Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 24352–24370.
DOI:10.3934/math.20251080
Received: 20 July 2025
Revised: 20 September 2025
Accepted: 10 October 2025
Published: 24 October 2025

*Research article*

# $C^*$-Algebra generated by $n$-orthogonal projections and its relationship to Toeplitz operators acting on the poly-Fock space

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**Abstract:** Starting from the system of orthogonal projections mapping the $n$-poly-Fock space $F_n^2(\mathbb{C})$ onto the true-poly-Fock components, we constructed a system of all-but-one orthogonal projections in generic position. We described the $C^*$-algebra generated by these projections via an isomorphism with a subalgebra of matrix-valued continuous functions on the two-point compactification of the real line. In addition, we studied the $C^*$-algebra generated by a Toeplitz operator with a horizontal symbol and the orthogonal projections from $F_n^2(\mathbb{C})$ onto the true-poly-Fock subspaces. An interesting fact in this work was that the $C^*$-algebras studied herein contain the $C^*$-algebra generated by all Toeplitz operators acting on $F_n^2(\mathbb{C})$ with horizontal symbols having limit values at $\pm\infty$. Our approach combines unitary equivalences induced by Bargmann-type transforms with a noncommutative Stone–Weierstrass-type argument to describe the algebras.

**Keywords:** Toeplitz operator; orthogonal projection; $C^*$-algebra; Fock space; polyanalytic function
**Mathematics Subject Classification:** 47A08, 47B32, 47B35, 47L15, 47L80

## 1. Introduction

$C^*$-Algebras generated by Toeplitz operators and orthogonal projections are an important topic in operator theory, owing to their connections with areas such as noncommutative geometry, quantization, and spectral analysis. In classical settings—such as the Hardy, Bergman, and Fock spaces—the structure of these algebras has been thoroughly explored, revealing deep links with both commutative and noncommutative function algebras [3, 6, 13, 16].

According to Balk [1], Kolossov introduced polyanalytic functions (1908), which have been studied in the context of plane elasticity. Polyanalytic functions also arise in other areas of mathematics such as partial differential equations and representation theory of Lie groups. For the fundamental properties of this kind of function, we refer to [1]. Poly-Fock spaces $F_n^2(\mathbb{C})$ and true-poly-Fock spaces $F_{(k)}^2(\mathbb{C})$, studied in [15], provide a natural generalization of the Fock space $F^2(\mathbb{C})$ by considering higher-order annihilation conditions. These spaces have become increasingly relevant as a setting where complex algebraic phenomena naturally emerge, particularly in the context of Toeplitz operators defined by certain classes of symbols. While Toeplitz operators with horizontal or vertical symbols have been studied in various poly-Fock settings (see [11, 12]), there has been comparatively little work on the $C^*$-algebras generated by combining Toeplitz operators with orthogonal projections. In this paper, we investigate exactly that setting, and we justify that certain $C^*$-algebras generated by Toeplitz operators can alternatively be studied from the point of view of $C^*$-algebras generated by orthogonal projections.

In [14], Vasilevski provided a general method for the construction of all-but-one orthoprojections in generic position. Using this approach we construct orthogonal projections acting on poly-Fock spaces. Indeed, we consider the orthogonal projections $B_{(j)}$ mapping $F_n^2(\mathbb{C})$ onto $F_{(j)}^2(\mathbb{C})$ $(j = \overline{1, n})$, and we construct an orthoprojection $P : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ given in terms of Toeplitz operators with horizontal symbols and acting on the Fock space $F^2(\mathbb{C})$. In this way, we obtain a system of all-but-one orthoprojections $P, B_{(1)}, \ldots, B_{(n)}$ in generic position. Unlike the approach developed in [14], we analyze the algebra generated by these projections and characterize its structure through an application of the noncommutative Stone–Weierstrass theorem [7].

Our second main goal is to describe the $C^*$-algebra generated by the $n$ orthogonal projections $B_{(j)}$ and a Toeplitz operator $T_{n,a_\beta} : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ with a specific horizontal symbol $a_\beta$.

In the latter part of the paper, we examine the relationship between the algebras studied herein and that generated by Toeplitz operators with horizontal symbols, demonstrating that both approaches yield closely related $C^*$-algebras.

Summarizing our main results:

- We construct a family of all-but-one orthogonal projections $P_T, \mathbf{e}_1, \ldots, \mathbf{e}_n : (F^2(\mathbb{C}))^n \to (F^2(\mathbb{C}))^n$ in generic position, and show that the $C^*$-algebra they generate is isomorphic to a subalgebra of $M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$.
- Through a system of unitary transformations based on Bargmann-type maps, we identify the Hilbert spaces $(F^2(\mathbb{C}))^n$ and $F_n^2(\mathbb{C})$. This procedure carries the study in the previous step into the poly-Fock space setting, and in this way, we get the collection of all-but-one orthoprojections $P, B_{(1)}, \ldots, B_{(n)}$ in generic position.
- We further investigate the $C^*$-algebra generated by one Toeplitz operator $T_{n,a_\beta} : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$, with a step-function symbol, together with the $n$ projections $B_{(j)} : F_n^2(\mathbb{C}) \to F_{(j)}^2(\mathbb{C})$, providing an explicit description of its structure.

The paper is organized as follows. In Section 2, we recall the construction of poly-Fock spaces and their realizations as $L_2$-spaces via Bargmann-type transforms. In Section 3, we review Toeplitz operators with horizontal symbols and acting on $F_n^2(\mathbb{C})$, as well as their associated spectral functions. In

Section 4, we introduce all-but-one orthogonal projections in generic position and acting on $(F^2(\mathbb{C}))^n$, we explore the $C^*$-algebra they generate, and we conclude the section by rephrasing the analysis of orthoprojections to the poly-Fock-space setting. Section 5 is dedicated to finding explicit integral representations of these projections through reproducing kernels. Finally, Section 6 is devoted to the study of the $C^*$-algebra generated by a single Toeplitz operator and $n$ projections acting on $F^2_n(\mathbb{C})$, as well as its connection to matrix-valued function algebras over the compactified real line.

## 2. Preliminaries

This section provides the fundamental background required for the developments that follow. We start by examining the structure of poly-Fock spaces $F^2_n(\mathbb{C})$, as developed in [15], and revisit the construction as nested subspaces within the space of square-integrable functions on the complex plane with respect to the Gaussian measure

$$d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dv(z),$$

where $dv(z) = dxdy$ is the usual Euclidean measure on $\mathbb{R}^2 = \mathbb{C}$ and $z = x + iy$. In particular, we recall the true-poly-Fock components and the associated orthogonal projections. We also introduce the Bargmann-type transforms that realize these spaces as standard $L_2$-spaces over the real line, allowing us to transfer operator-theoretic constructions to a more tractable setting. These findings provide the foundation for our subsequent analysis of Toeplitz operators and algebras generated by projections.

For $n \in \mathbb{Z}_+$, the **n**-poly-Fock space $F^2_n(\mathbb{C})$ is the set of all smooth functions $\varphi \in L_2(\mathbb{C}, d\mu)$ satisfying

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^n \varphi = 0.$$

In particular, $F^2_1(\mathbb{C}) = F^2(\mathbb{C})$ is the classical Fock space. Note that $F^2_n(\mathbb{C}) \subset F^2_{n+1}(\mathbb{C})$, and then the true-$n$-poly-Fock space $F^2_{(n)}(\mathbb{C})$ is defined as follows:

$$F^2_{(n)}(\mathbb{C}) = \begin{cases} F^2_1(\mathbb{C}), & \text{if } n = 1; \\ F^2_n(\mathbb{C}) \ominus F^2_{n-1}(\mathbb{C}), & \text{otherwise.} \end{cases}$$

Poly-Fock spaces as well as true-poly-Fock spaces are closed in $L_2(\mathbb{C}, d\mu)$, see [9, Proposition 3.3]. Let $B_{(n)}$ and $B_n$ be the orthogonal projections from $L_2(\mathbb{C}, d\mu)$ onto $F^2_{(n)}(\mathbb{C})$ and $F^2_n(\mathbb{C})$, respectively. The operator $B_{(n)}$ admits [15, Theorem 3.3] the integral representation

$$(B_{(n)}\varphi)(z) = \int_{\mathbb{C}} \varphi(w) q^{\{n\}}_w(z) d\mu(w), \quad w = u + iv,$$

where

$$q^{\{n\}}_w(z) = \frac{1}{(n-1)!} \left(-\frac{\partial}{\partial z} + \overline{z}\right)^{n-1} \left(-\frac{\partial}{\partial \overline{w}} + w\right)^{n-1} e^{z\overline{w}}, \quad z \in \mathbb{C}.$$

The poly-Fock spaces are related to Hermite functions, which are defined in terms of the Hermite polynomials:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2} = n! \sum_{k=0}^{[n/2]} c_{k,n} (2y)^{n-2k},$$

where

$$c_{k,n} = \frac{(-1)^k}{k!(n-2k)!}.$$

The system of Hermite functions

$$h_n(y) = (2^n n! \sqrt{\pi})^{-1/2} e^{-y^2/2} H_n(y), \quad n = 0, 1, \dots,$$

forms an orthonormal basis for the Hilbert space $L_2(\mathbb{R})$ of all square integrable functions with respect to the Lebesgue measure on $\mathbb{R}$.

We use $\mathcal{H}_n$ to denote the vectorial subspace generated by $h_n$. Then the orthogonal projection $P_{(n)}$ from $L_2(\mathbb{R})$ onto $\mathcal{H}_n$ is given by

$$(P_{(n)}\phi)(y) = \langle \phi, h_n \rangle h_n(y).$$

Let

$$\mathcal{H}_n^{\oplus} := \bigoplus_{k=0}^{n} \mathcal{H}_k.$$

Then $P_n := P_{(0)} \oplus \cdots \oplus P_{(n)}$ is the orthogonal projection from $L_2(\mathbb{R})$ onto $\mathcal{H}_n^{\oplus}$.

Next, we define three unitary operators $U_1, U_2, U_3$ whose composition $U = U_3 U_2 U_1$ allows us to identify the space $F^2_{(n)}(\mathbb{C})$ with $L_2(\mathbb{R})$. The operator $U_1$ maps $L_2(\mathbb{C}, d\mu)$ onto $L_2(\mathbb{R}^2, dxdy)$ by the rule

$$(U_1\varphi)(z) = \pi^{-1/2} e^{-\frac{z\bar{z}}{2}} \varphi(z).$$

The operator $U_2 = I \otimes \mathcal{F}$ acts on $L_2(\mathbb{R}^2, dxdy) = L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$, where $\mathcal{F}$ is the Fourier transform on $L_2(\mathbb{R})$ given by

$$(\mathcal{F} f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,$$

and $I$ denotes the identity operator. Finally, the operator $U_3 : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ is given by

$$(U_3 f)(x, y) = f\left(\frac{1}{\sqrt{2}}(x+y), \frac{1}{\sqrt{2}}(x-y)\right).$$

**Theorem 2.1.** *[15] The unitary operator $U = U_3 U_2 U_1 : L_2(\mathbb{C}, d\mu) \to L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}, dy)$ satisfies the following properties:*

1) The spaces $F^2_{(n)}(\mathbb{C})$ and $F^2_n(\mathbb{C})$ are mapped onto $L_2(\mathbb{R}) \otimes \mathcal{H}_{n-1}$ and $L_2(\mathbb{R}) \otimes \mathcal{H}_{n-1}^{\oplus}$, respectively.
2) The projections $B_{(n)}$ and $B_n$ are unitarily equivalent to the operators $UB_{(n)}U^{-1} = I \otimes P_{(n-1)}$ and $UB_nU^{-1} = I \otimes P_{n-1}$, respectively.

We now introduce isomorphisms mapping true-$n$-poly-Fock spaces into the space $L_2(\mathbb{R})$. These isomorphisms play the role of the Segal-Bargmann transform acting on the Fock space. First, we define the isometric embedding $R_{0,(n)} : L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$ by the rule

$$(R_{0,(n)} f)(x, y) = f(x) h_{n-1}(y).$$

The operator $R_{(n)} = R^*_{0,(n)} U : L_2(\mathbb{C}, d\mu) \to L_2(\mathbb{R})$, when restricted to the true-$n$-poly-Fock space, is an isometric isomorphism onto $L_2(\mathbb{R})$. This means

$$R^*_{(n)} R_{(n)} = B_{(n)} : L_2(\mathbb{C}, d\mu) \to F^2_{(n)}(\mathbb{C}),$$

$$R_{(n)}R^*_{(n)} = I : L_2(\mathbb{R}) \to L_2(\mathbb{R}).$$

As a consequence, the operator $R_{0,n} : (L_2(\mathbb{R}))^n \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$, defined by

$$(R_{0,n}f)(x,y) = \sum_{k=1}^{n} f_k(x)h_{k-1}(y) = [\mathbf{H}_n(y)]^T f(x),$$

is an isometric embedding, where $f = (f_1, \ldots, f_n)^T$. Here the symbol $T$ denotes matrix transposition, and

$$\mathbf{H}_n(y) = (h_0(y), \ldots, h_{n-1}(y))^T.$$

The adjoint operator $R^*_{0,n} : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}) \to (L_2(\mathbb{R}))^n$ is explicitly given by

$$(R^*_{0,n}\varphi)(x) = \int_{\mathbb{R}} \varphi(x,y)\mathbf{H}_n(y)dy.$$

Furthermore, the operator

$$R_n := R^*_{0,n}U : L_2(\mathbb{C}, d\mu) \to (L_2(\mathbb{R}))^n \tag{2.1}$$

isometrically maps $F^2_n(\mathbb{C})$ onto $(L_2(\mathbb{R}))^n$. Thus, it satisfies the following identities:

$$R^*_nR_n = B_n : L_2(\mathbb{C}, d\mu) \to F^2_n(\mathbb{C}),$$
$$R_nR^*_n = I : (L_2(\mathbb{R}))^n \to (L_2(\mathbb{R}))^n.$$

Straightforward calculations lead to the following results:

$$(R_n\varphi)(t) = \int_{\mathbb{C}} \varphi(z)e^{-\frac{\bar{z}^2+t^2}{2} + \sqrt{2}\bar{z}t}\widetilde{\mathbf{H}}_n\left(\frac{z+\bar{z}}{\sqrt{2}} - t\right) d\mu(z),$$

$$(R^*_nf)(z) = \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt}\left[\widetilde{\mathbf{H}}_n\left(\frac{z+\bar{z}}{\sqrt{2}} - t\right)\right]^T f(t)dt, \tag{2.2}$$

where $f \in (L_2(\mathbb{R}))^n$ and

$$\widetilde{\mathbf{H}}_n(t) = e^{t^2/2}\mathbf{H}_n(t).$$

## 3. Toeplitz operators with horizontal symbols

In this section, we review the definition and key properties of Toeplitz operators acting on the poly-Fock spaces with a focus on those defined by horizontal symbols (see [12] for details). These operators arise naturally in the study of translation-invariant structures on the complex plane and serve as building blocks for the $C^*$-algebras considered later in the paper. In [12], the authors characterize the Toeplitz operators with horizontal symbols via their unitary equivalence to multiplication operators on vector-valued $L_2$-spaces, and they describe the $C^*$-algebra generated by Toeplitz operators via matrix-valued functions, continuous on the two-point compactification of $\mathbb{R}$. In this context, each operator $R_{(n)}$ and

$R_n$ unitarily transforms Toeplitz operators into multiplication operators. To be precise, for a symbol $a \in L_\infty(\mathbb{C})$, the Toeplitz operator $T_{n,a} : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ is defined by

$$T_{n,a}(f) = B_n(af).$$

When $n = 1$, the operator $T_{1,a}$ is denoted by $T_a$.

A function $\varphi \in L_\infty(\mathbb{C})$ is said to be horizontal if, for every $h \in \mathbb{R}$,

$$\varphi(z - ih) = \varphi(z)$$

for almost all $z \in \mathbb{C}$, that is, there exists a function $a$ defined on $\mathbb{R}$ such that $\varphi(z) = a(\operatorname{Re} z)$. Throughout this work, we will be working only with bounded horizontal symbols. The Toeplitz operator $T_a : F^2(\mathbb{C}) \to F^2(\mathbb{C})$, with bounded horizontal symbol $a$, is unitarily equivalent to the multiplication operator

$$\gamma^a I = R_1 T_a R_1^*, \tag{3.1}$$

acting on $L_2(\mathbb{R})$, where

$$\gamma^a(x) = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right)(h_0(y))^2\, dy, \tag{3.2}$$

(see [4] for details). The corresponding Toeplitz operator $T_{n,a}$, acting on $F_n^2(\mathbb{C})$, is unitarily equivalent to the matrix-multiplication operator

$$\gamma^{n,a} I = R_n T_{n,a} R_n^*$$

acting on $(L_2(\mathbb{R}))^n$, where

$$\gamma^{n,a}(x) = (\gamma_{jk}^{n,a}(x))_{j,k=1,\dots,n} = \int_{\mathbb{R}} a\left(\frac{x+y}{\sqrt{2}}\right)[\mathbf{H}_n(y)]^T \mathbf{H}_n(y) dy, \tag{3.3}$$

(see [12] for details).

Denote by $L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$ the vector space consisting of all bounded measureble functions having limit values at the points $-\infty$ and $+\infty$, i.e., for each $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$, the following limits exist:

$$a_- := \lim_{x\to-\infty} a(x), \qquad a_+ := \lim_{x\to+\infty} a(x).$$

Let $\overline{\mathbb{R}} = [-\infty, \infty]$ be the set of extended real numbers, that is, $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$. Then a function $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$ can be seen as a function defined on $\overline{\mathbb{R}}$ with $a(\pm\infty) = a_\pm$. We identify $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$ with its extension $a(z) = a(x)$ to the complex plane $\mathbb{C}$, where $x = \operatorname{Re} z$. Elementary calculations show that

$$a_- I = \lim_{x\to-\infty} \gamma^{n,a}(x), \quad a_+ I = \lim_{x\to+\infty} \gamma^{n,a}(x),$$

for $a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})$, where $I$ denotes the identity matrix.

As notation, $C^*(\mathcal{F})$ stands for the $C^*$-algebra generated by a family $\mathcal{F}$ of bounded linear operators acting on a Hilbert space. As usual, $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ matrices with complex entries, thus $M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$ is the $C^*$-algebra of all continuous matrix-valued functions defined on $[-\infty, \infty]$.

**Theorem 3.1.** *[12] Let $\mathcal{T}_n^{-\infty,\infty}$ denote the family $\{T_{n,a} : a \in L_\infty^{\{-\infty,\infty\}}(\mathbb{R})\}$ of Toeplitz operators acting on $F_n^2(\mathbb{C})$. Then, the $C^*$-algebra $C^*(\mathcal{T}_n^{-\infty,\infty})$ is isomorphic and isometric to the $C^*$-algebra*

$$\mathcal{D}_n^{-\infty,\infty} = \{M \in M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}}) : M(\pm\infty) \in \mathbb{C}I\}.$$

The isomorphism maps each Toeplitz operator to its spectral function

$$T_{n,a} \longmapsto \gamma^{n,a} I.$$

## 4. $C^*$-Algebra generated by orthogonal projections

This section is devoted to the construction and analysis of a $C^*$-algebra generated by a system of all-but-one orthogonal projections in generic position [14]. Our motivation arises from the structural parallels between projection-generated algebras and those associated to Toeplitz operators with horizontal symbols. We begin by recalling the notion of generic position among projections, as introduced by Halmos [6] and further developed in [13, 14], and we adapt this framework to the poly-Fock-space setting. By selecting a suitable family of Toeplitz operators acting on $F^2(\mathbb{C})$, we construct an orthoprojection $P : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ that fulfills the genericity conditions in conjunction with the orthoprojections $B_{(j)}$, and establish an isomorphism between $C^*(P, B_{(1)}, \ldots, B_{(n)})$ and an algebra $\mathcal{D}^{1,n}$ of continuous matrix-valued functions. This construction provides a connection between algebras generated by projections and those arising from Toeplitz operators with horizontal symbols, a relationship we make precise in Theorems 3.1, 4.9, and 6.1.

Recall that a system of all-but-one orthogonal projections $P, Q_1, \ldots, Q_n$ are in generic position if they hold the following properties:

$$(\operatorname{Im} P)^\perp \cap \operatorname{Im} Q_j = \{0\} = \operatorname{Im} P \cap (\operatorname{Im} Q_j)^\perp, \quad j = 1, \ldots, n.$$

**Theorem 4.1** ( [14]). *For $k = 1, \ldots, n$, let $C_k : L \longrightarrow L$ be injective positive operators acting on a Hilbert space $L$. Suppose that these operators commute (i.e., $C_j C_k = C_k C_j$) and satisfy*

$$\sum_{k=1}^{n} C_k = I.$$

Then, the set of operators

i) $P = (C_j^{\frac{1}{2}} C_k^{\frac{1}{2}})_{j,k=1,\ldots,n} : L^n \longrightarrow L^n$,
ii) $Q_j := \operatorname{diag}(0, \ldots, I, \ldots, 0) : L^n \longrightarrow L^n$,

is a system of all-but-one orthogonal projections in generic position.

First, we construct orthogonal projections acting on $(L_2(\mathbb{R}))^n$ and satisfying the conditions in Theorem 4.1. One of our orthoprojections is given in terms of spectral functions of some Toeplitz operators acting on the Fock space (via (3.1)), and essentially we get a system of all-but-one orthogonal projections $P_T, \mathbf{e}_1, \ldots, \mathbf{e}_n$ in generic position acting on $(F^2(\mathbb{C}))^n$. The $C^*$-algebra generated by these projections is isomorphic to the algebra generated by a set of continuous matrix-valued functions. At this stage, we apply the noncommutative Stone-Weierstrass theorem. At the end of this section, we reach our goal using the fact that each bounded operator acting on $(F^2(\mathbb{C}))^n$ can be unitarily transformed in a bounded operator acting on $F_n^2(\mathbb{C})$. Although N. L. Vasilevski established a method to describe the $C^*$-algebra generated by a system of all-but-one orthogonal projections in generic position ( [14]), we follow another approach in our poly-Fock-space setting.

We start by constructing the first set of orthogonal projections. Let

$$a_1(z) = \chi_{(-\infty,1]}(x), \quad a_n(z) = \chi_{[n-1,\infty)}(x), \quad \text{and} \quad a_k(z) = \chi_{[k-1,k]}(x)$$

for $k = 2, \ldots, n-1$. According to (3.1), each Toeplitz operator $T_{a_k} : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ is unitarily equivalent to the multiplication operator by the corresponding function $\gamma^{a_k}$, given by (3.2). This fact implies that both operators $T_{a_k}$ and $\gamma^{a_k}I$ share some properties, such as positivity and injectivity, as shown in the following lemma.

**Lemma 4.2.** *Each operator $\gamma^{a_k}I$ is positive and injective on $L_2(\mathbb{R})$. Moreover, the following equality holds:*

$$\gamma^{a_1}I + \cdots + \gamma^{a_n}I = I. \tag{4.1}$$

*Proof.* The equality 4.1 follows directly from $a_1(z) + \cdots + a_n(z) \equiv 1$. On the other hand,

$$\gamma^{a_1}(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{2}-x} e^{-y^2} dy, \tag{4.2}$$

$$\gamma^{a_n}(x) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{2}(n-1)-x}^{\infty} e^{-y^2} dy, \tag{4.3}$$

$$\gamma^{a_k}(x) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{2}(k-1)-x}^{\sqrt{2}k-x} e^{-y^2} dy, \tag{4.4}$$

where $k = 2, \ldots, n-1$. Hence, $\gamma^{a_k}(x) > 0$ for all $x \in (-\infty, \infty)$, and then the operator $\gamma^{a_k}I$ is positive and injective. □

The last lemma implies that the operator $P_\gamma$, acting on $(L_2(\mathbb{R}))^n$, and given by

$$P_\gamma = \left( \sqrt{\gamma^{a_j}} \sqrt{\gamma^{a_k}} \right)_{j,k=1,\ldots,n} I,$$

is an orthogonal projection.

For $j = 1, \ldots, n$, let $Q_j$ be the orthogonal projection, acting on $(L_2(\mathbb{R}))^n$, and given by

$$Q_j(f_1, \ldots, f_n)^T = (0, \ldots, f_j, \ldots, 0)^T.$$

The following corollary follows immediately from Lemma 4.2 and Theorem 4.1.

**Corollary 4.3.** *The set of operators $P_\gamma, Q_1, \ldots, Q_n$ is a system of all-but-one orthogonal projections in generic position.*

As we mentioned above, the functions $\gamma^{a_1}, \ldots, \gamma^{a_n}$ arise from Toeplitz operators acting on the Fock space. Thus there exists a natural relation between the orthogonal projection $P_\gamma$ and an orthogonal projection acting on $(L_2(\mathbb{R}))^n$. To be precise, the operator that makes possible an identification between the spaces $(F^2(\mathbb{C}))^n$ and $(L_2(\mathbb{R}))^n$ is the following:

$$\mathbf{R} = \bigoplus_{k=1}^{n} R_1 : (L_2(\mathbb{C}, d\mu))^n \to (L_2(\mathbb{R}))^n,$$

where $R_1$ is given in (2.1) for $n = 1$. Furthermore, $\mathbf{R}$ satisfies:

$$\begin{aligned} \mathbf{R}^*\mathbf{R} &= \bigoplus_{k=1}^{n} B_{(1)} : (L_2(\mathbb{C}, d\mu))^n \to (F^2(\mathbb{C}))^n, \\ \mathbf{R}\mathbf{R}^* &= I : (L_2(\mathbb{R}))^n \to (L_2(\mathbb{R}))^n. \end{aligned}$$

Straightforward calculations show that $\mathbf{R}$ is given by

$$(\mathbf{R}\varphi)(t) = \frac{1}{\pi^{1/4}} \int_{\mathbb{C}} e^{-\frac{\overline{w}^2+t^2}{2} + \sqrt{2}\overline{w}t} \varphi(w)\, d\mu(w). \tag{4.5}$$

Furthermore,

$$(\mathbf{R}^* f)(z) = \frac{1}{\pi^{1/4}} \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} f(t) dt, \tag{4.6}$$

where $f \in (L_2(\mathbb{R}))^n$.

The operator $\mathbf{R}$ allows us to identify $B((F^2(\mathbb{C}))^n)$ with $B((L_2(\mathbb{R}))^n)$, where $B(\mathcal{H})$ denotes the algebra of bounded linear operators acting on $\mathcal{H}$. This isometric isomorphism is given by

$$T \mapsto \mathbf{R}T\mathbf{R}^*.$$

Considering this, let

$$\begin{aligned} \mathbf{e}_j &:= \mathbf{R}^* Q_j \mathbf{R}, \text{ for } j = 1, \ldots, n, \\ P_T &:= \mathbf{R}^* P_\gamma \mathbf{R}. \end{aligned} \tag{4.7}$$

**Lemma 4.4.** *The operators $P_T, \mathbf{e}_1, \ldots, \mathbf{e}_n : (F^2(\mathbb{C}))^n \to (F^2(\mathbb{C}))^n$ form a system of all-but-one orthogonal projections in generic position. Furthermore,*

$$P_T = \left( \sqrt{T_{a_k}} \sqrt{T_{a_l}} \right)_{k,l=1,\ldots,n},$$

*and $\mathbf{e}_j$ is the orthogonal projection given by*

$$\mathbf{e}_j f = (0, \ldots, f_j, \ldots, 0)^T, \tag{4.8}$$

*where $f = (f_1, \ldots, f_n)^T \in (F^2(\mathbb{C}))^n$.*

*Proof.* Straightforward calculations lead to (4.8). On the other hand, we have $R_1 T_{a_j} R_1^* = \gamma^{a_j} I$. Then, $\sqrt{T_{a_j}} = R_1^* \sqrt{\gamma^{a_j}} I R_1$, which implies that

$$P_T = \mathbf{R}^* P_\gamma I \mathbf{R} = \left( \sqrt{T_{a_j}} \sqrt{T_{a_k}} \right)_{j,k=1,\ldots,n}.$$

□

Note that $Q_j$ can be identified with the $n \times n$ matrix whose entries are zero except the $(j, j)$ entry which is 1. Thus, $Q_j Q_k = \delta_{jk} Q_j$ and $Q_1 + \cdots + Q_n = I$.

In this section, our goal is to describe the $C^*$-algebra

$$\mathcal{P}_n^{1,n} = C^*(P_T, \mathbf{e}_1, \ldots, \mathbf{e}_n) \tag{4.9}$$

generated by $P_T, \mathbf{e}_1, \ldots, \mathbf{e}_n$.

As previously mentioned, the $C^*$-algebra generated by $n$ orthogonal projections appears to have proximity with the $C^*$-algebra generated by Toeplitz operators with extended horizontal symbols. Instead of employing the methodologies outlined in [3, 6, 10, 13, 14], we will describe the $C^*$-algebra $\mathcal{P}_n^{1,n}$, defined in (4.9), by utilizing the noncommutative Stone-Weierstrass theorem. This approach allows us to envision the elements of this $C^*$-algebra as functions defined along the real number line, facilitating comparison with the $C^*$-algebra generated by Toeplitz operators with extended horizontal symbols.

The following commutative diagram shows some spaces and operators discussed above:

$$\begin{array}{ccc} (F^2(\mathbb{C}))^n & \xrightarrow{P_T,\ \mathbf{e}_j} & (F^2(\mathbb{C}))^n \\ \mathbf{R}\downarrow & & \downarrow\mathbf{R} \\ (L_2(\mathbb{R}))^n & \xrightarrow[P_\gamma,\ Q_j]{} & (L_2(\mathbb{R}))^n \end{array}$$

We take $E_{jk}$ to be the $n \times n$ matrix whose components are zero except the $(j, k)$ entry, which equals 1. Equations (4.2), (4.3), and (4.4) imply that $\gamma^{a_1}(-\infty) = \gamma^{a_n}(+\infty) = 1$, $\gamma^{a_1}(+\infty) = \gamma^{a_n}(-\infty) = 0$, and $\gamma^{a_k}(-\infty) = \gamma^{a_k}(+\infty) = 0$ for $k = 2, \ldots, n-1$. Therefore,

$$P_\gamma(-\infty) = E_{11} \quad \text{and} \quad P_\gamma(+\infty) = E_{nn}.$$

Consider the $C^*$-algebras

$$\mathcal{B}^{1,n} := C^*(P_\gamma, Q_1, \ldots, Q_n)$$

and

$$\mathcal{D}^{1,n} = \Big\{ M \in M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}}) : M(\pm\infty) \text{ are diagonal, and } \\ M_{jj}(-\infty) = M_{jj}(\infty) \ \ \forall j = \overline{2, n-1} \, \Big\}.$$

We will prove that $\mathcal{B}^{1,n} = \mathcal{D}^{1,n}$. Clearly $\mathcal{B}^{1,n}$ is a $C^*$-subalgebra of $\mathcal{D}^{1,n}$. We are dealing with $C^*$-bundles, allowing us to identify all pure states of $\mathcal{D}^{1,n}$. Actually $\mathcal{D}^{1,n}$ is a $C^*$-algebra consisting of continuous sections of the $C^*$-bundle $E = M_n(\mathbb{C}) \times \overline{\mathbb{R}}$, whose base space is $\overline{\mathbb{R}}$. The pure states of $\mathcal{D}^{1,n}$ can be obtained by restriction from those of $M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$. For $x_0 \in (-\infty, \infty)$, the evaluation map

$$\begin{array}{rccc} \pi_{x_0} : & \mathcal{D}^{1,n} & \to & M_n(\mathbb{C}) \\ & M & \longmapsto & M(x_0) \end{array}$$

is an $n$-dimensional irreducible representation. Thus, the pure states of $\mathcal{D}^{1,n}$ associated to $x_0 \in (-\infty, \infty)$ are given by

$$f_{x_0,v}(M) = \langle M(x_0)v, v \rangle, \quad M \in \mathcal{D}^{1,n},$$

where $v \in \mathbb{C}^n$ is a unimodular vector, see [8]. On the other hand, let $e_j$ be the canonical vector whose components are all zero except the $j$-th component which is 1, that is, $e_j = (0, \ldots, 1, \ldots, 0)^T$. The functionals

$$f_{\pm\infty,e_j}(M) = \langle M(\pm\infty)e_j, e_j \rangle, \quad j = 2, \ldots, n-1,$$

are pure states of $\mathcal{D}^{1,n}$ associated to the points $x_0 = \pm\infty$. In fact, these functionals are one-dimensional irreducible representations of $\mathcal{D}^{1,n}$. Note that $f_{-\infty,e_j} = f_{\infty,e_j}$ for $j = 2, \ldots, n-1$, thus $\mathcal{D}^{1,n}$ has only $n+2$ one-dimensional irreducible representations. In this way, all the pure states of $\mathcal{D}^{1,n}$ have been identified.

To prove that $\mathcal{B}^{1,n}$ is equal to $\mathcal{D}^{1,n}$, we use the noncommutative Stone-Weierstrass conjecture, which states the following: Let $\mathcal{B}$ and $\mathcal{A}$ be $C^*$-algebras with $\mathcal{B}$ contained in $\mathcal{A}$, and suppose that $\mathcal{B}$ separates all the pure states of $\mathcal{A}$ (and 0 if $\mathcal{A}$ is non-unital). Then $\mathcal{A} = \mathcal{B}$. This conjecture was proved by Kaplansky in [7] for completely continuous representations (CCR) $C^*$-algebras, see also [5] or Theorem 11.1.8 in [2]. Recall that a $C^*$-algebra $\mathcal{A}$ is called CCR or liminal if $\pi(\mathcal{A}) \subset K$ for every irreducible representation $\pi$ of $\mathcal{A}$ on a Hilbert space $H$, where $K$ is the ideal of all compact operators.

**Theorem 4.5.** *The $C^*$-algebra $\mathcal{P}_n^{1,n}$ is isomorphic and isometric to the $C^*$-algebra $\mathcal{D}^{1,n}$. Equivalently, the $C^*$-algebra $\mathcal{D}^{1,n}$ is generated by*

$$P_\gamma, Q_1, \ldots, Q_n.$$

Moreover, the map $\mathcal{P}_n^{1,n} \ni T \longmapsto \mathbf{R}T\mathbf{R}^* \in \mathcal{D}^{1,n}$ is an isometric isomorphism of $C^*$-algebras, where

$$P_T \longmapsto P_\gamma \quad \text{and} \quad \mathbf{e}_j \longmapsto Q_j.$$

*Proof.* Observe that $\mathcal{D}^{1,n}$ is a CCR $C^*$-algebra. The algebra $\mathcal{B}^{1,n}$ separates all pure states of $\mathcal{D}^{1,n}$ as shown in Lemmas 4.6, 4.7, and 4.8 below. Hence, by the noncommutative Stone-Weierstrass conjecture proved by I. Kaplansky for CCR $C^*$-algebras, we conclude that $\mathcal{D}^{1,n} = \mathcal{B}^{1,n}$. □

**Lemma 4.6.** *Let $v, w \in \mathbb{C}^n$ be unimodular vectors, and $x_0, x_1 \in (-\infty, \infty)$. Suppose that*

$$f_{x_0,v}(Q_j P_\gamma I Q_k) = f_{x_1,w}(Q_j P_\gamma I Q_k), \tag{4.10}$$

$$f_{x_0,v}(Q_j P_\gamma I Q_k P_\gamma I Q_j) = f_{x_1,w}(Q_j P_\gamma I Q_k P_\gamma I Q_j) \tag{4.11}$$

for all $j, k = 1, \ldots, n$. Then $x_0 = x_1$ and $v = \lambda w$, where $\lambda$ is a unimodular complex number.

*Proof.* From (4.10) and (4.11), we get

$$\langle Q_j(x_0)P_\gamma(x_0)Q_k(x_0)v, v\rangle = \langle Q_j(x_1)P_\gamma(x_1)Q_k(x_1)w, w\rangle,$$
$$\langle Q_j(x_0)P_\gamma(x_0)Q_k(x_0)P_\gamma(x_0)Q_j(x_0)v, v\rangle = \langle Q_j(x_1)P_\gamma(x_1)Q_k(x_1)P_\gamma(x_1)Q_j(x_1)w, w\rangle.$$

That is,

$$\sqrt{\gamma^{a_j}(x_0)\gamma^{a_k}(x_0)}v_k\overline{v_j} = \sqrt{\gamma^{a_j}(x_1)\gamma^{a_k}(x_1)}w_k\overline{w_j}, \tag{4.12}$$

$$\gamma^{a_j}(x_0)\gamma^{a_k}(x_0)|v_j|^2 = \gamma^{a_j}(x_1)\gamma^{a_k}(x_1)|w_j|^2. \tag{4.13}$$

For $j = k$, Eq (4.12) reduces to

$$\gamma^{a_j}(x_0)|v_j|^2 = \gamma^{a_j}(x_1)|w_j|^2.$$

Since $\gamma^{a_j}(x_0)$ and $\gamma^{a_j}(x_1)$ are different from zero for all $j = 1, \ldots, n$, then $v_j \neq 0$ if only if $w_j \neq 0$. Choose an index $j$ such that $v_j \neq 0$. Now, (4.13) means that

$$\gamma^{a_k}(x_0) = \gamma^{a_k}(x_1), \quad \forall k = 1, \ldots, n.$$

In particular $\gamma^{a_n}(x_0) = \gamma^{a_n}(x_1)$, that is,

$$\int_{\sqrt{2}(n-1)-x_0}^{\infty} (h_0(y))^2 dy = \int_{\sqrt{2}(n-1)-x_1}^{\infty} (h_0(y))^2 dy,$$

thus, $x_0 = x_1$. From (4.12), we get $v_k\overline{v_j} = w_k\overline{w_j}$, $\forall k = 1, \ldots, n$. Take $\lambda = \overline{w_j}/\overline{v_j}$. Then $v_k = \lambda w_k$ for all $k = 1, \ldots, n$, that is, $v = \lambda w$ with $|\lambda| = 1$. □

**Lemma 4.7.** *Let $v = (v_1, \ldots, v_n)^T$ be a unimodular vector, $x_0 \in (-\infty, \infty)$, and $x_1 = \pm\infty$. If $v_j \neq 0$, then*

$$f_{x_0,v}(Q_j P_\gamma I Q_j) \neq f_{x_1,e_k}(Q_j P_\gamma I Q_j), \quad k = 1, \ldots, n.$$

*Proof.* Since $P_\gamma(-\infty) = E_{11}$ and $P_\gamma(\infty) = E_{nn}$, we have

$$\begin{aligned} f_{-\infty,e_k}(Q_j P_\gamma I Q_j) &= \langle Q_j(-\infty)P_\gamma(-\infty)Q_j(-\infty)e_k, e_k\rangle \\ &= \langle E_{jj}E_{11}E_{jj}e_k, e_k\rangle \\ &= \delta_{j1}\delta_{k1}, \quad j, k = 1, \ldots, n. \end{aligned}$$

Similarly,

$$f_{\infty,e_k}(Q_j P_\gamma I Q_j) = \delta_{jn}\delta_{kn}, \quad j, k = 1, \ldots, n.$$

On the other hand, $f_{x_0,v}(Q_j P_\gamma Q_j) = \gamma^{a_j}(x_0)v_j\overline{v_j}$ with $\gamma^{a_j}(x_0) \in (0, 1)$. Suppose that $v_j \neq 0$. Then, $f_{x_0,v}(Q_j P_\gamma Q_j) \in (0, 1)$. Thus, $f_{x_0,v}(Q_j P_\gamma Q_j) \neq f_{x_1,e_k}(Q_j P_\gamma Q_j)$. □

**Lemma 4.8.** *The algebra $\mathcal{B}^{1,n}$ separates the following pure states of the algebra $\mathcal{D}^{1,n}$:*

$$f_{-\infty,e_1}, \ldots, f_{-\infty,e_n}, f_{+\infty,e_1}, f_{+\infty,e_n}.$$

*Proof.* Notice that

$$f_{-\infty,e_k}(Q_j) = \delta_{kj}, \tag{4.14}$$

thus $Q_j$ separates the pure states $f_{-\infty,e_j}$ and $f_{-\infty,e_k}$. On the other hand, $f_{+\infty,e_1}(Q_k) = \delta_{1k}$, and then by (4.14), we can see that $f_{-\infty,e_k}$ and $f_{+\infty,e_1}$ are separated by $Q_k$ for $k = 2, \ldots, n$. In the same way, $f_{+\infty,e_n}(Q_k) = \delta_{nk}$. Thus, by (4.14), $f_{-\infty,e_k}$ and $f_{+\infty,e_n}$ are separated by $Q_k$ for $k = 1, \ldots, n-1$.

Finally,

$$f_{-\infty,e_1}(P_\gamma I) = \langle E_{11}e_1, e_1\rangle = 1, \quad f_{+\infty,e_1}(P_\gamma I) = \langle E_{nn}e_1, e_1\rangle = 0,$$

$$f_{-\infty,e_n}(P_\gamma I) = \langle E_{11}e_n, e_n\rangle = 0, \quad f_{+\infty,e_n}(P_\gamma I) = \langle E_{nn}e_n, e_n\rangle = 1.$$

From this, $P_\gamma I$ separates $f_{-\infty,e_1}$ from $f_{+\infty,e_1}$, it separates $f_{-\infty,e_n}$ from $f_{+\infty,e_n}$, and it also separates $f_{+\infty,e_1}$ from $f_{+\infty,e_n}$. □

The algebra $\mathcal{P}_n^{1,n}$ is constructed with the purpose of obtaining a system of all-but-one orthogonal projections in generic position acting on $F_n^2(\mathbb{C})$, required to include the orthoprojections from $F_n^2(\mathbb{C})$ onto the true poly-Fock subspaces. Recall that $B(j)$ denotes the orthogonal projection from $L_2(\mathbb{C}, d\mu)$ onto $F_{(j)}^2(\mathbb{C})$; from now on, $B_{(j)}$ will be regarded as restricted to $F_n^2(\mathbb{C})$.

The Bargmann-type transform $R_n^* : (L_2(\mathbb{R}))^n \to F_n^2(\mathbb{C})$ accomplishes an adequate scheme in the study of the $C^*$-algebra generated by

$$P := R_n^* P_\gamma R_n, \; B_{(1)}, \ldots, B_{(n)},$$

and the relationship with the $C^*$-algebra generated by the Toeplitz operators with horizontal symbols. It is easy to see that $B_{(j)} = R_n^* Q_j R_n$, which means that the following diagram is commutative:

$$\begin{array}{ccc} (L_2(\mathbb{R}))^n & \xrightarrow{P_\gamma,\ Q_j} & (L_2(\mathbb{R}))^n \\ {\scriptstyle R_n^*}\downarrow & & \downarrow{\scriptstyle R_n^*} \\ F_n^2(\mathbb{C}) & \xrightarrow{P,\ B_{(j)}} & F_n^2(\mathbb{C}) \end{array}$$

**Theorem 4.9.** *The $C^*$-algebra $C^*(P, B_{(1)}, \dots, B_{(n)})$ is isomorphic to the $C^*$-algebra $\mathcal{D}^{1,n}$, and the isomorphism is given by $T \longmapsto R_n T R_n^*$. In particular,*

$$P \longmapsto P_\gamma, \quad B_{(j)} \longmapsto Q_j.$$

Up to now, we have the description of two $C^*$-algebras of bounded linear operators acting on the poly-Fock space $F_n^2(\mathbb{C})$, as indicated in Theorems 3.1 and 4.9. Under the same isomorphism $T \mapsto R_n T R_n^*$, the algebras $C^*(P, B_{(1)}, \dots, B_{(n)}) \cong \mathcal{D}^{1,n}$ and $C^*(\mathcal{T}_n^{-\infty,\infty}) \cong \mathcal{D}_n^{-\infty,\infty}$ look quite similar. Indeed, they are different from each other just by their representations corresponding to the points $x = \pm\infty$. In this sense, the algebra generated by Toeplitz operators with horizontal symbols can be generated by a system of all-but-one orthogonal projections in generic position.

## 5. Reproducing kernels

In this section, we get integral representations for $P_T$ and $P$, which are defined on $(F^2(\mathbb{C}))^n$ and $F_n^2(\mathbb{C})$, respectively. By expressing these projections as integral operators, we make explicit how their action depends on the geometry of the underlying Hilbert spaces and the structure of the associated spectral functions. The integral representations not only illustrates the analytic behavior of the projections but also emphasizes the connection between the projection-generated $C^*$-algebra and the classical theory of reproducing kernel Hilbert spaces.

With this in mind, we introduce the bounded continuous vector-valued function

$$M_n = (\sqrt{\gamma^{a_1}}, \dots, \sqrt{\gamma^{a_n}})^T : \mathbb{R} \longrightarrow \mathbb{R}^n.$$

Note that $P_\gamma = M_n M_n^T$, and that the Euclidean norm of the value $M_n(x) \in \mathbb{C}^n$ equals 1 for each $x \in \mathbb{R}$.

**Proposition 5.1.** *The following results hold:*

1) The operator $P_\gamma$ is the orthogonal projection from $(L_2(\mathbb{R}))^n$ onto the space of functions of the form $g = aM_n$ with $a \in L_2(\mathbb{R})$.
2) The operator $P_T$ is the orthogonal projection from $(L_2(\mathbb{C}))^n$ onto the space of vector-valued holomorphic functions of the form

$$h(z) = \frac{1}{\pi^{1/4}} \int_{\mathbb{R}} a(t) e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} M_n(t)\, dt,$$

where $a \in L_2(\mathbb{R})$ and $z = x + iy$.

*Proof.* Take $f = (f_1, \ldots, f_n)^T \in (L_2(\mathbb{R}))^n$, and then

$$g = P_\gamma f = M_n(M_n^T f) = (M_n^T f)M_n = \widetilde{a}M_n,$$

where $\widetilde{a} = M_n^T f = \sum_{k=1}^n \sqrt{\gamma^{a_k}} f_k \in L_2(\mathbb{R})$. If $a \in L_2(\mathbb{R})$, then $g := aM_n \in (L_2(\mathbb{R}))^n$, and

$$P_\gamma(g) = M_n M_n^T a M_n = aM_n(M_n^T M_n).$$

Since $M_n(x) \in \mathbb{C}^n$ has a Euclidean norm equal to 1 for each $x$, then the last equation simplifies to $P_\gamma(g) = g$. Thus $g \in \operatorname{Im} P_\gamma$ if and only if there exists $a \in L_2(\mathbb{R})$ such that $g = aM_n$ almost everywhere.

Since $P_T = \mathbf{R}^* P_\gamma \mathbf{R}$, the image of $P_T$ consists of all functions of the form $h = \mathbf{R}^* g$, where $g = aM_n$ and $a \in L_2(\mathbb{R})$. Finally, recall that $\mathbf{R}^* = \bigoplus_{k=1}^n R_1^*$, and the result follows from (2.2). □

**Theorem 5.2.** *The orthogonal projection $P_T : (F^2(\mathbb{C}))^n \to (F^2(\mathbb{C}))^n$ admits the integral representation*

$$(P_T\varphi)(z) = \int_{\mathbb{C}} K(z,w)\varphi(w)\, d\mu(w), \quad w = u + iv,$$

where

$$K(z,w) = \frac{1}{\pi^{1/2}} \int_{\mathbb{R}} e^{-t^2} e^{-\frac{z^2+\overline{w}^2}{2} + \sqrt{2}(z+\overline{w})t} P_\gamma(t)\, dt, \quad z = x + iy.$$

*Proof.* By Theorem 4.5, $P_T = \mathbf{R}^* P_\gamma \mathbf{R}$, where $\mathbf{R} = \bigoplus_{k=1}^n R_1 = R_1 I$. Take $\varphi \in (F^2(\mathbb{C}))^n$. Using (4.5) and (4.6), we get

$$\begin{aligned}
(P_T\varphi)(z) &= ([\mathbf{R}^* P_\gamma \mathbf{R}]\varphi)(z) \\
&= \frac{1}{\pi^{1/4}} \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} P_\gamma(t)(\mathbf{R}\varphi)(t)dt \\
&= \frac{1}{\pi^{1/2}} \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} P_\gamma(t) \int_{\mathbb{C}} e^{-\frac{\overline{w}^2+t^2}{2} + \sqrt{2}\overline{w}t} \varphi(w)\, d\mu(w)\, dt.
\end{aligned}$$

This proves the theorem. □

**Theorem 5.3.** *The orthogonal projection $P := R_n^* P_\gamma R_n : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$ admits the integral representation*

$$(P\varphi)(z) = \int_{\mathbb{C}} K^\gamma(z,w)\varphi(w)\, d\mu(w),$$

where $z = x + iy$, $w = u + iv$, and

$$K^\gamma(z,w) = \int_{\mathbb{R}} e^{-\frac{z^2+\overline{w}^2}{2} + \sqrt{2}(z+\overline{w})t - t^2} \left[\widetilde{\mathbf{H}}_n\left(\sqrt{2}x - t\right)\right]^T P_\gamma(t) \widetilde{\mathbf{H}}_n(\sqrt{2}u - t)\, dt.$$

*Proof.* Let $\varphi \in F_n^2(\mathbb{C})$. Then,

$$\begin{aligned}
(P\varphi)(z) &= (R_n^* P_\gamma R_n \varphi)(z) \\
&= \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} \left[\widetilde{\mathbf{H}}_n\left(\sqrt{2}x - t\right)\right]^T P_\gamma(t)(R_n\varphi)(t)dt \\
&= \int_{\mathbb{R}} e^{-\frac{z^2+t^2}{2} + \sqrt{2}zt} \left[\widetilde{\mathbf{H}}_n\left(\sqrt{2}x - t\right)\right]^T P_\gamma(t) \times
\end{aligned}$$

$$\int_{\mathbb{C}} \varphi(w) e^{-\frac{\overline{w}^2+t^2}{2}+\sqrt{2}\overline{w}t} \widetilde{\mathbf{H}}_n(\sqrt{2}u - t)\, d\mu(w) dt,$$

$$= \int_{\mathbb{C}} K^{\gamma}(z,w)\varphi(w)\, d\mu(w), \quad w = u + iv$$

where $K^{\gamma}$ is defined as indicated in this theorem. □

## 6. Algebra generated by a single Toeplitz operator and $n$ projections

We now focus on the $C^*$-algebra generated by a single Toeplitz operator $T_{n,a_\beta}$ together with the orthogonal projections $B_{(j)} : F_n^2(\mathbb{C}) \to F_{(j)}^2(\mathbb{C})$ for $j = 1, \ldots, n$. This setup allows us to analyze how minimal spectral data, encoded in a single symbol, interacts algebraically and topologically with the decomposition of the poly-Fock space, producing a $C^*$-algebra structure with a rich quantity of finite dimensional irreducible representations. As in previous sections, we characterize the resulting algebra via its image under the Bargmann-type transform, and show that it coincides with a matrix-valued function algebra with scalar boundary conditions—closely resembling the structure of the full Toeplitz algebra with horizontal symbols.

In what follows, we consider the horizontal symbol

$$a_\beta(z) = \chi_{[\beta,\infty)}(y),$$

and the corresponding Toeplitz operators $T_{n,a_\beta} : F_n^2(\mathbb{C}) \to F_n^2(\mathbb{C})$, where $\beta$ is fixed. According to (3.3), we have $\gamma^\beta I = R_n T_{n,a_\beta} R_n^*$, where

$$\gamma^\beta(x) := \gamma^{n,a_\beta}(x) = \int_{\sqrt{2}\beta - x}^{\infty} [H_n(y)]^T H_n(y) dy,$$

and $H_n(y) = (h_0(y), \ldots, h_{n-1}(y))$. This matrix-valued function satisfies

$$\lim_{x\to-\infty} \gamma^\beta(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \gamma^\beta(x) = I.$$

The following commutative diagram allows us to visualize the operators we are working with:

$$\begin{array}{ccc} (L_2(\mathbb{R}_+))^n & \xrightarrow{\ \gamma^\beta I,\ Q_j\ } & (L_2(\mathbb{R}_+))^n \\ {\scriptstyle R_n^*}\downarrow & & \downarrow{\scriptstyle R_n^*} \\ F_n^2(\mathbb{C}) & \xrightarrow[\ T_{n,a_\beta},\ B_{(j)}\ ]{} & F_n^2(\mathbb{C}) \end{array}$$

Introduce the $C^*$-algebras

$$\mathcal{D}_n = \{M \in M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}}) : M(\pm\infty) \text{ are diagonal matrices}\},$$

$$\mathcal{B}_n = C^*(\gamma^\beta, Q_1, \ldots, Q_n).$$

For the orthogonal projection $Q_j$ given in (4), we have

$$Q_j = R_n B_{(j)} R_n^*.$$

Of course $\mathcal{B}_n \subset \mathcal{D}_n$. We will prove that $\mathcal{B}_n$ separates all the pure states of $\mathcal{D}_n$. The algebra $\mathcal{D}_n$ consists of continuous sections of the $C^*$-bundle $E = M_n(\mathbb{C}) \times \overline{\mathbb{R}}$. Thus, each pure state of $\mathcal{D}_n$ has the form

$$f_{x_0,v}(M) = \langle M(x_0)v, v\rangle, \quad M \in \mathcal{D}_n,$$

or

$$f_{\pm\infty,e_j}(M) = \langle M(\pm\infty)e_j, e_j\rangle, \quad j = 1,\dots,n,$$

where $x_0 \in (-\infty,\infty)$ and $\|v\| = 1$.

**Theorem 6.1.** *The $C^*$-algebra $C^*(T_{n,a_\beta}, B_{(1)},\dots,B_{(n)})$, where $a_\beta(z) = \chi_{[\beta,\infty]}(y)$, is isomorphic and isometric to the $C^*$-algebra $\mathcal{D}_n$. Equivalently, the $C^*$-algebra $\mathcal{D}_n$ is generated by $\gamma^\beta, Q_1,\dots,Q_n$. Moreover, the map*

$$C^*(T_{n,a_\beta}, B_{(1)},\dots,B_{(n)}) \ni T \longmapsto R_n T R_n^* \in \mathcal{D}_n$$

*is an isometric isomorphism of $C^*$-algebras, where*

$$T_{n,a_\beta} \longmapsto \gamma^\beta \quad \text{and} \quad B_{(j)} \longmapsto Q_j.$$

*Proof.* Note that $\mathcal{D}_n$ is a CCR $C^*$-algebra. The $C^*$-algebra $\mathcal{B}_n$ separates all the pure states of $\mathcal{D}_n$ as shown in Lemmas 6.2, 6.3, and 6.4 below. By *the noncommutative Stone-Weierstrass conjecture* [7], we have that $\mathcal{D}_n = \mathcal{B}_n$. □

**Lemma 6.2.** *Let $v, w$ be unimodular vectors, and $x_0, x_1 \in (-\infty,\infty)$. Let $\gamma^\beta$ be the spectral function of the Toeplitz operator $T_{n,a_\beta}$, where $a_\beta(z) = \chi_{[\beta,\infty)}(y)$. Suppose that*

$$f_{x_0,v}(Q_j\gamma^\beta I Q_k) = f_{x_1,w}(Q_j\gamma^\beta I Q_k), \tag{6.1}$$
$$f_{x_0,v}(Q_j\gamma^\beta I Q_k\gamma^\beta I Q_l) = f_{x_1,w}(Q_j\gamma^\beta I Q_k\gamma^\beta I Q_l), \tag{6.2}$$

*for all $j,k,l = 1,\dots,n$. Then, $x_0 = x_1$ and $v = \lambda w$, where $\lambda$ is a unimodular complex number.*

*Proof.* Let $v = (v_1,\dots,v_n), w = (w_1,\dots,w_n)$ be unimodular vectors, and $x_0, x_1 \in (-\infty,\infty)$. Note that $Q_j\gamma^\beta(x)Q_k = \gamma^\beta_{jk}(x)E_{jk}$ and $E_{jk}E_{kl} = E_{jl}$. From (6.1) and (6.2), we obtain

$$\gamma^\beta_{jk}(x_0)v_k\overline{v_j} = \gamma^\beta_{jk}(x_1)w_k\overline{w_j},$$
$$\gamma^\beta_{jk}(x_0)\gamma^\beta_{kl}(x_0)v_l\overline{v_j} = \gamma^\beta_{jk}(x_1)\gamma^\beta_{kl}(x_1)w_l\overline{w_j},$$

for all $j,k,l = 1,\dots,n$. Since $a_\beta(z) = \chi_{[\beta,\infty)}(y)$,

$$\gamma^\beta_{jk}(x) = \int_{\sqrt{2}\beta - x}^{\infty} h_{j-1}(y)h_{k-1}(y)dy \neq 0 \ \forall x \in (-\infty,\infty).$$

Note that $v_k \neq 0$ if and only if $w_k \neq 0$. Thus,

$$\frac{\gamma^\beta_{jk}(x_0)}{\gamma^\beta_{jk}(x_1)} = \frac{w_k\overline{w_j}}{v_k\overline{v_j}}, \qquad \frac{\gamma^\beta_{jk}(x_0)}{\gamma^\beta_{jk}(x_1)}\frac{\gamma^\beta_{kl}(x_0)}{\gamma^\beta_{kl}(x_1)} = \frac{w_l\overline{w_j}}{v_l\overline{v_j}},$$

whenever $v_k\overline{v_j} \neq 0$ and $v_l\overline{v_j} \neq 0$. Choose $k$ such that $v_k \neq 0$, and take $j = l = k$. Then

$$\frac{w_k\overline{w_j}}{v_k\overline{v_j}}\frac{w_l\overline{w_k}}{v_l\overline{v_k}} = \frac{w_l\overline{w_j}}{v_l\overline{v_j}},$$

or

$$\frac{w_k\overline{w_k}}{v_k\overline{v_k}} = 1.$$

Therefore, $\gamma^{\beta}_{kk}(x_0) = \gamma^{\beta}_{kk}(x_1)$, that is,

$$\int_{\sqrt{2}\beta - x_0}^{\infty} (h_{k-1}(y))^2 dy = \int_{\sqrt{2}\beta - x_1}^{\infty} (h_{k-1}(y))^2 dy,$$

thus, $x_0 = x_1$. Consequently, $\gamma^{\beta}_{jk}(x_0)v_k\overline{v_j} = \gamma^{\beta}_{jk}(x_0)w_k\overline{w_j}$ for all $j$. That is, $v_k\overline{v_j} = w_k\overline{w_j}$ for all $j$. Let $\lambda = \overline{w_k}/\overline{v_k}$, and then $v = \lambda w$ with $|\lambda| = 1$. □

We proceed to separate the rest of the pure states of $\mathcal{D}_n$.

**Lemma 6.3.** *Let $e_j = (0, \ldots, 1, \ldots, 0)^T \in \mathbb{C}^n$, and $\gamma^\beta$ be the spectral function of the Toeplitz operator $T_{n,a_\beta}$, where $a_\beta(z) = \chi_{[\beta,\infty)}(y)$. Then*

$$f_{-\infty,e_j}(\gamma^\beta I) \neq f_{\infty,e_k}(\gamma^\beta I) \quad \text{for all} \quad j, k = 1, \ldots, n.$$

In addtion, the pure states $f_{\pm\infty,e_1}, \ldots, f_{\pm\infty,e_n}$ are separated by $Q_1, \ldots, Q_n$.

*Proof.* We have that $\gamma^\beta(-\infty) = 0$ and $\gamma^\beta(\infty) = I$, and then

$$f_{-\infty,e_j}(\gamma^\beta I) = 0 \quad \text{and} \quad f_{\infty,e_k}(\gamma^\beta I) = 1.$$

Thus, the pure states $f_{-\infty,e_j}$ and $f_{\infty,e_k}$ are separated by $\mathcal{B}_n$. On the other hand, $f_{-\infty,e_j}(Q_j) = 1$ and $f_{-\infty,e_k}(Q_j) = 0$ for $k \neq j$. Thus, $f_{-\infty,e_1}, \ldots, f_{-\infty,e_n}$ are separated by $Q_1, \ldots, Q_n$. □

**Lemma 6.4.** *Let $v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$ be a unimodular vector, $x_0 \in (-\infty, \infty)$, and $x_1 = \pm\infty$. Let $\gamma^\beta$ be the spectral function of the Toeplitz operator $T_{n,a_\beta}$, where $a_\beta(z) = \chi_{[\beta,\infty)}(y)$. If $v_j \neq 0$, then*

$$f_{x_0,v}(Q_j\gamma^\beta I Q_j) \neq f_{x_1,e_k}(Q_j\gamma^\beta I Q_j) \quad \text{for all } k = 1, \ldots, n.$$

*Proof.* Note that $f_{-\infty,e_k}(Q_j\gamma^\beta I Q_j) = 0$ and $f_{\infty,e_k}(Q_j\gamma^\beta I Q_j) = \delta_{jk}$. Choose $j$ such that $v_j \neq 0$. Since $\gamma^\beta_{jj}(x_0) \in (0, 1)$,

$$f_{x_0,v}(Q_j\gamma^\beta I Q_j) = \gamma^\beta_{jj}(x_0)|v_j|^2 \in (0, 1),$$

which means that $f_{x_0,v}$ and $f_{x_1,e_k}$ are separated by $\mathcal{B}_n$, where $x_1 = \pm\infty$. □

We conclude by summarizing the principal contributions of this work. We have described two $C^*$-algebras naturally arising in the context of the poly-Fock space $F^2_n(\mathbb{C})$. In Theorem 4.9, we analyzed the algebra $C^*(P, B_{(1)}, \ldots, B_{(n)})$ generated by a family of all-but-one orthogonal projections in generic position. We showed that this algebra is isomorphic to $\mathcal{D}^{1,n}$, which is a subalgebra of all $n \times n$-matrix-valued functions, continuous on the compactified real line. In parallel, Theorem 6.1

characterizes the $C^*$-algebra generated by a single Toeplitz operator $T_{n,a_\beta}$ together with the same family $\{B_{(j)}\}_{j=1}^n$ of projections, establishing an isomorphism with the algebra $\mathcal{D}_n$. Finally, Theorem 3.1 describes the algebra $C^*(\mathcal{T}_n^{-\infty,\infty})$ generated by Toeplitz operators, showing that it is isomorphic to $\mathcal{D}_n^{-\infty,\infty}$. These algebras—$\mathcal{D}^{1,n}$, $\mathcal{D}_n$, and $\mathcal{D}_n^{-\infty,\infty}$—are realized as subalgebras of $M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$ through a common isomorphism, and they differ only by their irreducible representations corresponding to $x_0 = \pm\infty$. This hierarchy reveals that algebras generated by Toeplitz operators can, in certain settings, be equivalently studied via families of orthogonal projections. This analysis is unified by the use of noncommutative Stone–Weierstrass-type arguments, which clarify the structural relationships among the algebras studied, and suggest a broader applicability within operator theory.

## Author contributions

Each author was equally involved in conceptualization, methodology, analysis, research, and writing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

All authors are grateful to the SECIHTI for its support. In particular, the first author was partially supported by the SECIHTI project Ciencia Frontera 61517.

## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References

1. M. B. Balk, *Polyanalitic functions*, Berlin: Akademie Verlag, 1991.
2. J. Dixmier, $C^*$-Algebras, Amsterdam: North-Holland Publishing Company, 1977.
3. R. G. Douglas, Local Toeplitz operators, *P. Lond. Math. Soc.*, **3** (1978), 243–272. https://doi.org/10.1112/plms/s3-36.2.243
4. K. Esmeral, N. Vasilevski, $C^*$-Algebra generated by horizontal Toeplitz operators on the Fock space, *Bol. Soc. Mat. Mex.*, **22** (2016), 567–582. https://doi.org/10.1007/s40590-016-0110-1
5. J. Glimm, A Stone–Weierstrass theorem for $C^*$-algebras, *Ann. Math.*, **72** (1960), 216–244. https://doi.org/10.2307/1970133 .
6. P. R. Halmos, Two subspaces, *Trans. Amer. Math. Soc.*, **144** (1969), 381–389. https://doi.org/10.1090/S0002-9947-1969-0251519-5
7. I. Kaplansky, The structure of certain operator algebras, *Trans. Amer. Math. Soc.*, **70** (1951), 219–255. https://doi.org/10.1090/S0002-9947-1951-0042066-0

8. T. K. Lee, Extreme points related to matrix algebras, *Kangweon-Kyungki Math. Jour.*, **9** (2001), 45–52.

9. E. A. Maximenko, A. M. Tellería-Romero, Radial operators on polyanalytic Bargmann–Segal–Fock spaces, In: *Operator algebras, Toeplitz operators and related topics*, Cham: Birkhäuser, 2020, 277–305. https://doi.org/10.1007/978-3-030-44651-2_18

10. G. K. Pedersen, Measure theory for $C^*$-algebras II, *Mat. Scand.*, **22** (1968), 63–74. https://doi.org/10.7146/math.scand.a-10871

11. J. Ramírez-Ortega, A. Sánchez-Nungaray, Toeplitz operators with vertical symbols acting on the poly-Bergman spaces of the upper half-plane, *Complex Anal. Oper. Theory*, **9** (2015), 1801–1817. https://doi.org/10.1007/s11785-015-0469-4

12. A. Sánchez-Nungaray, C. González-Flores, R. R. López-Martínez, J. L. Arroyo-Neri, Toeplitz operators with horizontal symbols acting on the poly-Fock space, *J. Funct. Space.*, **2018** (2018), 8031259. https://doi.org/10.1155/2018/8031259

13. N. L. Vasilevski, I. M. Spitkovsky, On the algebra generated by two projectors, *Dokl. Akad. Nauk. UKSSR, A*, **8** (1981), 10–13.

14. N. L. Vasilevski, $C^*$-Algebras generated by orthogonal projections and their applications, *Integr. Equ. Oper. Theory*, **31** (1998), 113–132. https://doi.org/10.1007/BF01203459

15. N. L. Vasilevski, Poly-Fock spaces, In: *Differential operators and related topics*, Basel: Birkhäuser, 2000, 371–386. https://doi.org/10.1007/978-3-0348-8403-7_28

16. N. L. Vasilevski, *Commutative algebras of Toeplitz operators on the Bergman space*, Basel: Birkhäuser, 2008. https://doi.org/10.1007/978-3-7643-8726-6

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