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Research article

Stability analysis of time-varying neutral stochastic pantograph equations with Markovian switching and multiple proportional delays

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Abstract: This paper rigorously investigates the stochastic stability of time-varying neutral stochastic pantograph systems with Markovian switching. Primarily, multiple proportional delays and time-varying coefficients are taken into account in the neutral hybrid stochastic systems. In order to surmount the complexities stemming from those factors, the Lyapunov functional approach and several stochastic analytical techniques are utilized, and a modified version of pantograph delays integral inequality is developed. Moreover, various stability criteria such as qth moment stability, qth moment asymptotic stability, qth exponential stability, almost sure asymptotic stability and almost sure exponential stability are put forward, where the upper bound of the diffusion operator can be a function with sign-changed time-varying coefficients rather than negative constants. Eventually, through several numerical examples, the theoretical results are strictly verified, demonstrating the feasibility and effectiveness of the proposed method.

Keywords: neutral stochastic pantograph systems; multiple proportional delays; time-varying coefficients; Markovian switching; stability
Mathematics Subject Classification: 34K40, 37H30, 93E15

1. Introduction

In recent years, stochastic differential equations have constituted a highly active research area, and as a significant branch of stochastic differential equations, neutral stochastic systems have also garnered extensive attention from researchers. However, in phenomena similar to power system faults, where there exist both delay-dependent current variation rates and topological switching, we must also account for Markovian switching systems. Therefore, in [1], Kolmanovskii et al. conducted the systematic study of neutral stochastic differential equations with Markovian switching, filling the theoretical gap for this class of hybrid systems. Subsequently, Mao et al. utilized the Lyapunov function method and stochastic analysis tools to further investigate the almost sure asymptotic stability of neutral Markovian switching systems and provided a systematic summary for such systems in [2–5]. Subsequently, some scholars focused on neutral systems in [6, 7], while others investigated Markovian switching systems in [8–10]. However, more and more scholars were integrating both frameworks to establish hybrid systems that better align with practical requirements. In [11], Liu and Xi proposed a bounded delay sampled-data controller in a neutral Markovian switching system, resolving synchronization challenges caused by stochastic switching topologies and delay coupling, thereby promoting the diversification of stochastic systems. Thereafter, in order to adapt to practical system requirements, Chen et al. used adaptive control to handle the stochastic coupled networks in [12], breaking through the delay constraints and achieving almost sure exponential stability. After that, in [13], Chen et al. dealt with limitations of traditional Halanay inequalities in handling stochastic disturbances and Markovian switching for neutral stochastic delay systems. In [14], Liu et al. further extended the research to scenarios with external perturbations by incorporating a Halanay inequality with time-varying coefficients.

In theoretical and practical research, time-varying is commonly used to describe and analyze dynamically evolving system characteristics over time, where coefficients are no longer fixed values but functions that vary with time. This concept serves as a fundamental pillar in fields such as statistics, econometrics, and engineering. Initially, in [15, 16], researchers discussed the stability conditions of linear and nonlinear time-varying systems. Gradually, due to the widespread existence of complex and dynamic scenarios in reality, Wu et al. solved the stability difficulties brought by time-varying conditions in multi-agent systems using the generalized Halanary inequality in [17]. Building on these foundations, studies in [18–23] further incorporated neutral terms to account for historical dependencies in state change rates, investigating the stability of neutral stochastic time-varying systems with Markovian switching. Furthermore, Wang et al. in [24] explored aperiodically intermittent control, while Huang et al. in [25] investigated the influence of impulsive effects, both extending stability analyses for neutral stochastic time-varying systems with Markovian switching. However, we can observe that previous research has predominantly focused on bounded time delays, and now we consider whether introducing unbounded time delays could enhance the applicability of these systems.

The phenomenon of unbounded delay is widespread in ecosystems, economic models, and power systems. Due to external environmental changes or internal complex mechanisms, the delay functions in such systems often exhibit unbounded characteristics, making traditional bounded delay models inadequate for accurately describing their dynamic features. Therefore, in [26], Lakshmikantham et al. conducted a systematic study on the theory of differential equations with unbounded delay. Thereafter, in [27], Xu et al. proposed a theoretical framework and controller design methodology for systems with unbounded delays, establishing key conditions for system stability. It is well established that proportional delay is a special form of unbounded delay, where the delay is proportional to time and belongs to unbounded time-varying delay. This delay exists in engineering and biological systems, most notably in neural signal propagation across synaptic networks and dynamic response analysis of inertial systems with energy dissipation mechanisms. In [28–32], researchers have progressively refined the stability theory framework for neutral stochastic pantograph differential equations. What's more, in [33], through stopping time techniques and Lyapunov estimates, Caraballo et al. proved the global existence and uniqueness of solutions for neutral pantograph systems with Lévy noise and Markovian switching and further established the h-stability theory of such systems. Notably, in [34], Zou et al. developed the stochastic LaSalle theorem combined with uniform continuity theory, resolving stability analysis complexities for neutral stochastic pantograph Markovian switching systems under unbounded delays and hybrid noise. To address proportional delays in applicable systems, in [35], Ruan et al. established a generalized stability framework for neutral stochastic quaternion-valued neural networks, extending exponential stability to polynomial stability, thereby opening new avenues for subsequent research. Subsequently, in [36], Ben Makhlouf et al. studied the stability of highly nonlinear neutral stochastic systems with Markovian switching and proportional delays by employing key inequalities, establishing mean-square exponential stability conditions under unbounded and multiple delays. It can be noticed that the aforementioned documents [28–36] mainly focused on the various stabilities, including exponential stability, polynomial stability, and stability with general decay rate of linear or highly nonlinear neutral stochastic systems with proportional delays. However, more general non-autonomous neutral proportional delay stochastic systems have not been explored concretely, and the upper coefficients of the $\mathcal{L}V$ operator generally are time-invariant. Meanwhile, the adopted approach is based on multiple Lyapunov functions. Naturally, some issues have arisen. First of all, for more general non-autonomous neutral proportional delay stochastic systems, how to analyze their various stochastic stability? Secondly, if the coefficients of the upper $\mathcal{L}V$ operator generally are time-varying and sign-changed, how to deal with this situation? Thirdly, how to develop a novel approach to examine the stochastic stability of these systems? It is evident that all these issues warrant further in-depth investigation.

Inspired by the aforementioned discussions, we focus on the stability analysis of a class of time-varying neutral stochastic systems with Markovian switching and multiple proportional delays. The main contributions are summarized below:

(i) In [21], Chen et al. investigated the stability of time-varying neutral hybrid systems, but the time-varying delays that they considered were bounded, and the obtained parameter η was also an invariant positive constant. This paper deliberates the case of unbounded multiple proportional delays, and the corresponding $\lambda(t)$ may be bounded or unbounded. Consequently, this paper can be regarded as an extension of the work [21].

(ii) In [29–36], the exponential stability and general decay rate stability of neutral-type proportional delay systems were investigated. Building upon this foundation, the present study employs generalized integral inequalities to analyze the stability of multiple proportional delay neutral systems with time-varying coefficients. Crucially, the upper bound conditions for the operator incorporate sign-changed time-varying functions, significantly amplifying the analytical complexity of system stability research.

(iii) Under varying conditions, the paper comprehensively examines multiple forms of stability, including qth moment stability, qth moment asymptotic stability, and qth exponential stability, further extending the analysis to investigate almost sure asymptotic and exponential stability.

The framework of this paper is divided into five sections. In Section 2, some necessary preliminaries and model descriptions are presented. Section 3 derives theoretical results on qth moment stability, qth moment asymptotic stability, qth exponential stability, and almost sure asymptotic and exponential stability for time-varying neutral stochastic systems with Markovian switching and multiple proportional delays. The validity and feasibility of the previous theoretical results are verified through two numerical examples in Section 4. In the last section, the relevant conclusions are summarized, and future research directions are proposed.

2. Preliminaries and model descriptions

Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\}$ be a complete probability space where $(\mathcal{F}_t)_{t\geq 0}$ is continuous on the right. $B(t)$ is an n-dimensional Brownian motion, and $B(t) = (B_1(t), B_2(t), ..., B_n(t))^T$ is defined on $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\}$. Subsequently, $\{\eta(t), t \in [t_0, +\infty)\}$ is a right-continuous Markov chain defined on $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\}$, taking values in a finite space $J = \{1, 2, ..., M\}$ with a generator $\Lambda = (\vartheta_{kj})_{M\times M}$ given by

$$P(\eta(t+\varpi) = j \mid \eta(t) = k) = \begin{cases} \vartheta_{kj}\varpi + o(\varpi), & k \neq j, \\ 1 + \vartheta_{kk}\varpi + o(\varpi), & k = j, \end{cases}$$

where $\varpi > 0$. If $k \neq j$, then $\vartheta_{kj} \geq 0$ is the transition rate from k to j, and $\vartheta_{kk} = -\sum_{k\neq j} \vartheta_{kj}$.

In this section, we consider the following neutral stochastic system with multiple proportional delays:

$$\begin{aligned} d\big[v(t) - C(t, v(\mu_1 t), \eta(t))\big] &= P(t, v(t), v(\mu_1 t), \cdots, v(\mu_m t), \eta(t))dt \\ &\quad + Q(t, v(t), v(\mu_1 t), \cdots, v(\mu_m t), \eta(t))dB(t), \end{aligned} \tag{2.1}$$

where $0 < \mu_i < 1$ and $\min\{\mu_i | i = 1, 2, \cdots, m\} = \bar{\mu}$, $\eta(t) \in J = \{1, 2, ..., M\}$. Moreover, we assume that $P : [t_0, +\infty) \times R^n \times R^n \times \cdots \times R^n \times J \longrightarrow R^n$, $Q : [t_0, +\infty) \times R^n \times R^n \times \cdots \times R^n \times J \longrightarrow R^{n\times l}$ and $C : [t_0, +\infty) \times R^n \times J \longrightarrow R^n$.

Let $D^{1,2}([t_0, +\infty)\times R^n\times J; [0, +\infty))$ denote a family of non-negative functions $U(t, v, k)$ on $[t_0, +\infty)\times R^n\times J$, which are once continuously differentiable and twice continuously differentiable with respect to t and v, respectively. Thus, for $v(t) = v_0 = (v_{01}, v_{02}, \cdots, v_{0n})^T \in R^n$ and $v(\mu_m t) = v_m(t)$, $\bar{v} = v_0 - C(t, v_1, k)$, we define

$$\begin{aligned} \mathcal{L}U(t, v, k) &= U_t(t, v_0 - C(t, v_1, k), k) + U_v(t, v_0 - C(t, v_1, k), k)P(t, v_0, v_1, ..., v_m, k) \\ &\quad + \frac{1}{2}\text{trace}(Q^T(t, v_0, v_1, ..., v_m, k)U_{vv}(t, v - C(t, v_1, k), k)Q(t, v_0, v_1, ..., v_m, k)) \\ &\quad + \sum_{j=1}^{M} \vartheta_{kj} U(t, v_0 - C(t, v_1, k), j), \end{aligned}$$

where $U_t = \frac{\partial U(t,v,k)}{\partial t}$, $U_v = (\frac{\partial U(t,v,k)}{\partial v_{01}}, ..., \frac{\partial U(t,v,k)}{\partial v_{0n}})$ and $U_{vv} = (\frac{\partial^2 U(t,v,k)}{\partial v_{0i}\partial v_{0j}})_{n\times n}$. Subsequently, the following assumptions are made for the aforementioned stochastic system.

Assumption 1. For $k \in J$, $\forall t \geqslant t_0$, $v_0, v_1, \cdots, v_m$, $v_0', v_1', \cdots, v_m' \in R^n$ and $\eta(t) = k$, there exists a continuous function $\sigma(t) > 0$ such that

$$\begin{aligned} &|P(t, v_0, v_1, ..., v_m, k) - P(t, v_0', v_1', ..., v_m', k)| \vee |Q(t, v_0, v_1, ..., v_m, k) - Q(t, v_0', v_1', ..., v_m', k)| \\ &\qquad \leqslant \sigma(t)(|v_0 - v_0'| + |v_1 - v_1'| + \cdots + |v_m - v_m'|). \end{aligned} \tag{2.2}$$

Meanwhile, we further suppose that $P(t, 0, 0, ..., 0, k) = 0$ and $Q(t, 0, 0, ..., 0, k) = 0$.

Assumption 2. For $k \in J$, there exists a function $\rho_k(t) > 0$ such that for $\forall t \geqslant t_0$, $v_1, v_1' \in R^n$, $\eta(t) = k$, we have

$$|C(t, v_1, k) - C(t, v_1', k)| \leqslant \rho_k(t)|v_1 - v_1'|, \tag{2.3}$$

where $\rho_k(t) \in (0,1)$ and $C(t,0,k) = 0$.

Remark 1. In Assumption 1, noting that $\sigma(t)$ is continuous, it means that system (1) satisfies the local Lipschitz condition and linear growth condition for arbitrary $t \in [\bar{\mu}t_0, T], T > t_0$, which prevents finite-time explosion of the unique solution to system (2.1), while Assumption 2 introduces a contraction property that constrains the neutral term. Under Assumptions 1 and 2, the existence and uniqueness of a global solution to system (2.1) can be guaranteed.

3. Main results

In this section, by employing stochastic analysis techniques, we address the problems of multiple proportional delays and time-varying delays in neutral pantograph stochastic systems with Markovian switching and establish novel criteria for both qth moment asymptotic stability and qth moment exponential stability of the system.

Theorem 3.1. Under Assumptions 1 and 2, if there exists a function $U(\cdot,\cdot,\cdot) \in D^{1,2}([t_0,+\infty) \times R^n \times J; [0,+\infty))$, constants $\beta_1 > 0$, $\beta_2 > 0$, and a series of functions $b_1(\cdot) : [\bar{\mu}t_0,+\infty) \to R$, $d_i(\cdot) : [\bar{\mu}t_0,+\infty) \to [0,+\infty)$, $i = (0,1,2,...,m)$ such that for all $v, v_0, v_1, v_2, \cdots, v_m \in R^n$, $t \geqslant t_0$, $\eta(t) = k \in J$ and $q \geqslant 2$,

(i) $\beta_1|v|^q \leqslant U(t,v,k) \leqslant \beta_2|v|^q$;

(ii) $\mathcal{L}U(t,v,k) \leqslant b_1(t)U(t,\bar{v},k) + \sum_{i=0}^m d_i(t)U(t,v_i,k)$, where $\bar{v} = v_0 - C(t,v_1,k) \in R^n$, $v_i = v(\mu_i t)$, $i = 1,2\cdots,m$;

(iii) $\rho_k(t) = e^{-\frac{\lambda(t)}{q}}\rho_k > 0$, $\bar{\rho} = \max\{\rho_k\} < 1$, where $\lambda(t) \geq \bar{\pi}(t) = \sup_{t\geqslant t_0}\sup_{\theta\in[-(1-\bar{\mu})t,0]}\int_{t+\theta}^t [-b_1(s) - \frac{\beta_2\sum_{i=0}^m d_i(s)}{\beta_1(1-\bar{\rho})^q}]ds \geq 0$;

furthermore, when

$$\delta_0 = \sup_{t\geqslant t_0}\int_{t_0}^t \bar{\delta}(s)ds < +\infty, \tag{3.1}$$

with $\bar{\delta}(t) = b_1(t) + \frac{\beta_2\left[d_0(t)+e^{\lambda(t)}\sum_{i=1}^m d_i(t)\right]}{\beta_1(1-\bar{\rho})^q}$, then system (2.1) is qth moment stable;
when

$$\int_{t_0}^{+\infty}\bar{\delta}(s)ds = -\infty, \tag{3.2}$$

then system (2.1) is qth moment asymptotically stable.

Proof. For system (2.1), by utilizing Itô's formula and condition (ii), it can be deduced that

$$\begin{aligned}\frac{d}{dt}[e^{-\int_{t_0}^t b_1(s)ds}E[U(t,\bar{v}(t),\eta(t))]] &= e^{-\int_{t_0}^t b_1(s)ds}\Big(-b_1(t)E[U(t,\bar{v}(t),\eta(t))] + \frac{d}{dt}E[U(t,\bar{v}(t),\eta(t))]\Big)\\ &= e^{-\int_{t_0}^t b_1(s)ds}\Big(-b_1(t)E[U(t,\bar{v}(t),\eta(t))] + E[\mathcal{L}U(t)(t,\bar{v}(t),\eta(t))]\Big)\\ &\leqslant e^{-\int_{t_0}^t b_1(s)ds}\Big[-b_1(t)E[U(t,\bar{v}(t),\eta(t))] + b_1(t)E[U(t,\bar{v}(t),\eta(t))]\\ &\quad + E[\sum_{i=0}^m d_i(t)U(t,v_i(t),\eta(t))]\Big]\end{aligned}$$

$$\leqslant e^{-\int_{t_0}^{t} b_1(s)ds} E[\sum_{i=0}^{m} d_i(t) u(t, v_i(t), \eta(t))].$$

Thereafter, integrating both sides of the above inequality and applying condition (i), we can derive

$$\begin{aligned} e^{-\int_{t_0}^{t} b_1(s)ds} E[U(t, \bar{v}(t), \eta(t))] &\leqslant e^{-\int_{t_0}^{t_0} b_1(s)ds} E[U(t_0, \bar{v}(t_0), \eta(t_0))] \\ &\quad + \int_{t_0}^{t} e^{-\int_{t_0}^{s} b_1(\xi)d\xi} E[\sum_{i=0}^{m} d_i(s) U(s, v_i(s), \eta(s))] ds \\ &\leqslant \beta_2 E|\bar{v}(t_0)|^q + \beta_2 \int_{t_0}^{t} e^{-\int_{t_0}^{s} b_1(\xi)d\xi} [d_0(s)|v(s)|^q + \sum_{i=1}^{m} d_i(s)|v(\mu_i s)|^q] ds. \end{aligned} \tag{3.3}$$

Hence, it follows from condition (i) that

$$E|\bar{v}(t)|^q \leqslant \frac{\beta_2}{\beta_1} E|\bar{v}(t_0)|^q e^{\int_{t_0}^{t} b_1(s)ds} + \frac{\beta_2}{\beta_1} \int_{t_0}^{t} e^{\int_{s}^{t} b_1(\xi)d\xi} [d_0(s)|v(s)|^q + \sum_{i=1}^{m} d_i(s)|v(\mu_i s)|^q] ds. \tag{3.4}$$

Whereafter, with $q \geqslant 1$ and the inequality $|a-b|^q \leqslant (|a|+|b|)^q \leqslant 2^{q-1}(|a|^q + |b|^q)$ established, under Assumption 2 and condition (iii), we are positioned to deduce that

$$\begin{aligned} E|\bar{v}(t_0)|^q &= E|v(t_0) - C(t_0, v(\mu_1 t_0), \eta(t_0))|^q \\ &\leqslant 2^{q-1}[E|v(t_0)|^q + E|C(t_0, v(\mu_1 t_0), \eta(t_0))|^q] \\ &\leqslant 2^{q-1}[E|v(t_0)|^q + E[\bar{\rho}^q |v(\mu_1 t_0)|^q]] \\ &\leqslant 2^{q-1}(1+\bar{\rho}^q) \sup_{\bar{\mu} t_0 \leqslant \theta \leqslant t_0} E[\varphi(\theta)]^q \\ &= 2^{q-1}(1+\bar{\rho}^q) H_1 \\ &= H_2, \end{aligned} \tag{3.5}$$

where $H_1 = \sup_{\bar{\mu} t_0 \leq \theta \leq t_0} E[\varphi(\theta)]^q$ and $H_2 = 2^{q-1}(1+\bar{\rho}^q)H_1$. Moreover, applying the inequality $(a+b)^q \leqslant \frac{1}{(1-\bar{\rho})^{q-1}} a^q + \frac{1}{\bar{\rho}^{q-1}} b^q$ in concert with Assumption 2, we have

$$\begin{aligned} |v(t)|^q &= |\bar{v}(t) + C(t, v(\mu_1 t), \eta(t))|^q \\ &\leqslant \frac{1}{(1-\bar{\rho})^{q-1}} |\bar{v}(t)|^q + (\frac{1}{\bar{\rho}})^{q-1} \bar{\rho}^q |v(\mu_1 t)|^q (e^{-\frac{\lambda(t)}{q}})^q \\ &\leqslant \frac{|\bar{v}(t)|^q}{(1-\bar{\rho})^{q-1}} + \bar{\rho} |v(\mu_1 t)|^q e^{-\lambda(t)}. \end{aligned}$$

Taking expectation on both sides of the inequality yields

$$E|v(t)|^q \leqslant \bar{\rho} e^{-\lambda(t)} E|v(\mu_1 t)|^q + \frac{E|\bar{v}(t)|^q}{(1-\bar{\rho})^{q-1}}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we derive

$$E|v(t)|^q \leqslant H_3 e^{\int_{t_0}^{t} b_1(s)ds} + \frac{\beta_2}{\beta_1 (1-\bar{\rho})^{q-1}} \int_{t_0}^{t} e^{\int_{s}^{t} b_1(\xi)d\xi} (\sum_{i=1}^{m} d_i(s)|v(\mu_i s)|^q + d_0(s)|v(s)|^q) ds$$

$$+\bar{\rho}e^{-\lambda(t)}E|v(\mu_1 t)|^q, \tag{3.7}$$

where $H_3 = \frac{\beta_2 H_2}{\beta_1(1-\bar{\rho})^{q-1}}$. For $\forall\varepsilon > 0$, we set that $H_4 = \max\{H_1 e^{\sup_{t\in[\bar{\mu}t_0,t_0]}\int_t^{t_0}\bar{\delta}(s)ds} + \varepsilon, \frac{H_3+\varepsilon}{1-\bar{\rho}}\}$. Subsequently, we will verify that

$$E|v(t)|^q < H_4 e^{\int_{t_0}^t \bar{\delta}(s)ds}, \quad t \geqslant \bar{\mu}t_0. \tag{3.8}$$

Demonstrably, when $t \in [\bar{\mu}t_0, t_0]$, we can obtain

$$E|v(t)|^q \leqslant H_1 < H_4 e^{\int_{t_0}^t \bar{\delta}(s)ds},$$

which indicates that the conclusion is valid. Subsequently, we employ reduction to absurdity to discuss the case for $t \geqslant t_0$. If (3.8) does not hold, then there exists a certain $t^* > t_0$ such that $E|v(t)|^q \leqslant H_4 e^{\int_{t_0}^t \bar{\delta}(s)ds}$, $t \in [\bar{\mu}t_0, t^*)$ and $E|v(t^*)|^q = H_4 e^{\int_{t_0}^t \bar{\delta}(s)ds}$. On the other hand, combining the above derivations, we conclude that

$$\begin{aligned}
E|v(t^*)|^q \leqslant\ & H_3 e^{\int_{t_0}^{t^*} b_1(s)ds} + \bar{\rho}e^{-\lambda(t^*)}E|v(\mu_1 t^*)|^q \\
& + \frac{\beta_2}{\beta_1(1-\bar{\rho})^{q-1}} \int_{t_0}^{t^*} e^{\int_s^{t^*} b_1(\xi)d\xi} (\sum_{i=1}^m d_i(s)|v(\mu_i s)|^q + d_0(s)|v(s)|^q)ds \\
\leqslant\ & H_3 e^{\int_{t_0}^{t^*} b_1(s)ds} + \bar{\rho}H_4 e^{\int_{t_0}^{\mu_1 t^*}\bar{\delta}(s)ds} e^{-\lambda(t^*)} \\
& + \frac{\beta_2}{\beta_1(1-\bar{\rho})^{q-1}} \int_{t_0}^{t^*} e^{\int_s^{t^*} b_1(\xi)d\xi} (\sum_{i=1}^m d_i(s)H_4 e^{\int_{t_0}^{\mu_1 s}\bar{\delta}(\xi)d\xi} + d_0(s)H_4 e^{\int_{t_0}^{s}\bar{\delta}(\xi)d\xi})ds \\
\leqslant\ & H_3 e^{\int_{t_0}^{t^*} b_1(s)ds} + \bar{\rho}H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(s)ds} e^{\int_{\mu_1 t^*}^{t^*} -\bar{\delta}(s)ds} e^{-\lambda(t^*)} \\
& + \frac{\beta_2}{\beta_1(1-\bar{\rho})^{q-1}} e^{\int_{t_0}^{t^*}\bar{\delta}(\xi)d\xi} \int_{t_0}^{t^*} e^{\int_s^{t^*} b_1(\xi)d\xi} e^{\int_{t_0}^{t^*} -\bar{\delta}(\xi)d\xi} (\sum_{i=1}^m d_i(s)H_4 e^{\int_{t_0}^{\mu_1 s}\bar{\delta}(\xi)d\xi} + d_0(s)H_4 e^{\int_{t_0}^{s}\bar{\delta}(\xi)d\xi})ds \\
\leqslant\ & H_3 e^{\int_{t_0}^{t^*} b_1(s)ds} + \bar{\rho}H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(s)ds} \\
& + H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(\xi)d\xi} \int_{t_0}^{t^*} e^{-\int_s^{t^*}[\bar{\delta}(\xi)-b_1(\xi)]d\xi} [\frac{\beta_2}{\beta_1(1-\bar{\rho})^{q-1}}(d_0(s) + \sum_{i=1}^m d_i(s)e^{\lambda(s)})]ds \\
\leqslant\ & H_3 e^{\int_{t_0}^{t^*} b_1(s)ds} + \bar{\rho}H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(s)ds} \\
& + (1-\bar{\rho})H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(\xi)d\xi)} \int_{t_0}^{t^*} e^{-\int_s^{t^*}(\bar{\delta}(\xi)-b_1(\xi))d\xi} [\bar{\delta}(s) - b_1(s)]ds \\
\leqslant\ & [H_3 - (1-\bar{\rho})H_4] e^{\int_{t_0}^{t^*} b_1(s)ds} + H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(s)ds} \\
<\ & H_4 e^{\int_{t_0}^{t^*}\bar{\delta}(s)ds},
\end{aligned} \tag{3.9}$$

which contradicts the foregoing result $E|v(t^*)|^q = H_4 e^{\int_{t_0}^t \bar{\delta}(s)ds}$. It means that assertion (3.8) holds. Furthermore, letting $\varepsilon \to 0$, we can derive the following expression from (3.8) that

$$E|v(t)|^q \leqslant H_5 e^{\int_{t_0}^t \bar{\delta}(s)ds}, \tag{3.10}$$

where $H_5 = \max\{H_1 e^{\sup\limits_{t\in[\bar{\mu}t_0,t_0]}\int_t^{t_0}\bar{\delta}(s)ds}, \frac{H_3}{1-\bar{\rho}}\}$. Therefore, the proof is complete.

When $\rho(k) = 0$, system (2.1) is reduced to the following form:

$$dv(t) = P(t, v(t), v(\mu_1 t), \cdots, v(\mu_m t), \eta(t))dt + Q(t, v(t), v(\mu_1 t), \cdots, v(\mu_m t), \eta(t))dB(t). \tag{3.11}$$

Consequently, according to Theorem 3.1, for system (3.11), we immediately derive the following assertion.

Corollary 1. Under Assumption 1, if there exists a function $U(\cdot,\cdot,\cdot) \in D^{1,2}([t_0, +\infty)\times R^n \times J; [0, +\infty))$, constants $\beta_1 > 0, \beta_2 > 0$, and a series of functions $b_1(\cdot) : [\bar{\mu}t_0, +\infty) \to R$, $d_i(\cdot) : [\bar{\mu}t_0, +\infty) \to [0, +\infty)$, $i = (0, 1, 2, ..., m)$ such that for all $v, v_0, v_1, v_2, \cdots, v_m \in R^n$, $t \geqslant t_0$, $\eta(t) = k \in J$ and $q \geqslant 2$,
(i) $\beta_1|v|^q \leqslant U(t, v, k) \leqslant \beta_2|v|^q$;
(ii) $\mathcal{L}U(t, v, k) \leqslant b_1(t)U(t, \bar{v}, k) + \sum_{i=0}^{m} d_i(t)U(t, v_i, k)$, where $v(t) = v_0 \in R^n$, $v_i = v(\mu_i t)$, $i = 1, 2\cdots, m$; furthermore, when

$$\delta_0 = \sup_{t\geqslant t_0}\int_{t_0}^{t}\bar{\delta}(s)ds < +\infty, \tag{3.12}$$

with $\lambda(t) \geq \bar{\pi}(t) = \sup\limits_{t\geqslant t_0}\sup\limits_{\theta\in[-(1-\bar{\mu})t,0]}\int_{t+\theta}^{t}[-b_1(s) - \frac{\beta_2\sum_{i=0}^{m}d_i(s)}{\beta_1(1-\bar{\rho})^q}]ds \geq 0$ and $\bar{\delta}(t) = b_1(t) + \frac{\beta_2\left[d_0(t)+e^{\lambda(t)}\sum_{i=1}^{m}d_i(t)\right]}{\beta_1}$, then system (2.1) is qth moment stable; when

$$\int_{t_0}^{+\infty}\bar{\delta}(s)ds = -\infty, \tag{3.13}$$

then system (3.11) is the qth moment asymptotically stable.

Remark 2. Notably, it can be noted from Theorem 3.1 that $\bar{\delta}(t) = b_1(t) + \frac{\beta_2\left[d_0(t)+e^{\lambda(t)}\sum_{i=1}^{m}d_i(t)\right]}{\beta_1(1-\bar{\rho})^q}$ and $\lambda(t) \geq \bar{\pi}(t) = \sup\limits_{t\geqslant t_0}\sup\limits_{\theta\in[-(1-\bar{\mu})t,0]}\int_{t+\theta}^{t}[-b_1(s) - \frac{\beta_2\sum_{i=0}^{m}d_i(s)}{\beta_1(1-\bar{\rho})^q}]ds \geq 0$. As a matter of fact, function $\lambda(t)$ characterizes the increment of the delay states with respect to current states, which implies $E|v(\mu_i t)|^q \leq E|v(t)|^q e^{\lambda(t)}, i \in \{1, 2, \cdots, m\}$. Combining with documents [17, 21, 24] and the proof procedure of Theorem 3.1, we can construct the function $\lambda(t)$. Since multiple unbounded proportional delays are introduced, the parameter θ satisfies $\theta \in [-(1 - \bar{\mu})t, 0]$, and the function $\lambda(t)$ may be a bounded constant or unbounded function. On the other hand, function $\bar{\delta}(t)$ depicts the change rate of system states. The function $\lambda(t)$ can be estimated by the given functions of conditions, while the function $\bar{\delta}(t)$ depends on the given functions and function $\lambda(t)$.

Theorem 3.2. Assume that the conditions in Theorem 3.1 hold and $\int_{t_0}^{t}\bar{\delta}(s)ds \leqslant -\gamma(t - t_0) + c_0$, $\gamma > 0$; then the system (2.1) is qth moment exponentially stable. In addition, if a continuous function $\tilde{\sigma}(\cdot) : [t_0, +\infty) \to [0, +\infty)$ satisfies conditions
(i) $(E|P(t, \varphi, k)|^q) \vee (E|Q(t, \varphi, k)|^q) \leqslant \hat{\sigma}(t) \sup\limits_{\theta\in[-(1-\bar{\mu})t,t]} E|\varphi(\theta)|^q$;
(ii) $\int_{t_0}^{+\infty}\hat{\sigma}(\xi)e^{-\rho\xi}d\xi \leqslant \Psi, \rho < \gamma$;
then system (2.1) is almost surely exponentially stable.

Proof. According to Theorem 3.1, we derive that

$$E|v(t)|^q \leqslant H_5 e^{\int_{t_0}^{t}\bar{\delta}(s)ds}. \tag{3.14}$$

Furthermore, since $\int_{t_0}^{t}\bar{\delta}(s)ds \leqslant -\gamma(t-t_0)+c_0$, substituting it into (3.14), we obtain

$$E|v(t)|^q \leqslant H_5 e^{c_0} e^{-\gamma(t-t_0)},$$

which indicates the qth moment exponentially stable of the system (2.1).

On the other hand, for any integer $\iota \geqslant 1$, it holds that

$$\begin{aligned} E\{\sup_{\xi\in[0,1]}|\bar{v}(t_0+\iota+\xi)|^q\} &\leqslant 3^{q-1}E|\bar{v}(t_0+\iota)|^q \\ &+ 3^{q-1}E\Big\{\Big(\int_{t_0+\iota}^{t_0+\iota+1}|P(t)|dt\Big)^q\Big\} + 3^{q-1}E\Big\{\sup_{\xi\in[0,1]}\Big|\int_{t_0+\iota}^{t_0+\iota+\xi}Q(t)dB(t)\Big|^q\Big\}. \end{aligned} \tag{3.15}$$

Subsequently, similar to [21], by combining Hölder's inequality and Burkholder-Davis-Gundy inequality, it follows that

$$\begin{aligned} E\{\sup_{\xi\in[0,1]}|\bar{v}(t_0+\iota+\xi)|^q\} &\leqslant 3^{q-1}E|\bar{v}(t_0+\iota)|^q \\ &+ 3^{q-1}E\Big\{\Big(\int_{t_0+\iota}^{t_0+\iota+1}|P(t)|dt\Big)^q\Big\} + 3^{q-1}E\Big\{\sup_{\xi\in[0,1]}\Big|\int_{t_0+\iota}^{t_0+\iota+\xi}Q(t)dB(t)\Big|^q\Big\} \\ &\leqslant 3^{q-1}\cdot 2^{q-1}(1+\bar{\rho}^q)E|v(t_0+\iota)|^q \\ &+ 3^{q-1}E\Big\{\int_{t_0+\iota}^{t_0+\iota+1}|P(t)|^q dt\Big\} + 3^{q-1}\Big(\frac{q^3}{2(q-1)}\Big)^{\frac{q}{2}}E\Big\{\Big(\int_{t_0+\iota}^{t_0+\iota+1}|Q(t)|^q dt\Big)\Big\} \\ &\leqslant 6^{q-1}(1+\bar{\rho}^q)H_5e^{c_0}e^{-\gamma\iota} \\ &+ 3^{q-1}\Big[\int_{t_0+\iota}^{t_0+\iota+1}\hat{\sigma}(s)e^{-\rho s}ds\Big]H_5e^{c_0}e^{\gamma t_0}e^{-(\gamma-\rho)\iota} \\ &+ 3^{q-1}\Big(\frac{q^3}{2(q-1)}\Big)^{\frac{q}{2}}\Big[\int_{t_0+\iota}^{t_0+\iota+1}\hat{\sigma}(s)e^{-\rho s}ds\Big]H_5e^{c_0}e^{\gamma t_0}e^{-(\gamma-\rho)\iota} \\ &\leqslant 6^{q-1}(1+\bar{\rho}^q)H_5e^{c_0}e^{-\gamma\iota} \\ &+ [3^{q-1}\Psi H_5e^{c_0+\gamma t_0}]e^{-(\gamma-\rho)\iota} + \Big[3^{q-1}\Big(\frac{q^3}{2(q-1)}\Big)^{\frac{q}{2}}\Psi H_5e^{c_0+\gamma t_0}\Big]e^{-(\gamma-\rho)\iota} \\ &\leqslant H_6e^{-(\gamma-\rho)\iota}, \end{aligned} \tag{3.16}$$

where $H_6 = 6^{q-1}(1+\bar{\rho}^q)H_5e^{c_0+\gamma t_0}e^{-\rho\iota}+3^{q-1}[1+(\frac{q^3}{2(q-1)})^{\frac{q}{2}}]\Psi H_5e^{c_0+\gamma t_0}$. Consequently, taking $\zeta\in(0,\gamma-\rho)$, we conclude that

$$P\{\sup_{t_0+\iota\leqslant t\leqslant t_0+\iota+1}|\bar{v}(t)|^q > e^{-(\gamma-\rho-\zeta)\iota}\} \leqslant H_6e^{-\zeta\iota}. \tag{3.17}$$

Moreover, since $\sum_{\iota=1}^{\infty}H_6e^{-\zeta\iota} < +\infty$, by the Borel-Cantelli lemma, there exists $\iota_0=\iota_0(\omega)$ such that when $\iota > \iota_0(\omega)$, we have

$$\sup_{\iota\leqslant t\leqslant\iota+1}|\bar{v}(t)|^q \leqslant H_6e^{-(\gamma-\rho-\zeta)\iota}, \quad \text{a.s.} \tag{3.18}$$

Letting $\zeta \to 0$, it yields that

$$\sup_{\iota \leqslant t \leqslant \iota+1} |\bar{v}(t)|^q e^{(\gamma-\rho)} \leqslant H_7, \tag{3.19}$$

where $H_7 = H_6 e^{\gamma-\rho}$. Therefore, based on the derivation in [4], we obtain

$$\lim_{t\to\infty} \sup \frac{\log |v(t)|}{t} \leqslant -\frac{\gamma-\rho}{q}, \quad \text{a.s.},$$

which signifies that system (2.1) is almost surely exponentially stable.

Remark 3. Recently, some achievements [21, 29–36] about neutral stochastic systems have been acquired. Particularly, in [21], the stability of time-varying neutral hybrid systems was investigated, but the time-varying delays that they considered are bounded, and the obtained parameter η also was an invariant positive constant, which cannot be directly extended to scenarios involving unbounded proportional delays. Hence, this paper deliberates the case of unbounded multiple proportional delays, and the corresponding $\lambda(t)$ may be bounded or unbounded. Besides, in [29–36], the exponential stability and general decay rate stability of neutral-type proportional delay systems were investigated. Building upon this foundation, our study employs generalized integral inequalities to analyze the stability of multiple proportional delay hybrid neutral systems with time-varying coefficients and random noise. Crucially, different from the results [29–36], the upper bound conditions for the operator incorporate sign-changed time-varying functions.

Remark 4. In [37], H_∞ and exponential stabilization of a class of nonlinear non-autonomous hybrid stochastic systems with constant delays were analyzed by utilizing the multiple Lyapunov functionals method and delay feedback control strategy. Furthermore, the upper bound of time delay was estimated accurately, and various stabilities with different speeds were discussed in detail. In [38], the author and collaborators focused on the exponential stability of highly nonlinear hybrid neutral stochastic systems with time-varying delays by constructing multiple degenerate functionals and relaxed the constraints of rigorous conditions. Although the approaches of the existing results in [37,38] are novel, the considered time delays are bounded. In our paper, it is extended to unbounded delay scenarios, especially proportional delays. What's more, we not only adopt the integral inequality method to overcome the complexities that proportional delays in neutral systems yield but also establish various stability criteria, allowing the upper bound of the diffusion operator to be a sign-changed time-varying function.

4. Numerical example

In this section, we present the following two numerical examples to verify the effectiveness of our theoretical findings.

Example 1. Consider the following time-varying neutral stochastic pantograph system with Markovian switching:

$$d[v(t) - C(t, v(\mu_1 t), \eta(t))] = P(t, v(t), v(\mu_1 t), \eta(t))dt + Q(t, v(t), v(\mu_1 t), \eta(t))dB(t), \tag{4.1}$$

where $\eta(t)$ is a Markov chain on the state space $J = \{1, 2\}$ with its generator

$$\Lambda = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}.$$

In this example, when $\mu_1 = \frac{3}{4}$, we specify $C(\cdot)$, $P(\cdot)$, $Q(\cdot)$ as follows:

$$P(t, v(t), v(\mu_1 t), 1) = -\frac{|\sin t| + 0.25}{1+t}[v(t) - \frac{1}{5}v(\frac{3}{4}t)] + \frac{0.5}{1+t}v(\frac{3}{4}t)e^{-\frac{\ln 2}{2}},$$

$$P(t, v(t), v(\mu_1 t), 2) = -\frac{|\sin t| + 0.2}{1+t}[v(t) - \frac{1}{5}v(\frac{3}{4}t)] + \frac{0.4}{1+t}v(\frac{3}{4}t)e^{-\frac{\ln 2}{2}},$$

$$Q(t, v(t), v(\mu_1 t), 1) = Q(t, v(t), v(\mu_1 t), 2) = \frac{0.2}{\sqrt{1+t}}v(t),$$

and

$$C(t, v(\mu_1 t), 1) = C(t, v(\mu_1 t), 2) = \frac{1}{5}v(\frac{3}{4}t)e^{-\frac{\ln 2}{2}}.$$

Choose $U(t, v, k) = \left[v(t) - \frac{1}{5}v(\frac{3}{4}t)\right]^2 = \bar{v}^2(t)$. Therefore, it can be concluded that

$$\mathcal{L}U(t, v, k) \leqslant -\frac{2|\sin t|}{1+t}U(\bar{v}(t), t, i) + \frac{0.5e^{-\ln 2}}{1+t}U(v(\frac{3}{4}t), t, k) + \frac{0.04}{1+t}U(v(t), t, k).$$

Noting that the coefficients of the considered neutral stochastic hybrid system are time-varying, and $-\frac{2|\sin t|}{1+t} + \frac{0.5e^{-\ln 2}}{1+t} + \frac{0.04}{1+t}$ is a sign-changed continuous function, which restricts the application of the existing results [29–33]. Moreover, it can be estimated that $\bar{\pi}(t) = \sup_{t \geqslant t_0} \sup_{\theta \in [-\frac{1}{4}t, 0]} \int_{t+\theta}^{t} [\frac{2|\sin s|}{1+s} - \frac{0.5e^{-\ln 2}+0.04}{1+s} \times \frac{25}{16}]ds \leqslant \int_{\frac{3}{4}t}^{t} \frac{2}{1+s}ds \leqslant \ln 2 = \lambda(t)$, which is a fixed constant.

We also calculate that

$$\begin{aligned}
\bar{\delta}(t) &= \int_{t_0}^{+\infty} [-\frac{2|\sin t|}{1+t} + \frac{0.5e^{-\ln 2}e^{\ln 2} + 0.04}{1+t} \times \frac{25}{16}]dt \\
&\leqslant -\int_{t_0}^{+\infty} \frac{1 - 0.54 \times \frac{25}{16}}{1+t}dt + \int_{t_0}^{+\infty} \frac{\cos 2t}{1+t}dt \\
&\leqslant -\int_{t_0}^{+\infty} \frac{0.1562}{1+t}dt + \int_{t_0}^{+\infty} \frac{\cos 2t}{1+t}dt = -\infty.
\end{aligned}$$

By Theorem 3.1, it can be inferred that the system is asymptotically stable in mean square. The Markov chain $\eta(t)$ and trajectories $v(t)$ and $v(\mu_1 t)$ of system (4.1) are plotted in Figures 1 and 2, respectively. Figure 1 illustrates the stochastic switching modes governed by the generator matrix, while Figure 2 demonstrates that the system state converges to zero asymptotically.

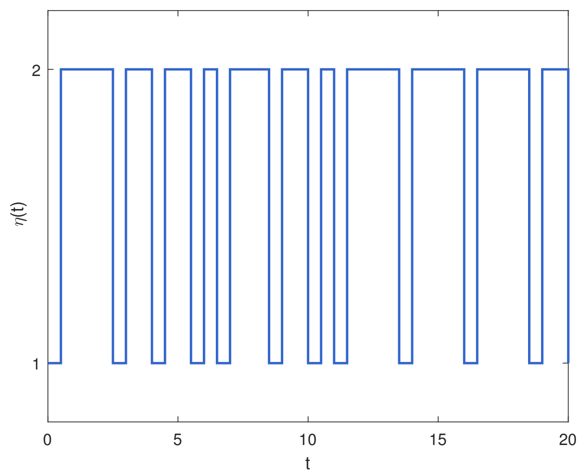

Figure 1. Switching modes of the Markov chain $\eta(t)$ of system (4.1).

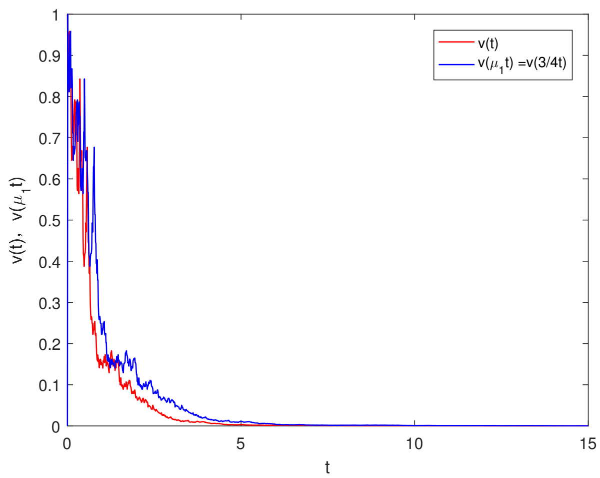

Figure 2. Trajectories $v(t)$ and $v(\mu_1 t)$ of system (4.1).

Example 2. Consider the following time-varying neutral stochastic pantograph system with Markovian switching:

$$d[v(t) - C(t, v(\mu_1 t), \eta(t))] = P(t, v(t), v(\mu_1 t), v(\mu_2 t), \eta(t))dt + Q(t, v(t), v(\mu_1 t), v(\mu_2 t), \eta(t))dB(t), \quad (4.2)$$

where $\eta(t)$ is a Markov chain on the state space $J = \{1, 2\}$ with its generator

$$\Lambda = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}.$$

In this example, when $\mu_1 = \frac{4}{5}$ and $\mu_2 = \frac{5}{6}$, we specify $C(\cdot)$, $P(\cdot)$, $Q(\cdot)$ as follows:

$$C(t, v(\mu_1 t), k) = \frac{1}{7} v(t) e^{-\frac{1}{10} t},$$

$$P(t, v(t), v(\mu_1 t), v(\mu_2 t), 1) = -1.2[v(t) - \frac{1}{7}v(\mu_1 t)] + 0.3\sin^2 tv(\mu_1 t)e^{-\frac{1}{10}t} + 0.3\sin^2 tv(\mu_2 t)e^{-\frac{1}{10}t},$$

$$P(t, v(t), v(\mu_1 t), v(\mu_2 t), 2) = -[v(t) - \frac{1}{7}v(\mu_1 t)] + 0.25\sin^2 tv(\mu_1 t)e^{-\frac{1}{10}t} + 0.25\sin^2 tv(\mu_2 t)e^{-\frac{1}{10}t},$$

$$Q(t, v(t), v(\mu_1 t), v(\mu_2 t), 1) = 1.2(v(t) - \frac{1}{7}v(\mu_1 t)),$$

and

$$Q(t, v(t), v(\mu_1 t), v(\mu_2 t), 2) = v(t) - \frac{1}{7}v(\mu_1 t).$$

Thereafter, setting $U(t, \bar{v}(t), \eta(t)) = (v(t) - \frac{1}{7}v(\mu_1 t))^2 = \bar{v}^2(t)$, we conclude that

$$\mathcal{L}U(t, v(t), 1) \leqslant (0.6\sin^2 t - 0.96)[v(t) - \frac{1}{7}v(\mu_1 t)]^2 + 0.3\sin^2 t[v^2(\mu_1 t) + v^2(\mu_2 t)]e^{-\frac{1}{5}t},$$

$$\mathcal{L}U(t, v(t), 2) \leqslant (0.5\sin^2 t - 1)[v(t) - \frac{1}{7}v(\mu_1 t)]^2 + 0.25\sin^2 t[v^2(\mu_1 t) + v^2(\mu_2 t)]e^{-\frac{1}{5}t}.$$

In summary, we have

$$\begin{aligned}\mathcal{L}U(t, v(t), \eta(t)) \leqslant\ & (0.6\sin^2 t - 0.96)U(t, v(t), v(\mu_1 t), v(\mu_2 t), \eta(t)) \\ & + 0.3\sin^2 t[v^2(\mu_1 t) + v^2(\mu_2 t)]e^{-\frac{1}{5}t}.\end{aligned}$$

Noting that the coefficients of the considered neutral stochastic hybrid system are time-varying, and the functions $0.6\sin^2 t - 0.96$ and $0.6\sin^2 t - 0.96 + 0.6\sin^2 te^{-\frac{1}{5}t}$ are two sign-changed continuous functions, which restricts the application of the existing results [29–33]. Consequently, we derive

$$\bar{\pi}(t) = \int_{\frac{4}{5}t}^{t} [0.96 - 0.6\sin^2 s - \frac{0.6\sin^2 s}{(1 - \frac{1}{7})^2}]ds \leqslant \frac{1}{5}t = \lambda(t),$$

where $\lambda(t)$ is a general function. Subsequently, we can obtain

$$\begin{aligned}\bar{\delta}(t) = & \int_{t_0}^{+\infty} [0.6\sin^2 t - 0.96 + \frac{0.6\sin^2 t}{(1 - \frac{1}{7})^2}]dt \\ & \leqslant -0.26(t - t_0) + c_0.\end{aligned}$$

By Theorem 3.2, it can be inferred that the system is exponentially stable in mean square and almost surely exponentially stable. Subsequently, Figure 3 depicts the stochastic switching modes determined by the generator matrix, while Figure 4 demonstrates rapid convergence to zero exponentially, which shows the consistency between theoretical achievements and numerical simulations.

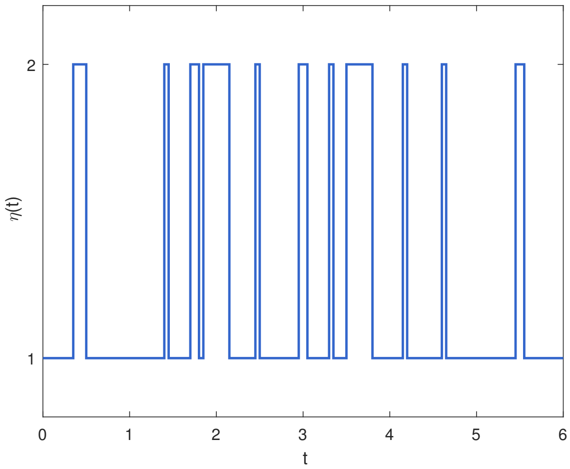

Figure 3. Switching modes of the Markov chain $\eta(t)$ of system (4.2).

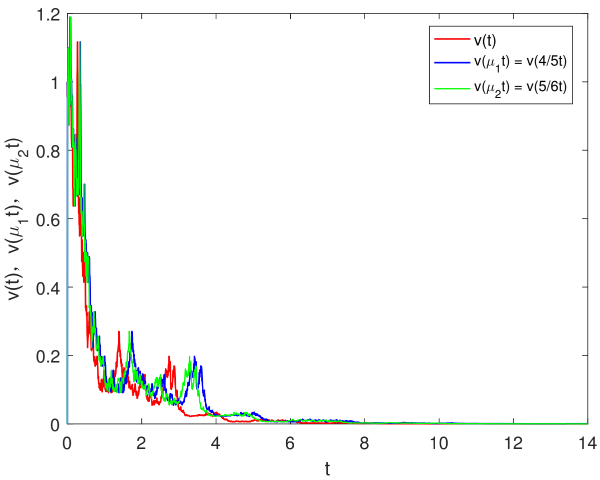

Figure 4. Trajectories $v(t)$, $v(\mu_1 t)$, and $v(\mu_2 t)$ of system (4.2).

5. Conclusions

This paper is concerned with the stability of time-varying neutral stochastic pantograph systems with Markovian switching and multiple proportional delays. First and foremost, by employing the Lyapunov functional method and diverse stochastic analysis techniques associated with developing a refined integral inequality for proportional delays, various stabilities of neutral hybrid stochastic systems have been discussed, which effectively overcomes the complexities arising from multiple proportional delays and time-varying coefficients. Furthermore, several stability criteria are established, including qth moment stability, qth moment asymptotic stability, qth exponential stability, and almost sure asymptotic and exponential stability, where the upper bound of the diffusion operator may be a function with time-varying sign-variable coefficients instead of negative constants.

Thereafter, numerical simulations validate the effectiveness and feasibility of our theoretical results. In the future, stability analysis and control issues of time-varying neutral hybrid stochastic pantograph systems with Lévy noise or impulsive effects are worthy of further exploration.

Author contributions

Yuhan Yu: Conceptualization, investigation, writing—original draft, software; Yinfang Song: Funding acquisition, methodology, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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