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Research article

Sharp global existence and orbital stability of standing waves for the Schrödinger-Hartree equation with partial confinement

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Abstract: This study is concerned with the sharp criterion of global existence and the orbital stability of standing waves for the Hartree equation in presence of a partial confinement. Using the scaling technique and constructing cross-invariant sets, we first derive the sharp threshold for global existence and blow-up of the solution in both L^2-critical and L^2-supercritical settings. Then, by taking advantage of the profile decomposition technique and concentration compact arguments together with variational methods, we explore the existence of normalized standing waves and show that these standing waves are orbitally stable in the L^2-subcritical, L^2-critical, and supercritical cases. Our conclusions complement and compensate some previous results.

Keywords: Hartree equation; partial confinement; global existence; sharp threshold; stability
Mathematics Subject Classification: 35A01, 35A15, 35B44, 35Q55

1. Introduction

In this study, we consider the global existence, blow-up, and orbital stability of standing waves for the following Schrödinger-Hartree equation in the presence of a partial confinement:

$$\begin{cases} i\varphi_t = -\Delta\varphi + \sum_{i=1}^{k} x_i^2\varphi + \alpha(|x|^{-\nu} * |\varphi|^2)\varphi, \ (t,x) \in [0,T) \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0, \ x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here, $*$ represents the standard convolution in $\mathbb{R}^N$, $N \geq 3$, $\varphi(t,x) : [0,T)\times\mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < T \leq \infty$, φ_0 is a given function, $1 \leq k < N$, $0 < \nu < min\{4, N\}$, and $\alpha < 0$.

The nonlinear Schrödinger equation (NLS), including external confinement, is widely applicable in physics, and is frequently used to describe the Bose-Einstein condensation(BEC) phenomenon [1–3]. Moreover, the Schrödinger equation with Hartree type nonlinearity is a fundamental model in quantum mechanics that describes the behavior of electrons in various systems. An intrinsic feature of the Hartree equation is that the convolution factor $|x|^{-\nu}$ acts as either a nonlocal kernel or response, thus describing the nonlocal feature to the nonlinearity of medium [4].

In the case of $\alpha > 0$, the Hartree equation with repulsive interactions is available to characterize BEC in gases with very weak two-body interactions, which were found in either $23Na$ or $87Rb$ atomic systems [5]. In the case of $\alpha < 0$, which represents the attractive interactions, one can observe BEC in the very weakly attractive two-body gas, such as in the $7Li$ atomic system, as long as the density of the gas in the trap is low enough [5].

In recent decades, there have been numerous significant works on the research of the NLS with complete harmonic confinement(i.e., $k = N$); see e.g., [6–8] and the references therein. Specially, Shu and Zhang [7, 8] derived the optimal threshold of the global existence for the NLS with power nonlinearities and complete harmonic potential using the cross-constrained variational method. Wang [9] investigated the optimal criterion of global existence and blow-up for the NLS with Hartree nonlinearity and harmonic potential by constructing cross-constrained invariant sets, and showed the strong instability of standing waves under some appropriate assumption on the frequency. Huang et al. [10] explored the sharp mass threshold of global existence and discussed the existence and orbital stability of standing waves for Eq (1.1) with harmonic potential in the critical case $\nu = 2$. Luo [11] researched the existence and stability/instability of normalized standing waves for the Hartree equation with or without harmonic potential. Feng [12] proved the global existence and blow-up of solutions to the generalized Hartree equation, and explored the stability and instability of standing waves. Alex et al. [13] constructed and classified the finite time blow-up solutions at the minimal mass threshold; additionally, they investigated the existence, orbital stability, and instability of standing waves by variational methods.

On the other hand, when $1 \le k < N$, Eq (1.1) turns to the NLS with a partial confinement. Model (1.1) with a partial confinement plays an important role in physics, especially in the description of nonlinear fluctuations and BEC. Increasing attention has been given to this kind of model from a mathematical perspective; for examples, see [14–16]. In particular, it is worth mentioning that Bellazzini et al. [14] undertook a comprehensive study on the existence, orbital stability, and some qualitative properties of standing waves for the NLS with L^2-supercritical power nonlinearity and a partial confinement for $N = 3$ and $k = 2$, and utilized the concentration compact principle to overcome the lack of compactness. Motivated by the outstanding work [14], the authors in [15, 16] improved the results to a nonlinear Schrödinger system with coupled power nonlinearities and a partial confinement. In the spirit of [14], in terms of the concentration compact arguments, Xiao et al. [17] showed the existence and orbital stability of standing waves for Eq (1.1) with a Hartree-type nonlinearity $(I_\alpha * |\varphi|^p)|\varphi|^{p-2}$ and a partial confinement in the L^2-supercritical case. However, their study only involved the case when $k = N - 1$, that is, the case with a harmonic confinement in $N - 1$ space directions. Meanwhile, for the more general case when $1 \le k < N$ with an L^2-subcritical and critical nonlinearities, they didn't give rigorous consideration and proof; this is one of the starting points of our study. Recently, in light of [14], Liu et al. [18] took a thorough consideration on the existence of stable standing waves for the inhomogeneous NLS with a partial harmonic potential in the L^2-subcritical, L^2-critical, and L^2-supercritical situations by taking advantage of the profile decomposition technique and concentration compact arguments. More recently, Hong and Jin [19] investigated the uniqueness and orbital stability of standing waves to the $3d$ cubic NLS with a strong $2d$ harmonic confinement by employing the dimension reduction. In [20], by means of the profile decomposition theory and cross-constrained variational method, Mo et al. took the strong instability of standing waves for the Hartree-type equation with a partial/complete harmonic confinement into account. Pan and Zhang [21] researched the dynamical properties of blow-up solutions for the NLS with a partial confinement and cubic nonlinearity for $k = 1, N = 2$. Gong and the second author in [22] discussed the sharp threshold of global existence and mass concentration properties to the blow-up solutions for the generalized Hartree equation with a complete and partial harmonic potential by constructing some cross-invariant sets and variational problems.

To the authors' knowledge, the sharp criterion of global existence and orbital stability issues of standing waves to the Hartree equation with a partial harmonic potential haven't been completely solved yet. Inspired by the aforementioned works [9–11, 14, 17, 18, 22], the main goal of this article is to address these problems to the Cauchy problem (1.1) and complement the corresponding results of [9–11, 17].

Before giving out the main conclusions of this study, let's first introduce some notations. Regarding Eq (1.1), we equip its energy space

$$\Sigma = \left\{ u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} \sum_{i=1}^{k} x_i^2 |u|^2 dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle_\Sigma = Re \int \left(u\bar{v} + \nabla u \cdot \nabla \bar{v} + \sum_{i=1}^{k} x_i^2 u\bar{v} \right) dx, \quad \forall u, v \in \Sigma,$$

and the corresponding norm is denoted by

$$||u||_\Sigma^2 = ||u||_2^2 + ||\nabla u||_2^2 + \int \sum_{i=1}^{k} x_i^2 |u|^2 dx, \quad \forall u \in \Sigma.$$

Meanwhile, Eq (1.1) enjoys a special solution known as the standing wave possessing the form $e^{i\varrho t}u(x)$, where $\varrho \in R$ is a frequency, and $u \in \Sigma$ is a nontrivial solution to the following elliptic equation:

$$-\Delta u + \varrho u + \sum_{i=1}^{k} x_i^2 u + \alpha(|x|^{-\nu} * |u|^2)u = 0. \tag{1.2}$$

The energy functional associated to Eq (1.1) is defined as follows:

$$E(u) = \int \left(|\nabla u|^2 + \sum_{i=1}^{k} x_i^2 |u|^2 + \frac{1}{2}\alpha(|x|^{-\nu} * |u|^2)|u|^2 \right) dx, \ u \in \Sigma. \tag{1.3}$$

The first part of this study is devoted to the criterion of sharp global existence in the L^2-critical and supercritical cases when $2 \leq \nu < min\{4, N\}$. For the L^2-critical case where $\nu = 2$, we first explore the sharp mass criterion for the existence of global and blow-up solutions using the Gagliardo-Nirenberg inequality and some scaling arguments. The conclusions are as follows.

Theorem 1.1. *Let $\nu = 2$ and $Q(x)$ be the positive radially symmetric ground state solution of Eq (1.13). If $\varphi_0 \in \Sigma$ and φ_0 satisfies*

$$\|\varphi_0\|_2 < \frac{1}{\sqrt{-\alpha}}\|Q(x)\|_2, \tag{1.4}$$

then the Cauchy problem (1.1) has a global and bounded solution $\varphi(t, x)$ in $C([0, \infty], \Sigma)$. Moreover, we have the following for any $0 \leq t < \infty$:

$$\int \Big(|\nabla\varphi|^2 + \sum_{i=1}^{k} x_i^2|\varphi|^2\Big)dx < \frac{E(\varphi_0)}{1 + \alpha\|Q(x)\|_2^{-2}(\int |\varphi_0|^2 dx)} + E(\varphi_0). \tag{1.5}$$

Theorem 1.2. *Let $Q(x)$ be the positive radially symmetric ground state solution of Eq (1.13), where $\nu = 2$. Then, for any $\varepsilon > 0$, there exists $\varphi_0 \in \Sigma$ that satisfies $\int |x|^2|\varphi_0|^2 dx < \infty$ such that*

$$\|\varphi_0\|_2^2 = \frac{1}{-\alpha}\|Q(x)\|_2^2 + \varepsilon,$$

and the solution $\varphi(t, x)$ of the Cauchy problem (1.1) blows up in finite time.

Remark 1.3. *(i) For the L^2-subcritical case $0 < \nu < 2$, by the local-well posed theory and using Gagliardo-Nirenberg and Young's inequalities, we are able to show the existence of the global solution to Eq (1.1) for any $\varphi_0 \in \Sigma$.*
(ii) From Theorems 1.1 and 1.2, we see that the ground state mass $\frac{1}{\sqrt{-\alpha}}\|Q(x)\|_2$ gives a sharp sufficient condition of global existence for the solution to Eq (1.1), which is the same as [10], in which the Hartree equation with a complete harmonic potential is considered.

For the L^2-supercritical situation when $2 \leq \nu < min\{4, N\}$, we explore the optimal threshold for blow-up and global solutions by proposing and studying several cross-invariant sets and constrained minimizing problems. In order to achieve this goal, for $u \in \Sigma$, we introduce three key functionals as follows:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2 + \sum_{i=1}^{k} x_i^2|u|^2)dx + \frac{1}{4}\alpha\int_{\mathbb{R}^N}(|x|^{-\nu} * |u|^2)|u|^2 dx, \tag{1.6}$$

$$S(u) = \int_{\mathbb{R}^N}|\nabla u|^2 + |u|^2 dx + \alpha\int_{\mathbb{R}^N}(|x|^{-\nu} * |u|^2)|u|^2 dx, \tag{1.7}$$

$$P(u) = \int_{\mathbb{R}^N}|\nabla u|^2 dx + \frac{\nu}{4}\alpha\int_{\mathbb{R}^N}(|x|^{-\nu} * |u|^2)|u|^2 dx. \tag{1.8}$$

Then, we denote the following two constrained minimizing problems:

$$d_M = \inf_{u\in M} I(u), \tag{1.9}$$

$$d_B = \inf_{u\in B} I(u), \tag{1.10}$$

where

$$M = \{u \in \Sigma \setminus \{0\}, P(u) = 0, S(u) < 0\},$$
$$B = \{u \in \Sigma \setminus \{0\}, S(u) = 0\}.$$

Let

$$d = \min\{d_M, d_B\}; \tag{1.11}$$

then, from Lemmas 2.2 and 2.3, one can conclude that $d > 0$.

We define the following manifolds:

$$\begin{aligned} K &= \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0, P(u) < 0\}, \\ K_+ &= \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0, P(u) > 0\}, \\ R_+ &= \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) > 0\}, \\ R_- &= \{u \in \Sigma \setminus \{0\}, I(u) < d, S(u) < 0\}. \end{aligned}$$

In Section 3, we will show that K, K_+, R_+, R_- *are invariant sets under the flow generated by Eq (1.1), that is, the solution* $\varphi(t,x)$ *of Eq (1.1) satisfies* $\varphi(t,x) \in K,\ K_+,\ R_+$ *or* R_- *for any* $t \in [0,T)$*, if* $\varphi_0 \in K, K_+, R_+$ *or* R_-.

The next two conclusions concern the sharp threshold of the global and blow-up solutions to Eq (1.1) in the mass-supercritical cases when $2 \leq \nu < min\{4, N\}$.

Theorem 1.4. *Let* $2 \leq \nu < min\{4, N\}$ *and* $\varphi_0 \in K_+ \bigcup R_+$*; then, the solution* $\varphi(t,x)$ *to Eq (1.1) globally exists in time* $t \in [0, \infty)$.

Theorem 1.5. *Let* $2 \leq \nu < min\{4, N\}$ *and assume* $\varphi_0 \in K$ *satisfies* $\int |x|^2 |\varphi_0|^2 dx < \infty$*; then, the solution* $\varphi(t,x)$ *to Eq (1.1) blows up in finite time.*

Remark 1.6. *(i) From the definitions of the invariant sets* K, K_+, R_+, R_-*, for* $2 \leq \nu < min\{4, N\}$*, it's obvious to see that*

$$\{u \in \Sigma \setminus \{0\}, I(u) < d\} = K_+ \cup R_+ \cup K,$$

which gives the sharp threshold of global existence if $|x|\varphi_0 \in L^2(\mathbb{R}^N)$.
(ii) For $k = N$ *in Eq (1.1), the conclusions of Theorems 1.4 and 1.5 remain valid. A novelty for our article lies in that the structures of invariant manifolds* K, K_+, R_+, R_- *are different from those of [9], that is, we derive a new sharp threshold of global existence for* $2 \leq \nu < min\{4, N\}$*, which differs from [9]. From this point of view, Theorems 1.4 and 1.5 can be viewed as a complement to [9].*

With regard to the Cauchy problem (1.1), our second interest focuses on the orbital stability of standing waves, which has gained increasing attention from both mathematicians and physicists and is defined below.

Definition 1.7. *The set* $\mathcal{A}$ *is orbitally stable if for any given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any initial data* φ_0 *which satisfies*

$$\inf_{u \in \mathcal{A}} \|\varphi_0 - u\|_\Sigma < \delta,$$

the corresponding solution $\varphi(t,x)$ *of Eq (1.1) globally exists and satisfies*

$$\inf_{u \in \mathcal{A}} \|\varphi(t,x) - u\|_\Sigma < \varepsilon, \ for \ \forall \ t > 0.$$

Based on this definition, in order to investigate the orbital stability of standing waves, we demand that the solutions of Eq (1.1) globally exist at least in the case when the initial value u_0 is sufficiently close to $\mathcal{A}$. In the L^2-subcritical case, all solutions for Eq (1.1) are global and bounded. However, in the L^2-critical and L^2-supercritical settings, according to local well-posedness theory of the NLS, the NLS with small initial data has a global and bounded solution, while for some large initial values, the solutions may blow up at a finite time. Therefore, to research the existence of stable standing waves, we take the following constrained minimalized problem inspired by Cazenave and Lions [23] into account:

$$m(c) = \inf_{u\in S(c)} E(u), \tag{1.12}$$

where $S(c) = \{u \in \Sigma, \|u\|_2 = c\}$, for $c > 0$.

In the L^2-subcritical case when $0 < \nu < 2$, or in the L^2-critical case when $\nu = 2$ and $0 < \sqrt{-\alpha c} < \|Q\|_2$, the Gagliardo-Nirenberg inequality yields that $E(u)$ is bounded from below on $S(c)$, where $Q(x)$ is the ground state of the elliptic equation with Hartree nonlinearity as follows:

$$-\Delta v + v - (|x|^{-\nu} * |v|^2)v = 0. \tag{1.13}$$

The existence of ground state solutions to (1.13) has been studied by previous authors, based on which we are able to study the orbital stability of standing waves by considering the minimization problem (1.12) in the mass-subcritical and critical situations. Regarding the stability issues of standing waves to (1.1), due to the presence of a partial confinement, we will face two major challenges: Is the absence of compactness and the Hartree nonlinearity $\alpha(|x|^{-\nu} * |\varphi|^2)\varphi$. First, since the embedding $\Sigma \hookrightarrow L^r$ with $r \in [2, \frac{2N}{N-2})$ is not compact, the general consideration is to apply the concentration compactness principle to overcome this difficulty. Second, due to $m(c) > 0$ and the Hartree nonlinearity $\alpha(|x|^{-\nu} * |\varphi|^2)\varphi$, the non-vanishing nature of the minimizing sequence is not easy to exclude. Actually, due to the presence of the nonlocal nonlinearity $\alpha(|x|^{-\nu} * |\varphi|^2)\varphi$, the general approach is to compare Eq (1.2) with its limiting equation

$$-\Delta u + \varrho u + \sum_{j=1}^{k} x_j^2 u = 0, \tag{1.14}$$

which does not have a non-trivial solution in Σ if $\varrho \geq -k$ (see [18]), in which an inhomogeneous NLS with a partial confinement was considered. Similarly, the conventional methods cannot be applied to our situation. To overcome these problems, we first attempt to apply the profile decomposition of bounded sequence in Σ to show the compactness of minimizing sequences for the minimization problem Eq (1.12) in the L^2-subcritical and critical regions. With this tool in hand, we are able to derive the existence of minimizers for the minimalized problem (1.12) and show the orbital stability of standing waves. In what follows, we denote the set of whole minimizers to (1.12) by the following:

$$\mathcal{M}_c = \{u \in S(c), E(u) = m(c)\}.$$

It is standard that for any $u_c \in \mathcal{M}_c$, there exists a $\omega_c \in R$ such that (u_c, ω_c) solves the stationary Eq (1.2) and $e^{i\omega_c t}u_c(x)$ is a standing wave solution of (1.1) with the initial data $u_0 = u_c$.

Theorem 1.8. *Suppose that either $c > 0$ if $0 < \nu < 2$ or $0 < \sqrt{-\alpha c} < \|Q\|_2$ if $\nu = 2$, where $Q(x)$ is the ground state solution to Eq (1.13). Then, $\mathcal{M}_c \neq \emptyset$ and is orbitally stable.*

Finally, we deal with the L^2-supercritical case when $2 \leq \nu < min\{4, N\}$. In this situation, the energy functional $E(u)$ is unbounded from below on $S(c)$. In fact, when $2 < \nu < min\{4, N\}$, by taking $u \in \Sigma$

such that $\|u\|_2 = c$, then we have the following:

$$E(u^\lambda) = \lambda^2\|\nabla u\|_2 + \lambda^{-2}\int\sum_{j=1}^{k} x_j^2|u|^2dx + \frac{\alpha}{2}\lambda^\nu\int(|x|^{-\nu} * |u|^2)|u|^2dx \to -\infty,$$

as $\lambda \to +\infty$, where $u^\lambda(x) = \lambda^{\frac{N}{2}}u(\lambda x)$. When $\nu = 2$ and $\sqrt{-\alpha}c > \|Q\|_{L^2}$, we set $u = \frac{c}{\|Q\|_{L^2}}Q$; then $\|u\|_{L^2} = c$, and we deduce the following from the pohožaev identity:

$$\begin{aligned}
E(u^\lambda) &= \lambda^2\|\nabla u\|_2^2 + \lambda^{-2}\int\sum_{j=1}^{k} x_j^2|u|^2dx + \frac{\alpha}{2}\lambda^2\int(|x|^{-\nu} * |u|^2)|u|^2dx \\
&= \lambda^2\frac{c^2}{\|Q\|_2^2}\|\nabla Q\|_2^2 + \lambda^{-2}\frac{c^2}{\|Q\|_2^2}\int\sum_{j=1}^{k} x_j^2|Q|^2dx + \frac{\alpha}{2}\lambda^2\frac{2}{\|Q\|_2^2}\|\nabla u\|_2^2\|u\|_2^2 \\
&= \lambda^2\frac{c^2}{\|Q\|_2^2}\|\nabla Q\|_2^2(1 + \alpha\frac{c^2}{\|Q\|_2^2}) + \lambda^{-2}\frac{c^2}{\|Q\|_2^2}\int\sum_{j=1}^{k} x_j^2|Q|^2dx \to -\infty,
\end{aligned}$$

as $\lambda \to +\infty$. Thus, we cannot directly derive the existence and orbital stability of standing waves for Eq (1.1) by considering the global minimization problem (1.12). Greatly inspired by [14, 18], we turn to consider the local minimization problem: for any given $r > 0$, define

$$m(c, r) = \inf_{u\in S(c)\bigcap B(r)} E(u), \tag{1.15}$$

where $B(r) = \{u \in \Sigma, \|u\|_\Sigma \le r\}$ and $\|u\|_\Sigma$ is given by

$$\|u\|_\Sigma^2 = \|\nabla u\|_2^2 + \int\sum_{j=1}^{k} x_j^2|u|^2dx. \tag{1.16}$$

It goes without saying that $m(c, r) > -\infty$ if $S(c)\bigcap B(r) \neq \emptyset$. Furthermore, there is no way of overcoming this difficulty by comparing it with the limiting Eq (1.14). One can actually solve the minimization problem (1.12) by proving the boundness of any translation sequence. Denote the set of all minimizers of (1.12) by the following:

$$\mathcal{M}_r(c) := \{u \in S(c)\bigcap B(r), E(u) = m(c, r)\}.$$

The main result of this situation is as follows.

Theorem 1.9. *Let* $1 \le k < N$ *and* $2 < \nu < min\{4, N\}$*; then, there exists* $r_0 \ge 2\sqrt{k}$*, such that for every given* $r \ge r_0$*, there exists a* C_r *with* $0 < C_r < 1$ *such that for any* $c \in (0, C_r)$*,*
(i) $\emptyset \neq \mathcal{M}_r(c) \subset S(c)\bigcap B(\frac{rc}{2})$*; and*
(ii) The set $\mathcal{M}_r(c)$ *is orbitally stable.*

Remark 1.10. *(i) In the case when* $k = N$*, the authors in [10, 11] applied the compact embedding and variational methods to study the existence and stability of normalized standing waves for the Hartree equation with a complete harmonic potential in the* L^2*-critical case when* $\nu = 2$ *and the supercritical case when* $2 < \nu < min\{4, N\}$*, respectively, and revealed the stabilizing effect of a complete harmonic*

potential on the standing waves. In the present study, the complete harmonic potential is replaced by a partial confinement, which results in the fact that the embedding $\Sigma \hookrightarrow L^r$ *with* $r \in [2, \frac{2N}{N-2})$ *loses compactness. We make full use of the profile decomposition principle and concentration compactness arguments to recover compactness and show the existence and orbital stability of normalized standing waves for* $\nu = 2$ *and* $2 < \nu < min\{4, N\}$*, respectively. The current study indicates that the partial confinement plays the same role as complete harmonic potential in [10, 11] for the Hartree equation. Our study extends and complements the corresponding results of [10, 11].*
(ii) For $k = N-1$*, [17] showed that the standing waves of Eq (1.1) with a Hartree-type nonlinearity* $(I_\alpha * |\varphi|^p)|\varphi|^{p-2}$ *and a partial confinement existed and were orbitally stable in the* L^2*-supercritical case with the aid of the concentration compactness principle. In contrast with [17], we consider the more general case when* $1 \le k < N$ *and conduct an exhaustive study on the orbital stability of standing waves by combining the profile decomposition principle and concentration compactness arguments, including the* L^2*-subcritical,* L^2*-critical, and supercritical cases (see Theorems 1.8 and 1.9). This is another novelty of the present paper. In addition, Theorems 1.8 and 1.9 remain valid for Eq (1.1) with a Choquard nonlinearity* $(I_\alpha * |\varphi|^p)|\varphi|^{p-2}$ *and a partial confinement for* $1 \le k < N$*. From this point of view, our results complement and compensate the corresponding ones of [17].*

Throughout this paper, C denotes various positive constants, which may vary from line to line. To simplify matter, we use $\int \cdot\, dx$ to represent $\int_{\mathbb{R}^N} \cdot\, dx$ and denote $\|u\|_p = \|u\|_{L^p(\mathbb{R}^N)} = (\int |u|^p dx)^{\frac{1}{p}}$ in this and subsequent sections.

The rest of this study is structured as follows: In Section 2, some notations and preliminaries are given; Section 3 is concerned with the sharp criterion of global existence and finite time blow-up to Eq (1.1); and the last section focuses on the orbital stability of standing waves.

2. Notations and preliminaries

To survey the criterion of global existence versus blow-up and the stability issues of standing waves, one requires the well-posedness to Eq (1.1), which can be proven based on Cazenave [6].

Proposition 2.1. *Suppose* $\varphi_0 \in \Sigma$ *and* $0 < \nu < min\{4, N\}$*; then, there exist* $T = T(\|u_0\|_\Sigma)$ *and a unique solution* $\varphi(t, x) \in C([0, T), \Sigma)$ *of Eq (1.1). Assume that the solution* $\varphi(t, x)$ *is well-defined on the maximal interval* $[0, T)$*. If* $T < \infty$*, then* $\lim_{t\to T} \|\varphi(t, x)\|_\Sigma = \infty$ *(blow-up). Moreover, for any* $t \in [0, T)$*, the following conservation laws of mass and energy hold:*

$$\|\varphi(t, x)\|_2 = \|\varphi_0\|_2, \tag{2.1}$$

$$E(\varphi(t, x)) = E(\varphi_0). \tag{2.2}$$

Lemma 2.2. *Let* $0 < \lambda < N$ *and* $s, r > 1$ *be constants such that*

$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2.$$

Assume that $g \in L^r(\mathbb{R}^N)$ *and* $h \in L^s(\mathbb{R}^N)$*. Then,*

$$\left|\int\int g(x)|x - y|^{-\lambda} h(y)dxdy\right| \le C(N, s, \lambda)\|g\|_r\|h\|_s. \tag{2.3}$$

For this lemma, the detailed proofs are available in [24]. By (2.3), we can derive the following convolution-type Gagliardo-Nirenberg inequality:

$$\int (|x|^{-\nu} * |u|^2)|u|^2 dx \leq C_\nu \int |\nabla u|^2 dx^{\frac{\nu}{2}} \int |u|^2 dx^{\frac{4-\nu}{2}}. \tag{2.4}$$

Following Weinstein [25] and Feng and Yuan [26], we are able to obtain the best constant in the inequality (2.4) by dealing with the existence of the minimizer to the functional as follows:

$$J(\nu) = \frac{\left(\int |\nabla u|^2 dx\right)^{\frac{\nu}{2}} \left(\int |u|^2 dx\right)^{\frac{4-\nu}{2}}}{\int (|x|^{-\nu} * |u|^2)|u|^2 dx}.$$

Lemma 2.3. *[26] The best constant in the convolution-type Gagliardo-Nirenberg inequality (2.4) is given by the following:*

$$C_\nu = \frac{4}{4-\nu}\left(\frac{4-\nu}{\nu}\right)^{\frac{\nu}{2}} \|Q(x)\|_2^{-2},$$

where $Q(x)$ is the ground state of the Hartree equation

$$-\Delta v + v - (|x|^{-\nu} * |v|^2)v = 0. \tag{2.5}$$

Especially, in the L^2-critical situation $\nu = 2$, $C_\nu = 2\|Q(x)\|_2^{-2}$.

It is well known that the ground state of (2.5) plays a pivotal role on the research of the global existence and blow-up dynamics for the NLS. In the following lemma, we recall some existing results and properties of the ground state solution to (2.5).

Lemma 2.4. *[27] Let $\nu \in (0, N)$ it follows that (2.5) admits a ground state solution $Q(x)$ in $H^1(\mathbb{R}^N)$. Every ground state $Q(x)$ of (2.5) is in $L^1 \cap C^\infty$, and there exists $x_0 \in \mathbb{R}^N$ and a monotone real function $\tau \in C^\infty(0, \infty)$ such that, $Q(x) = \tau(|x - x_0|)$ for every $x \in \mathbb{R}^N$. Moreover, the following Pohožaev identity holds:*

$$\frac{N-2}{2}\int |\nabla Q(x)|^2 dx + \frac{N}{2}\int |Q(x)|^2 dx = \frac{2N-\nu}{4}\int (|x|^{-\nu} * |Q(x)|^2)|Q(x)|^2 dx; \tag{2.6}$$

$$\int |\nabla Q(x)|^2 dx + \int |Q(x)|^2 dx = \int (|x|^{-\nu} * |Q(x)|^2)|Q(x)|^2 dx. \tag{2.7}$$

From (2.6) and (2.7), one has the following:

$$\|\nabla Q\|_2^2 = \frac{\nu}{4}\int (|x|^{-\nu} * |Q(x)|^2)|Q(x)|^2 dx. \tag{2.8}$$

By a direct computation, we can infer that if $Q(x)$ is a ground state solution of Eq (2.5), then $Q_\alpha(x) = \frac{1}{\sqrt{-\alpha}}Q(x)$ is the ground state solution of the following Hartree equation:

$$-\Delta u + u + \alpha(|x|^{-\nu} * |u|^2)u = 0, \quad \alpha < 0. \tag{2.9}$$

In order to study the blow-up phenomenon of Eq (1.1), we also need the following lemma, which can be derived based on the analysis of the virial functional $W(t) = \int |x|^2|\varphi(t,x)|^2 dx$ in light of [6, 25].

Proposition 2.5. *Assume that* $2 \le \nu < \min\{4, N\}$*; let* $\varphi_0 \in H^1(\mathbb{R}^N)$ *and* $|x|\varphi_0 \in L^2(\mathbb{R}^N)$*, and one of the following conditions are satisfied:*
Case 1. $E(\varphi_0) < 0$.
Case 2. $E(\varphi_0) = 0$ *and* $Im \int x\nabla\varphi_0\bar{\varphi}_0 dx < 0$.
Case 3. $E(\varphi_0) > 0$ *and* $Im \int x\nabla\varphi_0\bar{\varphi}_0 dx + \Big(2W(0)E(\varphi_0)\Big)^{\frac{1}{2}} \le 0$.
Then, the corresponding solution $\varphi(t, x)$ *of Eq (1.1) blows up in a finite time.*

Lemma 2.6. *[14]*

$$\Lambda_0 = \inf_{\int_{\mathbb{R}^N} |\omega|^2 dx = 1} \int_{\mathbb{R}^N} |\nabla\omega|^2 dx + \int \sum_{j=1}^{k} x_j^2 |\omega(x)|^2 dx,$$

and

$$\lambda_0 = \inf_{\int_{\mathbb{R}^N} |u|^2 dx_1 \cdots dx_k} \int_{\mathbb{R}^N} |\nabla_{x_1 \cdots x_k u|^2} dx_1 \cdots dx_k + \int_{\mathbb{R}^N} \sum_{j=1}^{k} x_j^2 |u|^2 dx_1 \cdots dx_k;$$

then, we have the following equality:

$$\Lambda_0 = \lambda_0.$$

Eventually, to investigate the compactness of the minimizing sequence, we first establish the profile decomposition of a bounded sequence in light of [18, 26] and recall the principle of concentration compactness.

Lemma 2.7. *Let* $1 \le k < N$ *and* $0 < \nu < \min\{4, N\}$*, and* $\{u_n\}$ *be a bounded sequence in* Σ*. Then, there exists a subsequence of* $\{u_n\}$ *(still denoted by* $\{u_n\}$ *), a family* $\{x_n^j\}_{n=1}^{\infty}$ *of sequences in* $\mathbb{R}^{N-k}$*, and a sequence* $\{U^j\}_{j=1}^{\infty}$ *in* Σ *such that the following hold:*
(i) for each $m \ne j$, $|x_n^m - x_n^j| \to +\infty$*, as* $n \to \infty$*;*
(ii) for each $l \ge 1$ *and* $x \in \mathbb{R}^N$*, we have*

$$u_n(x) = \sum_{j=1}^{l} \tau_{x_n^j} U^j(x) + r_n^l, \tag{2.10}$$

with $\limsup_{n\to\infty} \|r_n^l\|_q \to 0$ *as* $l \to \infty$ *for any* $q \in [2, \frac{2N}{N-2})$*. Here and in the following, we define* $\tau_y U(x) = U(x_1, \cdots, x_k, x_{k+1} - y_1, \cdots, x_N - y_{N-k})$ *for* $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ *and* $y = (y_1, \cdots, y_{N-k}) \in \mathbb{R}^{N-k}$*. Moreover,*

$$\|u_n\|_2^2 = \sum_{j=1}^{l} \|U^j\|_2^2 + \|r_n^l\|_2^2 + o(1), \tag{2.11}$$

$$\int \sum_{j=1}^{k} x_j^2 |u_n|^2 dx = \sum_{j=1}^{l} \int \sum_{i=1}^{k} x_i^2 |U^j|^2 dx + \int \sum_{j=1}^{k} x_j^2 |r_n^l|^2 dx + o(1), \tag{2.12}$$

$$\|\nabla u_n\|_2^2 = \sum_{j=1}^{l} \|\nabla U^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1), \tag{2.13}$$

$$\int (|x|^{-\nu} * |u_n|^2)|u_n|^2 dx = \sum_{j=1}^{l} \int (|x|^{-\nu} * |\tau_{x_n^j} u^j|^2)|\tau_{x_n^j} u^j|^2 dx + \int (|x|^{-\nu} * |r_n^l|^2)|r_n^l|^2 dx + o(1), \tag{2.14}$$

where $o(1) = o_n(1) \to 0$ *as* $n \to \infty$.

When removing the partial confinement, Zhang and Zhu [28] proposed a similar profile for the decomposition of a bounded sequence in $H^s(\mathbb{R}^N)(0 < s < 1)$ and applied it and the Gagliardo-Nirenberg inequality to investigate the orbital stability of standing waves for the nonlinear fractional Hartree equation.

Lemma 2.8. *[18] Let* $a > 0$, *and assume* $\{u_n\}_{n=1}^{\infty}$ *is a bounded sequence in* H^1 *which satisfies the following:*

$$\int_{\mathbb{R}^N} |u_n|^2 dx = a.$$

Then, there exists a subsequence $\{u_{n_j}\}_{j=1}^{\infty}$ *which satisfies one of the following three possibilities:*
(i) (Compactness) there exists $\{u_{n_j}\}_{j=1}^{\infty} \subset \mathbb{R}^N$ *such that* $|u_{n_j}(\cdot + y_{n_j})|^2$ *is compact, i.e.,*
$\forall \varepsilon > 0, \exists R < \infty$, $\int_{B_R(y_{n_j})} |u_{n_j}(x)|^2 dx \geq a - \varepsilon$;
(ii) (vanishing) $\lim_{j\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_{n_j}(x)|^2 dx = 0$ *for all* $R < \infty$; *and*
(iii) (dichotomy) there exists $b \in (0, a)$, *and* $u_{n_j}^{(1)}, u_{n_j}^{(2)} \subset H^1$ *such that*

$$|u_{n_j}^{(1)}| + |u_{n_j}^{(2)}| \leq |u_{n_j}|;$$

$$|\int_{\mathbb{R}^N} |u_{n_j}|^p dx - \int_{\mathbb{R}^N} |u_{n_j}^{(1)}|^p dx - \int_{\mathbb{R}^N} |u_{n_j}^{(2)}|^p dx| \to 0, \text{ as } j \to \infty \text{ for any } 2 \leq p < \tfrac{2N}{N-2};$$

$$\|u_{n_j}^{(1)}\|_2^2 \to b, \|u_{n_j}^{(2)}\|_2^2 \to a - b;$$

$$dist(Supp\ u_{n_j}^{(1)}, Supp\ u_{n_j}^{(2)}) \to \infty, \text{ as } j \to \infty;$$

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} (|\nabla u_{n_j}|^2 - |\nabla u_{n_j}{}^{(1)}|^2 - |\nabla u_{n_j}{}^{(2)}|^2) dx \geq 0.$$

3. Sharp criterion of global existence

In this section, we study the criterion of the global existence and blow-up to Eq (1.1) and verify the conclusions of Theorems 1.1 and 1.2, and Theorems 1.4 and 1.5.

3.1. The L^2-critical case

In this subsection, we demonstrate the global existence and blow-up of the solutions to Eq (1.1) in the L^2 critical case when $\nu = 2$ (i.e., Theorems 1.1 and 1.2).
Proof of Theorem 1.1. Let $\varphi(t, x)$ be the corresponding solution of Eq (1.1) in $C([0, T), \Sigma)$ with initial value $\varphi_0 \in \Sigma$. By (1.3), (2.1), (2.2), and Lemma 2.3, we obtain the following:

$$\begin{aligned} E(\varphi_0) = E(\varphi(t)) &= \int \Big(|\nabla\varphi|^2 + \sum_{i=1}^{k} x_i^2|\varphi|^2\Big)dx + \frac{\alpha}{2}\int (|x|^{-\nu} * |\varphi|^2)|\varphi|^2)dx \\ &\geq \int \Big(|\nabla\varphi|^2 + \sum_{i=1}^{k} x_i^2|\varphi|^2\Big)dx + \alpha\|Q(x)\|_2^{-2}\int |\nabla\varphi|^2 dx \int |\varphi|^2 dx \end{aligned}$$

$$= \int \sum_{i=1}^{k} x_i^2|\varphi|^2dx + \Big(1 + \alpha\frac{\|\varphi_0\|_2^2}{\|Q(x)\|_2^2}\Big)\|\nabla\varphi\|_2^2. \tag{3.1}$$

Based on (1.4) and (3.1), we can conclude that there exists C for $t \in [0, T)$ with $T < \infty$ such that

$$\int |\nabla\varphi|^2dx + \int \sum_{i=1}^{k} x_i^2|\varphi|^2dx \leq C.$$

Then, in accordance with Proposition 2.1, the solution $\varphi(t, x)$ is global and bounded in time. Furthermore, we have the following:

$$\|\nabla\varphi\|_2^2 < \frac{E(\varphi_0)}{1+\alpha\|Q(x)\|_2^{-2}(\int|\varphi_0|^2dx)}, \tag{3.2}$$

$$\int \sum_{i=1}^{k} x_i^2|\varphi|^2dx < E(\varphi_0). \tag{3.3}$$

Eventually, combining (3.2) and (3.3), we can conclude that (1.5) holds true.

Combining the variational character of the ground state solution of Eq (2.5), some scaling arguments, and energy conservation, we can show the existence of the blow-up solutions to Eq (1.1) for $\nu = 2$.

Proof of Theorem 1.2. For any $a > 1$ and $b > 0$, let $Q^{a,b}(x) = ab^{\frac{N}{2}}Q(bx)$. Depending on some scaling arguments, it follows that

$$\int |Q^{a,b}(x)|^2dx = a^2\int |Q(x)|^2dx, \tag{3.4}$$

$$\int |\nabla Q^{a,b}(x)|^2dx = a^2b^2\int |\nabla Q(x)|^2dx, \tag{3.5}$$

$$\int \sum_{i=1}^{k} x_i^2|Q^{a,b}(x)|^2dx = a^2b^{-2}\int \sum_{i=1}^{k} x_i^2|Q(x)|^2dx, \tag{3.6}$$

$$\int (|x|^{-\nu} * |Q^{a,b}(x)|^2)|Q^{a,b}(x)|^2dx = a^4b^2\int (|x|^{-\nu} * |Q(x)|^2)|Q(x)|^2dx. \tag{3.7}$$

Next, we set

$$a = \sqrt{\frac{\frac{1}{-\alpha}\int|Q(x)|^2dx + \varepsilon}{\int|Q(x)|^2dx}} > 1,\ \ b > \Big[\frac{\int\sum_{i=1}^{k} x_i^2|Q(x)|^2dx}{(a^2-1)\int|\nabla Q(x)|^2dx}\Big]^{\frac{1}{4}},$$

and $\varphi_0(x) = ab^{\frac{N}{2}}Q(bx)$; then, we have $\varphi_0(x) \in \Sigma$ and $\int|x|^2|\varphi_0|^2dx < \infty$. Indeed, by utilizing the exponential decay of ground state solution $Q(x)$ (see [27])

$$Q(|x|), \nabla Q(|x|) = O(|x|e^{-|x|}),\ as\ |x| \to \infty, \tag{3.8}$$

we conclude that $Q_{a,b}(x) \in L^2(\mathbb{R}^N)$, and so $\varphi_0 = ab^{\frac{N}{2}}Q(bx) \in H^1(\mathbb{R}^N)$ and $\int|x|^2|\varphi_0|^2dx < \infty$. Thus, we deduce that $\varphi_0 \in \Sigma$. Moreover, from (3.4), one has the following:

$$\int |\varphi_0|^2dx = \frac{1}{-\alpha}\int |Q(x)|^2dx + \varepsilon.$$

According to (1.3), (2.2), (2.8), and (3.5)–(3.7), we obtain the following:

$$\begin{aligned} E(\varphi) = E(\varphi_0) &= \int |\nabla\varphi_0|^2 + \sum_{i=1}^{k} x_i^2|\varphi_0|^2 dx + \frac{\alpha}{2}\int (|x|^{-\nu} * |\varphi_0|^2)|\varphi_0|^2 dx \\ &= a^2b^2 \int |\nabla Q|^2 dx + a^2b^{-2}\int \sum_{i=1}^{k} x_i^2|Q|^2 dx + \frac{\alpha}{2}a^4b^2 \int (|x|^{-\nu} * |Q|^2)|Q|^2 dx \\ &= a^2b^2 \int |\nabla Q|^2 dx + a^2b^{-2}\int \sum_{i=1}^{k} x_i^2|Q|^2 dx + \alpha a^4 b^2 \int |\nabla Q|^2 dx \\ &= (1+\alpha a^2)a^2b^2 \int |\nabla Q|^2 dx + a^2b^{-2}\int \sum_{i=1}^{k} x_i^2|Q|^2 dx \\ &= a^2b^2\Big((1+\alpha a^2)\int |\nabla Q|^2 dx + b^{-4}\int \sum_{i=1}^{k} x_i^2|Q|^2 dx\Big) \\ &< 0. \end{aligned}$$

Thus, it follows from Proposition 2.5 that the solution $\varphi(t,x)$ of Eq (1.1) blows up in a finite time.

3.2. The L^2-supercritical case

This part is concerned with the proof of Theorems 1.4 and 1.5.

Proposition 3.1. *Let* $2 \le \nu < min\{4, N\}$*; then,* $d > 0$.

Proof. We divide the proof into three steps: First, we demonstrate that M is not empty; second, we prove $d_M > 0$ using the Gagliardo-Nirenberg inequality and we prove $d_B > 0$ based on the continuity of the function; and finally it is convenient to justify $d > 0$ according to the definition of d.
Step 1. We prove $M \neq \emptyset$. According to Lemmas 2.3 and 2.4, there exists $u \in \Sigma \setminus \{0\}$ such that u is a solution of Eq (2.6). By multiplying both sides of Eq (2.6) with u and integrating over $\mathbb{R}^N$, we obtain the following:

$$\|\nabla u\|_2^2 + \|u\|_2^2 = -\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx. \tag{3.9}$$

It follows from (3.9) that $S(u) = 0$. Moreover, by taking the inner product of Eq (2.6) with $x \cdot \nabla u$, we have the following Pohozǎev identity:

$$\frac{2-N}{2}\|\nabla u\|_2^2 - \frac{N}{2}\|u\|_2^2 + \frac{\nu - 2N}{4}\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx = 0. \tag{3.10}$$

From (3.9) and (3.10), one has the following:

$$\frac{2-N}{2}\|\nabla u\|_2^2 + \frac{N}{2}\|u\|_2^2 + \frac{N}{2}\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx = \frac{\nu-2N}{4}\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx,$$
$$\|\nabla u\|_2^2 = -\frac{\nu}{4}\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx,$$

which implies that $P(u) = 0$. Thus, there exists $u \in \Sigma \setminus \{0\}$ such that $S(u) = 0$ and $P(u) = 0$.
Let $\phi(x) = \mu^{\frac{2+N-\nu}{2}} u(\mu x)$, $\mu > 0$. By some simple computations, we obtain the following

$$S(\phi(x)) = \mu^{4-\nu}\Big(\|\nabla u\|_2^2 + \alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx\Big) + \mu^{2-\nu}\|u\|_2^2,$$
$$P(\phi(x)) = \mu^{4-\nu}\Big(\|\nabla u\|_2^2 + \frac{\nu}{4}\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx\Big) = \mu^{4-\nu}P(u).$$

According to $P(u) = 0$, we have $P(\phi(x)) = 0$ for any $\mu > 0$. Moreover, in accordance with (3.9), one has the following:

$$\begin{aligned} S(\phi(x)) &= -\mu^{4-\nu}\|u\|_2^2 + \mu^{2-\nu}\|u\|_2^2 \\ &= (1-\mu^2)\mu^{2-\nu}\|u\|_2^2. \end{aligned}$$

Thus, there exists $\mu > 1$ such that $S(\phi(x)) < 0$ and $P(\phi(x)) = 0$, which implies $M \neq \varnothing$.
Step 2. We prove $d_M > 0$. Let $u \in M$; then, $S(u) < 0$ and $P(u) = 0$. Thus, $u \neq 0$. Since $P(u) = 0$, we have the following:

$$I(u) = (\frac{1}{2} - \frac{1}{\nu}) \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 + \sum_{i=1}^{k} x_i^2 |u|^2 dx. \tag{3.11}$$

It follows from $2 \le \nu < min\{4, N\}$ and $u \neq 0$ that $I(u) > 0$ for any $u \in M$. Thus, by (1.9), we obtain $d_M \ge 0$. In the following, we will divide the proof into two situations: The L^2-supercritical case and the L^2-critical case.

First, we consider the L^2-supercritical case when $2 < \nu < min\{4, N\}$. In this case, it follows from (2.3) that

$$\begin{aligned} \int (|x|^{-\nu} * |u|^2)|u|^2 dx &\le C\Big(\int |u(x)|^{\frac{4N}{2N-\nu}} dx\Big)^{\frac{2N-\nu}{N}} \\ &\le C\Big(\big(\int |\nabla u|^2 + |u|^2 dx\big)^{\frac{2N}{2N-\nu}}\Big)^{\frac{2N-\nu}{N}} \\ &= C\Big(\int |\nabla u|^2 + |u|^2 dx\Big)^2. \end{aligned}$$

Thus, due to $S(u) < 0$, one has

$$\int |\nabla u|^2 + |u|^2 dx < -\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx \le C(\int |\nabla u|^2 + |u|^2 dx)^2,$$

which yields that

$$\int |\nabla u|^2 + |u|^2 dx \ge C > 0.$$

Again, for $\nu > 2$, we have

$$I(u) \ge (\frac{1}{2} - \frac{1}{\nu}) \int |\nabla u|^2 + |u|^2 dx > (\frac{1}{2} - \frac{1}{\nu})C > 0, \ \textit{for any } u \in M, \tag{3.12}$$

which implies $d_M > 0$ for $2 < \nu < min\{4, N\}$.

Now, we deal with the L^2-critical case when $\nu = 2$. Assume $d_M = 0$; then, we infer that there exists a sequence $\{u_n\} \subset M$ such that $P(u_n) = 0$, $S(u_n) < 0$ and $I(u_n) \to 0$ as $n \to \infty$ with respect to the definition of d_M. Since $\nu = 2$, one can derive from (3.11) that

$$\int |u_n|^2 dx \to 0, \quad \int \sum_{i=1}^{k} x_i^2 |u_n|^2 dx \to 0, \quad as\, n \to \infty. \tag{3.13}$$

On the other hand, it follows from $S(u_n) < 0$ and (2.4) that

$$\int (|\nabla u_n|^2 + |u_n|^2) dx < -\alpha \int (|x|^{-\nu} * |u_n|^2)|u_n|^2 dx \leq C \int |\nabla u_n|^2 dx \int |u_n|^2 dx. \tag{3.14}$$

When n is sufficiently large, from (3.13), one has that

$$\int (|\nabla u_n|^2 + |u_n|^2) dx > C \int |\nabla u_n|^2 dx \int |u_n|^2 dx. \tag{3.15}$$

It is obvious that (3.15) contradicts (3.14). Thus, $d_M > 0$ for $\nu = 2$. In summary, we have $d_M > 0$ for $2 \leq \nu < min\{4, N\}$.

Step 3. We justify $d_B > 0$. For $u \in B$, we have $S(u) = 0$; then,

$$\int |\nabla u|^2 + |u|^2 dx = -\alpha \int (|x|^{-\nu} * |u|^2)|u|^2 dx,$$

which means

$$I(u) = \frac{1}{4} \int |\nabla u|^2 + |u|^2 dx + \frac{1}{2} \int \sum_{i=1}^{k} x_i^2 |u|^2 dx > 0.$$

Therefore, $d_B > 0$. This, together with Step 2, implies that the proposition holds true. □

Proposition 3.2. *The K, K_+, R_-, R_+ are invariant sets of Eq (1.1), that is, if $\varphi_0 \in K$, K_+, R_- or R_+, then the solution $\varphi(t, x)$ of Eq (1.1) also satisfies either $\varphi(t, x) \in K$, K_+, R_- or R_+ for any $t \in [0, T)$.*

Proof. First, we prove that $K \neq \emptyset$. According to the preceding discussion, we know that there exist $u \in \Sigma \setminus \{0\}$ such that u is a solution of Eq (2.9). It is clear that $S(u) = 0$ by multiplying both sides of Eq (2.9) by Δu. Moreover, from Eq (2.9), we have the Pohožaev identity as follows:

$$\frac{N-2}{2} \|\nabla u\|_2^2 + \frac{N}{2} \|\nabla u\|_2^2 = \frac{(\nu - 2N)\alpha}{4} \int (|x|^{-\nu} * |u|^2)|u|^2 dx, \tag{3.16}$$

which is obtained from multiplying Eq (3.16) by $x\nabla u$. Note that $S(u) = 0$; thus, $P(u) = 0$. Then, depending the definition on (1.8–1.10), one has the following:

$$\begin{aligned} I(\vartheta u) &= \frac{1}{2}\vartheta^2 \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2 + \sum_{i=1}^{k} x_i^2 |u|^2) dx + \frac{1}{4}\alpha\vartheta^4 \int_{\mathbb{R}^N} (|x|^{-\nu} * |u|^2)|u|^2 dx, \\ S(\vartheta u) &= \vartheta^2 \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx + \alpha\vartheta^4 \int_{\mathbb{R}^N} (|x|^{-\nu} * |u|^2)|u|^2 dx, \end{aligned}$$

$$P(\vartheta u) \;=\; \vartheta^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\nu}{4}\alpha\vartheta^4 \int_{\mathbb{R}^N} (|x|^{-\nu} * |u|^2)|u|^2 dx.$$

Since $d > 0$, for a large enough $\vartheta >> 1$, we always have that $S(\vartheta u) < 0$, $P(\vartheta u) < 0$, and $I(\vartheta u) < d$. In other words, $\vartheta u \in K$, and so $K \neq \emptyset$.

Next, we prove that K is an invariant set of Eq (1.1). Let $\varphi_0 \in \Sigma$ and $\varphi(t, x) = \varphi$ be the corresponding solution of Eq (1.1). On the basis of the conservation of mass and the conservation of energy, one has the following:

$$I(\varphi) = I(\varphi_0),\ for\ any\ t\ \in [0, T). \tag{3.17}$$

Thus, $I(\varphi_0) < d$ implies that $I(\varphi) < d$ for any $t \in [0, T)$.

(i) In the following, we demonstrate that $S(\varphi) < 0$ for any $t \in [0, T)$. If otherwise, by the continuity of $S(\varphi)$ on t, there exists $t_0 \in [0, T)$ such that $S(\varphi(t_0, x)) = 0$. By (3.17), we have $\varphi(t_0, x) \neq 0$. By the definition of B and (1.10), one has $\varphi(t_0, x) \in B$; then, $I(\varphi(t_0, x)) \geq d_B \geq d$. This is contradictory to $I(\varphi) < d$ for $t \in [0, T)$. Thus, $S(\varphi) < 0$ for all $t \in [0, T)$.

(ii) Next, we justify that $P(\varphi) < 0$ for any $t \in [0, T)$. Similarly, it is clear that there exists $t_0 \in [0, T)$ such that $P(\varphi(t_0, x)) = 0$ by the continuity of $P(\varphi)$ on t if $P(\varphi)$ is not constantly less than 0. From (i), we see that $\varphi(t, x) \in M$; thus, $I(\varphi(t, x)) \geq d_M \geq d$. This is contradictory to $I(\varphi) < d$ for $t \in [0, T)$. Thus, $P(\varphi) < 0$ for all $t \in [0, T)$.

Combining (i) and (ii), we get $\varphi(t, x) \in K$ for any $t \in [0, T)$. Similar to the proof above, we can also show that K_+, R_- , R_+ are invariant manifolds. □

Next, we shall apply the cross-constrained variational approach to investigate the sharp condition of global existence for Eq (1.1).

Proof of Theorem 1.4. First, we deal with the case where $\varphi_0 \in K_+$. According to Propositions 2.1 and 3.2, the initial-value problem (1.1) possesses a unique solution $\varphi(t, x) \in K_+$ for an arbitrary $t \in [0, T)$. Then, for all $t \in [0, T)$, we have $I(\varphi) < d$ and $P(\varphi) > 0$. This implies the following:

$$(\frac{1}{2} - \frac{1}{\nu}) \int |\nabla\varphi|^2 dx + \frac{1}{2} \int |\varphi|^2 + \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < d. \tag{3.18}$$

In the following, it is sufficient to give the proof on the global existence of the solution in two situations: The L^2-critical case and the L^2-supercritical case.

First, we discuss the L^2-critical case when $\nu = 2$. By (3.18), we obtain the following:

$$\frac{1}{2} \int |\varphi|^2 + \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < d. \tag{3.19}$$

Let $\varphi^\omega(t, x) = \omega^{\frac{2N-\nu}{4}} \varphi(t, \omega x)$; then, (1.8) gives us the following:

$$P(\varphi^\omega(x)) = \omega \int |\nabla\varphi|^2 dx + \frac{1}{2}\alpha \int (|x|^{-\nu} * |\varphi|^2)|\varphi|^2 dx.$$

Since $P(\varphi) > 0$, then one can find $0 < \omega_1 < 1$ such that $P(\varphi^{\omega_1}(x)) = 0$. According to (1.7) and (1.8), one has the following:

$$\omega_1^{\frac{4-\nu}{2}} \int |\nabla\varphi|^2 dx = -\frac{\nu}{4}\alpha \int (|x|^{-\nu} * |\varphi|^2)|\varphi|^2 dx. \tag{3.20}$$

Thus, by (3.19), we obtain the following:

$$I(\varphi^{\omega_1}(x)) < \omega_1^{\frac{-\nu}{2}} d.$$

For $S(\varphi^{\omega_1}(x))$, we will discuss the following two possibilities: $S(\varphi^{\omega_1}(x)) < 0$ and $S(\varphi^{\omega_1}(x)) \geq 0$. For the case when $S(\varphi^{\omega_1}(x)) < 0$ and $P(\varphi^{\omega_1}(x)) = 0$, then by the definitions of d and d_M, we obtain the following:

$$I(\varphi^{\omega_1}) \geq d_M \geq d > I(\varphi);$$

then, we have $I(\varphi) - I(\varphi^{\omega_1}) < 0$, i.e.,

$$(\frac{1}{2} - \frac{1}{\nu}\omega_1^{\frac{4-\nu}{2}}) \int |\nabla\varphi|^2 dx + \frac{1}{2}(1 - \omega_1^{-\frac{\nu}{2}}) \int |\varphi|^2 dx + \frac{1}{2}(1 - \omega_1^{-3}) \int \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < 0.$$

Thus, from (3.19), one has $\int |\nabla\varphi|^2 dx < \frac{2}{\omega_1} d$; then, we obtain the following:

$$\int |\nabla\varphi|^2 dx < C. \tag{3.21}$$

For the case when $S(\varphi^{\omega_1}(x)) \geq 0$, according to (3.20), one has the following:

$$I(\varphi^{\omega_1}) - \frac{1}{4}S(\varphi^{\omega_1}) = \frac{1}{4}\omega_1^{-\frac{\nu}{2}} \int |\varphi|^2 dx + \frac{1}{2}\omega_1^{-3} \int \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx + \frac{4-\nu}{4\nu}\omega_1^{\frac{4-\nu}{2}} \int |\nabla\varphi|^2 dx < \omega_1^{-\frac{\nu}{2}} d.$$

Therefore,

$$\int |\nabla\varphi|^2 dx < \frac{4\nu}{(4-\nu)\omega_1^2} d. \tag{3.22}$$

Thus, for $\nu = 2$, together with (3.21) and (3.22), we conclude that the solution $\varphi(t, x)$ is global in time by Proposition 2.1.

Next, when $2 < \nu < min\{4, N\}$, it follows from (3.18) that

$$\frac{1}{4}(\frac{\nu - 2}{2}) \int |\nabla\varphi|^2 dx + \frac{1}{2} \int |\varphi|^2 + \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < d,$$

which indicates that there exists $C > 0$ such that

$$\int |\nabla\varphi|^2 + \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < C.$$

Therefore, the solution $\varphi(t, x)$ is uniformly bounded in Σ for all $t \in [0, \infty)$. According to Proposition 2.1, it suffices to show that the solution $\varphi(t, x)$ to Eq (1.1) globally exists for $t \in [0, \infty)$.

Now, we consider $\varphi_0 \in R_+$. Let $\varphi_0 \in R_+$; then, $\varphi(t, x) \in R_+$ for $t \in [0, T)$, that is, $I(\varphi) < d, S(\varphi) > 0$ for $t \in [0, T)$, then one has the following:

$$\frac{1}{4} \int |\nabla\varphi|^2 + |\varphi|^2 dx + \frac{1}{2} \int \sum_{i=1}^{k} x_i^2 |\varphi|^2 dx < d.$$

Thus, the solution of $\varphi(t,x)$ of Eq (1.1) globally exists. This completes the proof.
Proof of Theorem 1.5. Suppose $\varphi_0 \in K$; from Proposition 3.2, we know that the solution $\varphi(t,x)$ of Eq (1.1) satisfies $\varphi(t,x) \in K$ for $t \in [0,T)$. We denote $\varphi(t,x) = \varphi$. For $W(t) = \int |x|^2|\varphi|^2dx$, in light of [6], from (1.1) and (2.2), we obtain the following:

$$W'(t) = 4Im\int x\cdot\nabla\varphi\overline{\varphi}dx,$$

$$W''(t) = 8E(\varphi_0) + \alpha(2\nu-4)\int(|x|^{-\nu} * |\varphi|^2)|\varphi|^2dx - 16\int\sum_{i=1}^{k} x_i^2|\varphi|^2dx.$$

In the L^2-supercritical case, according to $2 < \nu < min\{4, N\}$, if $(2\nu-4)\alpha < 0$, then one has the following:

$$W''(t) < 8P(\varphi),\ for\ \ t \in [0,T).$$

In the mass-critical case $\nu = 2$, $W''(t) = 8E(\varphi_0) - 16\int\sum_{i=1}^{k} x_i^2|\varphi|^2dx$. It follows from (1.9) that

$$W''(t) < 8P(\varphi),\ for\ \ t \in [0,T).$$

Thus, for $t \in [0,T)$, φ satisfies that $P(\varphi) < 0, S(\varphi) < 0$. For $\mu > 0$, take $\varphi_\mu = \mu^{\frac{2N-\nu}{4}}\varphi(\mu x)$; then,

$$S(\varphi_\mu) = \mu^{\frac{4-\nu}{2}}\|\nabla\varphi\|_2^2 + \mu^{\frac{-\nu}{2}}\|\varphi\|_2^2 + \alpha\int(|x|^{-\nu} * |\varphi|^2)|\varphi|^2dx,$$
$$P(\varphi_\mu) = \mu^{\frac{4-\nu}{2}}\|\nabla\varphi\|_2^2 + \frac{\nu\alpha}{4}\int(|x|^{-\nu} * |\varphi|^2)|\varphi|^2dx.$$

Since $2 \le \nu < min\{4, N\}$, $P(\varphi) < 0$, then there exists $\mu_1 > 1$ such that $P(\varphi_{\mu_1}) = 0$ by the continuity of $P(u)$, and for $\mu \in [1,\mu_1)$, $P(\varphi_\mu) < 0$. Moreover, $S(\varphi_\mu)$ may have the following two possibilities:
(i) $S(\varphi_\mu) < 0$ for $\mu \in [1,\mu_1]$; and
(ii) there exists $\mu_2 \in (1,\mu_1]$ such that $S(\varphi_{\mu_2}) = 0$.

For the case (i), we have $P(\varphi_{\mu_1}) = 0$ and $S(\varphi_{\mu_1}) < 0$; then, $\varphi_{\mu_1} \in M$, $I(\varphi_{\mu_1}) \ge d_M \ge d$. Furthermore, one has the following:

$$I(\varphi_\mu) = \frac{1}{2}\mu^{\frac{4-\nu}{2}}\|\nabla\varphi\|_2^2 + \frac{1}{2}\mu^{\frac{-\nu}{2}}\|\varphi\|_2^2 + \frac{1}{2}\mu^{\frac{2N-\nu}{2}}\int\sum_{i=1}^{k} x_i^2|\varphi(\omega x)|^2dx + \frac{\alpha}{4}\int(|x|^{-\nu} * |\varphi|^2)|\varphi|^2dx$$
$$= \frac{1}{2}\mu^{\frac{4-\nu}{2}}\|\nabla\varphi\|_2^2 + \frac{1}{2}\mu^{\frac{-\nu}{2}}\|\varphi\|_2^2dx + \frac{1}{2}\mu^{-\frac{\nu+4}{2}}\int\sum_{i=1}^{k} x_i^2|\varphi|^2dx + \frac{\alpha}{4}\int(|x|^{-\nu} * |\varphi|^2)|\varphi|^2dx.$$

$$I(\varphi) - I(\varphi_{\mu_1}) = \frac{1}{2}(1-\mu_1^{\frac{4-\nu}{2}})\|\nabla\varphi\|_2^2 + \frac{1}{2}(1-\mu_1^{-\frac{\nu}{2}})\|\varphi\|_2^2 + \frac{1}{2}(1-\mu_1^{-\frac{\nu+4}{2}})\int\sum_{i=1}^{k} x_i^2|\varphi|^2dx. \tag{3.23}$$

$$P(\varphi) - P(\varphi_{\mu_1}) = (1-\mu_1^{\frac{4-\nu}{2}})\|\nabla\varphi\|_2^2. \tag{3.24}$$

Taking that $\mu_1 > 1$ and $2 \le \nu < min\{4, N\}$ into account, we infer from (3.23) and (3.24) that

$$I(\varphi) - I(\varphi_{\mu_1}) \ge \frac{1}{2}(P(\varphi) - P(\varphi_{\mu_1})) = \frac{1}{2}P(\varphi). \tag{3.25}$$

For the case (ii), we have $S(\varphi_{\mu_2}) = 0$ and $P(\varphi_{\mu_2}) \leq 0$; then, $\varphi_{\mu_2} \in B$ and $I(\varphi_{\mu_2}) \geq d_B \geq d$. Using the similar procedure as case (i), one can derive the following:

$$I(\varphi) - I(\varphi_{\mu_2}) \geq \frac{1}{2}(P(\varphi) - P(\varphi_{\mu_2})) = \frac{1}{2}P(\varphi). \tag{3.26}$$

Since $I(\varphi_{\mu_1}) > d, I(\varphi_{\mu_2}) > d$, from (3.25) and (3.26), we obtain the following:

$$P(\varphi) < 2(I(\varphi_0) - d).$$

Then, by $I(\varphi) = I(\varphi_0), \varphi_0 \in K$, one has the following:

$$W''(t) < 8P(\varphi) < 16(I(\varphi_0) - d) < 0. \tag{3.27}$$

Then, by the convexity method introduced in [9], there must exist a time $0 < T < \infty$ such that $W(T) = 0$. Then, from Proposition 2.1, we have the following:

$$\lim_{t\to T} \|\varphi\|_\Sigma = \infty. \tag{3.28}$$

Thus, the proof is completed.

4. Orbital stability of standing waves

In this section, we focus on the orbital stability of normalized standing waves of (1.1). For further research, we first introduce the non-vanishing conclusion of the minimizing sequence.

Lemma 4.1. *Let $1 \leq k < N$ and $0 < \nu \leq 2$. Suppose $\{u_n\}$ is a minimizing sequence of (1.12); then, there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} \int (|x|^{-\nu} * |u_n|^2)|u_n|^2 dx > \delta. \tag{4.1}$$

Proof. Let us prove (4.1) by contradiction. If not, there exists a subsequence u_{n_j} such that

$$\lim_{j\to\infty} \int (|x|^{-\nu} * |u_{n_j}|^2)|u_{n_j}|^2 dx = 0.$$

Consequently, we obtain the following:

$$m(c) = \lim_{j\to\infty} E(u_{n_j}) = \lim_{j\to\infty} \int |\nabla u_{n_j}|^2 dx + \int \sum_{j=1}^{k} x_j^2 |u_{n_j}|^2 dx \geq \Lambda_0 c^2. \tag{4.2}$$

On the other hand, since the space $H = \{u \in H^1(\mathbb{R}^k), \int_{\mathbb{R}^k} \sum_{j=1}^{k} x_j^2 |u|^2 dx < \infty\}$ is compactly embedded in $L^2(\mathbb{R}^k)$, it is standard to show that λ_0 is achieved by some $\omega \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^k} |\omega|^2 dx = 1$. Let $\psi \in H^1(\mathbb{R}^{N-k})$ satisfy $\int_{\mathbb{R}^{N-k}} |\psi(x)|^2 dx = c^2$ and set

$$u_\lambda(x) = \omega(x_1, \cdots, x_k)\psi_\lambda(x_{k+1}, \cdots, x_N),$$

$$\psi_\lambda(x_{k+1}, \cdots, x_N) = \lambda^{\frac{N-k}{2}} \psi(\lambda x_{k+1}, \cdots, \lambda x_N).$$

Then, $u_\lambda \in S(c)$ for all $\lambda > 0$. It follows that

$$\begin{aligned} E(u_\lambda) &= \int |\nabla u_\lambda|^2 + \sum_{i=1}^{k} x_i^2 |u_\lambda|^2 dx + \frac{\alpha}{2}\int (|x|^{-\nu} * |u_\lambda|^2)|u_\lambda|^2 dx \\ &= I_1 + I_2 + \frac{\alpha}{2}\int (|x|^{-\nu} * |u_\lambda|^2)|u_\lambda|^2 dx, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} I_1 &= \int |\nabla u_\lambda|^2 dx \\ &= \int_{\mathbb{R}^k} |\nabla \omega|^2 |\psi_\lambda|^2 dx + \int_{\mathbb{R}^{N-k}} |\omega|^2 |\nabla \psi_\lambda|^2 dx \\ &= c^2 \int_{\mathbb{R}^k} |\nabla \omega|^2 dx_1 \cdots dx_k + \lambda^2 \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1}\cdots x_{N-k}} \psi_\lambda|^2 dx_{k+1} \cdots dx_N, \end{aligned}$$

$$\begin{aligned} I_2 &= \int \sum_{i=1}^{k} x_i^2 |u_\lambda|^2 dx \\ &= \int_{\mathbb{R}^k} \sum_{i=1}^{k} {x_i}^2 |\omega|^2 dx \int_{\mathbb{R}^{N-k}} |\psi_\lambda|^2 dx_{k+1} \cdots dx_N, \end{aligned}$$

which implies that (4.3) can be written as

$$\begin{aligned} E(u_\lambda) &= \Lambda_0 c^2 + \lambda^2 \int_{\mathbb{R}^{N-k}} |\nabla_{x_{k+1}\cdots x_{N-k}} \psi_\lambda|^2 dx_{k+1} \cdots dx_N + \frac{\alpha}{2}\int (|x|^{-\nu} * |\omega|^2 |\psi_\lambda|^2)|u_\lambda|^2 dx \\ &= \Lambda_0 c^2 + \lambda^2 \int_{\mathbb{R}^{N-k}} |\nabla_{\lambda_{x_{k+1}}\cdots x_N} \psi^2 dx_{k+1} \cdots dx_N \\ &+ \frac{\alpha}{2}\lambda^\nu \int \frac{|\omega(x)|^2 |\omega(y)|^2 |\psi_x|^2 |\psi_y|^2}{[(\lambda x_1 - \lambda y_1)^2 + \cdots + (\lambda x_k - \lambda y_k)^2 + (x_{k+1} - y_{k+1})^2 + \cdots + (x_N - y_N)^2]^{\frac{\nu}{2}}} dx \\ &< \Lambda_0 c^2, \end{aligned}$$

for a sufficiently small $\lambda > 0$. Notice that $u_\lambda \in B(r)$ for $\lambda > 0$ sufficiently small; consequently, we obtain the following:

$$m(c, r) \le E(u_\lambda) < \Lambda_0 c^2.$$

This is a contradiction with (4.2). This completes the proof. □

4.1. The L^2-subcritical and L^2-critical cases

Using the profile decomposition of bounded sequences in Σ, we can solve the variational problem (1.12) and obtain the following theorem.

Theorem 4.2. *Let $c > 0$ if $0 < \nu < 2$ or $0 < \sqrt{-\alpha}c < \|Q\|_2$ if $\nu = 2$, where $Q(x)$ is the ground state solution of Eq (1.13). Then, there must exist $u \in S(c)$ which satisfies $m(c) = E(u)$.*

Proof. We will prove this theorem in four steps.
Step 1. We prove that the minimalized problem (1.12) is well-defined and every minimizing sequence of (1.12) is bounded in Σ. For $u \in \Sigma$, from the inequality (3.13) that

$$\begin{aligned} E(u) &= \|\nabla u\|_2^2 + \int \sum_{i=1}^{k} x_i^2 |u|^2 dx + \frac{\alpha}{2} \int (|x|^{-\nu} * |u|^2)|u|^2 dx \\ &\geq \|u\|_{\dot{\Sigma}}^2 + \frac{\alpha}{2} C_{\nu,2} \|\nabla u\|_2^{\nu} \|u\|_2^{4-\nu}. \end{aligned}$$

For $0 < \nu < 2$, from Young's inequality, we deduce that for all $0 < \varepsilon < \frac{1}{2}$, there exists a constant $C(\varepsilon, C_{\nu,2}, c)$ such that

$$\frac{1}{2} C_{\nu,2} \|\nabla u\|_2^{\nu} \|u\|_2^{4-\nu} \leq \frac{1}{2} C_{\nu,2} \|u\|_{\dot{\Sigma}}^2 \|u\|_2^{4-\nu} \leq \varepsilon \|u\|_{\dot{\Sigma}}^2 + C(\varepsilon, C_{\nu,2}, c).$$

This implies that

$$E(u) \geq (1 + \alpha\varepsilon) \|u\|_{\dot{\Sigma}}^2 + \alpha C(\varepsilon, C_{\nu,2}, c). \tag{4.4}$$

When $\nu = 2$ and $0 < \sqrt{-\alpha c} < \|Q\|_2$, it follows from the inequality (3.13) that

$$\begin{aligned} E(u) &\geq \|u\|_{\dot{\Sigma}}^2 + \frac{\alpha}{\|Q\|_2^2} \|\nabla u\|_2^2 \|u\|_2^2 \\ &\geq \|u\|_{\dot{\Sigma}}^2 + \alpha \frac{\|u\|_2^2}{\|Q\|_2^2} \|u\|_{\dot{\Sigma}}^2 \\ &= \frac{\|Q\|_2^2 + \alpha c}{\|Q\|_2^2} \|u\|_{\dot{\Sigma}}^2 \\ &> 0. \end{aligned} \tag{4.5}$$

Therefore, $E(u)$ is bounded from below and the variational problem (1.12) is well-defined. Moreover, we see from (4.4) and (4.5) that every minimizing sequence of (1.12) is bounded in Σ.
Step 2. Applying the profile decomposition of bounded sequences in Σ, we shall prove that there exists only one term $U^{j_0} \neq 0$ in the decomposition (4.6). Applying Lemma 2.7 to the minimizing sequence $\{u_n\}_{n=1}^{\infty}$, u_n can be decomposed as follows:

$$u_n(x) = \sum_{j=1}^{l} \tau_{x_n^j} U^j(x) + r_n^l, \tag{4.6}$$

with $\limsup_{n\to\infty} \|r_n^l\|_q \to 0$ as $l \to \infty$ when $q \in [2, \frac{2N}{N-2})$. Injecting (4.6) into the energy functional $E(u_n)$, it follows from (4.6) and (2.9)–(2.14) that

$$E(u_n) = \sum_{j=1}^{l} E(\tau_{x_n^j} U^j) + E(r_n^l) + o(1), \ \textit{as } n \to \infty \textit{ and } l \to \infty. \tag{4.7}$$

For every $\tau_{x_n^j} U^j(x) (1 \leq j \leq l)$, taking the scaling transform $\tau_{x_n^j} U_{\lambda_j}^j(x) = \lambda_j \tau_{x_n^j} U^j(x)$ with $\lambda_j = \frac{c}{\tau_{x_n^j} \|U^j\|_2}$, it easily follows that

$$\|\tau_{x_n^j} U_{\lambda_j}^j\|_2 = c,$$

$$
\begin{aligned}
E(\tau_{x_n^j}U^j_{\lambda_j}) &= \|\nabla \tau_{x_n^j}U^j_{\lambda_j}\|_2^2 + \int V(x)|\tau_{x_n^j}U^j_{\lambda_j}|^2dx + \frac{\alpha}{2}\int (|x|^{-\nu} * |\tau_{x_n^j}U_{\lambda_j}|^2)|\tau_{x_n^j}U_{\lambda_j}|^2dx \\
&= {\lambda_j}^2\|\nabla \tau_{x_n^j}U^j\|_2^2 + {\lambda_j}^2\int V(x)|\tau_{x_n^j}U^j|^2dx + \frac{\alpha}{2}{\lambda_j}^4\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx \\
&= {\lambda_j}^2E(\tau_{x_n^j}U^j) + \frac{\alpha}{2}{\lambda_j}^2({\lambda_j}^2-1)\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx, \qquad (4.8)
\end{aligned}
$$

which means that

$$
E(\tau_{x_n^j}U^j) = \frac{E(\tau_{x_n^j}U^j_{\lambda_j})}{{\lambda_j}^2} - \frac{\alpha}{2}{\lambda_j}^2({\lambda_j}^2-1)\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx. \qquad (4.9)
$$

Similarly, one can get the estimate of $E(r_n^l)$ as follows:

$$
\begin{aligned}
E(r_n^l) &= \frac{\|r_n^l\|_2^2}{c^2}E(\frac{c}{\|r_n^l\|_2}r_n^l) + \frac{\alpha}{2}(1-\frac{c^2}{\|r_n^l\|_2^2})\int (|x|^{-\nu} * |r_n^l|^2)|r_n^l|^2dx + o(1) \\
&\geq \frac{\|r_n^l\|_2^2}{c^2}E(\frac{c}{\|r_n^l\|_2}r_n^l) + o(1). \qquad (4.10)
\end{aligned}
$$

Since $\|\tau_{x_n^j}U^j_{\lambda_j}\|_2 = \|\frac{c}{\|r_n^l\|_2}r_n^l\|_2 = c$, we deduce from the definition of $m(c)$ that

$$
E(\tau_{x_n^j}U^j_{\lambda_j}) \geq m(c),\ \ and\ E(\frac{c}{\|r_n^l\|_2}r_n^l) \geq m(c).
$$

Thus, we infer from (4.7)–(4.10) that

$$
\begin{aligned}
E(u_n) \geq& \sum_{j=1}^{l}\Big(\frac{E(\tau_{x_n^j}U^j_{\lambda_j})}{{\lambda_j}^2} - \frac{\alpha}{2}({\lambda_j}^2-1)\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx\Big) \\
&+ \frac{\|r_n^l\|_2^2}{c^2}E(\frac{c}{\|r_n^l\|_2}r_n^l) + o(1) \\
\geq& \sum_{j=1}^{l}\frac{m(c)}{{\lambda_j}^2} - \frac{\alpha}{2}\inf_{j\geq 1}({\lambda_j}^2-1)(\sum_{j=1}^{l}\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx) \\
&+ \frac{\|r_n^l\|_2^2}{c^2}E(\frac{c}{\|r_n^l\|_2}r_n^l) + o(1) \\
=& \sum_{j=1}^{l}\frac{\|U^j\|_2^2}{c^2} - \frac{\alpha}{2}\inf_{j\geq 1}({\lambda_j}^2-1)\Big(\sum_{j=1}^{l}\int (|x|^{-\nu} * |\tau_{x_n^j}U^j|^2)|\tau_{x_n^j}U^j|^2dx\Big) \\
&+ \frac{\|r_n^l\|_2^2}{c^2}E(\frac{c}{\|r_n^l\|_2}r_n^l) + o(1). \qquad (4.11)
\end{aligned}
$$

Since $\sum_{j=1}^{\infty}\|U^j\|_2^2$ is convergent, there exists $j_0 \geq 1$ such that

$$
\|U^{j_0}\|_2^2 = \sup_{j\geq 1}\|U^j\|_2^2,\ \ and\ \inf_{j\geq 1}\lambda_j = \lambda_{j_0} = \frac{c}{\|U^{j_0}\|_2}.
$$

Let $n \to \infty$ and $l \to \infty$ in (4.11); we deduce from Lemma 4.1 that

$$m(c) \geq m(c) - \frac{\alpha}{2}(\frac{c^2}{\|U^{j_0}\|_2} - 1)\delta,$$

which implies that $\|U^{j_0}\|_2^2 \geq c$. Hence, $\|U^{j_0}\|_2 = c$, and there exists only one term $U^{j_0} \neq 0$ in the decomposition (4.6). We consequently rewrite (4.6) as follows:

$$u_n(x) = \tau_{x_n^{j_0}} U^{j_0}(x) + r_n(x).$$

Due to $\|u_n\|_2 = \|U^{j_0}\|_2 + \|r_n\|_2 + o_n(1)$ and $\|u_n\|_2 = \|U^{j_0}\|_2 = c$, we get $\lim_{n\to\infty} \|r_n\|_2 = 0$. This shows that $r_n \to 0$ in L^2. This, together with the Gagliardo-Nirenberg inequality, implies that $\lim_{n\to\infty} \|r_n\|_{q+2}^{q+2} = 0$, for all $q \in (0, \frac{4}{N-2})$. We consequently obtain the following:

$$\int (|x|^{-\nu} * |r_n^l|^2)|r_n^l|^2 dx \to 0.$$

Applying the lower semi-continuity of norm, it follows that $\liminf_{n\to\infty} E(r_n) \geq 0$; thus,

$$\begin{aligned} \liminf_{n\to\infty} E(\tau_{x_n^{j_0}} U^{j_0}) &\leq \liminf_{n\to\infty} E(\tau_{x_n^{j_0}} U^{j_0}) + \liminf_{n\to\infty} E(r_n) \\ &\leq \liminf_{n\to\infty} \Big(E(\tau_{x_n^{j_0}} U^{j_0}) + E(r_n) \Big) \\ &= \liminf_{n\to\infty} E(u_n) = m(c). \end{aligned}$$

On the other hand, since $\|\tau_{x_n}^{j_0} U^{j_0}\|_2 = \|U^{j_0}\|_2 = c$ for all $n \geq 1$, we have $E(\tau_{x_n}^{j_0} U^{j_0}) \geq m(c)$ for all $n \geq 1$. Therefore,

$$\liminf_{n\to\infty} E(\tau_{x_n}^{j_0} U^{j_0}) = m(c).$$

Step 3. We show that the sequence $\{x_n^{j_0}\}$ is bounded. Indeed, if it is not true, then up to a subsequence, we assume that $|x_n^{j_0}| \to \infty$ as $n \to \infty$. Without a loss of generality, we assume that U^{j_0} is continuous and compactly supported. We have the following:

$$\int (|x|^{-\nu} * |\tau_{x_n^{j_0}} U^{j_0}|^2)|\tau_{x_n^{j_0}} U^{j_0}|^2 dx \to 0, \ as\ n \to \infty.$$

This yields the following:

$$\liminf_{n\to\infty} E(\tau_{x_n}^{j_0} U^{j_0}) = \|U^{j_0}\|_{\dot{\Sigma}} = m(c).$$

By the definition of $E(U^{j_0})$, we obtain the following:

$$E(U^{j_0}) - \frac{\alpha}{2}\int (|x|^{-\nu} * |U^{j_0}|^2)|U^{j_0}|^2 dx = m(c),$$

which means $E(U^{j_0}) < m(c)$, which is a contradiction with $E(U^{j_0}) \geq m(c)$ due to $\|U^{j_0}\|_2^2 = c^2$. Therefore, the sequence $\{x_n^{j_0}\} \subseteq \mathbb{R}^{N-k}$ is bounded and up to a subsequence, we assume that $x_n^{j_0} \to x^{j_0}$ in $\mathbb{R}^{N-k}$ as $n \to \infty$.

Step 4. Conclusion. Now, we write $u_n(x) = \widetilde{U}^{j_0}(x) + \widetilde{r_n}(x)$, where $\widetilde{U}^{j_0}(x) = \tau_{x_n^{j_0}} U^{j_0}(x)$ and $\widetilde{r_n}(x) = \tau_{x_n^{j_0}} U^{j_0}(x) - \tau_{x_n^{j_0}} U^{j_0}(x) + r_n(x)$. Using the fact that $\|u_n\|_2 = \|U^{j_0}\|_2 = c$, it is easy to see that

$$\widetilde{r_n} \rightharpoonup 0 \textit{ in } \Sigma \textit{ and } \widetilde{r_n} \rightharpoonup 0 \textit{ in } L^2(\mathbb{R}^N).$$

Consequently, we obtain the following:

$$E(u_n) = E(\widetilde{U}^{j_0}) + E(\widetilde{r_n}) + o_n(1).$$

Again, using the lower semi-continuity of norm and the fact $\lim_{n\to\infty} \int (|x|^{-\nu} * |\widetilde{r_n}|^2)|\widetilde{r_n}|^2 dx = 0$, we get that $\liminf_{n\to\infty} E(\widetilde{r_n}) \geq 0$. Therefore, using the fact that $\|\widetilde{U}^{j_0}\|_2^2 = c$, we infer the following:

$$\begin{aligned} m(c) = \liminf_{n\to\infty} E(u_n) &\geq \liminf_{n\to\infty} \left(E(\widetilde{U}^{j_0}) + E(\widetilde{r_n})\right) \\ &\geq E(\widetilde{U}^{j_0}) + \liminf_{n\to\infty} E(\widetilde{r_n}) \\ &\geq E(\widetilde{U}^{j_0}) \geq m(c). \end{aligned}$$

This implies $E(\widetilde{U}^{j_0}) = m(c)$ and concludes the proof. □

Now, we are in a position to show that the standing waves to Eq (1.1) are orbitally stable with the help of Theorem 4.2.

Proof of Theorem 1.8. First, we see that the solution $\varphi(t, x)$ of (1.1) globally exists from Theorem 4.2. By contradiction, suppose that there exist ε_0 and a sequence $\{\varphi_{0,n}\}_{n=1}^{\infty}$ such that

$$\inf_{u\in\mathcal{M}_c} \|\varphi_{0,n} - u\|_\Sigma < \frac{1}{n}, \tag{4.12}$$

and there exist $\{t_n\}_{n=1}^{\infty}$ such that the corresponding solution sequence $\{\varphi_n(t_n)\}_{n=1}^{\infty}$ of (1.1) satisfies

$$\inf_{u\in\mathcal{M}_c} \|\varphi_n(t_n) - u\|_\Sigma \geq \varepsilon_0. \tag{4.13}$$

Next, we show that there exists $v \in \mathcal{M}_c$ which satisfies the following:

$$\lim_{n\to\infty} \|\varphi_{0,n} - v\|_\Sigma = 0.$$

Indeed, by (4.12), there exists $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}_c$ such that

$$\|\varphi_{0,n} - v_n\|_\Sigma < \frac{2}{n}. \tag{4.14}$$

Due to $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}_c$, $\{v_n\}$ is a minimizing sequence to (1.12). By the argument of Theorem 4.2, there exists $v \in \mathcal{M}_c$ such that

$$\lim_{n\to\infty} \|v_n - v\|_\Sigma = 0. \tag{4.15}$$

Then, the claim immediately follows from (4.14) and (4.15). Hence,

$$\lim_{n\to\infty} \|\varphi_{0,n}\|_2 = \|v\|_2 = c, \lim_{n\to\infty} E(\varphi_{0,n}) = E(v) = m(c).$$

By (2.1) and (2.2), we obtain the following:

$$\lim_{n\to\infty} \|\varphi_n(t_n)\|_2 = c, \quad \lim_{n\to\infty} E(u_n(t_n)) = E(v) = m(c).$$

According to Theorem 4.2, $\{\varphi_n(t_n)\}_{n=1}^{\infty}$ is bounded in Σ. Set $\widetilde{\varphi_n} = \frac{c\varphi_n(t_n)}{\|\varphi_n(t_n)\|_2}$; then, $\|\widetilde{\varphi_n}\|_2 = c$ and

$$\begin{aligned}
E(\widetilde{\varphi_n}) &= \frac{c^2}{\|\varphi_n(t_n)\|_2^2}\|\nabla\varphi_n(t_n)\|_\Sigma^2 + \frac{c^2}{\|\varphi_n(t_n)\|_2^2}\int\sum_{i=1}^{k} x_i^2|\varphi_n(t_n)|^2dx \\
&+ \frac{\alpha}{2}\frac{c^4}{\|\varphi_n(t_n)\|_2^4}\int(|x|^{-\nu} * |\varphi_n(t_n)|^2)|\varphi_n(t_n)|^2dx \\
&= \frac{c^2}{\|\varphi_n(t_n)\|_2^2}E(\varphi_n(t_n)) + \frac{\alpha}{2}\frac{c^2}{\|\varphi_n(t_n)\|_2^2}(\frac{c^2}{\|\varphi_n(t_n)\|_2^2} - 1)\int(|x|^{-\nu} * |\varphi_n(t_n)|^2)|\varphi_n(t_n)|^2dx,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} E(\widetilde{\varphi_n}) = E(\varphi_n(t_n)) = m(c).$$

Therefore, $\widetilde{\varphi_n}$ also becomes a minimizing sequence to (1.12). Then, by the argument of Theorem 4.2, there exists $\widetilde{v} \in \mathcal{M}_c$ such that

$$\widetilde{\varphi_n} \to \widetilde{v} \; in \; \Sigma.$$

From the definition of $\widetilde{\varphi_n}$, it follows that

$$\widetilde{\varphi_n} - \varphi_n(t_n) \to 0 \; in \; \Sigma.$$

Consequently, we obtain the following:

$$\varphi_n(t_n) \to \widetilde{v} \; in \; \Sigma,$$

which contradicts (4.13). Thus, we complete the proof.

4.2. The L^2-supercritical case

Lemma 4.3. *Let $1 \le k < N$ and $2 < \nu < min\{4, N\}$. Then, there exists $r_0 \ge 2\sqrt{k}$ such that for every given $r \ge r_0$, there exists $C_r \in (0, 1)$ such that $\forall c < C_r$,*

$$\inf_{u\in S(c)\bigcap B(\frac{rc}{2})} E(u) < \inf_{u\in S(c)\bigcap(B(r)\backslash B(rc))} E(u). \tag{4.16}$$

Proof. First, we claim that

$$S(c)\bigcap B(\frac{rc}{2}) \neq \emptyset. \tag{4.17}$$

Indeed, let $u \in \Sigma$ be such that $\|u\|_2 = 2$ and $\|u\|_\Sigma^2 = r_0$. Then, for all $c > 0$, taking $u_c = \frac{c}{2}u$, we have $\|u_c\|_2 = c$ and $\|u_c\|_\Sigma^2 = \frac{r_0 c}{2} < \frac{rc}{2}$, $\forall r > r_0$, namely $u_c \in S(c)\bigcap B(\frac{rc}{2})$; thus, (4.17) is verified.

Next, we prove $r_0 \ge 2\sqrt{k}$. Let $u \in S(c)\bigcap B(\frac{rc}{2})$; by a similar argument as (4.17), we have the following:

$$\|u\|_2^2 = c^2 \le \frac{2}{k}\sum_{j=1}^{k}\|x_j u\|_2\|u_{x_j}\|_2$$

$$
\begin{aligned}
&\le \max_{1\le j\le k}\frac{1}{k}\Big(\sum_{j=1}^{k}\|x_j u\|_2^2+\sum_{j=1}^{k}\|u_{x_j}\|_2^2\Big)\\
&\le \frac{1}{k}\|u\|_{\dot\Sigma}^2\\
&\le \frac{r^2c^2}{4k}.
\end{aligned}
$$

This implies $r \ge 2\sqrt{k}$. Finally, we prove (4.16). We deduce from the Gagliardo-Nirenberg inequality that

$$
\begin{aligned}
E(u) &\ge \|u\|_{\dot\Sigma}^2+\frac{\alpha}{2}C_{\nu,2}\|\nabla u\|_2^{\nu}\|u\|_2^{4-\nu}\\
&= \|u\|_{\dot\Sigma}^2+\frac{\alpha}{2}C_{\nu,2}c^{4-\nu}\|\nabla u\|_2^{\nu}\\
&\ge \|u\|_{\dot\Sigma}^2+\frac{\alpha}{2}C_{\nu,2}c^{4-\nu}\|u\|_{\dot\Sigma}^{\nu}\\
&= \alpha_c(\|u\|_{\dot\Sigma}),\ \forall u\in S(c),\\
E(u) &\le \|u\|_{\dot\Sigma}^2=\beta_c(\|u\|_{\dot\Sigma}),\ \forall u\in S(c),
\end{aligned}
$$

where $\varepsilon = 4-\nu, \delta = \nu-2$ and

$$
\alpha_c(t)=t^2(1+\frac{\alpha}{2}C_{\nu,2}c^{4-\nu}t^{\nu-2})=t^2(1+\frac{\alpha}{2}C_{\nu,2}c^{\varepsilon}t^{\delta}),\quad \beta_c(t)=t^2.
$$

It is sufficient to prove that there exists $0 < C_r << 1$ such that

$$
\beta_c(\frac{cr}{2})=\frac{c^2r^2}{4}<\frac{5}{16}c^2r^2\le \inf_{t\in(cr,c)}\alpha_c(t),\ \forall c<C_r,
$$

which completes the proof of lemma. □

Lemma 4.4. *Assume that* $1 \le k < N$ *and* $2 < \nu < min\{4, N\}$. *Let* $r > 0$ *and* $C_r > 0$ *be as in Lemma 4.3; then, for any* $0 < c_2 < c_1 < C_r$, *we have the following:*

$$
m(c,r) < m(\sqrt{c_1^2-c_2^2}, r)+m(c_2,r).
$$

Proof. First, let $\{V_n\} \subset S(c_2)\bigcap B(r)$ be a minimizing sequence of (1.12) (i.e., $\lim_{n\to\infty} E(V_n) = m(c_2, r)$). Applying Lemma 4.3, we have $V_n \in B(\frac{c_2 r}{2})$ for sufficiently large n and

$$
\frac{c_1}{c_2}V_n\in S(c_1)\bigcap B(\frac{c_1 r}{2})\subset S(c_1)\bigcap B(r).
$$

Thus, we deduce from Lemma 4.1 that

$$
\begin{aligned}
m(c,r)\le E(\frac{c_1}{c_2}V_n) &= (\frac{c_1}{c_2})^2\int|\nabla V_n|^2+\sum_{i=1}^{k}x_i^2|V_n|^2dx+\frac{\alpha}{2}(\frac{c_1}{c_2})^4\int(|x|^{-\nu}*|V_n|^2)|V_n|^2dx\\
&= (\frac{c_1}{c_2})^2E(V_n)+\frac{\alpha}{2}(\frac{c_1}{c_2})^2[(\frac{c_1}{c_2})^2-1]\int(|x|^{-\nu}*|V_n|^2)|V_n|^2dx
\end{aligned}
$$

$$\leq (\frac{c_1}{c_2})^2 m(c_2, r) + \frac{\alpha}{2}(\frac{c_1}{c_2})^2[(\frac{c_1}{c_2})^2 - 1]\delta + o_n(1)$$
$$< (\frac{c_1}{c_2})^2 m(c_2, r),$$

for sufficiently large n. This implies that

$$m(c_1, r) = \frac{{c_1}^2 - {c_2}^2}{{c_1}^2} m(c_1, r) + \frac{{c_2}^2}{{c_1}^2} m(c_1, r) < m(\sqrt{{c_1}^2 - {c_2}^2}, r) + m(c_2, r).$$

Thus, the proof is complete. □

Proof of Theorem 1.9. (i) Let $\{u_n\} \subset S(c) \bigcap B(r)$ be a minimizing sequence of (1.15). We shall apply the principle of concentration compactness to show that there exists a subsequence $\{u_{n_j}\}$ and $u \in \Sigma \setminus \{0\}$ such that

$$u_{n_j} \to u \ in \ \Sigma.$$

In particular, $u \in \mathcal{M}_r(c)$. We proceed as follows.

Step 1. We prove that the vanishing case does not occur. If not, by Lion's Lemma, we have the following:

$$u_{n_j} \to 0, \ for \ all \ q \in (2, \frac{2N}{N-2}).$$

In addition, we apply Lemma 2.2 by taking $s = r = \frac{2N}{2N-\nu}$ and $g(x) = \|u_{n_j}(x)\|_2^2, \ h(y) = \|u_{n_j}(y)\|_2^2$,

$$\int\int \frac{|u_{n_j}(x)|^2 |u_{n_j}(y)|^2}{|x-y|^\nu} dxdy \leq C \||u_{n_j}(x)|^2\|_{\frac{2N}{2N-\nu}} \||u_{n_j}(y)|^2\|_{\frac{2N}{2N-\nu}}$$
$$\leq C\|u_{n_j}(x)\|^4_{\frac{4N}{2N-\nu}},$$

according to the interpolation inequality for $u \in \Sigma$. Note that $P_1 = \frac{4}{N}, 2 < \nu < min\{4, N\}$. One can find that $0 < \theta = \frac{2(P_1+2)}{2NP_1} < 1$ such that $\frac{1}{\frac{4N}{2N-\nu}} = \frac{\theta}{2+P_1} + \frac{1-\theta}{2}$ and

$$\int\int \frac{|u_{n_j}(x)|^2 |u_{n_j}(y)|^2}{|x-y|^\nu} dxdy \leq C\|u_{n_j}(x)\|^4_{\frac{4N}{2N-\nu}} \leq C\|u_{n_j}\|_2^{4(1-\theta)} \|u_{n_j}\|^{4\theta}_{2+\frac{4}{N}} \to 0,$$

as $j \to \infty$. This is a contradiction with (4.1).

Step 2. We show that the dichotomy case can not occur. If not, there exist $u_{n_j}^{(1)}$ and $u_{n_j}^{(2)}$ such that

$$d_{n_j} = dist\{Suppu_{n_j}^{(1)}, Suppu_{n_j}^{(2)}\} \to \infty$$
$$\delta_{n_j} = \int_{\mathbb{R}^N} \sum_{i=1}^k x_i^2 |u_{n_j}^{(1)}|^2 dx \to a^2$$
$$\eta_{n_j} = \int_{\mathbb{R}^N} \sum_{i=1}^k x_i^2 |u_{n_j}^{(2)}|^2 dx \to c^2 - a^2,$$

as $j \to \infty$.

By a similar argument as Lemma 2.7, we have the following:

$$\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}|^2dx=\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}^{(1)}|^2dx+\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}^{(2)}|^2dx+o_j(1).$$

Consequently, from Lemma 2.8, we obtain the following:

$$\begin{aligned}m(c,r)&=\lim_{j\to\infty}\int_{\mathbb{R}^N}|\nabla u_{n_j}|^2dx+\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}|^2dx+\frac{\alpha}{2}\int(|x|^{-\nu}*|u_{n_j}|^2)|u_{n_j}|^2dx\\&\geq\lim_{j\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla u_{n_j}^{(1)}|^2dx+\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}^{(1)}|^2dx+\frac{\alpha}{2}\int(|x|^{-\nu}*|u_{n_j}^{(1)}|^2)|u_{n_j}^{(1)}|^2dx\Big)\\&+\lim_{j\to\infty}\Big(\int_{\mathbb{R}^N}|\nabla u_{n_j}^{(2)}|^2dx+\int_{\mathbb{R}^N}\sum_{i=1}^{k}x_i^2|u_{n_j}^{(2)}|^2dx+\frac{\alpha}{2}\int(|x|^{-\nu}*|u_{n_j}^{(2)}|^2)|u_{n_j}^{(2)}|^2dx\Big)+o_j(1)\\&\geq m(a,r)+m(\sqrt{c^2-a^2},r)+o_j(1).\end{aligned}$$

Letting $j\to\infty$, we obtain that $m(c,r)\geq m(a,r)+m(\sqrt{c^2-a^2},r)$, which is a contradiction with (4.1). Hence, the dichotomy can not occur.

Applying the concentration compact argument, there exist $\omega\in\Sigma\setminus\{0\}$, a subsequence $\{u_{n_j}\}$, and a sequence $\{y_{n_j}\}\subset\mathbb{R}^{N-k}$ such that

$$\tau_{y_{n_j}}u_{n_j}\to\omega\ in\ L^q\ ,\ for\ \ all\ \ q\in[2,\frac{2N}{N-2}).\tag{4.18}$$

Step 3. Conclusion. First, demonstrate that the sequence $\{y_{n_j}\}$ is bounded. Indeed, if it's not true, then up to a subsequence, we assume that $|y_{n_j}|\to\infty$ as $n\to\infty$. Consequently, deduce from (4.17), we deduce the following:

$$\int(|x|^{-\nu}*|u_{n_j}|^2)|u_{n_j}|^2dx=\int\tau_{y_{n_j}}(|x|^{-\nu}*|\tau_{y_{n_j}}u_{n_j}|^2)|\tau_{y_{n_j}}u_{n_j}|^2dx\to0,$$

as $j\to\infty$. This yields the following:

$$\|\omega\|_{\Sigma}\leq\lim_{j\to\infty}\|\tau_{y_{n_j}}u_{n_j}\|_{\Sigma}^2=\lim_{j\to\infty}E(u_{n_j})=m(c,r).$$

By the definition of $E(\omega)$, we obtain the following:

$$E(\omega)-\frac{\alpha}{2}\int(|x|^{-\nu}*|\omega|^2)|\omega|^2dx=\|\omega\|_{\Sigma}^2\leq m(c,r),$$

which implies that $E(\omega)<m(c,r)$, which is a contradiction with $E(\omega)\geq m(c,r)$ due to $\|\omega\|_2=c$. Therefore, the sequence $\{y_{n_j}\}\subset\mathbb{R}^{N-k}$ is bounded, and up to a subsequence, we assume that $y_{n_j}\to y_0$ in $\mathbb{R}^{N-k}$ as $j\to\infty$. Consequently, from (4.18), we deduce that for all $q\in[2,\frac{2N}{N-2})$,

$$\|u_{n_j}-\tau_{-y_0}\omega\|_L^q\leq\|u_{n_j}-\tau_{-y_{n_j}}\omega\|_L^q+\|\tau_{-y_{n_j}}\omega-\tau_{-y_0}\omega\|_L^q\to0,$$

as $j\to\infty$. Let $u(x)=\tau_{-y_0}\omega(x)$; it follows from $u\in S(c)\bigcap B(r)$ that

$$m(c,r)=\liminf_{n\to\infty}E(u_{n_j})\geq E(u)\geq m(c,r).$$

Therefore, $E(u)=m(c,r)$ and $u_{n_j}\to u_j$ in Σ as $j\to\infty$. This completes the proof.

Lemma 4.5. *Let $1 \leq k < N$ and $2 < \nu < min\{4, N\}$. Set $r > 0$ and $C_r > 0$ as in Lemma 4.3; then, there exists $\delta > 0$ such that for any $\varphi_0 \in \Sigma$ and $\inf_{u \in \mathcal{M}_r(c)} \|\varphi_0 - u\|_\Sigma < \delta$, the maximal solution $\varphi(t, x)$ of (4.17) with the initial data φ_0 globally exists.*

Proof. We denote the right hand of (4.16) by $\overline{B}$. Since the energy functional $E(u)$ is continuous with respect to $\varphi_0 \in \Sigma$, we deduce from $E(u) = m(c, r) < \overline{B}$ and $\|u\|_{\dot{\Sigma}} < \frac{rc}{2}$ that there exists $\delta > 0$ such that for any $\varphi_0 \in \Sigma$ and $\|\varphi_0 - u\|_\Sigma < \delta$, we have the following:

$$E(\varphi_0) < \overline{B} \text{ and } \|\varphi_0\|_{\dot{\Sigma}} < \frac{rc}{2}.$$

Now, we prove this claim by contradiction. If not, there exists $u_0 \in \Sigma$ such that $\|\varphi_0 - u\|_\Sigma < \delta$ and the corresponding solution blows up in a finite time. By continuity, there exist $T_1 > 0$ such that $\|\varphi(T_1)\|_{\dot{\Sigma}} \geq r$. We now consider the initial $\widetilde{\varphi_0} = \frac{c}{\|\varphi_0\|_2}\varphi_0$. If $\delta > 0$ is sufficiently small, then we have $\widetilde{\varphi_0} \in S(c)$ and $E(\widetilde{\varphi_0}) < \overline{B}$. If $c \leq \|\varphi_0\|_2$, then $\|\widetilde{\varphi_0}\|_{\dot{\Sigma}} \leq \|\varphi_0\|_{\dot{\Sigma}} < r$. If $c > \|\varphi_0\|_2$, due to $0 < c < C_r < 1$, then we have the following:

$$\widetilde{\varphi_0} < \frac{c}{\|\varphi_0\|_2}\frac{rc}{2} < r.$$

This implies that $\widetilde{\varphi_0} \in S(c) \bigcap B(r)$. Since the solution of (1.1) continuously depends on the initial data and $\|\varphi(T_1)\|_{\dot{\Sigma}} > r$, there exist $T_2 > 0$ such that $\|\widetilde{\varphi(T_2)}\|_{\dot{\Sigma}} > r$, where $\widetilde{\varphi_t}$ is the solution of (1.1) with the initial data $\widetilde{\varphi_0}$. Consequently, we infer from the continuity that there exist $T_3 > 0$ such that $\|\widetilde{\varphi(T_3)}\|_{\dot{\Sigma}} = r$, which implies $\widetilde{\varphi(T_3)} \in S(c) \bigcap (B(r) \setminus B(rc))$. It follows from Lemma 4.3 that

$$\overline{B} > E(\widetilde{\varphi_0}) = E(\widetilde{\varphi(T_3)}) \geq \inf_{v \in S(c) \bigcap (B(r) \setminus B(rc))} E(v) = \overline{B},$$

which gives a contradiction. Thus, the proof is completed. □

Proof of Theorem 1.9. (ii) Now, we show that $\mathcal{M}_r(c)$ is orbitally stable. Let's argue by contradiction. We suppose that there exist $\varepsilon > 0$, a sequence of initial data $\{\varphi_{n,0}\} \subset X$, and a sequence $\{t_n\} \subset R$ such that the solution $\varphi_n(t)$ with $\varphi_n(0) = \varphi_{n,0}$ satisfies the following:

$$\lim_{n \to \infty} \inf_{u \in \mathcal{M}_r(c)} \|\varphi_{n,0} - u\|_\Sigma = 0 \text{ and } \inf_{u \in \mathcal{M}_r(c)} \|\varphi_n(t_n) - u\|_\Sigma \geq \varepsilon\,. \tag{4.19}$$

By a similar argument as in (4.15), there exists $V \in \mathcal{M}_r(c)$ such that

$$\lim_{n \to \infty} \|\varphi_{n,0} - V\|_\Sigma = 0.$$

Next, we set $\widetilde{\varphi_n} = \frac{c}{\|\varphi_n(t_n)\|_2}\varphi_n(t_n)$. Since $V \in S(c) \bigcap B(\frac{rc}{2})$, we obtain

$$\widetilde{\varphi_n} \in S(c) \bigcap B(r),$$

and

$$\lim_{n \to \infty} E(\widetilde{\varphi_n}) = \lim_{n \to \infty} E(\varphi_n(t_n)) = \lim_{n \to \infty} E(\varphi_{n,0}) = E(V) = m(c, r),$$

which yields that $\widetilde{\varphi_n}$ is a minimizing sequence of (1.15). Due to the compactness of minimizing sequences of (1.15), there exists $\widetilde{\varphi} \in \mathcal{M}_r(c)$ such that

$$\widetilde{\varphi_n} \to \widetilde{\varphi} \ in \ \Sigma.$$

According to the definition of $\widetilde{\varphi_n}$, one has that

$$\widetilde{\varphi_n} - \varphi_n(t_n) \to 0 \ in \ \ \Sigma.$$

Thus we derive the following:

$$\widetilde{\varphi_n(t_n)} \to \widetilde{\varphi} \ in \ \Sigma,$$

which contradicts to (4.19). This completes the proof.

5. Conclusions

In this paper, we investigated the sharp global existence of the solution and the stability property of standing waves for the Schrödinger-Hartree equation with partial confinement. To be specific, for $2 \le \nu < min\{4, N\}$, we constructed some novel cross-invariant manifolds and variational problems to analyze the sharp criterion for global existence, that is, the solution $\varphi(t, x)$ for Eq (1.1) globally exists in time $t \in [0, \infty)$ if the initial data $\varphi_0 \in K_+ \cup R_+$, or the solution $\varphi(t, x)$ that corresponds to problem (1.1) blows up in a finite time if the initial data $\varphi_0 \in K$ and $|x|\varphi_0 \in L^2(\mathbb{R}^{\mathbb{N}})$. Especially in the critical $\nu = 2$, we obtained two different characterizations on the criterion of global existence versus blow-up. Additionally, by utilizing the profile decomposition technique, we showed the existence of orbitally stable standing waves for $c > 0$ if $0 < \nu < 2$ or $0 < \sqrt{-\alpha c} < \|Q\|_2$ if $\nu = 2$. Finally, we utilized the concentration compactness principle to demonstrate the orbital stability of standing waves in the L^2-supercritical case $2 < \nu < min\{4, N\}$.

Author contributions

Feiyan Lei: Methodology, investigation, formal analysis, writing-original manuscript; Hui Jian: Conceptualization, methodology, supervision, funding acquisition, writing-review and editing. All authors have read and agreed to the submitted version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that no conflict of competing interests exists.

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