(b_x,\varphi_x)\,\kappa(t)dt-k\int_0^T((-a+b+\alpha c_x),\varphi)\,\kappa(t)dt\\&-\mu_2\int_0^T\left(|b_t|^{q-2}b_t,\varphi\right)\kappa(t)dt+\int_0^T(g_2,\varphi)\,\kappa(t)dt,\\-\rho h\int_0^T(c_t,\psi)\,\xi'(t)dt=&-EI\int_0^T(c_{xx},\psi_{xx})\,\xi(t)dt-\alpha k\int_0^T((-a+b+\alpha c_x),\psi_x)\,\xi(t)dt\\&-\mu_3\int_0^T\left(|c_t|^{r-2}c_t,\psi\right)\xi(t)dt+\int_0^T(g_3,\psi)\,\xi(t)dt.\end{aligned}$$

Thus, we get

$$a_{tt} \in L^2\left([0,T];H^{-1}(0,L)\right),\ \ b_{tt} \in L^2\left([0,T];H^{-1}(0,L)\right),\ c_{tt} \in L^2\left([0,T];H_{\otimes}^{-2}(0,L)\right). \tag{3.29}$$

Using (3.18) and (3.29), we arrive at

$$\begin{cases} a_t \in L^2\left((0,T);L^2(0,L)\right), & a_{tt} \in L^2\left([0,T];H^{-1}(0,L)\right), \\ b_t \in L^2\left((0,T);L^2(0,L)\right), & b_{tt} \in L^2\left([0,T];H^{-1}(0,L)\right), \\ c_t \in L^2\left((0,T);L^2(0,L)\right), & c_{tt} \in L^2\left([0,T];H_{\otimes}^{-2}(0,L)\right). \end{cases} \tag{3.30}$$

On the account of Lemma 2.1, we get

$$a_t \in C([0,T);H^{-1}(0,L)),\ b_t \in C([0,T);H^{-1}(0,L))\ \text{ and } c_t \in C([0,T),H_{\otimes}^{-2}(0,L)). \tag{3.31}$$

Therefore, $a_t^j(x,0), b_t^j(x,0)$ and $c_t^j(x,0)$ all makes sense and so

$$\begin{cases} a_t^j(x,0) \longrightarrow a_t(x,0) \in H^{-1}(0,L), \\ b_t^j(x,0) \longrightarrow b_t(x,0) \in H^{-1}(0,L), \\ c_t^j(x,0) \longrightarrow c_t(x,0) \in H_{\otimes}^{-2}(0,L). \end{cases}$$

Recalling (3.3) leads to the following

$$\begin{cases} a_t^j(x,0) = a_1^j(x) \longrightarrow a_1(x) \in L^2(0,L), \\ b_t^j(x,0) = v_1^j(x) \longrightarrow b_1(x) \in L^2(0,L), \\ c_t^j(x,0) = c_1^j(x) \longrightarrow c_1(x) \in L^2(0,L). \end{cases}$$

Hence,

$$a_t(x,0) = a_1(x),\ \ b_t(x,0) = b_1(x),\ \ c_t(x,0) = c_1(x),\ \text{in}\ \ (0,L). \tag{3.32}$$

Uniqueness. Let $(\tilde{a}\ ,\tilde{b},\ \tilde{c})$ and $(\hat{a},\ \hat{b},\ \hat{c})$ be two solutions of (3.25), (3.28) and (3.32), then

$$(a,b,c) = \left(\tilde{a}-\hat{a},\tilde{b}-\hat{b},\tilde{c}-\hat{c}\right)$$

satisfies for any

$$(\phi,\varphi,\psi) \in H_0^1(0,L)\times H_0^1(0,L)\times H_{\otimes}^2(0,L),$$

$$\begin{cases} \rho_1 h_1 \displaystyle\int_0^L a_{tt}\phi dx + E_1 h_1 \int_0^L a_x\phi_x dx - k\int_0^L (-a+b+\alpha c_x)\phi dx \\ \qquad\qquad + \mu_1 \displaystyle\int_0^L \left(|\tilde{a}_t|^{p-2}\tilde{a}_t - |\hat{a}_t|^{p-2}\hat{a}_t\right)\phi dx = 0, \\ \rho_3 h_3 \displaystyle\int_0^L b_{tt}\varphi dx + E_3 h_3 \int_0^L b_x\varphi_x dx + k\int_0^L (-a+b+\alpha c_x)\varphi dx \\ \qquad\qquad + \mu_2 \displaystyle\int_0^L \left(|\tilde{b}_t|^{q-2}\tilde{b}_t - |\hat{b}_t|^{q-2}\hat{a}_t\right)\phi dx = 0, \\ \rho h \displaystyle\int_0^L c_{tt}\psi dx + EI\int_0^L c_{xx}\psi_{xx} dx + \alpha k\int_0^L (-a+b+\alpha c_x)\psi_x dx \\ \qquad\qquad + \mu_3 \displaystyle\int_0^L \left(|\tilde{c}_t|^{r-2}\tilde{c}_t - |\hat{c}_t|^{r-2}\hat{c}_t\right)\phi dx = 0. \end{cases} \tag{3.33}$$

Notice that, for all (ϕ, φ, ψ) in $C_0^\infty\left((0,L)\times(0,T)\right)$, system (3.33) is valid. Thus, by density argument, the system (3.33) also holds for all $(\phi, \varphi, \psi) \in L^2\left((0,L)\times(0,T)\right)$. Therefore, replacing (ϕ, φ, ψ) by (a_t, b_t, c_t) in the system (3.33), we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\left[\rho_1 h_1\|a_t\|^2 + E_1 h_1\|a_x\|^2\right] - k\int_0^L (-a+b+\alpha c_x)a_t dx \\ \qquad\qquad + \mu_1 \int_0^L \left(|\tilde{a}_t|^{p-2}\tilde{a}_t - |\hat{a}_t|^{p-2}\hat{a}_t\right)(\tilde{a}_t - \hat{a}_t)dx = 0, \\ \frac{1}{2}\frac{d}{dt}\left[\rho_3 h_3\|b_t\|^2 + E_3 h_3\|b_x\|^2\right] + k\int_0^L (-a+b+\alpha c_x)b_t dx \\ \qquad\qquad + \mu_2 \int_0^L \left(|\tilde{b}_t|^{q-2}\tilde{b}_t - |\hat{b}_t|^{q-2}\hat{a}_t\right)(\tilde{b}_t - \hat{b}_t)dx = 0, \\ \frac{1}{2}\frac{d}{dt}\left[\rho h\|c_t\|^2 + EI\|c_{xx}\|^2\right] + \alpha k\int_0^L (-a+b+\alpha c_x)c_{xt} dx \\ \qquad\qquad + \mu_3 \int_0^L \left(|\tilde{c}_t|^{r-2}\tilde{c}_t - |\hat{c}_t|^{r-2}\hat{c}_t\right)(\tilde{c}_t - \hat{c}_t)dx = 0. \end{cases} \tag{3.34}$$

Adding the equations in system (3.34) and recalling Lemma 2.2, we arrive at

$$\tilde{\Phi}'(t) \leq 0, \quad \text{for all } \ t \in [0,\infty), \tag{3.35}$$

where

$$\begin{aligned} \tilde{\Phi}(t) =& \frac{1}{2}\left[\rho_1 h_1\|a_t\|^2 + E_1 h_1\|a_x\|^2 + \rho_3 h_3\|b_t\|^2 + E_3 h_3\|b_x\|^2\right] \\ &+ \frac{1}{2}\left[\rho h\|c_t\|^2 + EI\|c_{xx}\|^2 + k\|(-a+b+\alpha c_x)\|^2\right]. \end{aligned}$$

Integrating (3.35) over $(0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\left[\rho_1 h_1\|a_t\|^2 + E_1 h_1\|a_x\|^2 + \rho_3 h_3\|b_t\|^2 + E_3 h_3\|b_x\|^2\right] \\ &\quad + \frac{1}{2}\left[\rho h\|c_t\|^2 + EI\|c_{xx}\|^2 + k\|(-a+b+\alpha c_x)\|^2\right] \leq 0. \end{aligned}$$

Hence,

$$(\tilde{a}, \tilde{b}, \tilde{c}) = (\hat{a}, \hat{b}, \hat{c}).$$

□

4. Asymptotic stability

We study the stability of Problems (1.6)–(1.8) when $g_1 \equiv g_2 \equiv g_3 \equiv 0$. That is, we consider

$$\begin{cases} \rho_1 h_1 a_{tt} - E_1 h_1 a_{xx} - k(-a+b+\alpha c_x) + \mu_1 |a_t|^{p-2}a_t = 0, & \text{in } (0,L)\times(0,T), \\ \rho_3 h_3 b_{tt} - E_3 h_3 b_{xx} + k(-a+b+\alpha c_x) + \mu_2 |b_t|^{q-2}b_t = 0, & \text{in } (0,L)\times(0,T), \\ \rho h c_{tt} + EI c_{xxxx} - \alpha k(-a+b+\alpha c_x)_x + \mu_3 |c_t|^{r-2}c_t = 0, & \text{in } (0,L)\times(0,T), \end{cases} \tag{4.1}$$

satisfying

$$\begin{cases} a(0,t) = b(0,t) = c(0,t) = c_{xx}(0,t) = 0, \ t \geq 0, \\ a(L,t) = b(L,t) = c(L,t) = c_{xx}(L,t) = 0, \ \ t \geq 0, \end{cases} \tag{4.2}$$

and the initial data

$$\begin{cases} a(x,0) = a_0(x), \ b(x,0) = b_0(x), \ c(x,0) = c_0(x), \ x \in (0,L), \\ a_t(x,0) = a_1(x), \ b_t(x,0) = b_1(x), \ c_t(x,0) = c_1(x), \ x \in (0,L). \end{cases} \tag{4.3}$$

The energy of Problems (4.1)−(4.3) is represented by the functional

$$\begin{aligned} \Phi(t) =& \frac{1}{2}\left[\rho_1 h_1 \|a_t\|^2 + E_1 h_1 \|a_x\|^2 + \rho_3 h_3 \|b_t\|^2 + E_3 h_3 \|b_x\|^2\right] \\ &+ \frac{1}{2}\left[\rho h \|c_t\|^2 + EI \|c_{xx}\|^2 + k\|(-a + b + \alpha c_x)\|^2\right]. \end{aligned} \tag{4.4}$$

The next lemma shows that the system (4.1)−(4.3) is dissipative.

Lemma 4.1. *Let* (a, b, c) *be the unique solution of Problems* (4.1)−(4.3). *Then, the functional* (4.4) *satisfies*

$$\frac{d}{dt}\Phi(t) = -\mu_1 \int_0^L |a_t|^p dx - \mu_2 \int_0^L |b_t|^q dx - \mu_3 \int_0^L |c_t|^r dx \leq 0, \ \forall t \geq 0. \tag{4.5}$$

Proof. The proof of (4.5) follows by first multiplying the three equations in (4.1) with a_t, b_t, and c_t, respectively, then applying integration by parts to each term over $(0, L)$ and summing the equations. □

The stability result for Problems (4.1)−(4.3) is given by

Theorem 4.1. *Let*

$$(a_0, a_1, b_0, b_1, c_0, c_1) \in H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_\otimes^2(0,L) \times L^2(0,L)$$

be given and assume $2 \leq p, q, r < \infty$. *Then, for all* $t \geq 0$, *the energy functional* (4.4) *satisfies,*

$$\Phi(t) \leq \begin{cases} Ce^{-\lambda t}, & \text{if } \ (p,q,r) = (2,2,2), \\ C\,(1+t)^{-\frac{2}{(\varpi-2)}}, & \text{if } \ (p,q,r) \neq (2,2,2), \end{cases} \tag{4.6}$$

where $\varpi = \max\{p, q, r\}$, *and* C, λ *are positive constants.*

Proof. We begin by multiplying the three equations in (4.1) by $a\Phi^\sigma(t)$, $b\Phi^\sigma(t)$, and $c\Phi^\sigma(t)$, respectively. Integrating the results over the interval $(0, L) \times (s, T)$, we get

$$\begin{cases} \displaystyle\int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho_1 h_1 a_{tt} a - E_1 h_1 a_{xx} a - k(-a + b + \alpha c_x)a + \mu_1 |a_t|^{p-2} a_t a\Big)\, dx dt = 0, \\ \displaystyle\int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho_3 h_3 b_{tt} b - E_3 h_3 b_{xx} b + k(-a + b + \alpha c_x)b + \mu_2 |b_t|^{q-2} b_t b\Big)\, dx dt = 0, \\ \displaystyle\int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho h c_{tt} c + EI c_{xxxx} c - \alpha k(-a + b + \alpha c_x)_x c + \mu_3 |c_t|^{r-2} c_t c\Big)\, dx dt = 0, \end{cases} \tag{4.7}$$

for $T > s$, and a constant $\sigma > 0$ to be declared later. Integration by parts and routine calculations leads to

$$\begin{cases} \int_s^T \Phi^\sigma(t) \int_0^L \Big[\rho_1 h_1 \Big((a_t a)_t - a_t^2\Big) + E_1 h_1 a_x^2 + k(-a + b + \alpha c_x)(-a)\Big] dxdt \\ \qquad\qquad\qquad + \mu_1 \int_s^T \Phi^\sigma(t) \int_0^L |a_t|^{p-2} a_t a dxdt = 0, \\ \int_s^T \Phi^\sigma(t) \int_0^L \Big[\rho_3 h_3 \Big((b_t b)_t - b_t^2\Big) + E_3 h_3 b_x^2 + k(-a + b + \alpha c_x) b\Big] dxdt \\ \qquad\qquad\qquad + \mu_2 \int_s^T \Phi^\sigma(t) \int_0^L |b_t|^{q-2} b_t b dxdt = 0, \\ \int_s^T \Phi^\sigma(t) \int_0^L \Big[\rho h \Big((c_t c)_t - c_t^2\Big) + EI c_{xx}^2 + k(-a + b + \alpha c_x)(\alpha c_x)\Big] dxdt \\ \qquad\qquad\qquad + \mu_3 \int_s^T \Phi^\sigma(t) \int_0^L |c_t|^{r-2} c_t c dxdt = 0. \end{cases} \tag{4.8}$$

It follows that

$$\begin{cases} \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho_1 h_1 a_t^2 + E_1 h_1 a_x^2 + k(-a + b + \alpha c_x)(-a)\Big) dxdt \\ \quad + \int_s^T \Phi^\sigma(t) \int_0^L \rho_1 h_1 (a a_t)_t dxdt - \frac{3}{2} \int_s^T \Phi^\sigma(t) \int_0^L \rho_1 h_1 a_t^2 dxdt \\ \quad + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L E_1 h_1 a_x^2 dxdt + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L k(-a + b + \alpha c_x)(-a) dxdt \\ \quad + \mu_1 \int_s^T \Phi^\sigma(t) \int_0^L |a_t|^{p-2} a_t a dxdt = 0, \\ \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho_3 h_3 b_t^2 + E_3 h_3 b_x^2 + k(-a + b + \alpha c_x) b\Big) dxdt \\ \quad + \int_s^T \Phi^\sigma(t) \int_0^L \rho_3 h_3 (b b_t)_t dxdt - \frac{3}{2} \int_s^T \Phi^\sigma(t) \int_0^L \rho_3 h_3 b_t^2 dxdt \\ \quad + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L E_3 h_3 b_x^2 dxdt + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L k(-a + b + \alpha c_x) b dxdt \\ \quad + \mu_2 \int_s^T \Phi^\sigma(t) \int_0^L |b_t|^{q-2} b_t b dxdt = 0, \\ \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L \Big(\rho h c_t^2 + EI c_{xx}^2 + k(-a + b + \alpha c_x)(\alpha c_x)\Big) dxdt \\ \quad + \int_s^T \Phi^\sigma(t) \int_0^L \rho h (c c_t)_t dxdt - \frac{3}{2} \int_s^T \Phi^\sigma(t) \int_0^L \rho h c_t^2 dxdt \\ \quad + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L EI c_{xx}^2 dxdt + \frac{1}{2} \int_s^T \Phi^\sigma(t) \int_0^L k(-a + b + \alpha c_x)(\alpha c_x) dxdt \\ \quad + \mu_3 \int_s^T \Phi^\sigma(t) \int_0^L |c_t|^{r-2} c_t c dxdt = 0. \end{cases} \tag{4.9}$$

Addition of the equations in (4.9) and recalling (4.4), we get

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt = &-\int_s^T \Phi^{\sigma}(t)\int_0^L (\rho_1 h_1 (aa_t)_t + \rho_3 h_3 (bb_t)_t + \rho h (cc_t)_t)\, dxdt\\
&+\frac{3}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \left(\rho_1 h_1 a_t^2 + \rho_3 h_3 b_t^2 + \rho h c_t^2\right) dxdt\\
&-\frac{1}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \left(E_1 h_1 a_x^2 + E_3 h_3 b_x^2 + EI c_{xx}^2\right) dxdt\\
&-\frac{1}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L k\|(-a+b+\alpha c_x)\|^2 dxdt\\
&-\mu_1\int_s^T \Phi^{\sigma}(t)\int_0^L |a_t|^{p-2}a_t a dxdt - \mu_2\int_s^T \Phi^{\sigma}(t)\int_0^L |b_t|^{q-2}b_t b dxdt\\
&-\mu_3\int_s^T \Phi^{\sigma}(t)\int_0^L |c_t|^{r-2}c_t c dxdt.\end{aligned}$$

From this, we get

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \le &\underbrace{-\int_s^T \frac{d}{dt}\left(\int_0^L \Phi^{\sigma}(t)\left(\rho_1 h_1 a a_t + \rho_3 h_3 b b_t + \rho h c c_t\right) dx\right) dt}_{\psi_1}\\
&\underbrace{+\sigma\int_s^T \Phi^{\sigma-1}(t)\Phi'(t)\int_0^L \left(\rho_1 h_1 a a_t + \rho_3 h_3 b b_t + \rho h c c_t\right) dxdt}_{\psi_2}\\
&\underbrace{+\frac{3}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \left(\rho_1 h_1 a_t^2 + \rho_3 h_3 b_t^2 + \rho h c_t^2\right) dxdt}_{\psi_3}\\
&\underbrace{-\mu_1\int_s^T \Phi^{\sigma}(t)\int_0^L |a_t|^{p-2}a_t a dxdt}_{\psi_4}\underbrace{-\mu_2\int_s^T \Phi^{\sigma}(t)\int_0^L |b_t|^{q-2}b_t b dxdt}_{\psi_5}\\
&\underbrace{-\mu_3\int_s^T \Phi^{\sigma}(t)\int_0^L |c_t|^{r-2}c_t c dxdt}_{\psi_6}.\end{aligned}\tag{4.10}$$

Now, we estimate the terms $\psi_1 - \psi_6$ in (4.10). For ψ_1, using Young's inequality, we have

$$\begin{aligned}|\psi_1| \le &\left|\int_0^L \Phi^{\sigma}(s)\left(\rho_1 h_1 a(x,s)a_t(x,s) + \rho_3 h_3 b(x,s)b_t(x,s) + \rho h c(x,s)c_t(x,s)\right) dx\right|\\
&+\left|\int_0^L \Phi^{\sigma}(T)\left(\rho_1 h_1 a(x,T)a_t(x,T) + \rho_3 h_3 b(x,T)b_t(x,T) + \rho h c(x,T)c_t(x,T)\right) dx\right|\\
\le &\frac{1}{2}\Phi^{\sigma}(s)\left[\rho_1 h_1(\|a(s)\|^2 + \|a_t(s)\|^2) + \rho_3 h_3(\|b(s)\|^2 + \|b_t(s)\|^2)\right]\\
&+\frac{1}{2}\Phi^{\sigma}(s)\rho h(\|c(s)\|^2 + \|c_t(s)\|^2) + \frac{1}{2}\Phi^{\sigma}(T)\rho_1 h_1(\|a(T)\|^2 + \|a_t(T)\|^2)\\
&+\frac{1}{2}\Phi^{\sigma}(T)\left[\rho_3 h_3(\|b(T)\|^2 + \|b_t(T)\|^2) + \rho h(\|c(T)\|^2 + \|c_t(T)\|^2)\right].\end{aligned}\tag{4.11}$$

Using (2.4) and Poincaré inequality, it follows that

$$\begin{aligned}|\psi_1| \leq &\frac{1}{2}\Phi^{\sigma}(s)\left[\rho_1 h_1(C_p\|a_x(s)\|^2 + \|a_t(s)\|^2) + \rho_3 h_3(C_p\|b_x(s)\|^2 + \|b_t(s)\|^2)\right]\\ &+ \frac{1}{2}\Phi^{\sigma}(s)\rho h(C_e\|c_{xx}(s)\|^2 + \|c_t(s)\|^2) + \frac{1}{2}\Phi^{\sigma}(T)\rho_1 h_1(C_p\|a_x(T)\|^2 + \|a_t(T)\|^2)\\ &+ \frac{1}{2}\Phi^{\sigma}(T)\left[\rho_3 h_3(C_p\|b_x(T)\|^2 + \|b_t(T)\|^2) + \rho h(C_e\|c_{xx}(T)\|^2 + \|c_t(T)\|^2)\right],\end{aligned}$$

where C_e and C_p are the embedding and Poincaré constants, respectively. Knowing that $\Phi'(t) \leq 0$, we obtain

$$|\psi_1| \leq C\left(\Phi^{\sigma+1}(T) + \Phi^{\sigma+1}(s)\right) \leq \tilde{C}\Phi^{\sigma+1}(s). \tag{4.12}$$

For ψ_2, recalling $\Phi'(t) \leq 0$, we have

$$\begin{aligned}|\psi_2| &\leq -\sigma\int_s^T \Phi^{\sigma-1}(t)\Phi'(t)\int_0^L |(\rho_1 h_1 a a_t + \rho_3 h_3 b b_t + \rho h c c_t)|dxdt\\ &\leq -\frac{\sigma}{2}\int_s^T \Phi^{\sigma-1}(t)\Phi'(t)\int_0^L (\rho_1 h_1(a^2 + a_t^2) + \rho_3 h_3(b^2 + b_t^2) + \rho h(c^2 + c_t^2))dxdt\\ &\leq -C\int_s^T \Phi^{\sigma-1}(t)\Phi'(t)\int_0^L (a_x^2 + a_t^2 + b_x^2 + b_t^2 + c_{xx}^2 + c_t^2)dxdt\\ &\leq -C\int_s^T \Phi^{\sigma}(t)\Phi'(t)dt\\ &= -C\int_s^T \left(\Phi^{\sigma+1}(t)\right)' dt \leq C\Phi^{\sigma+1}(s).\end{aligned} \tag{4.13}$$

Similarly, using (4.5) ψ_3 becomes

$$\begin{aligned}|\psi_3| \leq &\left|\frac{3}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \rho_1 h_1 a_t^2 dxdt\right| + \left|\frac{3}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \rho_3 h_3 b_t^2 dxdt\right|\\ &+ \left|\frac{3}{2}\int_s^T \Phi^{\sigma}(t)\int_0^L \rho h c_t^2 dxdt\right|\\ \leq &\, C\int_s^T \Phi^{\sigma}(t)\left(\int_0^L a_t^p dx\right)^{\frac{2}{p}} dt + C\int_s^T \Phi^{\sigma}(t)\left(\int_0^L b_t^q dx\right)^{\frac{2}{q}} dt\\ &+ C\int_s^T \Phi^{\sigma}(t)\left(\int_0^L c_t^r dx\right)^{\frac{2}{r}} dt\\ \leq &\, C\underbrace{\int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{p}} dt}_{\phi_1} + C\underbrace{\int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{q}} dt}_{\phi_2}\\ &+ C\underbrace{\int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{r}} dt}_{\phi_3}.\end{aligned} \tag{4.14}$$

Next, we consider different cases to estimate the terms ϕ_1 , ϕ_2, and ϕ_3. For ϕ_1, we have
Case 1. $p = 2$.

$$|\phi_1| \leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right) dt \leq C\Phi^{\sigma+1}(s). \tag{4.15}$$

Case 2. $p > 2$. We set $p^* = \frac{p}{2}, q^* = \frac{p}{p-2}$, then with $\epsilon > 0$ and young's inequality, the estimate below holds

$$\begin{aligned} |\phi_1| &\leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{p}} dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma p}{p-2}} dt + C_\epsilon \int_s^T (-\Phi'(t))dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma p}{p-2}} dt + C_\epsilon \Phi(s). \end{aligned} \tag{4.16}$$

For ϕ_2, we get
Case 1. $q = 2$.

$$|\phi_2| \leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right) dt \leq C\Phi^{\sigma+1}(s). \tag{4.17}$$

Case 2. $q > 2$. We set $p^* = \frac{q}{2},\ q^* = \frac{q}{q-2}$, by young's inequality, for any $\epsilon > 0$, we get

$$\begin{aligned} |\phi_2| &\leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{q}} dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma q}{q-2}} dt + C_\epsilon \int_s^T (-\Phi'(t))dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma q}{q-2}} dt + C_\epsilon \Phi(s). \end{aligned} \tag{4.18}$$

Similarly, for ϕ_3, we get
Case 1. $r = 2$.

$$|\phi_3| \leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right) dt \leq C\Phi^{\sigma+1}(s). \tag{4.19}$$

Case 2. $r > 2$. We set $p^* = \frac{r}{2},\ q^* = \frac{r}{r-2}$, then using young's inequality, we obtain any $\epsilon > 0$

$$\begin{aligned} |\phi_3| &\leq C \int_s^T \Phi^{\sigma}(t)\left(-\Phi'(t)\right)^{\frac{2}{r}} dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma r}{r-2}} dt + C_\epsilon \int_s^T (-\Phi'(t))dt \\ &\leq \epsilon \int_s^T (\Phi(t))^{\frac{\sigma r}{r-2}} dt + C_\epsilon \Phi(s). \end{aligned} \tag{4.20}$$

For ψ_4, we make use of Poincaré and Young's inequalities with $p^* = \frac{p}{p-1},\ q^* = p$, recalling (4.4) and (4.5), we have for any $\epsilon > 0$,

$$\begin{aligned}
|\psi_4| &\leq \mu_1 \int_s^T \Phi^\sigma(t) \int_0^L |a_t|^{p-1}|a| dx dt \\
&\leq \mu_1 \int_s^T \Phi^\sigma(t) \left(\epsilon \int_0^L |a|^p dx + C_\epsilon \int_0^L |a_t|^p dx \right) dt \\
&\leq \mu_1 \epsilon \int_s^T \int_0^L \Phi^\sigma(t) |a|^p dx dt + C_\epsilon \int_s^T \Phi^\sigma(t) \left(-\Phi'(t) \right) dt \\
&\leq C_p \mu_1 \epsilon \int_s^T \Phi^\sigma(t) \left(\int_0^L a_x^2 dx \right)^{\frac{p}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq C_p \mu_1 \epsilon \int_s^T \Phi^\sigma(t) \left(2\Phi(t) \right)^{\frac{p}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq \epsilon \left(C_p \mu_1 \left(\Phi(0) \right)^{\frac{p}{2}-1} \right) \int_s^T \Phi^{\sigma+1}(t) dt + C_\epsilon \Phi^{\sigma+1}(s).
\end{aligned} \tag{4.21}$$

Similarly, with $p^* = \frac{q}{q-1}$, $q^* = q$, we estimate ψ_5 as follows

$$\begin{aligned}
|\psi_5| &\leq \mu_2 \int_s^T \Phi^\sigma(t) \int_0^L |b_t|^{q-1}|b| dx dt \\
&\leq \mu_2 \int_s^T \Phi^\sigma(t) \left(\epsilon \int_0^L |b|^q dx + C_\epsilon \int_0^L |b_t|^q dx \right) dt \\
&\leq \mu_2 \epsilon \int_s^T \int_0^L \Phi^\sigma(t) |b|^q dx dt + C_\epsilon \int_s^T \Phi^\sigma(t) \left(-\Phi'(t) \right) dt \\
&\leq C_q \mu_2 \epsilon \int_s^T \Phi^\sigma(t) \left(\int_0^L b_x^2 dx \right)^{\frac{q}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq C_q \mu_2 \epsilon \int_s^T \Phi^\sigma(t) \left(2\Phi(t) \right)^{\frac{q}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq \epsilon \left(C_q \mu_2 \left(\Phi(0) \right)^{\frac{q}{2}-1} \right) \int_s^T \Phi^{\sigma+1}(t) dt + C_\epsilon \Phi^{\sigma+1}(s).
\end{aligned} \tag{4.22}$$

In the same way with $p^* = \frac{r}{r-1}, q^* = r$, we estimate ψ_6 as follows

$$\begin{aligned}
|\psi_6| &\leq \mu_3 \int_s^T E^p(t) \int_0^L |c_t|^{r-1}|c| dx dt \\
&\leq \mu_3 \int_s^T \Phi^\sigma(t) \left(\epsilon \int_0^L |c|^r dx + C_\epsilon \int_0^L |c_t|^r dx \right) dt \\
&\leq \mu_2 \epsilon \int_s^T \int_0^L \Phi^\sigma(t) |b|^r dx dt + C_\epsilon \int_s^T \Phi^\sigma(t) \left(-\Phi'(t) \right) dt \\
&\leq C_{e,r} \mu_3 \epsilon \int_s^T \Phi^\sigma(t) \left(\int_0^L c_{xx}^2 dx \right)^{\frac{r}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq C_{e,r} \mu_3 \epsilon \int_s^T \Phi^\sigma(t) \left(2\Phi(t) \right)^{\frac{r}{2}} dt + C_\epsilon \Phi^{\sigma+1}(s) \\
&\leq \epsilon \left(C_{e,r} \mu_3 \left(\Phi(0) \right)^{\frac{r}{2}-1} \right) \int_s^T \Phi^{\sigma+1}(t) dt + C_\epsilon \Phi^{\sigma+1}(s).
\end{aligned} \tag{4.23}$$

We conclude this proof by discussing two cases.
Case 1. $(p, q, r) = (2, 2, 2)$.
In this case, using (4.10), (4.12), (4.13), (4.15), (4.17), (4.19), and (4.21)–(4.23), we have

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq C\Phi^{\sigma+1}(s) + \epsilon\left(C_p\mu_1 + C_q\mu_2 + C_{e,r}\mu_3\right)\int_s^T \Phi^{\sigma+1}(t)dt + C_\epsilon\Phi^{\sigma+1}(s). \tag{4.24}$$

Thus, we get

$$\left(1 - \epsilon\left(C_p\mu_1 + C_q\mu_2 + C_{e,r}\mu_3\right)\right)\int_s^T \Phi^{\sigma+1}(t)dt \leq (C + C_\epsilon)\Phi^{\sigma+1}(s). \tag{4.25}$$

By choosing ϵ very small in (4.25) such that

$$\left(1 - \epsilon\left(C_p\mu_1 + C_p\mu_2 + C_e\mu_3\right)\right) > 0,$$

we arrive at

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq \tilde{C}\Phi^{\sigma+1}(s), \ \forall T > s > 0. \tag{4.26}$$

Therefore, taking $\sigma = 0$ and letting $T \longrightarrow +\infty$, we obtain

$$\int_s^{+\infty} \Phi(t)dt \leq \tilde{C}\Phi(s), \ \forall s > 0. \tag{4.27}$$

Applying Lemma 2.3 part $(2.2)_2$, we obtain $(4.6)_1$.
Case 2. $(p, q, r) \neq (2, 2, 2)$.
Subcase 2.1. $p = q = r > 2$.
We use (4.10), (4.12), (4.13), (4.16), (4.18), (4.20)–(4.23) to get

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \leq& C\Phi^{\sigma+1}(s) + C_\epsilon\Phi^{\sigma+1}(s) + C_\epsilon\Phi(s) + 3\epsilon\int_s^T (\Phi(t))^{\frac{\sigma p}{p-2}}\,dt \\ &+ \epsilon\left(C_p\mu_1 + C_q\mu_2 + C_{e,r}\mu_3\right)(\Phi(0))^{\frac{p}{2}-1}\int_s^T \Phi^{\sigma+1}(t)dt.\end{aligned} \tag{4.28}$$

We then choose σ such that

$$\frac{\sigma p}{p-2} = \sigma + 1, \ \text{ which gives } \sigma = \frac{p}{2} - 1. \tag{4.29}$$

Therefore, (4.28) leads to

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \leq& C\Phi^{\sigma+1}(s) + C_\epsilon\Phi^{\sigma+1}(s) + C_\epsilon\Phi(s) \\ &+ \epsilon\left(3 + C_p\mu_1 + C_q\mu_2 + C_{e,r}\mu_3\right)(\Phi(0))^{\frac{p}{2}-1}\int_s^T \Phi^{\sigma+1}(t)dt.\end{aligned} \tag{4.30}$$

Next, selecting ϵ sufficiently small such that

$$1 - \epsilon\left(3 + C_p\mu_1 + C_q\mu_2 + C_{e,r}\mu_3\right)(\Phi(0))^{\frac{p}{2}-1} > 0,$$

and knowing $\sigma = \frac{p}{2} - 1$, we get

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq C\Phi^{\sigma+1}(s) + C\Phi(s) \leq \tilde{C}\Phi(s), \ \forall\, T > s > 0. \tag{4.31}$$

By letting $T \longrightarrow +\infty$ and recalling (4.29), we obtain

$$\int_s^{+\infty} \Phi^{\frac{r}{2}}(t)dt \leq \tilde{C}\Phi(s), \ \forall s > 0. \tag{4.32}$$

Thus, applying Lemma 2.3 part $(2.2)_1$, we obtain $(4.6)_2$.

Subcase 2.2. $p > q \geq r \geq 2$ or $p > r \geq q \geq 2$ or $q > p \geq r \geq 2$ or $q > r \geq p \geq 2$ or $r > p \geq q \geq 2$ or $r > q \geq p \geq 2$.

Here, it is sufficient to consider $p > q \geq r \geq 2$. The other options can be treated in exactly the same way. Thus, we have the following subsubcases:

Subsubcase 2.2.1. $p > q = r = 2$.

It follows from (4.10), (4.12), (4.13), (4.16), (4.17), (4.19), and (4.21)–(4.23) that

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \leq& C\Phi^{\sigma+1}(s) + C_\epsilon\Phi^{\sigma+1}(s) + C_\epsilon\Phi(s) + \epsilon\int_s^T (\Phi(t))^{\frac{\sigma p}{P-2}}\, dt \\ &+ \epsilon\left(C_p\mu_1\,(\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 + C_e\mu_3\right)\int_s^T \Phi^{\sigma+1}(t)dt.\end{aligned} \tag{4.33}$$

Thus,

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \leq& C\Phi^{\sigma+1}(s) + C_\epsilon\Phi^{\sigma+1}(s) + C_\epsilon\Phi(s) \\ &+ \epsilon\left(1 + C_p\mu_1\,(\Phi(0))^{\frac{p}{2}-1} + C_q\mu_2 + C_{e,r}\mu_3\right)\int_s^T \Phi^{\sigma+1}(t)dt.\end{aligned} \tag{4.34}$$

Choosing ϵ to be very small and satisfying

$$\epsilon\left(1 + C_p\mu_1\,(\Phi(0))^{\frac{p}{2}-1} + C_q\mu_2 + C_{e,r}\mu_3\right) < 1,$$

and selecting σ as in (4.29) leads to

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq C\Phi^{\sigma+1}(s) + C\Phi(s) \leq \tilde{C}\Phi(s), \ \forall\, T > s > 0. \tag{4.35}$$

Letting $T \longrightarrow +\infty$, we arrive at

$$\int_s^{+\infty} \Phi^{\frac{p}{2}}(t)dt \leq \tilde{C}\Phi(s), \ \forall s > 0. \tag{4.36}$$

Thus, the application Lemma 2.3 part $(2.2)_1$, we obtain a polynomial decay result in $(4.6)_2$.

Subsubcase 2.2.2. $p > q > r = 2$.

Again, from (4.10), (4.12), (4.13), (4.16), (4.18), (4.19), and (4.21)–(4.23), we get

$$\begin{aligned}\int_s^T \Phi^{\sigma+1}(t)dt \leq& C\Phi^{\sigma+1}(s) + C_\epsilon\Phi^{\sigma+1}(s) + C_\epsilon\Phi(s) \\ &+ \epsilon\int_s^T (\Phi(t))^{\frac{\sigma p}{p-2}}\, dt + \epsilon\int_s^T (\Phi(t))^{\frac{\sigma q}{q-2}}\, dt \\ &+ \epsilon\left(C_p\mu_1\,(\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2\,(\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3\right)\int_s^T \Phi^{\sigma+1}(t)dt.\end{aligned} \tag{4.37}$$

With choice of $\sigma > 0$ in (4.29), we see that $\frac{\sigma q}{q-2} > \sigma + 1$, since $p > q > r = 2$, thus we obtain

$$\begin{aligned} \int_s^T \Phi^{\sigma+1}(t)dt \leq & C\Phi^{\sigma+1}(s) + C_\epsilon \Phi^{\sigma+1}(s) + C_\epsilon \Phi(s) \\ & + \epsilon\Big(2 + C_p\mu_1 (\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 (\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3\Big) \int_s^T \Phi^{\sigma+1}(t)dt. \end{aligned} \tag{4.38}$$

Thus, we select ϵ small so that

$$1 - \epsilon\Big(2 + C_p\mu_1 (\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 (\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3\Big) > 0,$$

and since $\sigma = \frac{p}{2} - 1$, we arrive at

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq C\Phi^{\sigma+1}(s) + C\Phi(s) \leq \tilde{C}\Phi(s),\ \forall\, T > s > 0. \tag{4.39}$$

Letting $T \longrightarrow +\infty$ leads to

$$\int_s^{+\infty} \Phi^{\frac{p}{2}}(t)dt \leq \tilde{C}\Phi(s),\ \forall s > 0. \tag{4.40}$$

Using Lemma 2.3 part $(2.2)_1$, we obtain $(4.6)_2$.

Subsubcase 2.2.3. $p > q > r > 2$.

A combination of (4.10), (4.12), (4.13), (4.16), (4.18), and (4.20)–(4.23) leads to

$$\begin{aligned} \int_s^T \Phi^{\sigma+1}(t)dt \leq & C\Phi^{\sigma+1}(s) + C_\epsilon \Phi^{\sigma+1}(s) + C_\epsilon \Phi(s) \\ & + \epsilon \int_s^T (\Phi(t))^{\frac{\sigma p}{p-2}}\, dt + \epsilon \int_s^T (\Phi(t))^{\frac{\sigma q}{q-2}}\, dt + \epsilon \int_s^T (\Phi(t))^{\frac{\sigma r}{r-2}}\, dt \\ & + \epsilon\Big(C_p\mu_1 (\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 (\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3 (\Phi(0))^{\frac{r}{2}-1}\Big) \int_s^T \Phi^{\sigma+1}(t)dt. \end{aligned} \tag{4.41}$$

Again, keeping in mind the choice of $\sigma > 0$ in (4.29), we obtain $\frac{\sigma q}{q-2} > \sigma + 1$, $\frac{\sigma r}{r-2} > \sigma + 1$ since $p > q > r > 2$. Therefore, we obtain

$$\begin{aligned} \int_s^T \Phi^{\sigma+1}(t)dt \leq & C\Phi^{\sigma+1}(s) + C_\epsilon \Phi^{\sigma+1}(s) + C_\epsilon \Phi(s) \\ & + \epsilon\Big(3 + C_p\mu_1 (\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 (\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3 (\Phi(0))^{\frac{r}{2}-1}\Big) \int_s^T \Phi^{\sigma+1}(t)dt. \end{aligned} \tag{4.42}$$

By selecting ϵ small such that

$$\epsilon\Big(3 + C_p\mu_1 (\Phi(0))^{\frac{p}{2}-1} + C_p\mu_2 (\Phi(0))^{\frac{q}{2}-1} + C_e\mu_3 (\Phi(0))^{\frac{r}{2}-1}\Big) < 1$$

leads to

$$\int_s^T \Phi^{\sigma+1}(t)dt \leq C\Phi^{\sigma+1}(s) + C\Phi(s) \leq \tilde{C}\Phi(s),\ \forall\, T > s > 0. \tag{4.43}$$

Letting $T \longrightarrow +\infty$ we get

$$\int_s^{+\infty} \Phi^{\frac{p}{2}}(t)dt \leq \tilde{C}\Phi(s),\ \forall s > 0. \tag{4.44}$$

Using Lemma 2.3 part $(2.2)_1$, we obtain $(4.6)_2$. This completes the proof. □

5. Numerical analysis and solutions

Our objective of this section is to numerically validate the theoretical results obtained in Theorem 4.1.

5.1. Numerical analysis

First, we utilize the Finite Difference Method (FDM) (see also [6]) to discretize system (4.1). This can be achieved by letting a_i^{n+1} and b_i^{n+1} represent the longitudinal displacement of the top and bottom layers, respectively, and c_i^{n+1} the transverse displacement of the beam, at future time step $(n+1)\Delta t$ and position $i\Delta x$, where Δt and Δx are the time and space steps, respectively. The discretized system is as follows:

$$\begin{cases} \rho_1 h_1 \left[\dfrac{a_i^{n+1}-2a_i^n+a_i^{n-1}}{\Delta t^2}\right] - E_1 h_1 \left[\dfrac{a_{i+1}^n - 2a_i^n + a_{i-1}^n}{\Delta x^2}\right] \\ \qquad - k\left[-a_i^n + b_i^n + \alpha\left(\dfrac{c_i^n - c_{i-1}^n}{\Delta x}\right)\right] + \mu_1 \left|\dfrac{a_i^n - a_i^{n-1}}{\Delta t}\right|^{p-2}\left(\dfrac{a_i^n - a_i^{n-1}}{\Delta t}\right) = 0, \\ \rho_3 h_3 \left[\dfrac{b_i^{n+1}-2b_i^n+b_i^{n-1}}{\Delta t^2}\right] - E_3 h_3 \left[\dfrac{b_{i+1}^n - 2b_i^n + b_{i-1}^n}{\Delta x^2}\right] \\ \qquad + k\left[-a_i^n + b_i^n + \alpha\left(\dfrac{c_i^n - c_{i-1}^n}{\Delta x}\right)\right] + \mu_2 \left|\dfrac{b_i^n - b_i^{n-1}}{\Delta t}\right|^{q-2}\left(\dfrac{b_i^n - b_i^{n-1}}{\Delta t}\right) = 0, \\ \rho h \left[\dfrac{c_i^{n+1}-2c_i^n+c_i^{n-1}}{\Delta t^2}\right] + EI \left[\dfrac{c_{i+2}^n - 4c_{i+1}^n + 6c_i^n - 4c_{i-1}^n + c_{i-2}^n}{\Delta x^4}\right] \\ \qquad - \alpha k\left[-\left(\dfrac{a_i^n - a_{i-1}^n}{\Delta x}\right) + \left(\dfrac{b_i^n - b_{i-1}^n}{\Delta x}\right) + \alpha\left(\dfrac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{\Delta x^2}\right)\right] \\ \qquad + \mu_3 \left|\dfrac{c_i^n - c_i^{n-1}}{\Delta t}\right|^{r-2}\left(\dfrac{c_i^n - c_i^{n-1}}{\Delta t}\right) = 0. \end{cases} \tag{5.1}$$

Solving for a_i^{n+1}, b_i^{n+1}, and c_i^{n+1} in (5.1), we obtain

$$\begin{cases} a_i^{n+1} = 2a_i^n - a_i^{n-1} + z_1^c\left[a_{i+1}^n - 2a_i^n + a_{i-1}^n\right] + z_2^c\left[-a_i^n + b_i^n\right] + z_3^c\left[c_i^n - c_{i-1}^n\right] \\ \qquad - z_4^c\left|a_i^n - a_i^{n-1}\right|^{p-2}\left(a_i^n - a_i^{n-1}\right), \\ b_i^{n+1} = 2b_i^n - b_i^{n-1} + m_1^c\left[b_{i+1}^n - 2b_i^n + b_{i-1}^n\right] - m_2^c\left[-a_i^n + b_i^n\right] - m_3^c\left[c_i^n - c_{i-1}^n\right] \\ \qquad - m_4^c\left|b_i^n - b_i^{n-1}\right|^{q-2}\left(b_i^n - b_i^{n-1}\right), \\ c_i^{n+1} = 2c_i^n - c_i^{n-1} - d_1^c\left[c_{i+2}^n - 4c_{i+1}^n + 6c_i^n - 4c_{i-1}^n + c_{i-2}^n\right] - d_2^c\left(a_i^n - a_{i-1}^n\right) \\ \qquad + d_2^c\left(b_i^n - b_{i-1}^n\right) + d_3^c\left[c_{i+1}^n - 2c_i^n + c_{i-1}^n\right] - d_4^c\left|c_i^n - c_i^{n-1}\right|^{r-2}\left(c_i^n - c_i^{n-1}\right), \end{cases} \tag{5.2}$$

where

$$z_1^c = \frac{E_1}{\rho_1}\frac{\Delta t^2}{\Delta x^2}, \quad z_2^c = \frac{k\Delta t^2}{\rho_1 h_1}, \quad z_3^c = \frac{k\alpha}{\rho_1 h_1}\frac{\Delta t^2}{\Delta x}, \quad z_4^c = \frac{\mu_1}{\rho_1 h_3 \Delta t^{p-3}(2^{p-1})},$$

$$m_1^c = \frac{E_3}{\rho_3}\frac{\Delta t^2}{\Delta x^2},\ \ m_2^c = \frac{k\Delta t^2}{\rho_3 h_3},\ \ m_3^c = \frac{k\alpha}{\rho_3 h_3}\frac{\Delta t^2}{\Delta x},\ \ m_4^c = \frac{\mu_1}{\rho_3 h_3 \Delta t^{q-3}(2^{q-1})},$$
$$d_1^c = \frac{EI}{\rho h}\frac{\Delta t^2}{\Delta x^4},\ \ d_2^c = \frac{k\alpha}{\rho h}\frac{\Delta t^2}{\Delta x},\ \ d_3^c = \frac{k\alpha^2}{\rho h}\frac{\Delta t^2}{\Delta x^2},\ \ d_4^c = \frac{\mu_1}{\rho h \Delta t^{r-3}(2^{r-1})}.$$

We note that the numerical solution in (5.2) is stable if

$$\begin{aligned} &2 - 2z_1^c - z_2^c \geq 0,\\ &2 - 2m_1^c - m_2^c \geq 0,\\ &2 - 6d_1^c - 2d_3^c \geq 0 \end{aligned}$$

simultaneously. Hence, for stability, we require that

$$\Delta t \leq \min\left\{\Delta x\sqrt{\frac{\rho_1}{E_1}},\ \Delta x\sqrt{\frac{\rho_3}{E_3}},\ \Delta x^2\sqrt{\frac{\rho h}{EI}}\right\}. \tag{5.3}$$

5.2. Numerical solutions

Second, using the resulting numerical scheme in (5.2), we present a numerical solution to problem (1.6). In addition, we validate the decay result concerning the energy functional $\Phi(t)$, and the estimates obtained in Section 4, through numerical simulations. To do these, we consider three different lengths of the beam, which are, $L = 1$, $L = 2$, and $L = 6$; then, different parameter values for ρ_1, ρ_3, the damping coefficients μ_1, μ_2, μ_3, and damping exponents p, q, r. Throughout, we simulate with four different initial displacements, namely,

$$a(x,0) = 20\sin\left(\frac{\pi}{4}x\right),\ b(x,0) = 20\sin\left(\frac{\pi}{4}x\right) \text{ and } c(x,0) = 20\sin\left(\frac{\pi}{4}x\right), \tag{5.4}$$

$$a(x,0) = 20\sin\left(\frac{3\pi}{40}x\right),\ b(x,0) = 20\sin\left(\frac{3\pi}{40}x\right) \text{ and } c(x,0) = 20\sin\left(\frac{3\pi}{40}x\right), \tag{5.5}$$

$$a(x,0) = 20\sinh\left(\frac{7\pi}{4}x\right),\ b(x,0) = 20\sinh\left(\frac{7\pi}{4}x\right) \text{ and } c(x,0) = 20\sinh\left(\frac{7\pi}{4}x\right), \tag{5.6}$$

$$a(x,0) = 50\cosh\left(\frac{7\pi}{4}x\right),\ b(x,0) = 30\cosh\left(\frac{7\pi}{4}x\right) \text{ and } c(x,0) = 10\cosh\left(\frac{7\pi}{4}x\right), \tag{5.7}$$

and

$$a_t(x,0) = 0,\ b_t(x,0) = 0,\ \text{ and }\ c_t(x,0) = 0,$$

as initial velocities.

Instance 1. ($L = 1$) Considering a beam of length $L = 1$, we present numerical solutions for the problem 1.6 under consideration. The effect and influence of each of the parameters, the damping coefficients and damping exponents, on the decay of solutions are presented and analyzed in Figures 2–10. It is observed that the parameters ρ_1, ρ_3, the damping coefficients μ_1, μ_2, μ_3, and the damping exponents p, q, r greatly influence the wave amplitudes, oscillations, and speed of decay of the solution to zero. First, we use the initial displacement (5.5) for Figures 2–5. Second, we use the initial displacement (5.6) for Figures 6–9. Third, we use the initial displacement (5.7) for Figure 10.

In this instance, we demonstrate different simulations of the critical exponent cases $p = 2,\ q, r > 0$ or $q = 2,\ p, r > 0$ or $r = 2,\ p, q > 0$ or $p = q = 2,\ r > 0$ or $p = r = 2,\ q > 0$ or $q = r = 2,\ p > 0$, which are presented in Figures 2–10.

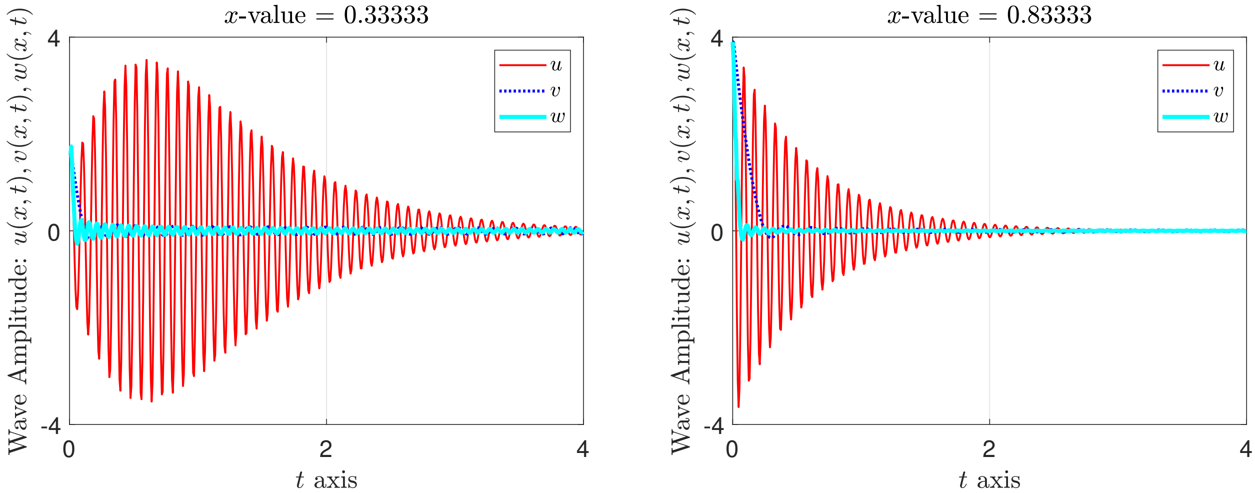


Figure 2. Solution for different x values, $\rho_1 = 0.01, \rho_3 = 0.02, \mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2, q = 3.5, r = 3$. (a) $a(\frac{2}{6}, t), b(\frac{2}{6}, t), c(\frac{2}{6}, t)$, (b) $a(\frac{5}{6}, t), b(\frac{5}{6}, t), c(\frac{5}{6}, t)$.

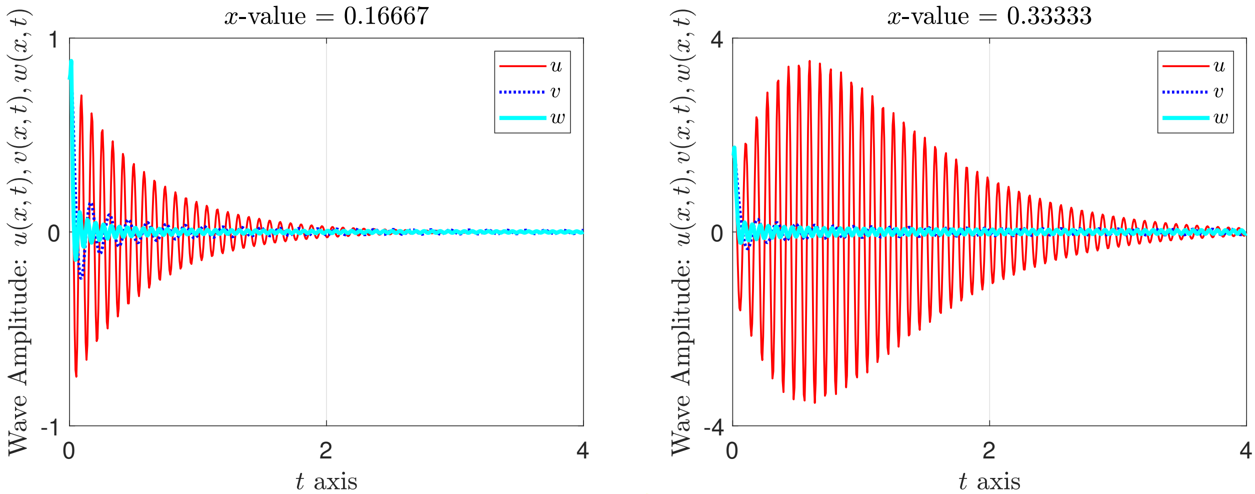
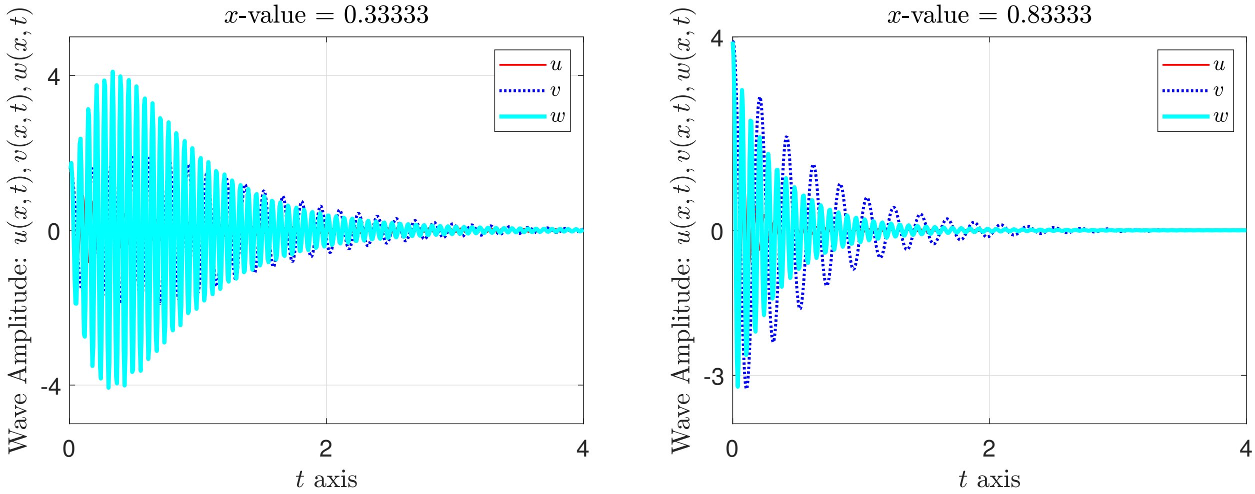
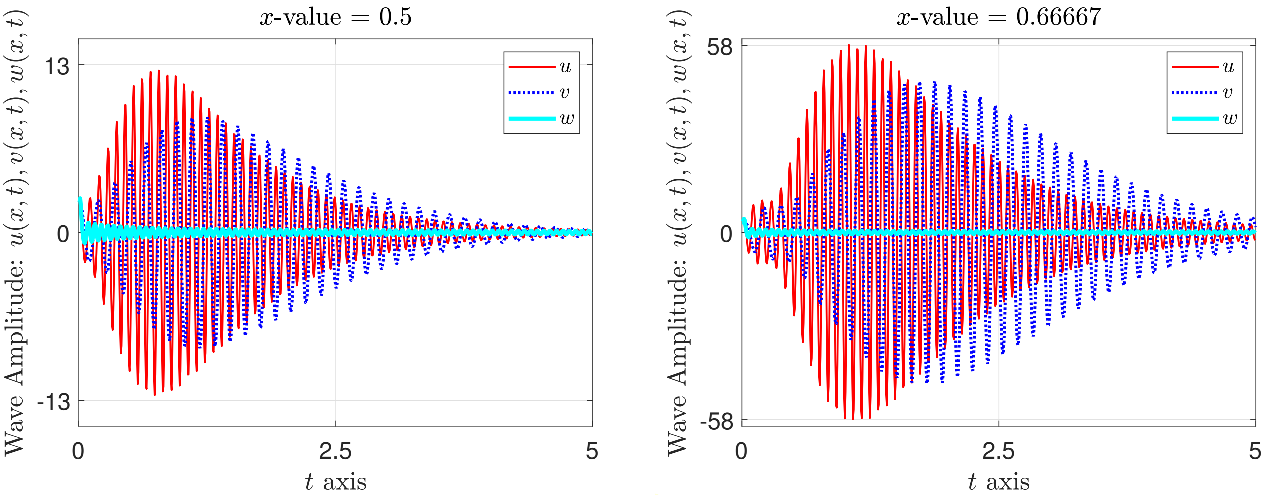


Figure 3. Solution for different x values, $\rho_1 = 0.01, \rho_3 = 0.02, \mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2, q = 3, r = 3.05$. (a) $a(\frac{1}{6}, t), b(\frac{1}{6}, t), c(\frac{1}{6}, t)$, (b) $a(\frac{2}{6}, t), b(\frac{2}{6}, t), c(\frac{2}{6}, t)$.



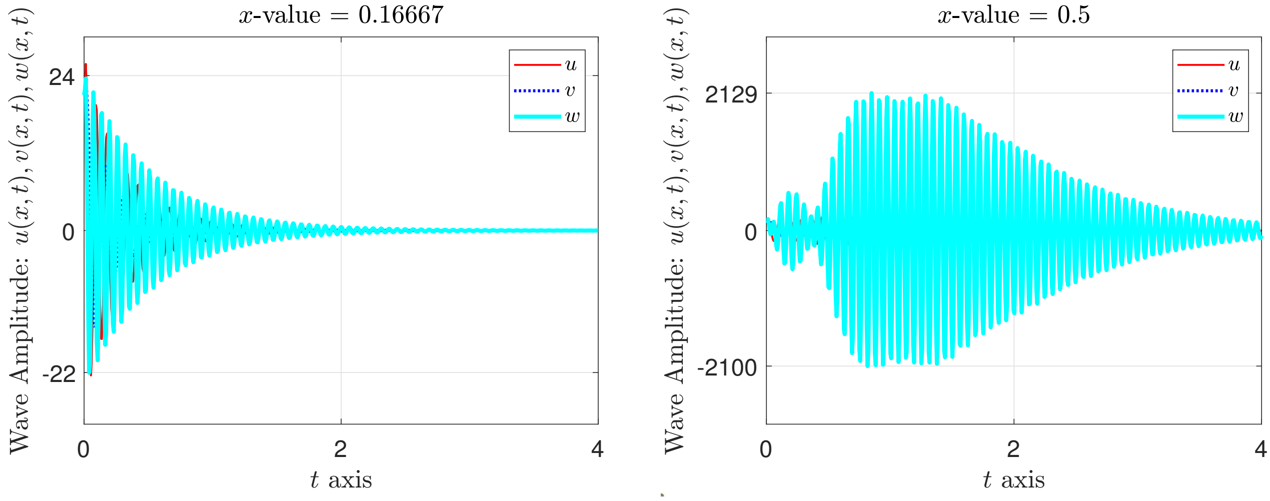
(a) **(b)**

Figure 4. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2.5, q = 2, r = 2.1$. (a) $a(\frac{2}{6}, t), b(\frac{2}{6}, t), c(\frac{2}{6}, t)$, (b) $a(\frac{5}{6}, t), b(\frac{5}{6}, t), c(\frac{5}{6}, t)$.



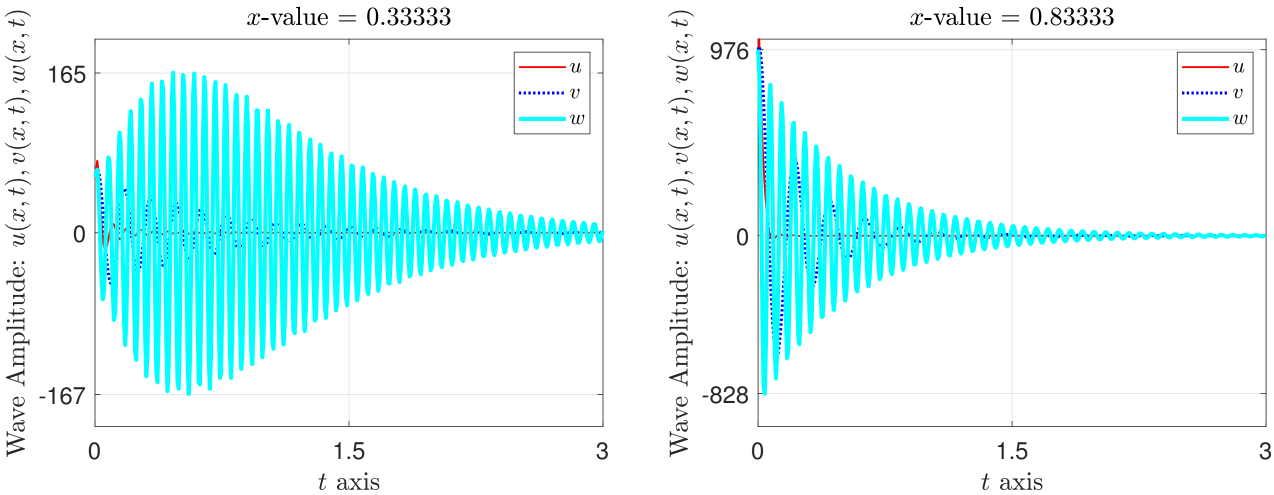
(a) **(b)**

Figure 5. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2.1, q = 2, r = 2.7$. (a) $a(\frac{1}{2}, t), b(\frac{1}{2}, t), c(\frac{1}{2}, t)$, (b) $a(\frac{2}{3}, t), b(\frac{2}{3}, t), c(\frac{2}{3}, t)$.



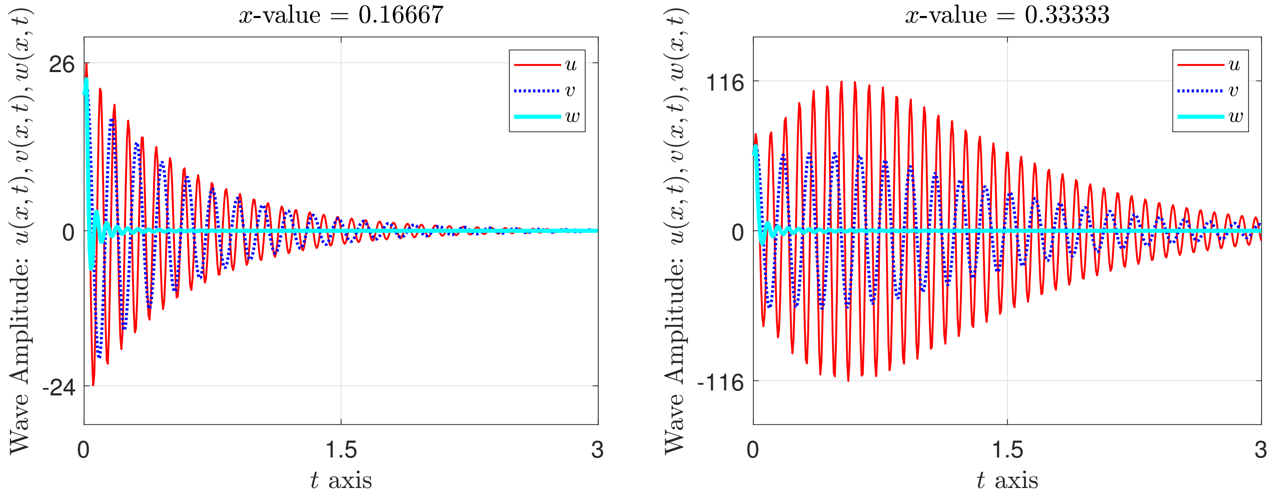
(a) **(b)**

Figure 6. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2.1, q = 2.2, r = 2$. (a) $a(\frac{1}{6}, t), b(\frac{1}{6}, t), c(\frac{1}{6}, t)$, (b) $a(\frac{1}{2}, t), b(\frac{1}{2}, t), c(\frac{1}{2}, t)$.



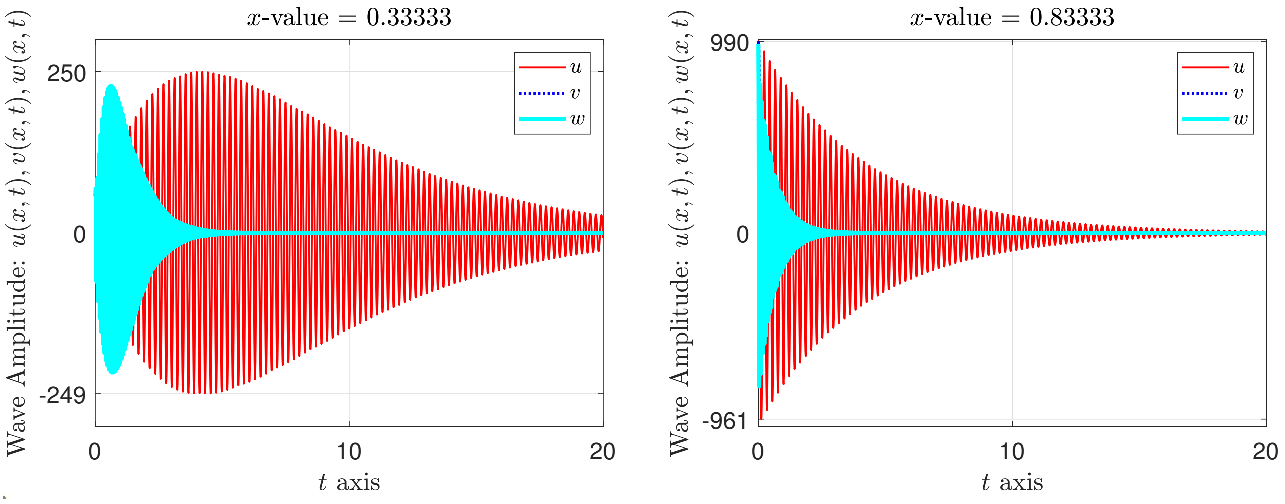
(a) **(b)**

Figure 7. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = 2.4, q = 2.1, r = 2$. (a) $a(\frac{1}{3}, t), b(\frac{1}{3}, t), c(\frac{1}{3}, t)$, (b) $a(\frac{5}{6}, t), b(\frac{5}{6}, t), c(\frac{5}{6}, t)$.



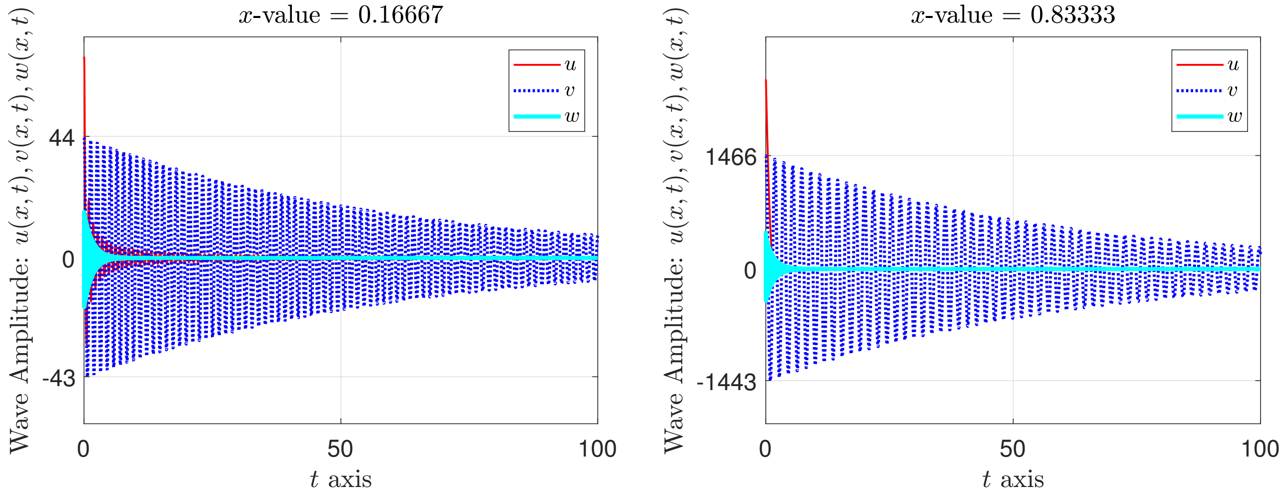
(a) **(b)**

Figure 8. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$, $p = q = 2, r = 2.45$. (a) $a(\frac{1}{6}, t), b(\frac{1}{6}, t), c(\frac{1}{6}, t)$, (b) $a(\frac{1}{3}, t), b(\frac{1}{3}, t), c(\frac{1}{3}, t)$.



(a) **(b)**

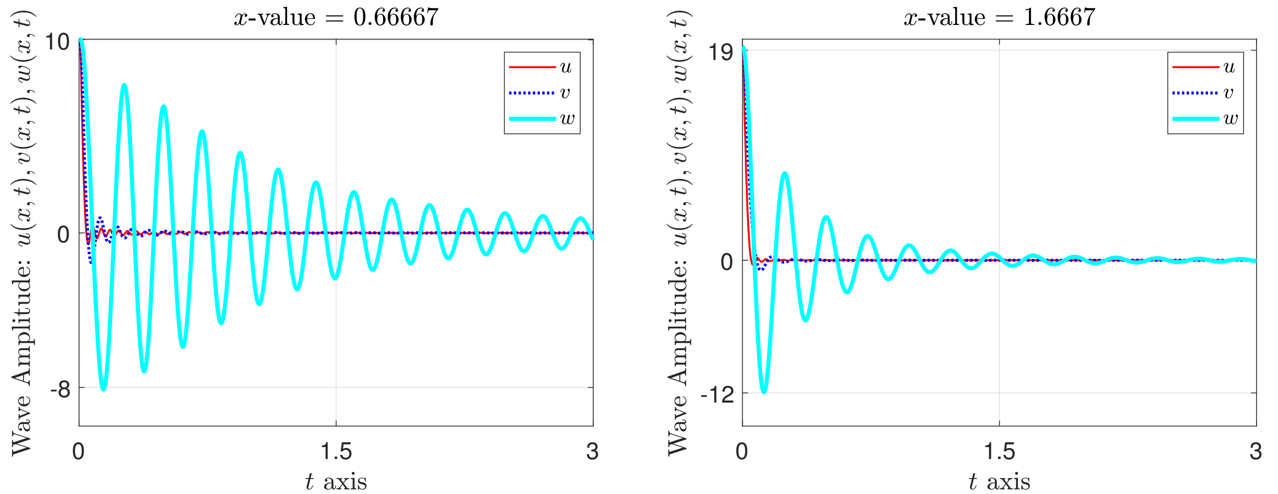
Figure 9. Solution for different x values, $\rho_1 = 0.07$, $\rho_3 = 0.03$, $\mu_1 = 10, \mu_2 = 5, \mu_3 = 25$, $p = r = 2, q = 2.55$. (a) $a(\frac{1}{3}, t), b(\frac{1}{3}, t), c(\frac{1}{3}, t)$, (b) $a(\frac{5}{6}, t), b(\frac{5}{6}, t), c(\frac{5}{6}, t)$.



(a) **(b)**

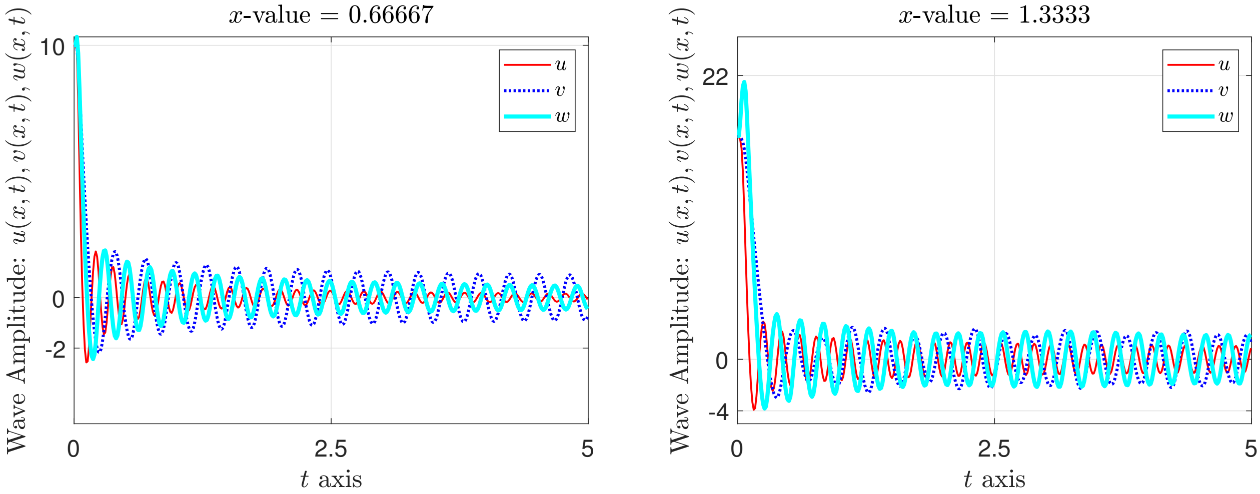
Figure 10. Solution for different x values, $\rho_1 = 1.1$, $\rho_3 = 1.5$, $\mu_1 = 5, \mu_2 = 15, \mu_3 = 10$, $q = r = 2, p = 3.1$. (a) $a(\frac{1}{6}, t), b(\frac{1}{6}, t), c(\frac{1}{6}, t)$, (b) $a(\frac{5}{6}, t), b(\frac{5}{6}, t), c(\frac{5}{6}, t)$.

Instance 2. ($L = 2$) Considering a beam of length $L = 2$, we present numerical solutions for the problem (1.6) under consideration. Using the initial displacement (5.4), the effect and influence of each of the parameters, the damping coefficients, and damping exponents on the decay of solutions are presented and analyzed in Figures 11–13. It is observed that the parameters ρ_1, ρ_3, the damping coefficients μ_1, μ_2, μ_3, and the damping exponents p, q, r greatly influence the wave amplitudes, oscillations, and speed of decay of the solution to zero.



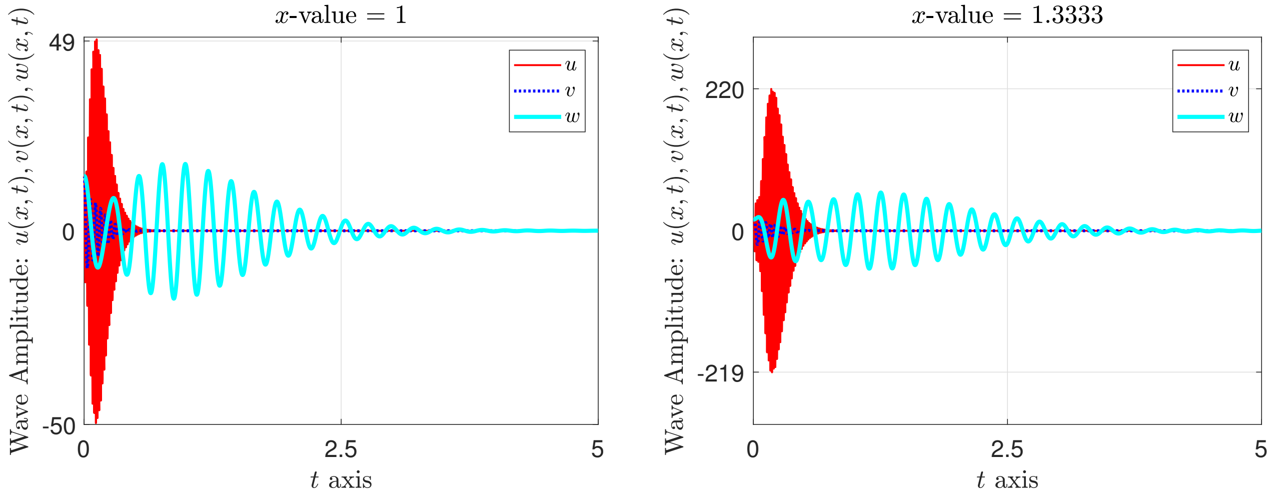
(a) **(b)**

Figure 11. Solution for different x values, $\rho_1 = 0.001, \rho_3 = 0.002, \mu_1 = 10, \mu_2 = 15, \mu_3 = 20$, $p = 2.6, q = 2.4, r = 2.3$. (a) $a(\frac{2}{3}, t), b(\frac{2}{3}, t), c(\frac{2}{3}, t)$, (b) $a(\frac{5}{3}, t), b(\frac{5}{3}, t), c(\frac{5}{3}, t)$.



(a) (b)

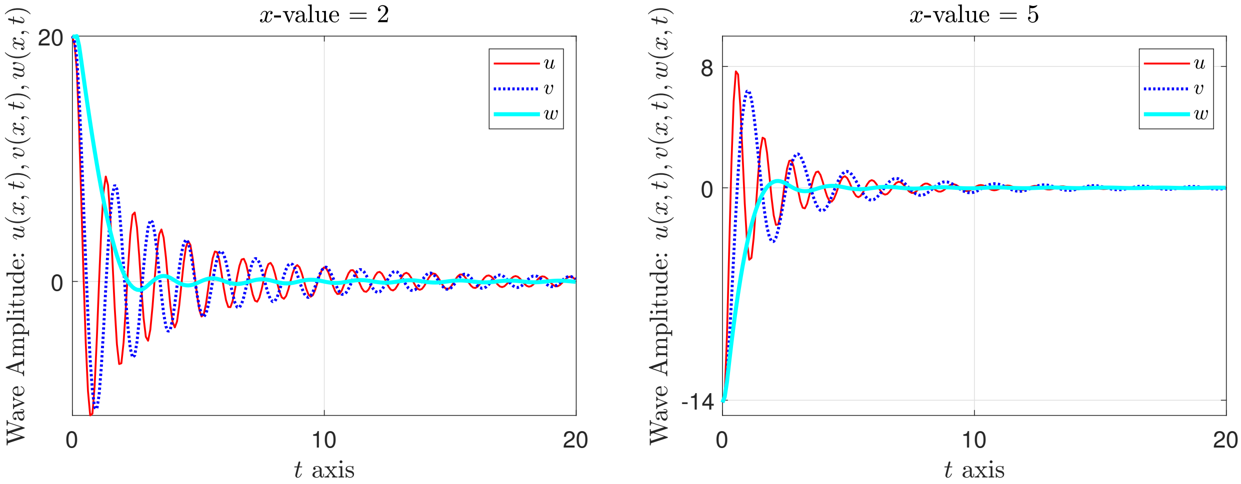
Figure 12. Solution for different x values, $\rho_1 = 0.01, \rho_3 = 0.02, \mu_1 = 20, \mu_2 = 20, \mu_3 = 30, p = 2.7, q = 3.5, r = 3.0$. (a) $a(\frac{2}{3}, t), b(\frac{2}{3}, t), c(\frac{2}{3}, t)$, (b) $a(\frac{4}{3}, t), b(\frac{4}{3}, t), c(\frac{4}{3}, t)$.



(a) (b)

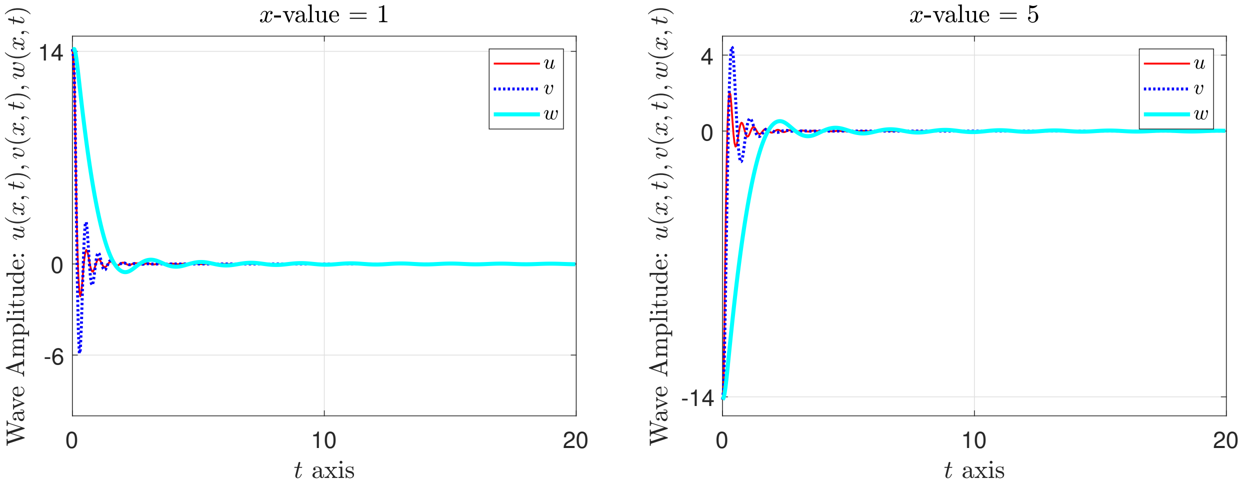
Figure 13. Solution for different x values, $\rho_1 = 0.001, \rho_3 = 0.002, \mu_1 = 10, \mu_2 = 20, \mu_3 = 30, p = q = r = 2$. (a) $a(1, t), b(1, t), c(1, t)$, (b) $a(\frac{4}{3}, t), b(\frac{4}{3}, t), c(\frac{4}{3}, t)$.

Instance 3. ($L = 6$) Next, we consider a beam of length $L = 6$, and present numerical solutions for problem (3.18) under consideration. Similarly, using the initial displacement (5.4), we observe the effect and influence of each of the parameters, the damping coefficients, and damping exponents, on the decay of solutions and associated energy functional, as presented in Figures 14–16. It is observed that the parameters ρ_1, ρ_3, the damping coefficients μ_1, μ_2, μ_3, and the damping exponents p, q, r greatly influence the wave amplitudes, oscillations, and speed of decay of the solution to zero.



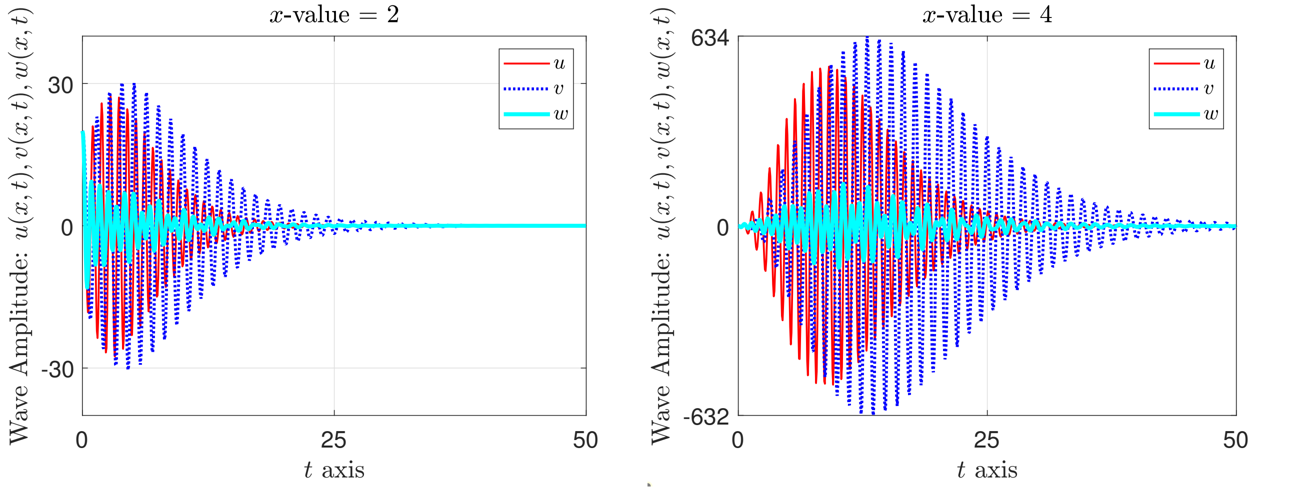
(a) **(b)**

Figure 14. Solution for different x values, $\rho_1 = 0.05$, $\rho_3 = 0.05$, $\mu_1 = 20, \mu_2 = 20, \mu_3 = 50$, $p = 2.4, q = 2.4, r = 2.6$. (a) $a(2,t), b(2,t), c(2,t)$, (b) $a(5,t), b(5,t), c(5,t)$.



(a) **(b)**

Figure 15. Solution for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 20, \mu_2 = 30, \mu_3 = 60$, $p = 2.4, q = 2.1, r = 2.7$. (a) $a(1,t), b(1,t), c(1,t)$, (b) $a(5,t), b(5,t), c(5,t)$.

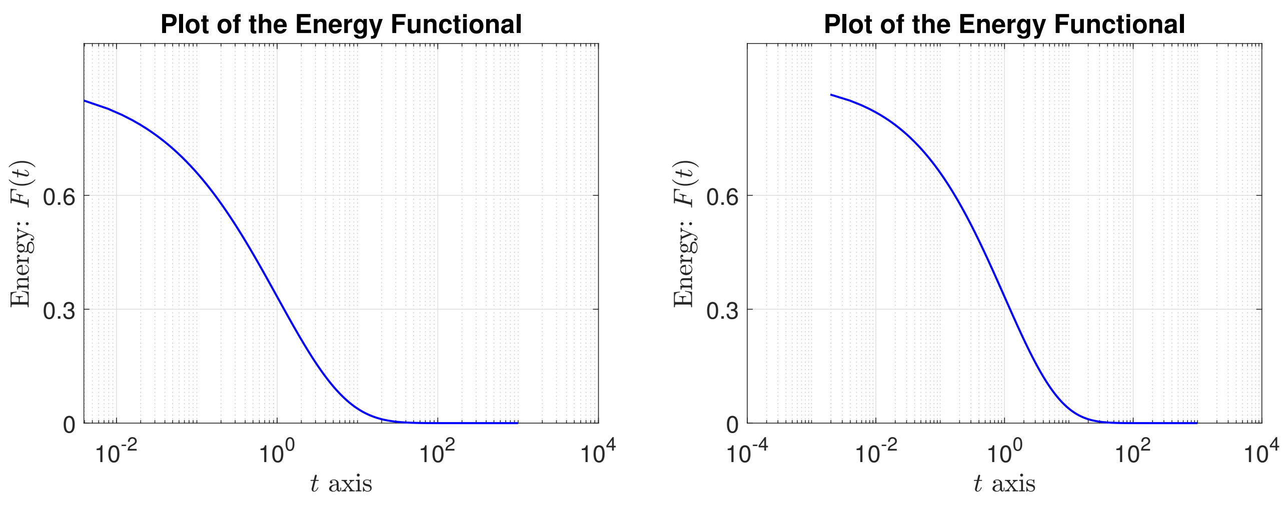


(a) **(b)**

Figure 16. Solution for different x values, $\rho_1 = 0.15, \rho_3 = 0.15, \mu_1 = 30, \mu_2 = 20, \mu_3 = 10$, $p = q = r = 2$. (a) $a(2,t), b(2,t), c(2,t)$, (b) $a(5,t), b(5,t), c(5,t)$.

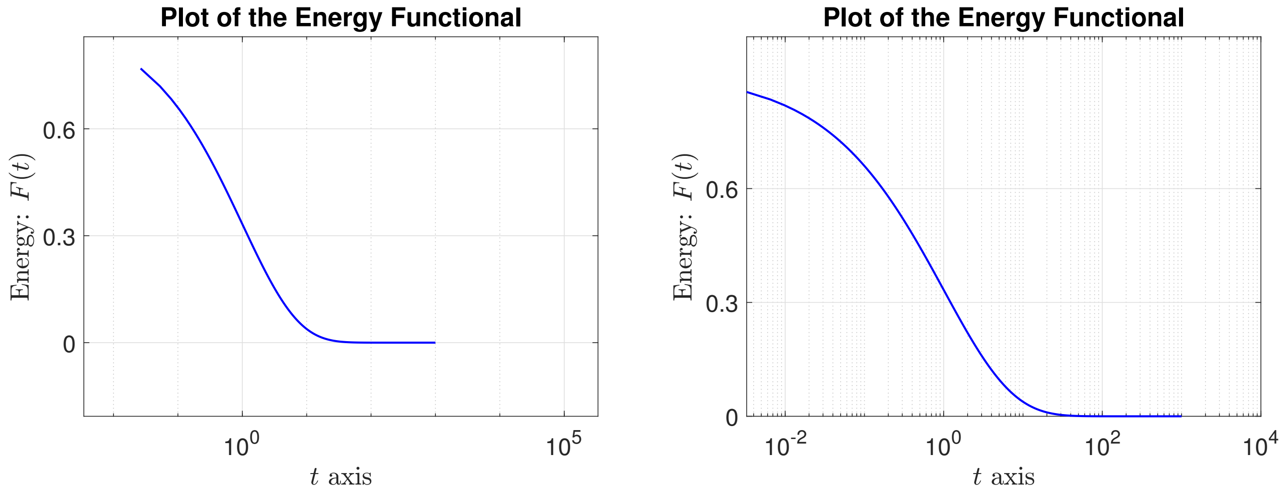
The energy

We present, in Figures 17–21, the energy decay for a beam of length $L = 1$ and correspond the numerical solutions presented in Figures 2–10 to the cases considered in Instance 1. Moreover, Figures 22–24 show numerical simulations of the energy for different lengths of the beam $L = 2$ and $L = 6$, and for different damping exponents p, q, r. We observe the decay of the energy $\Phi(t)$ to zero in all scenarios.



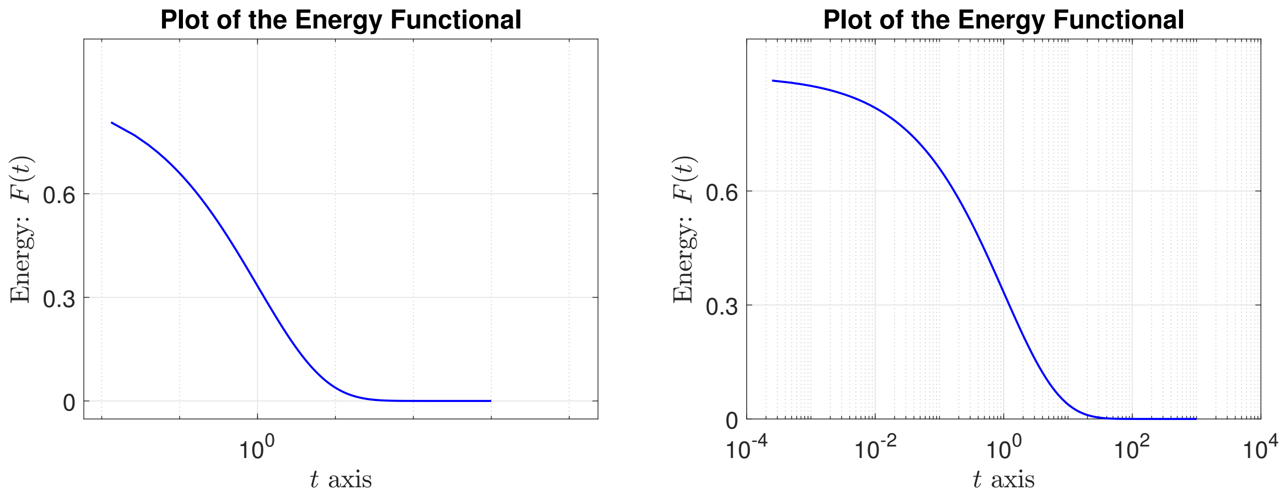
(a) **(b)**

Figure 17. Energy decay corresponding to the numerical solutions in Figures 2 and 3, respectively. (a) $L = 1 \;:\; p = 2, q = 3.5, r = 3$, (b) $L = 1 \;:\; p = 2, q = 3, r = 3.05$.



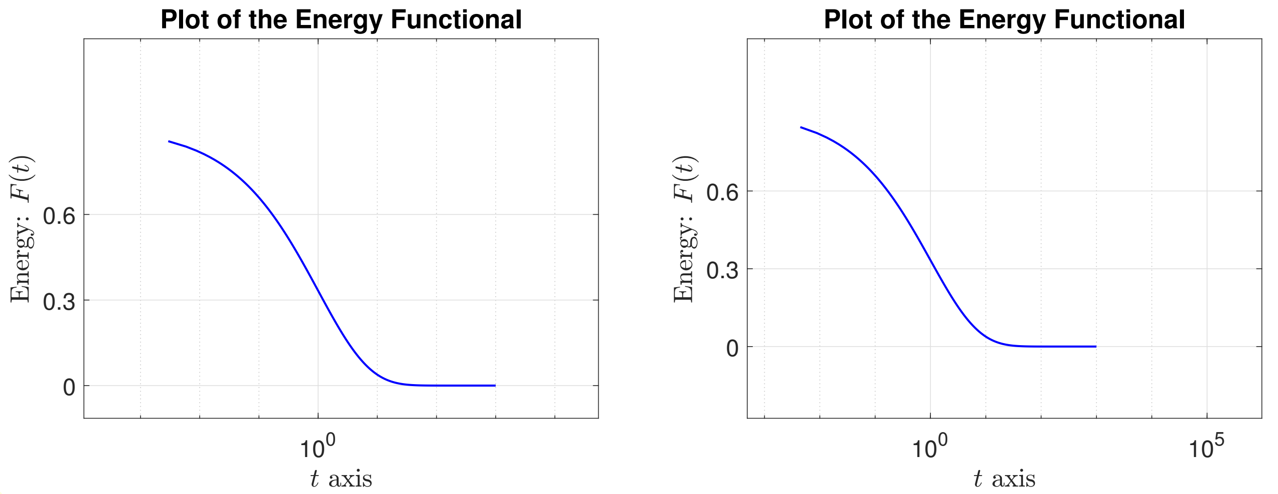
(a) **(b)**

Figure 18. Energy decay corresponding to the numerical solutions in Figures 4 and 5, respectively. (a) $L = 1\ :\ p = 2.5, q = 2, r = 2.1$, (b) $L = 1\ :\ p = 2.1, q = 2, r = 2.7$.



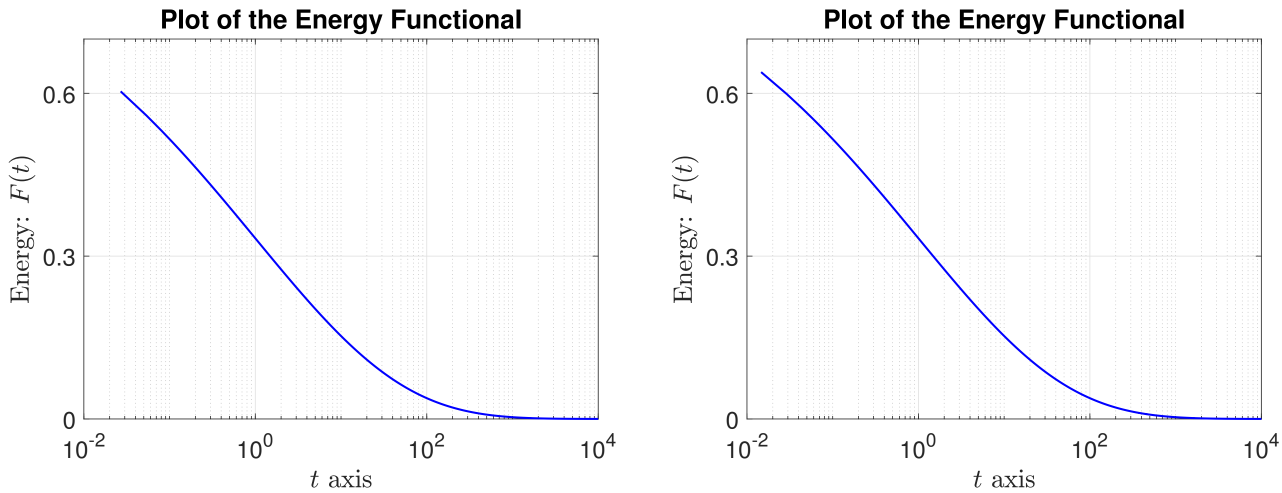
(a) **(b)**

Figure 19. Energy decay corresponding to the numerical solutions in Figures 6 and 7, respectively. (a) $L = 1\ :\ p = 2.1, q = 2.2, r = 2$, (b) $L = 1\ :\ p = 2.4, q = 2.1, r = 2$.



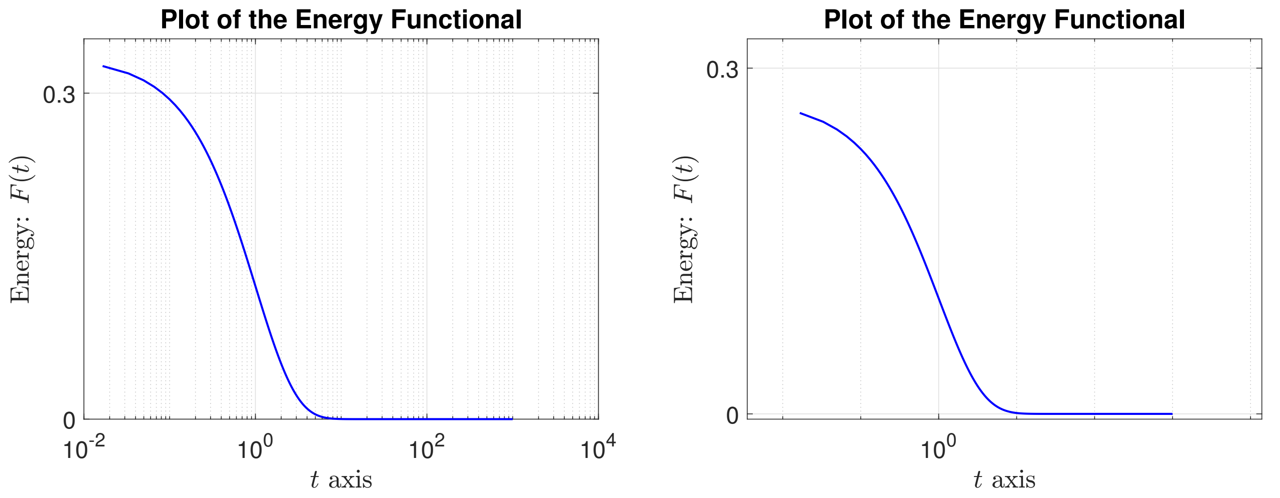
(a) **(b)**

Figure 20. Energy decay corresponding to the numerical solutions in Figures 8 and 9, respectively. (a) $L = 1 \ : \ p = q = 2, r = 2.45$, (b) $L = 1 \ : \ p = r = 2, q = 2.55$.



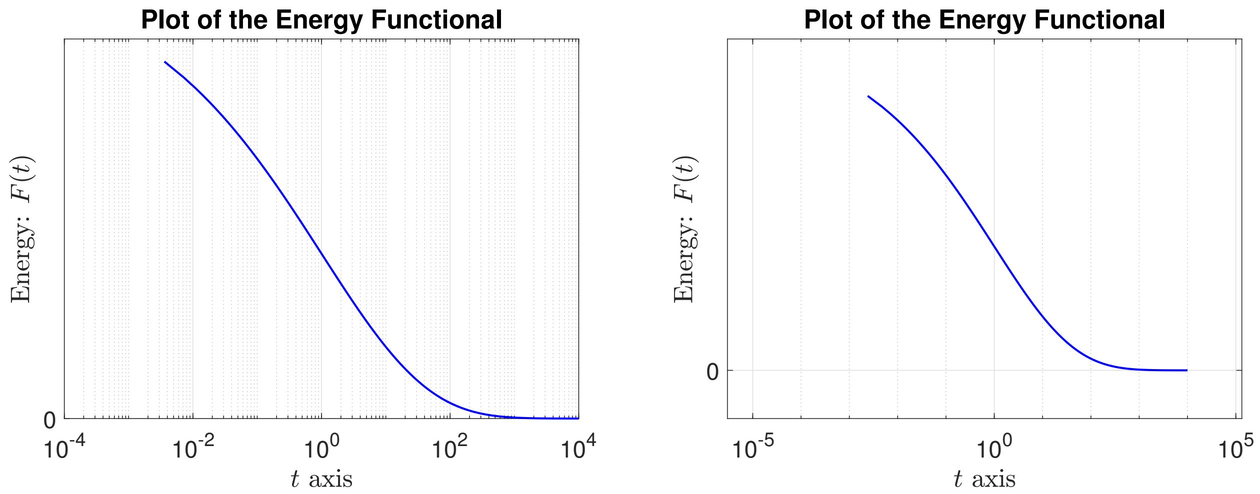
(a) **(b)**

Figure 21. Energy decay corresponding to the numerical solutions in Figure 10. (a) $L = 1 \ : \ p = 3.1, q = r = 2$, (b) $L = 1 \ : \ p = 2.7, q = 2.7, r = 2$.



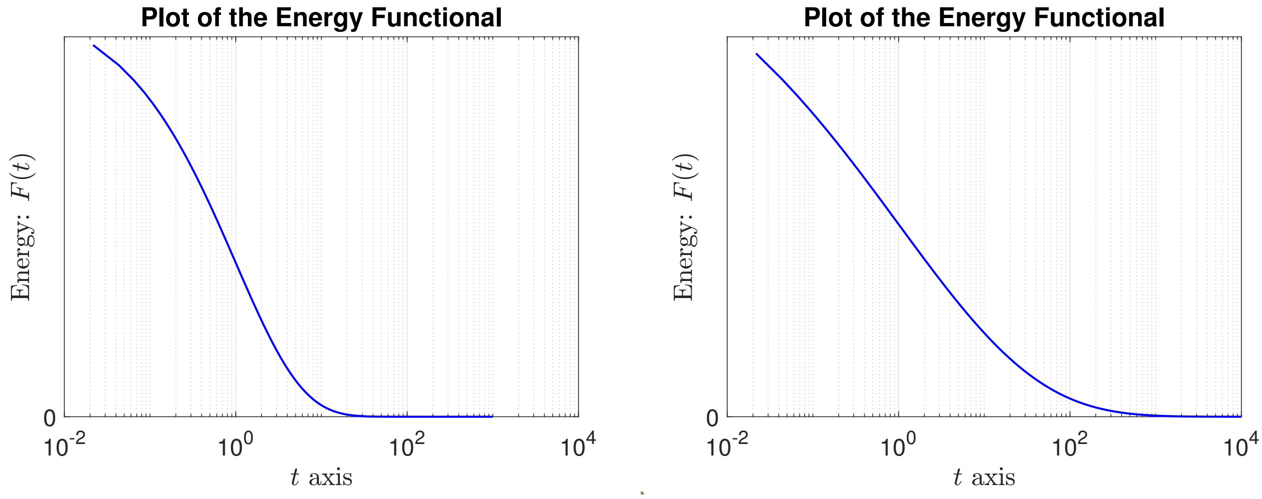
(a) **(b)**

Figure 22. Energy decay for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$. (a) $L = 2 : p = q = r = 2$, (b) $L = 6 : p = q = r = 2$.



(a) **(b)**

Figure 23. Energy decay for different x values, $\rho_1 = 0.01$, $\rho_3 = 0.02$, $\mu_1 = 10, \mu_2 = 20, \mu_3 = 30$. (a) $L = 2 : p = 2.6, q = 2.4, r = 2.3$, (b) $L = 6 : p = 2.6, q = 2.4, r = 2.3$.



(a) (b)

Figure 24. Energy decay for different x values, $\rho_1 = 0.1, \rho_3 = 0.2, \mu_1 = 10, \mu_2 = 20, \mu_3 = 30$. (a) $L = 2\ :\ p = 2.7, q = 3.5, r = 3.0$, (b) $L = 6\ :\ \ p = 2.7, q = 3.5, r = 3.0$.

6. Conclusions

In this article, we have shown that the Non-linear Rao-Nakra beam model (1.6)–(1.8) is well-posed. We also showed that without external forces, the Rao-Nakra system (4.1)–(4.3) is stable exponentially or polynomially depending on the range of the nonlinearities. Furthermore, we presented numerical examples to reinforce our theoretical analysis.

Author contributions

Soh Edwin Mukiawa: Conceptualization, data curation, formal analysis, writing-original draft; Cyril Dennis Enyi: Investigation, methodology, writing-original draft, software; Johnson D. Audu: Conceptualization, writing-review, software, validation; Hasan A. Almutairi: Methodology, writing-review, editing, resources. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no potential conflict of interests.

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