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The properties of the Laplacian permanental polynomials of graphs

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Abstract: In this paper, some properties of the Laplacian permanental polynomials of graphs are given. First, we provide a formula to evaluate the coefficients of the Laplacian permanental polynomial. Following this, we derive some interesting derivative properties of the Laplacian permanental polynomial. In addition, the recursive relations on the Laplacian permanental coefficients of some subdivision graphs are deduced.

Keywords: Laplacian permanental polynomial; permanental polynomial
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1. Introduction

The graph G considered here is simple and undirected. The vertex set of G is $V(G) = \{v_1, \ldots, v_n\}$, and $v_i \sim v_j$ means that v_i and v_j are adjacent. The degree of the vertex v_i in G is denoted by $d_G(v_i)$. The diagonal matrix $D(G)$ is the one with $d_G(v_i)$ as the entry (i, i). The *adjacency matrix* $A(G) = (a_{ij})_{n\times n}$ of G is a (0,1) matrix with $a_{ij} = 1$ if and only if v_i and v_j are joined by an edge. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$.

The *permanental polynomial* $\pi(G, x)$ of G is defined as

$$\mathrm{per}(xI - A(G)) = \sum_{i=0}^{n} b_i x^{n-i},$$

where I is an identity matrix of order n. For a matrix $A = (a_{ij})_{n\times n}$,

$$\mathrm{per}(A) = \sum_{\sigma\in\Gamma_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where Γ_n denotes the set of all the permutations of $\{1, 2, \cdots, n\}$. It was shown in [11] that $\mathrm{per}(A) = \sum_{j=1}^{n} a_{ij}\mathrm{per}(A_{ij})$, where A_{ij} is the matrix obtained from A by deleting the i-th row and j-th column. Moreover, it holds that $\mathrm{per}(-A) = (-1)^n\mathrm{per}(A)$.

The *Laplacian permanental polynomial* $\psi(G, x)$(or $\psi(L(G), x)$) of a graph G is [13]

$$\mathrm{per}(xI - L(G)) = \sum_{i=0}^{n} c_i x^{n-i}. \tag{1.1}$$

It is obvious that $c_0 = 1$ and $c_n = (-1)^n\mathrm{per}(L(G))$. For the graph K_0 that has no vertices and edges, $\psi(K_0, x) = 1$.

Since the characteristic polynomial fails to distinguish certain graphs, the permanental polynomial was introduced to differentiate cospectral graphs. However, for trees at least, the permanental polynomial offers no additional discriminative power beyond that of the characteristic polynomial [13]. Given that the Laplacian matrix possesses certain advantages over the adjacency matrix [2], researchers have explored the permanent of Laplacian matrix and its associated Laplacian permanental polynomial on the coefficients [8] and roots [16]. Up to now, we have not known for which graphs the Laplacian permanental polynomials are the same. Extensive studies have been done on the permanental polynomials. The relations between the coefficients of the permanental polynomial and the characteristic polynomial were discussed in [5, 6], and the relations between the matching polynomial and the permanental polynomial were given in [7]. In [9], the extremal problem for the coefficient sum of the permanental polynomials of extremal hexagonal chains were determined. The per-spectra of a complete graph with respect to removing the edges were considered in [15]. In addition, the computation of the permanental polynomials of some graphs by the skew-characteristic polynomial was investigated in [17, 18].

Compared with these extensive research on permanental polynomials, relatively little work has been done on Laplacian permanental polynomials. To our knowledge, the existing research on this topic falls into two main categories: Studies focusing on computing the permanents of Laplacian matrices and establishing optimal lower bounds [1, 4] and upper bounds [3], and investigations into the coefficients and roots of Laplacian permanental polynomials. For example, Merris [12] obtained the coefficients of the Laplacian permanental polynomials for trees. Liu and Wu [10] proved a recursive formula for computing the Laplacian permanental polynomial of graphs under edge deletion and vertex deletion operations, as shown below.

In the following, $L_S(G)$ denotes the principal submatrix of L(G) formed by deleting the row and column corresponding to all vertices of $S \subseteq V(G)$.

Theorem 1.1. *[10] (1) Let v be a vertex of a graph G, $C_G(v)$ be the set of cycles of G containing v, and N(v) be the set of vertices of G adjacent to v. Then*

$$\psi(G, x) = (x - d(v))\psi(L_v(G), x) + \sum_{u \in N(v)} \psi(L_{uv}(G), x) + 2 \sum_{C \in C_G(v)} \psi(L_{V(C)}(G), x).$$

(2) Let $e = (u, v)$ be an edge of a graph G and let $C_G(e)$ be the set of cycles containing e in G. Let G-e be the graph obtained by deleting the edge e from G. Then

$$\psi(G, x) = \psi(L(G - e), x) - \psi(L_u(G - e), x) - \psi(L_v(G - e), x) + 2\psi(L_{uv}(G), x) + 2 \sum_{C \in C_G(e)} \psi(L_{V(C)}(G), x).$$

Computing the permanent of a square matrix is a #P-complete problem even for a (0, 1) matrix [14]. Thus, it is difficult to compute the permanental polynomials and Laplacian permanental polynomials of graphs. In 2019, Liu [8] gave a combinatorial expressions for the first five coefficients of the (signless) Laplacian permanental polynomial. Merris et al. established the general formula for the coefficients of permanental polynomials [13]. Motivated by this, in this paper, we try to give a general expression for the coefficients of the Laplacian permanental polynomials of a graph in Section 2. On the basis of this result, in Section 3, we show some derivative properties of the Laplacian permanental polynomial of a graph. In Section 4, we derive the recursive relations for the Laplacian permanental polynomials of graphs with respect to the subdivision operation.

2. The coefficients of Laplacian permanental polynomials

For a graph G, we use G^* to denote the graph obtained from G by attaching a loop for each vertex of G, called the *loop graph* of G. We use L_v to denote the loop incident on the vertex v of G^*. A *linear subgraph* U_i of G^* is a subgraph of G^* on i vertices such that each component is a cycle, an edge, or a loop. For a linear subgraph U_i of G^*, let $d(U_i)$ be the product of the degrees of the vertices in G which lie in the loops of U_i. Let $c(U_i)$ be the number of cycles in U_i, and let $o(U_i)$ be the number of cycles of odd length in U_i. As early as the 1980s [2, 13], a formula to compute the coefficients of the permanental polynomial was given as

$$b_i = (-1)^i \sum_{F_i \subset G} 2^{c(F_i)} \qquad \text{for} \quad 1 \leq i \leq n,$$

where F_i is a subgraph of G on i vertices, each component of F_i is an edge or a cycle, and $c(F_i)$ denotes the number of cycles of F_i.

In the following, we show that there is a similar result for the coefficients of Laplacian permanental polynomials.

Theorem 2.1. *For the Laplacian permanental polynomial of a graph G on n vertices,*

$$c_i = (-1)^i \sum_{U_i \subset G^*} (-1)^{o(U_i)} 2^{c(U_i)} d(U_i) \qquad \text{for} \quad 1 \leq i \leq n,$$

where the sum includes all linear subgraphs U_i of the loop graph G^*.

Proof. By the definition of Laplacian permanental polynomial,

$$c_i = (-1)^i \sum \text{per}(L_i(G)),$$

where $L_i(G)$ is the principle submatrix of $L(G)$ of order i, and the sum is over all such values of $L_i(G)$. We first prove that

$$c_n = (-1)^n \sum_{U_n \subset G^*} (-1)^{o(U_n)} 2^{c(U_n)} d(U_n).$$

Suppose that $L(G) = (a_{ij})_{n\times n}$. By the definition of permanence, $c_n = (-1)^n \text{per}(L(G)) = (-1)^n \sum_{\sigma\in\Gamma_n} \prod_{i=1}^{n} a_{i\sigma(i)}$, where Γ_n is the set of all the permutations of $\{1, 2, \cdots, n\}$. Each term $a_{1\sigma_1} \cdots a_{n\sigma_n} \neq 0$ if and only if each $a_{i\sigma_i} \neq 0$. We know that each term $a_{1\sigma_1} \cdots a_{n\sigma_n}$ is a product of cyclic products. For these cyclic products, any product is one of the following.

(a) A cycle in G^*. Each cycle in G^* corresponds to two opposite cyclic permutations with a product 1 if C is of even length and with a product -1 if C is of odd length.

(b) An edge in G^*. Each edge in G^* corresponds to a cyclic permutation of length 2 with a product 1.

(c) A loop in G^*. Each loop in G^* corresponds to a cyclic permutation of length 1 with a product that equals the degree of the corresponding vertex in G.

According to this, each nonzero cyclic product of a term $a_{1\sigma_1}\cdots a_{n\sigma_n}$ of per($L(G)$) corresponds to a cycle or an edge or a loop of a linear subgraph U_n of G^*, and vice versa. Thus, we find $c_n = (-1)^n \sum_{U_n \subset G^*}(-1)^{o(U_n)}2^{c(U_n)}d(U_n)$.

By a similar analysis, we derive $c_i = (-1)^i \sum_{U_i \subset G^*}(-1)^{o(U_i)}2^{c(U_i)}d(U_i)$.

According to Theorem 2.1, we immediately obtain the following result for a regular graph.

Corollary 2.1. *If G is a regular graph of degree r, then*

$$c_i = (-1)^i \sum_{U_i \subset G^*}(-1)^{o(U_i)}2^{c(U_i)}r^{l(U_i)},$$

where $l(U_i)$ is the number of loops in a linear subgraph U_i.

3. The derivative property of the Laplacian permanental polynomial

In this section, we show the derivative property of the Laplacian permanental polynomials for general graphs, which leads to a way to compute such polynomials in terms of the polynomials of subgraphs.

For a graph G and a set S of vertices in G, $G - S$ denotes the graph obtained from G by deleting all the vertices of S. We use $(G - S)_G$ to denote the half-edge graph of $G - S$, which means that for any vertex v of $(G - S)_G$ if v is adjacent to a vertex u of S in G, then there is a half-edge incident to v in $(G - S)'$. Thus, the degree of each vertex v of $(G - S)_G$ is the same as the degree of v in the graph G. If S is a single vertex v, we use $(G - v)_G$ as $(G - S)_G$.

Theorem 3.1. *Let G be a graph with n vertices $\{v_1, \cdots, v_n\}$. Then*

$$\frac{d\psi(G,x)}{dx} = \sum_{i=1}^{n} \psi((G - v_i)_G, x). \tag{3.1}$$

Proof. Suppose that $\psi(G, x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$. Then

$$\frac{d\psi(G,x)}{dx} = nc_0x^{n-1} + (n-1)c_1x^{n-2} + \cdots + 2c_{n-2}x + c_{n-1}. \tag{3.2}$$

Let $\psi((G - v_i)_G, x) = c_0^i x^{n-1} + c_1^i x^{n-2} + \cdots + c_{n-2}^i x + c_{n-1}^i$. By Theorem 2.1,

$$c_k^i = (-1)^k \sum_{U_k \subset (G-v_i)_G^*}(-1)^{o(U_k)}2^{c(U_k)}d(U_k) \qquad \text{for} \quad 1 \le i \le n \text{ and } 0 \le k \le n-1.$$

Thus

$$\sum_{i=1}^{n} c_k^i = (-1)^k \sum_{i=1}^{n} \sum_{U_k \subset (G-v_i)_G^*}(-1)^{o(U_k)}2^{c(U_k)}d(U_k). \tag{3.3}$$

Each linear subgraph U_k of G^* has k vertices, without loss of generality, denoted as $\{v_{i_1}, \cdots, v_{i_k}\}$. It can be seen that U_k is not a linear subgraph of $(G - v_{i_r})^*_G$ for $r = 1, \cdots, k$. However, U_k is a linear subgraph of $(G - v_t)^*_G$, where $v_t \in V(G)$ and $t \notin \{i_1, \cdots, i_k\}$. Therefore, each U_k belongs to $(n - k)$ half-edge subgraphs $(G - v_i)^*_G$, so the sum in Eq (3.3) counts each linear subgraph U_k of G^* exactly $(n - k)$ times. Thus

$$\sum_{i=1}^{n} c_k^i = (-1)^k (n-k) \sum_{U_k \subset G^*} (-1)^{o(U_k)} 2^{c(U_k)} d(U_k)$$
$$= (n-k) c_k.$$

Following Eq (3.2), we obtain

$$\frac{d\psi(G,x)}{dx} = \sum_{i=1}^{n} \psi((G - v_i)_G, x).$$

In the following, we use C_n to denote a cycle on n vertices and use P_n to denote a path on n vertices. **Note 1:** In Table 1, we show the Laplacian permanental polynomials of the cycle C_5 and the path $(C_5 - v)_{C_5}$. We can see that the derivative of $\psi(C_5, x)$ is equal to $5\psi((C_5 - v)_{C_5}, x)$. That is, $\psi'(C_5, x) = 5\psi((C_5 - v)_{C_5}, x)$. This verifies the result of Theorem 3.1.

Table 1. The Laplacian permanental polynomials of cycles C_5 and P_4.

Graphs	**The Laplacian permanental polynomials**
C_5	$\psi(C_5, x) = x^5 - 10x^4 + 45x^3 - 110x^2 + 145x - 80$
Derivative	$\psi'(C_5, x) = 5x^4 - 40x^3 + 135x^2 - 220x + 145$
P_4	$\psi((C_5 - v)_{C_5}, x) = x^4 - 8x^3 + 27x^2 - 44x + 29$
Sum	$5\psi((C_5 - v)_{C_5}, x) = 5x^4 - 40x^3 + 135x^2 - 220x + 145$

Corollary 3.1. *Let G be a graph with n vertices* $\{v_1, \cdots, v_n\}$*. Then*

$$\psi(G,x) = \int_0^x \sum_{i=1}^{n} \psi((G - v_i)_G, t)dt + \psi(G, 0).$$

Proof. Integrating both sides of Eq (3.1), we obtain

$$\psi(G,x) = \int_0^x \sum_{i=1}^{n} \psi((G - v_i)_G, t)dt + C, \tag{3.4}$$

where C is a constant number.

Set $x = 0$. It holds for the left-hand side of Eq (3.4) that $\psi(G, 0) = \sum_{k=0}^{n} c_k x^{n-k} = c_n$. On the right-hand side of Eq (3.4), $\int_0^0 \sum_{i=1}^{n} \psi((G - v_i)_G, t)dt = 0$. Thus, we have $C = \psi(G, 0)$.

In a similar approach, we deduce the following result for the permanental polynomial.

Corollary 3.2. *Let G be a graph with n vertices* $\{v_1, \cdots, v_n\}$*. Then*

$$\pi(G,x) = \int_0^x \sum_{i=1}^{n} \pi(G - v_i, t)dt + \pi(G, 0).$$

Let $\mathscr{S}_k(V) = \{S | S \subseteq V(G) \text{ and } |S| = k\}$, where $|S|$ denotes the number of vertices in S. Then Theorem 3.1 can be generalized as shown below.

Theorem 3.2. *Let G be a graph with the vertex set V. Then*

$$\frac{d^k \psi(G, x)}{dx^k} = k! \sum_{S \in \mathscr{S}_k(V)} \psi((G - S)_G, x). \tag{3.5}$$

Proof. By Theorem 3.1, we know that Eq (3.5) holds for $k = 1$.

We use induction on k. By the induction hypothesis, we have

$$\frac{d^{k-1} \psi(G, x)}{dx^{k-1}} = (k-1)! \sum_{S' \in \mathscr{S}_{k-1}(V)} \psi((G - S')_G, x).$$

Following this, it holds that

$$\frac{d^k \psi(G, x)}{dx^k} = (k-1)! \sum_{S' \in \mathscr{S}_{k-1}(V)} \frac{d\psi((G - S')_G, x)}{dx}. \tag{3.6}$$

It follows from Theorem 3.1 that

$$\begin{aligned} \frac{d\psi((G - S')_G, x)}{dx} &= \sum_{i=1}^{|V(G)|-|S'|} \psi(((G - S')_G - v_i)_{(G-S')_G}, x) \\ &= \sum_{i=1}^{|V(G)|-|S'|} \psi((G - S' - v_i)_G, x), \end{aligned} \tag{3.7}$$

where each v_i is a vertex of $G - S'$ for $S' \in \mathscr{S}_{k-1}(V)$.

For each $S \in \mathscr{S}_k(V)$, let $\{v_{j_1}, \cdots, v_{j_k}\}$ denote the vertices in S. Then there are k different $S' \in \mathscr{S}_{k-1}(V)$ such that $S' \cup \{v_{j_t}\} = S$, where $t \in \{1, \cdots, k\}$. Thus, combining Eqs (3.6) and (3.7), we have

$$\begin{aligned} \frac{d^k \psi(G, x)}{dx^k} &= (k-1)! \sum_{S' \in \mathscr{S}_{k-1}(V)} \sum_{i=1}^{|V(G)|-|S'|} \psi((G - S' - v_i)_G, x) \\ &= k! \sum_{S \in \mathscr{S}_k(V)} \psi((G - S)_G, x). \end{aligned}$$

A similar result holds for the permanental polynomial of a graph.

Corollary 3.3. *Let G be a graph with the vertex set V. Then*

$$\frac{d^k \pi(G, x)}{dx^k} = k! \sum_{S \in \mathscr{S}_k(V)} \pi(G - S, x).$$

In the following, we show the relation between the permanental polynomial and the Laplacian permanental polynomial of a graph.

Theorem 3.3. *Let G be a graph with a set V of n vertices. Then*

$$\psi(G,x) = (-1)^n \sum_{k=0}^{n} \sum_{S\in \mathscr{S}_k(V)} d_G(S)\pi(G-S,-x),$$

where $d_G(S) = \prod_{v\in S} d_G(v)$*, and* $d_G(S) = 1$ *if* $S = \emptyset$*.*

Proof. Suppose that $V = \{v_1, \cdots, v_n\}$, and let $d_i = d_G(v_i)$ for each vertex v_i. Then

$$\begin{aligned}\psi(G,x) =& \mathrm{per}(xI - L(G))\\ =& \mathrm{per}(xI - D(G) + A(G))\\ =& \mathrm{per}\begin{pmatrix} x-d_1 & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & x-d_2 & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & x-d_3 & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & x-d_n\end{pmatrix}.\end{aligned}$$

By the expansion property of permanent, it holds that

$$\psi(G,x) = \mathrm{per}\begin{pmatrix} x & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & x-d_2 & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & x-d_2 & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & x-d_n\end{pmatrix} + \mathrm{per}\begin{pmatrix} -d_1 & 0 & 0 & \cdots & 0\\ a_{21} & x-d_2 & a_{23} & \cdots & a_{2n}\\ a_{31} & a_{32} & x-d_2 & \cdots & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \cdots & x-d_n\end{pmatrix}.$$

If we repeatedly apply this property, after the n-th step, we find that $\psi(G,x)$ is the sum of 2^n permanents. It is written as

$$\begin{aligned}\psi(G,x) =& \mathrm{per}(xI + A(G)) + \sum_{1\le i\le n}(-d_i)\mathrm{per}(xI + A(G-v_i))\\ &+ \sum_{1\le i<j\le n}(-d_i)(-d_j)\mathrm{per}(xI + A(G-v_i-v_j)) + \cdots +\\ &\sum_{1\le i_1<i_2<\cdots<i_{n-1}\le n}(-d_{i_1})(-d_{i_2})\cdots(-d_{i_{n-1}})\mathrm{per}(xI + A(G-v_{i_1}-v_{i_2}-\cdots-v_{i_{n-1}}))\\ &+ (-d_1)(-d_2)\cdots(-d_n)\mathrm{per}(xI + A(G-v_1-v_2-\cdots-v_n)).\end{aligned}$$

Since $\mathrm{per}(-A) = (-1)^n\mathrm{per}(A)$, the equation above can be expressed as

$$\begin{aligned}\psi(G,x) =& (-1)^n\mathrm{per}(-xI - A(G)) + \sum_{1\le i\le n}(-d_i)(-1)^{n-1}\mathrm{per}(-xI - A(G-v_i))\\ &+ \sum_{1\le i<j\le n}(-d_i)(-d_j)(-1)^{n-2}\mathrm{per}(-xI - A(G-v_i-v_j)) + \cdots\\ &+ \sum_{1\le i_1<i_2<\cdots<i_{n-1}\le n}(-d_{i_1})(-d_{i_2})\cdots(-d_{i_{n-1}})(-1)^{n-(n-1)}\mathrm{per}(-xI - A(G-v_{i_1}-v_{i_2}-\cdots-v_{i_{n-1}}))\\ &+ (-d_1)(-d_2)\cdots(-d_n)(-1)^{n-n}\mathrm{per}(-xI - A(G-v_1-v_2-\cdots-v_n)).\end{aligned}$$

Let $S \subseteq V$. The product of the degrees of vertices in S is denoted as $d_G(S)$, i.e. $d_G(S) = \prod_{v\in S} d_G(v)$. Thus

$$\begin{aligned}\psi(G,x) =&(-1)^n\Big[\mathrm{per}(-xI - A(G)) + \sum_{S\in\mathscr{S}_1(V)} d_G(S)\mathrm{per}(-xI - A(G-S))+\\ &\sum_{S\in\mathscr{S}_2(V)} d_G(S)\mathrm{per}(-xI - A(G-S)) + \cdots + \sum_{S\in\mathscr{S}_n(V)} d_G(S)\mathrm{per}(-xI - A(G-S))\Big]\\ =&(-1)^n\sum_{k=0}^{n}\Big[\sum_{S\in\mathscr{S}_k(V)} d_G(S)\pi(G-S,-x)\Big].\end{aligned}$$

According to Corollary 3.3 and Theorem 3.3, we derive the consequence below.

Corollary 3.4. *Let G be a r-regular graph with n vertices. Then*

$$\psi(G,x) = (-1)^n \sum_{k=0}^{n} \frac{r^k}{k!}\frac{d^k\pi(G,y)}{dy^k}\Big|_{y=-x}.$$

4. The Laplacian permanental polynomials of subdivision graphs

In this section, we consider graphs with a cut edge and unicyclic graphs. The recursive relations of the coefficients of the Laplacian permanental polynomials of some subdivision graphs will be given.

Theorem 4.1. *Let G_0 be a graph on n vertices and let $e = (u,v)$ be a cut edge of G_0. Let G_t be the graph obtained from G_0 by inserting t vertices into the edge e for $t = 1,2$. Then*

$$c_{i+2}(G_2) = c_i(G_0) - 2c_{i+1}(G_1) + c_{i+2}(G_1) \quad \textit{for} \quad 0 \le i \le n-1, \tag{4.1}$$

where $c_i(G_k)$ is the value c_i of $\psi(G_k,x) = \sum_{i=0}^{n+k} c_i x^{n+k-i}$ for $k = 0,1,2$.

Proof. By Theorem 2.1, it holds that

$$c_{i+2}(G_2) = (-1)^{i+2}\sum_{U_{i+2}\subset G_2^*}(-1)^{o(U_{i+2})}2^{c(U_{i+2})}d(U_{i+2}),$$

where G_k^* denotes the loop graph of G_k.

Suppose that G_1 (respectively, G_2) is obtained from G_0 by inserting the vertex x (respectively, vertices x and y) into the edge $e = (u,v)$, as shown in Figure 1.

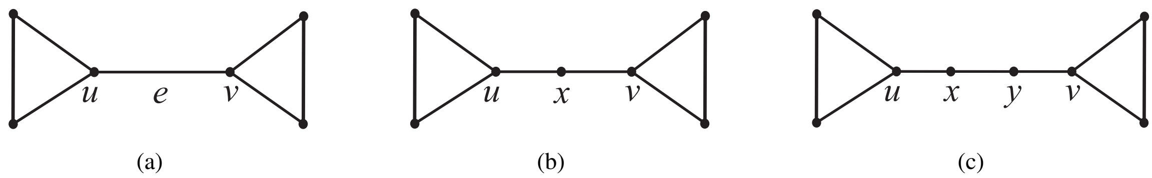

Figure 1. (a) G_0, (b) G_1, and (c) G_2.

We can see that (x, y) is a cut edge of G_2. Let U_{i+2} be a linear subgraph of G_2^*, and suppose that U_{i+2} contributes m to $c_{i+2}(G_2)$. Then the following cases exist for U_{i+2}.

Case 1. $(x, y) \in U_{i+2}$. Since U_{i+2} contains the edge (x, y), U_{i+2} must not contain the edge (u, x) and (y, v). Thus $U_i' = U_{i+2} - x - y$ is a linear subgraph of G_0^* and it does not contain the edge $e = (u, v)$. Moreover, U_i' contributes $(-1)^i(-1)^{o(U_i')}2^{c(U_i')}d(U_i') = (-1)^{i+2}(-1)^{o(U_{i+2})}2^{c(U_{i+2})}d(U_{i+2}) = m$ to $c_i(G_0)$.

Case 2. $(x, y) \notin U_{i+2}$, $L_x \in U_{i+2}$, and $L_y \in U_{i+2}$. Let $U_{i+1}' = U_{i+2} - L_y$. Then U_{i+1}' is a linear subgraph of G_1^*, which contains the loop L_x, and U_{i+1}' contributes $(-\frac{1}{2}m)$ to $c_{i+1}(G_1)$. This means that the contribution of U_{i+2} to $c_{i+2}(G_2)$ is (-2) times the contribution of U_{i+1}' to $c_{i+1}(G_1)$.

Case 3. $(x, y) \notin U_{i+2}$, $L_x \in U_{i+2}$, and $L_y \notin U_{i+2}$. Under this condition, we consider the following cases of the edge (y, v).

Case 3.1. $(y, v) \notin U_{i+2}$. In this case, $U_{i+2}' = U_{i+2}$ is a linear subgraph of G_1^*, which contains the loop L_x, and it contributes m to $c_{i+2}(G_1)$.

Case 3.2. $(y, v) \in U_{i+2}$. In this case, $U_{i+1}' = U_{i+2} - L_x - y - v + (x, v)$ is a linear subgraph of G_1^*, which contains the edge (x, v), and it contributes $(-\frac{1}{2}m)$ to $c_{i+1}(G_1)$.

Case 4. $(x, y) \notin U_{i+2}$, $L_x \notin U_{i+2}$, and $L_y \in U_{i+2}$. Under this condition, we consider the following cases of the edge (u, x).

Case 4.1. $(u, x) \notin U_{i+2}$. In this case, $U_{i+1}' = U_{i+2} - L_y$ is a linear subgraph of G_1^*, which does not contain the loop L_x, as well as the edges (u, x) and (y, v), and it contributes $(-\frac{1}{2}m)$ to $c_{i+1}(G_1)$.

Case 4.2. $(u, x) \in U_{i+2}$. In this case, $U_{i+1}' = U_{i+2} - L_y$ is a linear subgraph of G_1^*, which does not contain L_x, but contains the edge (u, x). It contributes $(-\frac{1}{2}m)$ to $c_{i+1}(G_1)$.

Case 5. $(x, y) \notin U_{i+2}$, $L_x \notin U_{i+2}$, and $L_y \notin U_{i+2}$. In this case, we need to consider the following:

Case 5.1. $(u, x) \in U_{i+2}$ and $(y, v) \in U_{i+2}$. In this case, $U_i' = U_{i+2} - u - x - y - v + (u, v)$ is a linear subgraph of G_0^*, which contains the edge (u, v). It contributes m to $c_i(G_0)$.

Case 5.2. $(u, x) \in U_{i+2}$ and $(y, v) \notin U_{i+2}$. In this case, $U_{i+2}' = U_{i+2}$ is a linear subgraph of G_1^*, which contains the edge (u, x) but not L_x. It contributes m to $c_{i+2}(G_1)$.

Case 5.3. $(u, x) \notin U_{i+2}$ and $(y, v) \in U_{i+2}$. In this case, $U_{i+2}' = U_{i+2} - y - v + (x, v)$ is a linear subgraph of G_1^*, which contains the edge (x, v) but not L_x. It contributes m to $c_{i+2}(G_1)$.

Case 5.4. $(u, x) \notin U_{i+2}$ and $(y, v) \notin U_{i+2}$. In this case, $U_{i+2}' = U_{i+2}$ is a linear subgraph of G_1^*, which does not contain the loop L_x and the edges (u, x) and (y, v). It contributes m to $c_{i+2}(G_1)$.

Consider the contributions of the linear subgraphs U_{i+2} in Case 1 and Case 5.1. Each linear subgraph U_{i+2} of G_2^* in these two cases corresponds to a linear subgraph U_i' of G_0^*, and vice versa. Since $e = (u, v)$ is a cut edge of G_0, e is either a component of a linear subgraph of G_0^* or it is not. Thus, we find that

$$(-1)^{i+2} \sum_{U_{i+2} \in \text{ Cases 1 and 5.1}} (-1)^{o(U_{i+2})}2^{c(U_{i+2})}d(U_{i+2}) = (-1)^i \sum_{U_i' \subset G_0^*} (-1)^{o(U_i')}2^{c(U_i')}d(U_i') = c_i(G_0). \tag{4.2}$$

For the linear subgraphs U_{i+2} in Cases 2, 3.2, 4.1, and 4.2, each linear subgraph U_{i+2} of G_2^* corresponds to a linear subgraph U_{i+1}' of G_1^*, and vice versa. Since x is a cut vertex of G_1^*, a linear subgraph U_{i+1}' of G_1^* either contains the loop L_x or it does not. If $L_x \notin U_{i+1}'$, then U_{i+1}' may contain either (u, x) or (x, v), or none of them. Thus, it holds that

$$\begin{aligned} &(-1)^{i+2} \sum_{U_{i+2} \in \text{ Cases 2, 3.2, 4.1, and 4.2}} (-1)^{o(U_{i+2})}2^{c(U_{i+2})}d(U_{i+2}) \\ =&(-1)^i 2(-1)^{i+1} \sum_{U_{i+1}' \subset G_1^*} (-1)^{o(U_{i+1}')}2^{c(U_{i+1}')}d(U_{i+1}') = -2c_{i+1}(G_1). \end{aligned} \tag{4.3}$$

Now, we consider the contributions of the linear subgraphs U_{i+2} in Case 3.1 and Cases 5.2–5.4. It can be found that

$$\begin{aligned}&(-1)^{i+2}\sum_{U_{i+2}\in\text{ Case 3.1 and Cases 5.2–5.4}}(-1)^{o(U_{i+2})}2^{c(U_{i+2})}d(U_{i+2})\\=&(-1)^{i+2}\sum_{U'_{i+2}\subset G_1^*}(-1)^{o(U'_{i+2})}2^{c(U'_{i+2})}d(U'_{i+2})=c_{i+2}(G_1).\end{aligned}\tag{4.4}$$

Combining Eqs (4.2), (4.3) and (4.4), we have

$$c_{i+2}(G_2)=c_i(G_0)+(-)^i 2c_{i+1}(G_1)+c_{i+2}(G_1)\quad\text{for}\quad 0\le i\le n-1.$$

Note 2: In Table 2, we show the Laplacian permanental polynomials of the paths P_4, P_5, and P_6. The path P_5 (respectively, P_6) is obtained from P_4 by inserting one vertex (respectively, two vertices) to some edge of P_4. We can see that Eq (4.1) holds for the coefficients of their permanental polynomials.

Table 2. The Laplacian permanental polynomials of the paths P_4, P_5, and P_6.

Graphs	The Laplacian permanental polynomials
P_4	$\psi(P_4,x)=x^4-6x^3+16x^2-20x+10$
P_5	$\psi(P_5,x)=x^5-8x^4+29x^3-56x^2+57x-24$
P_6	$\psi(P_6,x)=x^6-10x^5+46x^4-120x^3+185x^2-158x+58$

5. Conclusions

Given the difficulty of computing the permanent, for a general graph G, we find a way to compute the coefficients of the Laplacian permanental polynomial of G in terms the linear subgraphs of the loop graph of G. Our result is a generalization of the results given in [8]. On the basis of our coefficient formula, we give an equation for the derivative of the Laplacian permannetal polynomial of G (Theorem 3.1). Following this, we show that the Laplacian permannetal polynomial of a G can be calculated by the integral of the Laplacian permannetal polynomial of the subgraphs of G (Corollary 3.2). Moreover, we establish the relation between the Laplacian permanental polynomial of G and the permanental polynomials of the subgraphs of G (Theorem 3.6). Compared with the known result in Theorem 1.1, our results provide new ways to calculate the Laplacian permanental polynomial.

Moreover, we considered a graph with a cut edge. We derive the relations between the coefficients of the Laplacian permanental polynomials of subdivision graphs with a cut edge (subdividing on the cut edge). For further research, one may derive more results on the Laplacian permanental polynomials of subdivision graphs without a cut edge. Moreover, one may use these general results on the Laplacian permanental polynomial to investigate the extremal problem of the coefficient sum of the Laplacian permanental polynomials of a linear hexagonal system.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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