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Research article

Construction of reversible MDS codes in the Rosenbloom-Tsfasman metric

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Abstract: Maximum distance separable (MDS) codes are a type of error-correcting codes that aim to optimize the shortest possible distance between codewords. These codes are useful in situations where error correction is critical, such as data storage or communication systems, and they can be found in a variety of domains, including information theory, cryptography, and reliable data transmission. The concept of MDS codes is fundamental in designing robust and efficient error-correcting codes that can withstand the challenges posed by noisy communication channels or unreliable storage systems. The Rosenbloom-Tsfasman (RT) metric provides a framework for constructing codes optimized for error correction, and reversible codes leverage this to maximize their error-correction capabilities. This study explored the characteristics of reversible MDS codes in the RT-metric by analyzing the structure of different types of generator matrices. It also established various properties of these codes, such as the conditions under which certain reversible MDS codes were self-dual over $\mathbb{F}_2$ and $\mathbb{F}_q$. In addition, this study proposed several constructions for reversible MDS codes in the RT-metric.

Keywords: reversible code; self-dual code; MDS code; Linear code and RT-metric
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1. Introduction

Reversible codes, introduced by Massey [1], are crucial in data transmission and storage due to their ability to ensure lossless information retrieval. They are effective in correcting solid burst errors and enhancing transmission efficiency [2, 3]. These codes have broad applications in mathematics, cryptography [4, 5], and DNA coding theory [6, 7]. Notably, DNA code construction utilizes reversible codewords derived from linear codes [8]. Reversible codes form an important subclass of BCH (Bose-Chaudhuri-Hocquenghem) codes, highlighting a strong connection between the two. Additionally, they are closely linked to linear complementary dual (LCD) codes, introduced by Massey [9]. Yang and Massey further established that a cyclic code is an LCD code if, and only if, it is reversible [10].

This relationship among BCH codes, reversible codes, and LCD codes underscores their significance and broad applicability across various fields.

In coding theory context, a particular class of non-Hamming metric, called the Rosenbloom-Tsfasman metric (in short, RT-metric), was initially introduced by Rosenbloom and Tsfasman [11]. Subsequently, this metric found application in the theory of uniform distributions, where it was explored by Martin and Stinson [12] and Skriganov [13]. Due to its characterization as a generalization of the classical Hamming metric, the RT-metric quickly captured the interest of coding theorists, leading to extensive research on codes equipped with this metric. Numerous aspects of codes in the RT-metric have been investigated by researchers. These studies have primarily focused on establishing various bounds [14], exploring linearity [15, 16], investigating weight distribution and MacWilliam's identities [17–20], analyzing groups of automorphisms [21], studying maximum distance separability [22], enumerating burst errors [23–25], examining covering properties [26], exploring normality [27], studying construction of self-dual codes [28], and the existence of LCD codes [29] over various algebraic structures. This comprehensive research on the properties and characteristics of codes under the RT-metric has contributed significantly to the understanding and advancement of coding theory. The diverse range of topics studied in this context demonstrates the versatility and importance of the RT-metric in coding theory and related fields. Despite its relevance, to the best of our knowledge reversible MDS codes remain unexplored as far as the RT-metric is concerned.

Owing to the intriguing distinction of reversible MDS codes, there is a compelling need for a thorough investigation of these codes in the RT-metric, particularly focusing on their existence. This paper seeks to address this issue by examining the existence of reversible MDS codes and further exploring their properties if they are found to exist. These codes are effective in handling asymmetric and burst errors in delay-sensitive communication systems under the RT-metric [11]. The algebraic structure of reversible MDS codes enables efficient decoding, enhances error detection and correction, and provides cryptographic advantages. These properties render them suitable for applications in communication, cryptography, and data storage.

The remainder of this paper is organized as follows. Section 2 presents the basic definitions and concepts that are essential to the results discussed in subsequent sections. In Section 3, we introduce some results for certain special matrices, which are useful for a better understanding of reversible MDS codes. In Section 4, we establish the conditions for MDS codes to be reversible in the RT-metric and provide a deeper understanding of their structure. In Section 5, we discuss the properties of self-dual reversible MDS codes using the RT-metric. Section 6 establishes the necessary and sufficient conditions for reversible MDS codes to become LCD codes. In Section 7, we present some construction methods. Finally, Section 8 concludes the paper.

2. Preliminaries

The RT-distance between two vectors $b = (b_1, b_2, \ldots, b_n)$ and $c = (c_1, c_2, \ldots, c_n)$ in space $\mathbb{F}_q^n$ is determined by the maximum index i where the corresponding components of b and c differ, provided that $1 \leq i \leq n$. This is expressed as $d_{RT}(b, c) = \max\{i \mid b_i \neq c_i\}$. Subsets of $\mathbb{F}_q^n$ equipped with this metric are called q-ary RT-metric codes, or simply q-ary codes in the RT-metric. If these subsets also form vector spaces, they are referred to as linear RT-metric codes.

For a k-dimensional linear code $\mathscr{C} \subseteq \mathbb{F}_q^n$, the generator matrix G, which has dimensions $k \times n$, contains rows that form the basis of $\mathscr{C}$. A set of k linearly independent columns from G is referred to as the information set for $\mathscr{C}$.

To establish MacWilliams-type identities for codes in the RT-metric, a specialized inner product in the space $Mat_{m\times n}(\mathbb{F}_q)$ was introduced in [12]. This inner product is crucial for studying codes in the RT-metric, as it leads to significant results, such as the fact that the dual of an MDS code under this inner product is also an MDS.

For vectors $b = (b_1, b_2, \ldots, b_n)$ and $c = (c_1, c_2, \ldots, c_n)$ in $\mathbb{F}_q^n$, their inner product is defined as

$$\langle b, c\rangle = \langle c, b\rangle = \sum_{i=1}^{n} b_i c_{n-i+1},$$

where all arithmetic operations are performed over the finite field $\mathbb{F}_q$. Then, the dual $\mathscr{C}^\perp$ of the code $\mathscr{C}$ can be defined as

$$\mathscr{C}^\perp = \{\alpha \in \mathbb{F}_q^n | \langle \alpha, \beta\rangle = 0 \; for\; all\, \beta \in \mathscr{C}\}.$$

This inner product plays a central role in shaping the properties and duality of codes within the RT-metric framework.

An RT-metric code $\mathscr{C}$ is classified based on specific structural properties. It is called self-orthogonal if it is entirely contained within its dual code $\mathscr{C}^\perp$. A code is considered self-dual when it satisfies the equality $\mathscr{C} = \mathscr{C}^\perp$. In contrast, an LCD code is characterized by having no nonzero codewords in common with its dual. An $[n, k, d_{RT}]_q$ code in the RT-metric that attains the Singleton bound is considered an MDS code, meaning $d_{RT} = n - k + 1$. Additionally, a code $\mathscr{C}$ is termed reversible if, for every vector $(b_1, b_2, \ldots, b_n)$ in $\mathscr{C}$, its reversed form $(b_n, b_{n-1}, \ldots, b_2, b_1)$ also belongs to $\mathscr{C}$. A code $\mathscr{C}$ is called a reversible MDS code in the RT-metric if it is both reversible and MDS.

Let $\mathcal{R} = (r_{ij})_{\mathfrak{m}\times\mathfrak{n}}$ be a matrix of dimensions $\mathfrak{m} \times \mathfrak{n}$. Throughout this study, we adopt the following notations (see Table 1):

Table 1. Symbols and their meanings.

Symbol	Meaning
J_n	This matrix has ones along its anti-diagonal and zeros elsewhere.
$Flip(\mathcal{R})$	It is obtained by reversing the order of the columns in $\mathcal{R}$ and is defined as $Flip(\mathcal{R}) = \mathcal{R}J_n = (r_{i,n-j+1})_{m\times n}$.
$\mathcal{R}^T$	The transpose of a matrix $\mathcal{R}$, obtained by interchanging its rows and columns, defined as $\mathcal{R}^T = (r_{ji})_{n\times m}$.
$\mathcal{R}_k$	Represents the submatrix consisting of the first k columns of the generator matrix in standard form, i.e., $G = [A\|I_k]$.
$\mathcal{R}^F$	The flip-transpose of a matrix $\mathcal{R}$, obtained by reflecting $\mathcal{R}$ across its anti-diagonal, defined as $\mathcal{R}^F = (r_{n-j+1,m-i+1})_{n\times m}$.
$J_m\mathcal{R}$	The row-reversed form of $\mathcal{R}$, obtained by reversing the order of its rows, given by $\text{Rev}(\mathcal{R}) = (r_{m-i+1,j})_{m\times n}$.
$\mathcal{R}^S$	The flip-diagonal transpose of $\mathcal{R}$, obtained by reversing both rows and columns, defined as $\mathcal{R}^S = J_m\mathcal{R}J_n = (r_{m-i+1,n-j+1})_{m\times n}$.
$SMDS$	Self-dual MDS.
$SRMDS$	Self-dual Reversible MDS.
$RMDS$	Reversible MDS.

A square matrix $\mathcal{R}$ of order n is classified based on its properties as follows:

- $\mathcal{R}$ is orthogonal if it satisfies $\mathcal{R}\mathcal{R}^T = I_n = \mathcal{R}^T\mathcal{R}$, meaning its transpose is also its inverse.
- $\mathcal{R}$ is symmetric if it is equal to its transpose, that is, $\mathcal{R} = \mathcal{R}^T$.
- $\mathcal{R}$ is persymmetric if it remains unchanged when reflected across its anti-diagonal, that is, $\mathcal{R} = \mathcal{R}^F$.
- $\mathcal{R}$ is centrosymmetric if it remains unchanged under a flip-diagonal transpose, that is, $\mathcal{R} = \mathcal{R}^S$.

A matrix is referred to as a block-circulant [30] if each row is obtained by shifting the previous row from one position to the right. A 4×4 block-circulant matrix can be expressed as

$$BCirc(\kappa, \lambda, \mu, \nu) = \begin{bmatrix} \kappa & \lambda & \mu & \nu \\ \nu & \kappa & \lambda & \mu \\ \mu & \nu & \kappa & \lambda \\ \lambda & \mu & \nu & \kappa \end{bmatrix},$$

where $\kappa, \lambda, \mu, \nu$ are square matrices of size $n \times n$.

A block-circulant matrix is

$$BCirc(\alpha, \chi) = \begin{bmatrix} \alpha & \chi \\ \chi & \alpha \end{bmatrix},$$

where $\alpha = BCirc(\kappa, \lambda), \chi = BCirc(\mu, \nu)$ are two block-circulant matrices and $\kappa, \lambda, \mu, \nu$ are any $n \times n$ square matrices, which is called a 2×2 doubly block-circulant matrix, that is,

$$DBCirc(\kappa, \lambda, \mu, \nu) = \begin{bmatrix} \kappa & \lambda & \mu & \nu \\ \lambda & \kappa & \nu & \mu \\ \mu & \nu & \kappa & \lambda \\ \nu & \mu & \lambda & \kappa \end{bmatrix}.$$

3. Some results on certain special matrices

Lemma 3.1. *Let $\mathcal{R}$ be a square matrix of order n. Then, $(Flip(\mathcal{R}))^2 = I_n$ if, and only if, $\mathcal{R}^{-1} = \mathcal{R}^S$.*

Proof. The proof of Lemma 3.1 is straightforward, based on notations and basic algebraic manipulation. □

Lemma 3.2. *Let $\mathcal{R}$ be a symmetric matrix of order n. Then, $(Flip(\mathcal{R}))^2 = I_n$ if, and only if, $\mathcal{R}^{-1} = \mathcal{R}^S = \mathcal{R}^F$.*

Proof. The proof of Lemma 3.2 is straight based on notations and basic algebraic manipulation. □

Lemma 3.3. *Let $\mathcal{R}$ be an invertible matrix of order n. Then, $(Flip(\mathcal{R}))^2 = \mathcal{R}^2$ if, and only if, $\mathcal{R}$ is centrosymmetric.*

Proof.

$$\begin{aligned} \text{Consider } (Flip(\mathcal{R}))^2 = \mathcal{R}^2 &\Leftrightarrow (\mathcal{R}J_n)^2 = \mathcal{R}^2 \\ &\Leftrightarrow (\mathcal{R}J_n)(\mathcal{R}J_n) = \mathcal{R}\mathcal{R} \end{aligned}$$

$$\Leftrightarrow \mathcal{R}(J_n\mathcal{R}J_n) = \mathcal{R}\mathcal{R}$$
$$\Leftrightarrow \mathcal{R}^S = \mathcal{R}$$
$$\Leftrightarrow \mathcal{R}\ is\ centrosymmetric.$$

□

Lemma 3.4. *Let $\mathcal{R}$ be a square matrix of order n. Then, $\mathcal{R}\mathcal{R}^S = \mathcal{R}^S\mathcal{R}$ if, and only if, $\mathcal{R}\mathcal{R}^S$ is centrosymmetric.*

Proof. The proof of Lemma 3.4 follows from the definition of the centrosymmetric matrix. □

Lemma 3.5. *Let $\mathcal{R}$ be an $n \times n$ square matrix. If any two of the following conditions are true, then the third condition is also true:*
(i) $\mathcal{R}$ is an involutory matrix;
(ii) $(Flip(\mathcal{R}))^2 = I_n$;
(iii) $\mathcal{R}$ is centrosymmetric.

Proof. We first demonstrate that (i) and (ii) imply that (iii). Let $\mathcal{R}$ be an involutory matrix and $(Flip(\mathcal{R}))^2 = I_n$. Then,

$$\begin{aligned}(Flip(\mathcal{R}))^2 = I_n &\Leftrightarrow (\mathcal{R}J_n)^2 = I_n\\ &\Leftrightarrow (\mathcal{R}J_n)^{-1} = (\mathcal{R}J_n)\\ &\Leftrightarrow J_n{}^{-1}\mathcal{R} = \mathcal{R}J_n\ \ (\because \mathcal{R}\ is\ involutory)\\ &\Leftrightarrow \mathcal{R} = \mathcal{R}^S\\ &\Leftrightarrow\ \mathcal{R}\ is\ centrosymmetric.\end{aligned}$$

Now, we show that conditions (ii) and (iii) together imply (i). That is, suppose $(Flip(\mathcal{R}))^2 = I_n$ and $\mathcal{R}$ is centrosymmetric. Then, from (ii) and (iii), it follows that $\mathcal{R} = \mathcal{R}^S$ and $(Flip(\mathcal{R}))^2 = I_n$.

$$\begin{aligned}(Flip(\mathcal{R}))^2 = I_n &\Leftrightarrow (\mathcal{R}J_n)^2 = I_n\\ &\Leftrightarrow (\mathcal{R}J_n)(\mathcal{R}J_n) = I_n\\ &\Leftrightarrow \mathcal{R}(J_n\mathcal{R}J_n) = I_n\\ &\Leftrightarrow \mathcal{R}\mathcal{R}^S = I_n\\ &\Leftrightarrow \mathcal{R}^2 = I_n\ \ (\because \mathcal{R}\ is\ centrosymmetric)\\ &\Leftrightarrow \mathcal{R}^{-1} = \mathcal{R}\\ &\Leftrightarrow \mathcal{R}\ is\ involutory.\end{aligned}$$

This proves that $\mathcal{R}$ is involutory. □

4. RMDS codes in the RT-metric

Theorem 4.1. *Let $\mathscr{C}$ be an RMDS code of dimension k and length n, where $k < n/2$ with respect to the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid Y \mid I_k]$, then $\mathcal{R}_k$ is invertible.*

Proof. On the contrary, suppose that $\mathcal{R}_k$ is not invertible. Then, there exist at least two codewords in $\mathscr{C}$ whose first k coordinates are identical;

$$x = (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{n-k}, x_{n-k+1}, \dots, x_n) \in \mathscr{C},$$

and

$$z = (x_1, x_2, \dots, x_k, z_{k+1}, \dots, z_{n-k}, z_{n-k+1}, \dots, z_n) \in \mathscr{C}.$$

As $\mathscr{C}$ is reversible, $Flip(x), Flip(z) \in \mathscr{C}$. If we assume that $\mathcal{R}_k$ is singular, then the difference $Flip(x) - Flip(z) \in \mathscr{C}$, and by the assumption of reversibility, its reverse will have a maximum RT-weight of $n-k$. As $\mathscr{C}$ is an MDS code, this leads to a contradiction. Therefore, $\mathcal{R}_k$ is invertible. □

Remark 4.1. *Consider an irreversible MDS code $\mathscr{C}$ of dimension k and length n, where $k < n/2$ in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid Y \mid I_k]$, then $\mathcal{R}_k$ does not need to be invertible.*

Remark 4.2. *Consider an MDS code $\mathscr{C}$ of dimension k and length n, where $k < n/2$ in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid Y \mid I_k]$ where $\mathcal{R}_k$ is invertible, then $\mathscr{C}$ does not need to be a reversible code in the RT-metric.*

Example 4.1. Let $\mathscr{C}$ be a $[8, 3, 6]$ linear code in the RT-metric over $GF(2)$, whose generator matrix is

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible but $\mathscr{C}$ is not reversible. This is clearly the MDS code. Hence, a linear $[8, 3, 6]$ MDS code with an invertible $\mathcal{R}_k$ does not need to be reversible over $GF(2)$. This is an example to support Remark 4.2.

Example 4.2. Let $\mathscr{C}$ be a $[7, 2, 6]$ linear code in the RT-metric over $GF(3)$, whose generator matrix is given by

$$G = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 2 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible but $\mathscr{C}$ is not reversible. This is clearly the MDS code. Hence, a linear $[7, 2, 6]$ MDS code with invertible $\mathcal{R}_k$ does not need to be reversible over $GF(3)$. This is an example to support Remark 4.2.

Theorem 4.2. *Let $\mathscr{C}$ be an RMDS code of dimension k and length n in the RT-metric, where $k < n/2$. Then, $G = [\mathcal{R}_k|Y|I_k]$ is a generator matrix of $\mathscr{C}$ if, and only if, $G' = [I_k|Y^S|\mathcal{R}_k^S]$ is also a generator matrix of $\mathscr{C}$.*

Proof. Since $\mathscr{C}$ is reversible, the flip of generator matrix G of $\mathscr{C}$ is also a $Flip(G) = [J_k|Flip(Y)|Flip(\mathcal{R}_k)]$, which also generates $\mathscr{C}$; since J_k is non-singular, $J_k.Flip(G)$ is also a generator matrix of $\mathscr{C}$. That is, $G' = J_k.Flip(G) = [J_k^2|J_kFlip(Y)|J_kFlip(\mathcal{R}_k)] = [I_k|Y^S|\mathcal{R}_k^S]$ is a generator matrix of $\mathscr{C}$. □

Theorem 4.3. *Let $\mathscr{C}$ be an MDS code of dimension k and length n in the RT-metric, where $k < n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid Y \mid I_k]$, then $\mathscr{C}$ is reversible if, and only if, it satisfies one of the following:*

(i) $Y = \mathcal{R}_k Y^S$ *and* $(Flip(\mathcal{R}_k))^2 = I_k$*;*

(ii) $Y = \mathcal{R}_k Y^S$ *and* $\mathcal{R}_k^{-1} = \mathcal{R}_k^S$*,*
or $\mathcal{R}_k \mathcal{R}_k^S = \mathcal{R}_k^S \mathcal{R}_k = I_k$*.*

Proof. (i) Suppose there is the RMDS code $\mathscr{C}$ of dimension k and length n in the RT-metric, where $k < n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid Y \mid I_k]$, then $Flip(G) = [J_k | Flip(Y) | Flip(\mathcal{R}_k)]$ also generates $\mathscr{C}$ as well. According to Theorem 4.1, $\mathcal{R}_k$ is non-singular, as is $Flip(\mathcal{R}_k)$. Thus, $(Flip(\mathcal{R}_k))G$ is a generator matrix of $\mathscr{C}$.

$$\begin{aligned}(Flip(\mathcal{R}_k))G &= [(Flip(\mathcal{R}_k))J_k | Flip(\mathcal{R}_k) Flip(Y) | (Flip(\mathcal{R}_k))(Flip(\mathcal{R}_k))] \\ &= [\mathcal{R}_k | \mathcal{R}_k Y^S | (Flip(\mathcal{R}_k))^2].\end{aligned}$$

We note that the row vectors of $(Flip(\mathcal{R}_k))G$ and those of $[\mathcal{R}_k | Y | I_k]$ generate the same code $\mathscr{C}$; since both the matrices agree in the first k columns, this implies $Y = \mathcal{R}_k Y^S$ and $(Flip(\mathcal{R}_k))^2 = I_k$.

Conversely, let $Y = \mathcal{R}_k Y^S$, $(Flip(\mathcal{R}_k))^2 = I_k$, and $\mathscr{C}$ be an MDS code. Suppose, for the sake of contradiction, that $\mathscr{C}$ is not reversible. This implies that $Flip(G)$ is not a generator matrix of $\mathscr{C}$. Since $Flip(\mathcal{R}_k)$ is non-singular, the product $Flip(\mathcal{R}_k)\, Flip(G)$ would also fail to be a generator matrix of $\mathscr{C}$. However,

$$Flip(\mathcal{R}_k)\, Flip(G) = [\, Flip(\mathcal{R}_k)\, J_k \mid Flip(\mathcal{R}_k)\, Flip(Y) \mid (Flip(\mathcal{R}_k))^2 \,] = [\, \mathcal{R}_k \mid Y \mid I_k \,],$$

which is indeed a generator matrix of $\mathscr{C}$. This contradicts our assumptions. Therefore, $\mathscr{C}$ must be reversible.

(ii)

$$\begin{aligned}\text{Consider } (Flip(\mathcal{R}_k))^2 = I_k &\Leftrightarrow (\mathcal{R}_k J_k)(\mathcal{R}_k J_k) = I_k \\ &\Leftrightarrow \mathcal{R}_k (J_k \mathcal{R}_k J_k) = \mathcal{R}_k \mathcal{R}_k^S = I_k \\ &\Leftrightarrow J_k (\mathcal{R}_k J_k)(\mathcal{R}_k J_k) J_k = I_k \\ &\Leftrightarrow (J_k \mathcal{R}_k J_k) \mathcal{R}_k = \mathcal{R}_k^S \mathcal{R}_k = I_k \\ &\Leftrightarrow \mathcal{R}_k^{-1} = \mathcal{R}_k^S.\end{aligned}$$

□

Theorem 4.4. *Let $\mathscr{C}$ be an MDS code of dimension k and length n in the RT-metric, where $k = n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k \mid I_k]$, then $\mathscr{C}$ is reversible if, and only if, it satisfies one of the following:*

(i) $(Flip(\mathcal{R}_k))^2 = I_k$*;*

(ii) $\mathcal{R}_k^{-1} = \mathcal{R}_k^S$ *or* $\mathcal{R}_k \mathcal{R}_k^S = \mathcal{R}_k^S \mathcal{R}_k = I_k$*.*

Proof. This proof follows by the same notations in Theorem 4.3. □

Theorem 4.5. *Let $\mathscr{C}$ be an RMDS code of dimension k and length n in the RT-metric, where $n > k > n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[Y_{(k,n-k)} | I_k]$ and $\mathcal{R}_k = [Y_{(k,n-k)} | I_{(k,1:2k-n)}]$, then $\mathcal{R}_k$ is invertible.*

Proof. This proof follows by the same notations in Theorem 4.1.

□

Lemma 4.1. *Let* $\mathcal{R}_k = \left[Y_{(k,n-k)} | \frac{I_{2k-n}}{O_{(n-k,2k-n)}}\right]$ *be a* $k \times k$ *square matrix and* $\mathcal{P}_{(k,n-k)} = \left[\frac{O_{(2k-n,n-k)}}{J_{n-k}}\right]_{(k,n-k)}$ *be a rectangular matrix, where* n, k $(n > k > n/2) \in \mathbb{N}$. *Then,* $Flip(\mathcal{R}_k) * \mathcal{P}_{(k,n-k)} = Y_{(k,n-k)}$.

Proof. Suppose $Y_{(k,n-k)} = \left[\frac{Y'_{(2k-n,n-k)}}{Y''_{(n-k,n-k)}}\right]_{(k,n-k)}$ and $\mathcal{P}_{(k,n-k)} = \left[\frac{O_{(2k-n,n-k)}}{J_{n-k}}\right]_{(k,n-k)}$. Then,

$$\begin{aligned}
Flip(\mathcal{R}_k) * \mathcal{P}_{(k,n-k)} &= \left[\begin{array}{c|c} J_{2k-n} & Flip(Y')_{(2k-n,n-k)} \\ \hline O_{(n-k,2k-n)} & Flip(Y'')_{(n-k,n-k)} \end{array}\right] . \left[\begin{array}{c} O_{(2k-n,n-k)} \\ \hline J_{n-k} \end{array}\right] \\
&\quad \text{where } Y' = Y(1 : 2k-n, :) \quad \text{and} \quad Y'' = Y(2k-n+1 : k, :) \\
&= \left[\begin{array}{c} O_{(2k-n,n-k)} + Y'_{(2k-n,n-k)} \\ \hline O_{(n-k,n-k)} + Y''_{(n-k,n-k)} \end{array}\right] \\
&= Y_{(k,n-k)}.
\end{aligned}$$

□

Theorem 4.6. *Let* $\mathscr{C}$ *be an RMDS code of dimension* k *and length* n *in the RT-metric, where* $n > k > n/2$. *If the generator matrix of* $\mathscr{C}$ *is represented as* $[Y_{(k,n-k)}|I_k]$, *then* $[I_k|Y^S_{(k,n-k)}]$ *is also a generator matrix of* $\mathscr{C}$.

Proof. This proof follows by the same notations in Theorem 4.2.

□

Theorem 4.7. *Let* $\mathscr{C}$ *be an RMDS code of dimension* k *and length* n *in the RT-metric, where* $n > k > n/2$. *If the generator matrix of* $\mathscr{C}$ *is represented as* $[Y_{(k,n-k)}|I_k]$ *and* $\mathcal{R}_k = \left[Y_{(k,n-k)} | \frac{I_{(2k-n,2k-n)}}{O_{(n-k,2k-n)}}\right]$, *then* $\mathscr{C}$ *is reversible if, and only if, the matrix* $\mathcal{R}_k$ *satisfies the following:*
(i) $(Flip(\mathcal{R}_k))^2 = I_k$;
(ii) ${\mathcal{R}_k}^{-1} = {\mathcal{R}_k}^S$ *or* $\mathcal{R}_k{\mathcal{R}_k}^S = {\mathcal{R}_k}^S\mathcal{R}_k = I_k$.

Proof. Suppose that $\mathscr{C}$ is an RMDS code with the form $[Y_{(k,n-k)}|I_k]$ and $\mathcal{R}_k = \left[Y_{(k,n-k)} | \frac{I_{(2k-n,2k-n)}}{O_{(n-k,2k-n)}}\right]$. Then, the flipped generator matrix G of $\mathscr{C}$ with $Flip(G) = [J_k|Flip(Y_{(k,n-k)})]$ also generates $\mathscr{C}$ as well. By Theorem 4.5, $\mathcal{R}_k$ is non-singular and so is $Flip(\mathcal{R}_k)$. Thus, $Flip(\mathcal{R}_k).Flip(G)$ is a generator matrix of $\mathscr{C}$.

$$\begin{aligned}
Flip(\mathcal{R}_k).\,Flip(G) &= Flip(\mathcal{R}_k).\left[\begin{array}{c|cc} O_{(2k-n,n-k)} & J_{2k-n} & Flip(Y') \\ J_{n-k} & O_{(n-k,2k-n)} & Flip(Y'') \end{array}\right] \\
&\quad \text{where } Y' = Y(1 : 2k-n, :) \quad \text{and} \quad Y'' = Y(2k-n+1 : k, :) \\
&= Flip(\mathcal{R}_k).\left[\mathcal{P}_{(k,n-k)} \,\middle|\, Flip(\mathcal{R}_k)\right] \\
&= \left[Flip(\mathcal{R}_k).\mathcal{P}_{(k,n-k)} \,\middle|\, (Flip(\mathcal{R}_k))^2\right] \\
&= \left[Y_{(k,n-k)} \,\middle|\, (Flip(\mathcal{R}_k))^2\right] \qquad (\because \text{ from Lemma 4.1}).
\end{aligned}$$

Note that the row vectors of $Flip(\mathcal{R}_k).Flip(G)$ and those of $[Y_{(k,n-k)}|I_k]$ generate $\mathscr{C}$. This implies $(Flip(\mathcal{R}_k))^2 = I_k$.
(ii) Clearly, $(Flip(\mathcal{R}_k))^2 = I_k$ if, and only if, ${\mathcal{R}_k}^{-1} = {\mathcal{R}_k}^S$. □

Example 4.3. *Let $\mathscr{C}$ be a $[6,4,3]$ linear code in the RT-metric over $GF(2)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible and $(Flip(\mathcal{R}_k))^2 = I_k$; $\mathscr{C}$ is reversible. This is clearly an RMDS code. Hence, a linear $[6,4,3]$ MDS code with invertible $\mathcal{R}_k$ is reversible over $GF(2)$. This is an example to support Theorem 4.7.

Example 4.4. *Let $\mathscr{C}$ be a $[4,3,2]$ linear code in the RT-metric over $GF(3)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible and $(Flip(\mathcal{R}_k))^2 = I_k$; $\mathscr{C}$ is reversible. This is clearly an RMDS code. Hence, a linear $[4,3,2]$ MDS code with invertible $\mathcal{R}_k$ is reversible over $GF(3)$. This is an example to support Theorem 4.7.

Example 4.5. *Let $\mathscr{C}$ be a $[3,2,2]$ linear code in the RT-metric over $GF(5)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible and $(Flip(\mathcal{R}_k))^2 = I_k$; $\mathscr{C}$ is reversible. This is clearly an RMDS code. Hence, a linear $[3,2,2]$ MDS code with invertible $\mathcal{R}_k$ is reversible over $GF(5)$. This is an example to support Theorem 4.7.

Remark 4.3. *Consider an MDS code $\mathscr{C}$ of dimension k and length n in the RT-metric, where $k > n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[Y_{(k,n-k)}|I_k]$, then $\mathcal{R}_k$ need not be invertible.*

Remark 4.4. *Consider an MDS code $\mathscr{C}$ of dimension k and length n in the RT-metric, where $k > n/2$. If the generator matrix of $\mathscr{C}$ is represented as $[Y_{(k,n-k)}|I_k]$ and $\mathcal{R}_k$ is an invertible, then $\mathscr{C}$ does not need to be a reversible code in the RT-metric, as demonstrated in the following examples.*

Example 4.6. *Let $\mathscr{C}$ be a $[6,4,3]$ linear code in the RT-metric over $GF(2)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible, and $\mathscr{C}$ is irreversible. This is clearly the MDS code. Hence, a linear $[6,4,3]$ MDS code with an invertible $\mathcal{R}_k$ does not need to be reversible over $GF(2)$. This is an example to support Remark 4.4.

Example 4.7. *Let $\mathscr{C}$ be a $[4,3,2]$ linear code in the RT-metric over $GF(3)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible, whereas $\mathscr{C}$ is not reversible. This is clearly the MDS code. Hence, a linear $[4,3,2]$ MDS code with invertible $\mathcal{R}_k$ does not need to be reversible over $GF(3)$. This is an example to support Remark 4.4.

Example 4.8. *Let $\mathscr{C}$ be a $[3,2,2]$ linear code in the RT-metric over $GF(5)$, whose generator matrix is given by*

$$G = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Here, $\mathcal{R}_k$ is invertible, whereas $\mathscr{C}$ is not reversible. This is clearly the MDS code. Hence, a linear $[3,2,2]$ MDS code with invertible $\mathcal{R}_k$ does not need to be reversible over $GF(5)$. This is an example to support Remark 4.4.

5. SRMDS codes in the RT-metric

In this section, we explore the properties of self-dual reversible MDS (SRMDS) codes in the RT-metric. These codes form a distinctive class of linear error-correcting codes that integrate three significant features: self-duality, reversibility, and MDS. The combination of these properties ensures robust error detection and correction capabilities while maintaining optimal code parameters. Moreover, SRMDS codes are of considerable importance in various domains, including cryptography, data storage systems, and DNA computing, where reliability, efficiency, and structural symmetry are crucial.

5.1. Binary SRMDS codes in the RT-metric

Theorem 5.1. *Let $\mathscr{C}$ be an $[n = 2k, k, d_{RT}]$ binary MDS code of dimension k and length n in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$, then $\mathscr{C}$ is self-dual code if, and only if, matrix $\mathcal{R}_k$ satisfies the following:*

(i) $Flip(\mathcal{R}_k)$ is symmetric;

(ii) $\mathcal{R}_k^S = \mathcal{R}_k^T$;

(iii) $\mathcal{R}_k$ is persymmetric.

Proof. In [28], $GG^\diamond = 0 \Leftrightarrow \mathscr{C}$ is self-dual where $G^\diamond = (Flip(G))^T$.

$$\begin{aligned}(i)\text{ Suppose } \mathscr{C} \text{ is self-dual} &\Leftrightarrow GG^\diamond = 0 \\ &\Leftrightarrow [\mathcal{R}_k|I_k][J_k|Flip(\mathcal{R}_k)]^T = 0 \\ &\Leftrightarrow Flip(\mathcal{R}_k) + (Flip(\mathcal{R}_k))^T = 0 \\ &\Leftrightarrow Flip(\mathcal{R}_k) = (Flip(\mathcal{R}_k))^T.\end{aligned}$$

$$(ii)\text{ Consider } \; Flip(\mathcal{R}_k) = (Flip(\mathcal{R}_k))^T$$

$$\Leftrightarrow \mathcal{R}_k J_k = J_k \mathcal{R}_k^T$$
$$\Leftrightarrow J_k \mathcal{R}_k J_k = J_k J_k \mathcal{R}_k^T$$
$$\Leftrightarrow \mathcal{R}_k^S = \mathcal{R}_k^T.$$

$$(iii) \text{ Consider } \mathcal{R}_k J_k = J_k \mathcal{R}_k^T$$
$$\Leftrightarrow \mathcal{R}_k J_k J_k = J_k \mathcal{R}_k^T R_k$$
$$\Leftrightarrow \mathcal{R}_k = \mathcal{R}_k^F.$$

□

Theorem 5.2. *Let $\mathscr{C}$ be a $[n = 2k, k, d_{RT}]$ binary RMDS code of dimension k and length n in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$, then $\mathscr{C}$ is a self-dual code if, and only if, matrices $\mathcal{R}_k$ and $Flip(\mathcal{R}_k)$ are orthogonal.*

Proof. Assume that $\mathscr{C}$ is self-dual in the RT-metric. According to Theorem 5.1, $\mathscr{C}$ is the SMDS code in the RT-metric $\Leftrightarrow$ $Flip(\mathcal{R}_k)$ is symmetric $\Leftrightarrow$ $\mathcal{R}_k$ is persymmetric. By Theorem 4.3, $\mathscr{C}$ is the RMDS code in the RT-metric $\Leftrightarrow$ $(Flip(\mathcal{R}_k))^2 = I_n$. From Theorems 4.3 and 5.1, $Flip(\mathcal{R}_k)$ is orthogonal. By [31, Lemma 3.2] , $\mathcal{R}_k$ *is persymmetric* and $(Flip(\mathcal{R}_k))^2 = I_k$, then $\mathcal{R}_k$ is orthogonal. □

Remark 5.1. *Consider a $[n = 2k, k, d_{RT}]$ binary SMDS code $\mathscr{C}$ of dimension k and length n in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$, then $\mathcal{R}_k$ and $Flip(\mathcal{R}_k)$ need not be orthogonal.*

Example 5.1. *Consider a $[4, 2, 3]$ linear code $\mathscr{C}$ over $GF(2)$ in the RT-metric, whose generator matrix is given by*

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, this is a binary SMDS code in the RT-metric, in which $\mathcal{R}_k$ and $Flip(\mathcal{R}_k)$ are not orthogonal. It should also be noted that this code is irreversible. Hence, an MDS linear code $[4, 2, 1]$ is a self-dual (not reversible) code over $GF(2)$, where $\mathcal{R}_k$ and $Flip(\mathcal{R}_k)$ need not be orthogonal in the RT-metric. This is an example to support Remark 5.1.

5.2. Nonbinary SRMDS codes in the RT-metric

Theorem 5.3. *Let $\mathscr{C}$ be an $[n = 2k, k, d_{RT}]$ nonbinary MDS code $\mathscr{C}$ of dimension k and length n in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$, then $\mathscr{C}$ is a self-dual code if, and only if, the matrix $\mathcal{R}_k$ satisfies the following:*

- *(i)* $Flip(\mathcal{R}_k)$ *is skew-symmetric;*
- *(ii)* $\mathcal{R}_k^S = -\mathcal{R}_k^T$;
- *(iii)* $\mathcal{R}_k^F = -\mathcal{R}_k$.

Proof. In [28], $GG^\diamond = 0 \Leftrightarrow$ $\mathscr{C}$ is self-dual, where $G^\diamond = (Flip(G))^T$.

$$(i) \text{ Suppose } \mathscr{C} \text{ is self-dual } \Leftrightarrow GG^\diamond = 0$$
$$\Leftrightarrow [\mathcal{R}_k|I_k][J_k|Flip(\mathcal{R}_k)]^T = 0$$

$$\Leftrightarrow Flip(\mathcal{R}_k) + (Flip(\mathcal{R}_k))^T = 0$$
$$\Leftrightarrow Flip(\mathcal{R}_k) = -(Flip(\mathcal{R}_k))^T.$$

$$(ii) \text{ Consider } Flip(\mathcal{R}_k) = -(Flip(\mathcal{R}_k))^T$$
$$\Leftrightarrow \mathcal{R}_k J_k = -J_k \mathcal{R}_k^T$$
$$\Leftrightarrow J_k \mathcal{R}_k J_k = -J_k J_k \mathcal{R}_k^T$$
$$\Leftrightarrow \mathcal{R}_k^S = -\mathcal{R}_k^T.$$

$$(iii) \text{ Consider } \mathcal{R}_k J_k = -J_k \mathcal{R}_k^T$$
$$\Leftrightarrow \mathcal{R}_k J_k J_k = -J_k \mathcal{R}_k^T J_k$$
$$\Leftrightarrow \mathcal{R}_k = -\mathcal{R}_k^F.$$

□

Theorem 5.4. *Let $\mathscr{C}$ be an $[n = 2k, k, d_{RT}]$ nonbinary MDS code of odd dimension k and length n in the RT-metric. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$ and $\mathcal{R}_k$ is non-singular, then $\mathscr{C}$ cannot be self-dual.*

Proof. Assume that $\mathscr{C}$ is self-dual in the RT-metric. According to Theorem 5.3, $Flip(\mathcal{R}_k)$ is skew-symmetric. For $k \in \mathbb{N}$, if k is odd, then all $k \times k$ skew-symmetric matrices are singular. This contradicts the fact that A_k is non-singular. Hence, an MDS code with odd dimensions over $\mathbb{F}_q (q > 2)$ cannot be self-dual in the RT-metric. □

Theorem 5.5. *Let $\mathscr{C}$ be an $[n = 2k, k, d_{RT}]$ nonbinary RMDS code of dimension k and length n. If the generator matrix of $\mathscr{C}$ is represented as $[\mathcal{R}_k|I_k]$, then $\mathscr{C}$ is a self-dual code if, and only if, it satisfies the following:*
(i) If k is odd, then the reversible code $\mathscr{C}$ cannot be self-dual.
(ii) If k is even, then $Flip(\mathcal{R}_k)$ is a skew-symmetric involutory matrix and $det(Flip(\mathcal{R}_k)) = (-1)^{k/2}$.

Proof. (i) By Theorems 4.1 and 5.4, $\mathcal{R}_k$ is non-singular, and code $\mathscr{C}$ cannot be self-dual. Therefore, an RMDS code with odd dimensions over $\mathbb{F}_q (q > 2)$ cannot be self-dual.

(ii) By Theorem 4.3 and 5.3, "A code $\mathscr{C}$ is SRMDS with even dimension if, and only if, $Flip(\mathcal{R}_k)$ is skew-symmetric involutory matrix". The determinant of an involutory matrix over $\mathbb{F}_q$ is ± 1, and the determinant of a non-singular skew-symmetric matrix over $\mathbb{F}_q$ with an even order is 1 or $q - 1$. Hence, the determinant of the skew-symmetric involutory matrix is $(-1)^{k/2}$.

□

Example 5.2. *Consider a $[6, 3, 4]$ linear RMDS code $\mathscr{C}$ over $GF(3)$ in the RT-metric, whose generator matrix is given by*

$$G = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, $(Flip(\mathcal{R}_k))^2 = I_k$; $Flip(\mathcal{R}_k)$ is not skew-symmetric because a skew-symmetric matrix with odd order is always singular. Hence, a $[6, 3, 4]$ linear RMDS code over $GF(3)$ in the RT-metric with odd dimensions cannot be self-dual. This is an example to support Theorem 5.4.

Example 5.3. *Consider a* $[12, 6, 7]$ *linear RMDS code* $\mathscr{C}$ *over* $GF(5)$ *in the RT-metric, whose generator matrix is given by*

$$G = \begin{bmatrix} 3\,0\,0\,0\,0\,0 & 1\,0\,0\,0\,0\,0 \\ 0\,2\,0\,0\,0\,0 & 0\,1\,0\,0\,0\,0 \\ 0\,0\,2\,0\,0\,0 & 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,3\,0\,0 & 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,3\,0 & 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,2 & 0\,0\,0\,0\,0\,1 \end{bmatrix}.$$

Here, $(Flip(\mathcal{R}_k))^2 = I_k$ *and* $Flip(\mathcal{R}_k)$ *is skew-symmetric. Hence, a* $[12, 6, 7]$ *linear RMDS code with an even dimension over* $GF(5)$ *in the RT-metric is self-dual. This is an example to support Theorem 5.5.*

Example 5.4. *Consider a* $[12, 6, 7]$ *linear RMDS code* $\mathscr{C}$ *over* $GF(17)$ *in the RT-metric, whose generator matrix is given by*

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 13 & 0 & 1\,0\,0\,0\,0\,0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0\,1\,0\,0\,0\,0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0\,0\,1\,0\,0\,0 \\ 0 & 0 & 0 & 13 & 0 & 0 & 0\,0\,0\,1\,0\,0 \\ 13 & 0 & 0 & 0 & 0 & 0 & 0\,0\,0\,0\,1\,0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0\,0\,0\,0\,0\,1 \end{bmatrix}.$$

Here, $(Flip(\mathcal{R}_k))^2 = I_k$ *and* $Flip(\mathcal{R}_k)$ *is skew-symmetric. Hence, a* $[12, 6, 7]$ *linear RMDS code with an even dimension over* $GF(17)$ *in the RT-metric is self-dual. This is an example to support Theorem 5.5.*

6. LCD MDS codes in the RT-metric

In this section, we discuss the properties of LCD MDS codes in terms of the RT -metric. These codes offer robust error- correction capabilities, ensuring reliable communication, even in the presence of noise or interference. The RT-metric allows for a thorough assessment of these codes, considering both the error correction and compression rates.

Theorem 6.1. *Let* $\mathscr{C}$ *be an MDS code of dimension* k *and length* n *in the RT-metric, where* $k < n/2$. *If the generator matrix of* $\mathscr{C}$ *is represented as* $[\mathcal{R}_k|Y|I_k]$, *then* $\mathscr{C}$ *is an LCD MDS code if, and only if,* $Flip(\mathcal{R}_k) + YY^{\diamond} + (Flip(\mathcal{R}_k))^T$ *is invertible.*

Proof. From [29, Theorem 3.4], "$\mathscr{C}$ is LCD if, and only if, $GG^{\diamond}$ *is non-singular*", and $GG^{\diamond} = Flip(\mathcal{R}_k) + YY^{\diamond} + (Flip(\mathcal{R}_k))^T$ is non-singular. □

Theorem 6.2. *Let* $\mathscr{C}$ *be an MDS code of dimension* k *and length* n *in the RT-metric, where* $n > k > n/2$. *If the generator matrix of* $\mathscr{C}$ *is represented as* $[Y_{(k,n-k)}|I_k]$ *and* $\mathcal{R}_k = \left[Y_{(k,n-k)}|\frac{I_{2k-n}}{O_{(n-k,2k-n)}}\right]$, *then* $\mathscr{C}$ *is an LCD MDS code if, and only if,* $Flip(\mathcal{R}_k) + \left[\frac{O_{(2k-n,k)}}{Y^{\diamond}}\right]$ *is invertible.*

Proof. From [29, Theorem 3.4], "$\mathscr{C}$ is LCD if, and only if, $GG^{\diamond}$ *is non-singular*", and $GG^{\diamond} = Flip(\mathcal{R}_k) + \left[\frac{O_{2k-n,k}}{Y^{\diamond}}\right]$ is invertible. □

Theorem 6.3. *Let* $\mathscr{C}$ *be an* $[n = 2k, k, d_{RT}]$ *MDS code of dimension* k *and length* n *in the RT-metric. If the generator matrix of* $\mathscr{C}$ *is represented as* $[\mathcal{R}_k|I_k]$ *where* $\mathcal{R}_k$ *is non-singular persymmetric over* $\mathbb{F}_q (q > 2)$, *then* $\mathscr{C}$ *is LCD MDS in the RT-metric.*

Proof. From [29, Theorem 3.4], "$\mathscr{C}$ is an LCD code if, and only if, $GG^{\diamond}$ is non-singular".

$$\begin{aligned}\text{Consider } GG^{\diamond} &= [\mathcal{R}_k|I_k].[R_k|Flip(\mathcal{R}_k)]^T\\ &= Flip(\mathcal{R}_k) + (Flip(\mathcal{R}_k))^T \qquad (\because\ \mathcal{R}_k \text{ is persymmetric })\\ &= 2 * Flip(\mathcal{R}_k) \text{ is non-singular.}\end{aligned}$$

□

7. Construction of RMDS codes in the RT-metric

In this section, we present methods for constructing RMDS codes from MDS codes of smaller dimensions and lengths.

Theorem 7.1. *Let $\mathcal{P}$ be any $n \times n$ square matrix and $\mathscr{C}$ be an $[s = 2n, k = n, d_{RT}]$ MDS code in the RT-metric with generator matrix in the form $[\mathcal{P}|I_n]$. Then,*

$$G = \begin{bmatrix} J_n - \mathcal{P} & \mathcal{P} - J_n & J_n - \mathcal{P} & \mathcal{P} & I_n & O & O & O \\ J_n - \mathcal{P} & \mathcal{P} - J_n & -\mathcal{P} & J_n + \mathcal{P} & O & I_n & O & O \\ J_n - \mathcal{P} & \mathcal{P} & J_n - \mathcal{P} & \mathcal{P} - J_n & O & O & I_n & O \\ -\mathcal{P} & J_n + \mathcal{P} & J_n - \mathcal{P} & \mathcal{P} - J_n & O & O & O & I_n \end{bmatrix}$$

generates an RMDS code $[s = 8n, k = 4n, d_{RT}]$ of length $8n$.

Theorem 7.2. *Let $G_1 = [L_1|I_{4n}]$ and $G_2 = [L_2|I_{4n}]$ be the MDS codes in the RT-metric, where $L_1 = Flip(L_2)$ and $L_2 = Circ(I_n, \mathcal{P}, O_n, \mathcal{P})$, and where $\mathcal{P}$ is an $n \times n$ binary square matrix. Then,*

(i) $G_1 = [L_1|I_{4n}]$ *generates an RMDS code of length $8n$ in the RT-metric.*
(ii) *If $\mathcal{P}$ is symmetric, then $G_1 = [L_1|I_{4n}]$ generates an SRMDS code of length $8n$ in the RT-metric.*
(iii) *If $\mathcal{P}$ is persymmetric, then $G_2 = [L_2|I_{4n}]$ generates an SMDS code of length $8n$ in the RT-metric.*
(iv) *If $\mathcal{P}$ is centrosymmetric, then $G_2 = [L_2|I_{4n}]$ generates an RMDS code of length $8n$ in the RT-metric.*
(v) *If $\mathcal{P}$ is bisymmetric, then $G_2 = [L_2|I_{4n}]$ generates an SRMDS code of length $8n$ in the RT-metric.*

Theorem 7.3. *Let $G_1 = [L_1|I_{4n}]$ and $G_2 = [L_2|I_{4n}]$ be the MDS codes in the RT-metric, where $L_1 = Flip(L_2)$ and $L_2 = DBCirc(I_n, \mathcal{P}, \mathcal{P}, O_n)$, and where $\mathcal{P}$ is an $n \times n$ binary square matrix. Then,*

(i) $G_1 = [L_1|I_{4n}]$ *generates an RMDS code of length $8n$ in the RT-metric.*
(ii) *If $\mathcal{P}$ is symmetric, then $G_2 = [L_2|I_{4n}]$ generates an SMDS code of length $8n$ in the RT-metric.*
(iii) *If $\mathcal{P}$ is centrosymmetric, then G_1 and G_2 generate an SRMDS code.*

Theorem 7.4. *Let $G_1 = [L_1|I_{4n}]$ and $G_2 = [L_2|I_{4n}]$ be the MDS codes in the RT-metric, where $L_1 = Flip(L_2)$ and $L_2 = DBCirc(O_n, \mathcal{P}, \mathcal{P}, J_n)$, and where $\mathcal{P}$ is an $n \times n$ binary square matrix. Then,*

(i) $G_1 = [L_1|I_{4n}]$ *generates an RMDS code of length $8n$ in the RT-metric.*
(ii) *If $\mathcal{P}$ is symmetric, then $G_1 = [L_1|I_{4n}]$ generates an SRMDS code of length $8n$ in the RT-metric.*
(iii) *If $\mathcal{P}$ is orthogonal, then $G_2 = [L_2|I_{4n}]$ generates an SMDS code of length $8n$ in the RT-metric.*

Theorem 7.5. *Let* $G_1 = [L_1|I_{4n}]$ *and* $G_2 = [L_2|I_{4n}]$ *be the MDS codes in the RT-metric, where* $L_1 = Flip(L_2)$ *and* $L_2 = DBCirc(I_n, \mathcal{P}, \mathcal{P}^{-1}, \mathcal{P} + \mathcal{P}^{-1})$*, and* $\mathcal{P}$ *is any* $n \times n$ *binary square invertible matrix. Then,*

(i) $G_1 = [L_1|I_{4n}]$ *generates an RMDS code of length* $8n$ *in the RT-metric.*

(ii) If $\mathcal{P}$ *is symmetric, then* $G_2 = [L_2|I_{4n}]$ *generates an SMDS code of length* $8n$ *in the RT-metric.*

(iii) If $\mathcal{P}$ *is centrosymmetric, then* G_1 *and* G_2 *generate a RMDS code.*

Theorem 7.6. *Let* $G = [\mathcal{P}|I_n]$ *be an RMDS code in the RT-metric,* $G_1 = [L_1|I_{4n}]$ *and* $G_2 = [L_2|I_{4n}]$ *be MDS codes in the RT-metric, where* $L_1 = Flip(L_2)$ *and* $L_2 = DBCirc(I_n, \mathcal{P}, \mathcal{P}^{-1}, \mathcal{P}+\mathcal{P}^{-1})$*, and where* $\mathcal{P}$ *is an* $n \times n$ *binary square invertible matrix. Then,*

(i) $G_1 = [L_1|I_{4n}]$ *generates an RMDS code of length* $8n$ *in the RT-metric.*

(ii) If $\mathcal{P}$ *is symmetric, then* $G_1 = [L_1|I_{4n}]$ *generates an SRMDS code of length* $8n$ *in the RT-metric.*

(iii) If $\mathcal{P}$ *is orthogonal, then* $G_2 = [L_2|I_{4n}]$ *generates an SMDS code of length* $8n$ *in the RT-metric.*

Theorem 7.7. *Let* $G = [\mathcal{P}|I_n]$ *be an RMDS code in the RT-metric, and let* $G_1 = [L_1|I_{4n}]$ *and* $G_2 = [L_2|I_{4n}]$ *be MDS codes in the RT-metric, where* $L_1 = Flip(L_2)$ *and* $L_2 = DBCirc(O_n, \mathcal{P}, \mathcal{P}, I_n)$*, and where* $\mathcal{P}$ *is any* $n \times n$ *binary square matrix. Then,* $G_1 = [L_1|I_{4n}]$ *generates an SRMDS code of length* $8n$ *in the RT-metric.*

8. Conclusions

We investigated the properties of reversible MDS codes in the RT-metric using the structures of various types of generator matrices. Furthermore, several properties including the necessary and sufficient conditions for certain reversible MDS codes to be self-dual over $\mathbb{F}_2$ in particular, and over $\mathbb{F}_q$ in general, were established. Finally, certain methods for constructing reversible MDS codes within the RT-metric were introduced.

Author contributions

Conceptualization: Bodigiri Sai Gopinadh formulated the initial research problem and developed an overarching mathematical framework. Validation: Venkatrajam Marka independently verified the correctness of the mathematical results, played a pivotal role in the problem discussion, and provided continuous supervision. Both authors read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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