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Research article

Study on pest management SI epidemic model with instantaneous and non-instantaneous impulsive effects

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Abstract: In this study, we propose and analyze a Susceptible-Infected (SI) epidemic model applied to pest management, focusing on the nonlinear release of infected pests and an instantaneous pulse of pesticide spraying. Additionally, the mortality rates of both susceptible and infected pests following the pesticide application are modeled as non-instantaneous pulses. Utilizing the comparison theorem for pulse differential equations and Floquet theory, we derive a threshold condition for the eradication of susceptible pests. We also demonstrate that all solutions are uniformly ultimately bounded. Furthermore, we establish conditions for the globally asymptotic stability of the pest-free boundary periodic solution and the permanence of the system. Finally, numerical simulations are conducted to verify the theoretical findings, and the key parameters affecting the pest extinction threshold were obtained, thereby providing a solid theoretical basis for the development of effective pest management strategies.

Keywords: pest management SI epidemic model; instantaneous and non-instantaneous impulse effects; persistence; nonlinearly releasing infectious insects
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1. Introduction

In recent years, pest-related losses have become widespread, with economic damages from pests and diseases emerging as one of the major threats we face today. According to the *Scientific Review of the Impact of Climate Change on Plant Pests and Diseases* published by the Food and Agriculture Organization of the United Nations, up to 40% of global crop production is lost to pests each year. Every year, plant diseases result in global economic losses exceeding 220 billion dollars, while the economic damage attributed to invasive insect species exceeds 70 billion dollars. How to solve the economic losses caused by pests and diseases is an urgent problem that countries around the world need to solve.

Integrated pest management (IPM) employs a multifaceted strategy—encompassing biological, physical, chemical, and other complementary methods—to reduce both economic damage and ecological disturbance. Current pest control strategies primarily include chemical methods, such as pesticide sprays [1], and biological methods, such as the release of natural predators [2]. A considerable body of research has focused on constructing mathematical models to simulate and refine strategies for the control of pests and plant diseases, which have proposed pest regulation frameworks that rely exclusively on chemical interventions [3–5]. Several studies have indicated that prolonged and repeated application of the same pesticide can lead to the emergence of resistance among pest populations [6–8]. Based on these findings, other studies have focused on exploring the dynamic behaviour of pest management frameworks that incorporate biological control strategies [9–11]. However, the application of a biological control often relies on artificial breeding, a process that is usually accompanied by high economic costs [12, 13]. Therefore, plenty of researchers have developed IPM that incorporates both chemical control (pesticide application) and biological control strategies. Two primary categories of integrated control strategies are currently available: one involves the concurrent deployment of insecticides and natural enemies [14], and the other entails the sequential application of insecticides and release of natural enemies [15, 16]. Building on these existing approaches, several studies [17–19] have incorporated switching systems into pest management practices, enabling the implementation of time-divided operations for insecticide spraying and natural enemy release. Specifically, some mathematical models were formulated to describe the combined strategy of releasing infected pests (serving as natural predators) and applying pesticides at distinct impulsive moments [14, 15].

Liu et al. [14] considered a Lotka–Volterra prey–predator framework incorporating both impulsive predator introductions and scheduled pesticide applications, described by the following system:

$$\begin{cases} \left.\begin{aligned} \frac{dx(v)}{dv} &= x(v)(r - ax(v)) - bx(v)y(v), \\ \frac{dy(v)}{dv} &= -dy(v) + cx(v)y(v), \end{aligned}\right\} & v \neq n\omega, \\ \left.\begin{aligned} \Delta x(v) &= -\mu_1 x(v), \\ \Delta y(v) &= -\mu_2 y(v) + \mu, \end{aligned}\right\} & v = n\omega,\ n = 1,2,\dots, \end{cases}$$

where $x(v)$ and $y(v)$ represent the pest and natural predator populations at impulsive moments $v = n\omega$, respectively; r is the intrinsic growth rate of the pest population; b characterizes the coefficient associated with intraspecific competition; $\Delta x(v) = x(v^+) - x(v)$ and $\Delta y(v) = y(v^+) - y(v)$; $\mu \geq 0$ is the quantity of predators released at the impulsive moments $v = n\omega$; $0 \leq \mu_1 < 1$ and $0 \leq \mu_2 < 1$ denote the proportions of prey and predator, respectively, removed by spraying pesticides at the impulsive moments $v = n\omega$; ω denotes the period of the impulsive intervention; and $n \in \mathbb{Z}_+ = \{1, 2, \dots\}$.

Jiao et al. [15] proposed a pest management model based on the Susceptible-Infected (SI) framework, which integrates biological and chemical interventions. In their approach, susceptible and infected pests are targeted using pulsed pesticide applications, while infected individuals, representing harmless natural enemies, are introduced at distinct impulsive moments, thereby leading to the establishment of the following SI pest management model:

$$\begin{cases} \frac{\mathrm{d}S(v)}{\mathrm{d}v} = rS(v)\left(1 - \frac{S(v)+\theta I(v)}{K}\right) - \beta S(v)I(v), & v \neq (n+\gamma-1)\omega,\ v \neq n\omega, \\ \frac{\mathrm{d}I(v)}{\mathrm{d}v} = \beta S(v)I(v) - wI(v), & v \neq (n+\gamma-1)\omega,\ v \neq n\omega, \\ \Delta S(v) = -\mu_1 S(v), & v = (n+\gamma-1)\omega, \\ \Delta I(v) = -\mu_2 I(v), & v = (n+\gamma-1)\omega, \\ \Delta S(v) = 0, & v = n\omega, \\ \Delta I(v) = \mu, & v = n\omega, \qquad 0 < \gamma < 1,\ n = 1, 2\ldots, \end{cases}$$

where K is the pest-carrying capacity; $0 < \theta < 1$ represents the competitive ability of infected pests relative to susceptible pests; $0 \leq \mu_1 < 1$ and $0 \leq \mu_2 < 1$ denote the proportions of susceptible and infected pests removed by pesticide application at times $v = (n+\gamma-1)\omega$, respectively; $\mu \geq 0$ is the amount of infective pests released at times $v = n\omega$; $\Delta S(v) = S(v^+) - S(v)$ and $\Delta I(v) = I(v^+) - I(v)$; and ω is the period of the impulsive effect.

As the research continued to deepen, some scholars have found that spraying pesticides will be accompanied by sublethal effects [19, 20], delayed effects [21], pest resistance [22], residual effects [21, 23], and so on. In order to be better aligned with the actual situation, other scholars have considered switching dynamics models involving diffusion [20, 24], impulsive dredging and pulse inputting [25], harvests [27–29], births [30], etc., under both instantaneous and noninstantaneous effects. In addition, some scholars have considered hibernation switching systems [26, 31]. In particular, Jiao et al. [27] investigated a stage-structured single-population model incorporating transient and nontransient impulsive harvesting and birth pulse:

$$\begin{cases} \left.\begin{aligned} \frac{dx_1(v)}{dv} &= -(c_1+d_1)x_1(v), \\ \frac{dx_2(v)}{dv} &= c_1 x_1(v) - d_2 x_2(v), \end{aligned}\right\} v \in (n\omega, (n+\gamma)\omega], \\ \left.\begin{aligned} \Delta x_1(v) &= -\mu_1 x_1(v), \\ \Delta x_2(v) &= -\mu_2 x_2(v), \end{aligned}\right\} v = (n+\gamma)\omega, \\ \left.\begin{aligned} \frac{dx_1(v)}{dv} &= -(c_2+d_3)x_1(v) - E_1 x_1(v), \\ \frac{dx_2(v)}{dv} &= c_2 x_1(v) - d_4 x_2(v) - E_2 x_2(v), \end{aligned}\right\} v \in ((n+\gamma)\omega, (n+1)\omega], \\ \left.\begin{aligned} \Delta x_1(v) &= x_2(v)(a - b x_2(v)), \\ \Delta x_2(v) &= 0, \end{aligned}\right\} v = (n+1)\omega. \end{cases}$$

For a biological explanation of this model, we refer to the literature in [27].

Considering the limited resources in pest management, related researchers have studied problems such as pulse release and harvesting through nonlinear methods. Drawing on reference [32], Zhou et al. [33] developed a predator-prey model that incorporates nonlinear harvesting and implements constant release across distinct time intervals. Meanwhile, Wu et al. [34, 35] investigated an age-structured aquaculture management model, which integrates nonlinear release and constant ratio harvesting, with both practices applied either simultaneously or at separate time points. Jiao et al. [36] centred their research on a switched predator-prey breeding management model featuring nonlinear release and pulse-delay harvesting. Separately, Zhou et al. [37] examined an IPM system that combines nonlinear harvesting and release operations conducted at different times. Wu et al. [38] have also contributed to this field by studying two natural enemy species, focusing on their nonlinear pulse release and the interspecific cooperation within an IPM framework. Building on the foundational work presented in references [26, 32], Sun et al. [39] have further advanced the research by developing a switched host-parasite model that includes nonlinear release mechanisms. Li and Tang [32] put forward an in IPM approach that employs nonlinear pulse techniques for the release of natural enemies and the application of pesticides:

$$\left\{\begin{array}{l}\left.\begin{array}{l}\frac{dx(v)}{dv}=x(v)g(x(v))-p(x(v))y(v),\\ \frac{dy(v)}{dv}=cp(x(v))y(v)-Dy(v),\end{array}\right\}v\neq n\omega,\\ \left.\begin{array}{l}\Delta x(v)=-\frac{\delta x(v)}{x(v)+h}x(v),\\ \Delta y(v)=\frac{\lambda}{1+\theta y(v)},\end{array}\right\}v=n\omega.\end{array}\right.$$

For a biological explanation of this model, we refer to the literature in [32].

Most existing studies assume that pesticides kill pests instantly, and at a rate which is proportional to the size of the pest population. However, in reality, pesticides have sublethal effects, delayed effects, residual effects, and so on, so that pests are not killed immediately after application, and parameters such as the birth rate and the mortality rate are affected accordingly. That is, pesticides have an instantaneous pulse effect on pests, followed by a continuous non-instantaneous effect within a certain period of time. Therefore, this paper proposes an SI-type integrated pest management model that takes these delayed effects of pesticide spraying into acocunt. In addition, infected pests are released as biological control in the form of instantaneous pulses which, to avoid a waste of resources, are of nonlinear nature. The combination of these issues has not yet been considered in previous pest management models. In view of the various effects of pesticides, a noninstantaneous pulse switching system model is established to explore the SI IPM. Meanwhile, by releasing and spraying at separate times, the mutual influence of these two disturbances is effectively circumvented. The results established in this study offer a rigorous theoretical foundation for the formulation of efficient pest management strategies.

This paper is organized into six principal parts. The first part constitutes the introduction, the second part introduces the basic model, and the third part focuses on proving basic properties of the solutions, and the global stability of the all-infected solution. The fourth part uses Floquet theory and the comparison theorem of impulsive differential equations to analyse the dynamics of the switching system model and obtain sufficient conditions for pest extinction and persistence. The fifth part presents the results of numerical simulations as well as a discussion, and is followed by the conclusion.

2. Model formulation

Inspired by the previous studies [15, 27, 32], we have established the following pest management SI epidemic model with pulse effects under a switching strategy:

$$\left\{\begin{array}{l}\left.\begin{array}{l}\frac{dS(v)}{dv}=r_1S(v)(1-\frac{S(v)+\theta_1I(v)}{K_1})-\beta_1S(v)I(v),\\ \frac{dI(v)}{dv}=\beta_1S(v)I(v)-d_1I(v),\end{array}\right\}v\in(n\omega,(n+\gamma)\omega],\\ \left.\begin{array}{l}\Delta S(v)=-\mu_1S(v),\\ \Delta I(v)=-\mu_2I(v),\end{array}\right\}v=(n+\gamma)\omega,\\ \left.\begin{array}{l}\frac{dS(v)}{dv}=r_2S(v)(1-\frac{S(v)+\theta_2I(v)}{K_2})-\beta_2S(v)I(v)-h_1S(v),\\ \frac{dI(v)}{dv}=\beta_2S(v)I(v)-d_2I(v)-h_2I(v),\end{array}\right\}v\in((n+\gamma)\omega,(n+1)\omega],\\ \left.\begin{array}{l}\Delta S(v)=0\\ \Delta I(v)=\frac{\mu_{\max}}{1+\alpha I(v)},\end{array}\right\}v=(n+1)\omega,\end{array}\right. \tag{2.1}$$

where $S(v)$ and $I(v)$ represent the populations of susceptible pests and infected pests at time v, respectively. The parameters $\beta_1 > 0$ and $\beta_2 > 0$ denote the transmission rates, while $d_1 > 0$ and $d_2 > 0$ correspond to the natural death rates of infected pests. The intrinsic growth rates of the noninfected pest population are given by $r_1 > 0$ and $r_2 > 0$, and the environmental carrying capacities in the time intervals $(n\omega, (n+\gamma)\omega]$ and $((n+\gamma)\omega, (n+1)\omega$ are represented by $K_1 > 0$ and $K_2 > 0$, respectively. Infected pests do not reproduce. The impulsive variations in the population are described by $\Delta S(v) = S(v^+) - S(v)$ and $\Delta I(v) = I(v^+) - I(v)$. The parameters $\theta_1, \theta_2 \in (0,1)$ characterize the competitive ability of infected pests relative to susceptible pests. The parameter $\gamma \in (0,1)$ determines the pesticide application times. The parameters $0 \le \mu_1 < 1$ and $0 \le \mu_2 < 1$ denote the instantaneous mortality rates on noninfected and infected pests, respectively, imposed by pesticide spraying at times $v = (n+\gamma)\omega$, with $n \in \mathbb{Z}_+$. In addition, $0 \le h_1 < 1$ and $0 \le h_2 < 1$ represent the increased mortalitiy rates on the noninfected and infected pests, respectively, caused by the lingering pesticide effects over the time intervals $((n+\gamma)\omega, (n+1)\omega]$. The function $\frac{\mu_{\max}}{1+\alpha I(v)} \ge 0$ models the density-dependent release rate of infected pests as natural enemies at times $v = (n+1)\omega$, where $\mu_{\max} > 0$ denotes the maximum release capacity and $\alpha > 0$ is a shape parameter. This function ensures that the release rate approaches 0 as $I(v) \to +\infty$, and approaches $\mu_{\max}$ as $I(v) \to 0$, indicating the reduced release of natural enemies when their existing population is large. Finally, the parameter ω denotes the impulsive period.

We will establish that the system (2.1) possesses a boundary periodic solution corresponding to the extinction of susceptible pests, which is GAS. Under appropriate combinations of the modifiable parameters $\omega, \gamma, \mu_{max}$ and α, the entire pest population transitions to the infected class and is ultimately eradicated. Therefore, the combined use of biological and chemical control methods can lead to complete pest elimination, a state referred to as pest-extinction. Throughout the analysis, it is assumed that the ecological harm caused by the natural enemies is significantly less than that caused by pests.

3. Preliminary

Let $F = (F_1, F_2)^T$ denote map defined by the right-hand side of the system (2.1). The solution of the system (2.1), denoted by $z(v) = (S(v), I(v))$, is a piecewise continuous function $z : \mathbb{R}_+ \to \mathbb{R}^2_+$, where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \{z \in \mathbb{R}^2 : z \ge 0\}$. The function $z(v)$ is continuous on the intervals $(n\omega, (n+\gamma)\omega]$ and $((n+\gamma)\omega, (n+1)\omega]$ for all $n \in \mathbb{Z}_+$, with $0 < \gamma < 1$. According to [40], the global existence and uniqueness of solutions to the system (2.1) are ensured by the smoothness of the function F.

Let $V : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$. The function V is said to belong to the class V_0 if the following conditions are satisfied:

(1) V is continuous on $(n\omega, (n+\gamma)\omega] \times \mathbb{R}^2_+$ and on $((n+\gamma)\omega, (n+1)\omega] \times \mathbb{R}^2_+$ for all $z \in \mathbb{R}^2_+$ and $n \in \mathbb{Z}_+$, and the limits

$$\lim_{(v,y)\to((n+\gamma)\omega^+, z)} V(v, y) = V((n+\gamma)\omega^+, z)$$

and

$$\lim_{(v,y)\to((n+1)\omega^+, z)} V(v, y) = V((n+1)\omega^+, z)$$

exist.

(2) V is locally Lipschitz in z.

Definition 3.1. *If $V \in V_0$, then for $(v, z) \in (n\omega, (n+\gamma)\omega] \times \mathbb{R}_+^2$ and $(v, z) \in ((n+\gamma)\omega, (n+1)]\omega \times \mathbb{R}_+^2$, the upper right-hand derivative of $V(v, z)$ with respect to the impulsive differential system* (2.1) *is defined as:*

$$D^+V(v, z) = \lim_{h\to 0^+} \sup \frac{1}{h} \left[V(v + h, z + hF(v, z)) - V(v, z)\right].$$

Lemma 3.1. *[40] Consider a function $m \in PC^1(\mathbb{R}^+, \mathbb{R})$ that fulfills the following conditions:*

$$\begin{cases} \frac{dm}{dv} \leq p(v)m(v) + q(v), & v \geq v_0, v \neq v_k; k = 1, 2, \dots, \\ m(v_k^+) \leq d_k m(v_k) + b_k, & v = v_k, \end{cases}$$

where $p, q \in C(\mathbb{R}^+, \mathbb{R})$, $d_k \geq 0$ for all $k \in \mathbb{Z}_+$, and b_k are the given constants. Suppose the following:

(1) The sequence v_k satisfies $0 \leq v_0 < v_1 < v_2 < \dots$, with $\lim_{k\to\infty} v_k = \infty$;

(2) The function m belongs to $PC^1(\mathbb{R}^+, \mathbb{R})$, the space of piecewise continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$, such that m is continuously differentiable on each interval $(v_{k-1}, v_k]$ and left-continuous at each v_k for $k \in \mathbb{Z}_+$.

Then, for all $v \geq v_0$, the following inequality holds:

$$\begin{aligned} m(v) \leq & m(v_0) \prod_{v_0 < v_k < v} d_k \exp\Big(\int_{v_0}^{v} p(\mu)\, d\mu\Big) \\ & + \sum_{v_0 < v_k < v} \Big(\prod_{v_k < v_j < v} d_j \exp\Big(\int_{v_0}^{v} p(\mu)\, d\mu\Big)\Big) b_k \\ & + \int_{v_0}^{v} \prod_{\mu < v_k < v} d_k \exp\Big(\int_{\mu}^{v} p(\sigma)\, d\sigma\Big) q(\mu)\, d\mu, \quad v \geq v_0. \end{aligned}$$

Lemma 3.2. *(Positivity of solutions) Assume that $z(v) = (S(v), I(v))^\intercal$ is a solution to the system* (2.1) *with $S(0^+) \geq 0$ and $I(0^+) \geq 0$, in which case, $S(v) \geq 0$ and $I(v) \geq 0$ for all $v \geq 0$. Moreover, if $S(0^+) > 0$ and $I(0^+) > 0$, then $S(v) > 0$ and $I(v) > 0$.*

Proof. Initially, taking the continuity property of $S(v)$ into account, we obtain

$$S(v) = S(0^+)(1 - \mu_1)e^{\psi(v)},$$

for $\gamma\omega < v \leq \omega$

$$\psi(v) = \int_0^{\gamma\omega} \left(r_1(1 - \frac{S(s) + \theta_1 I(s)}{K_1}) - \beta_1 I(s)\right) ds + \int_{\gamma\omega}^{v} \left(r_2(1 - \frac{S(s) + \theta_2 I(s)}{K_2}) - \beta_2 I(s) - h_1\right) ds,$$

where, the positivity of $S(v)$ is contingent upon $S(0^+)$.

Similar to $S(v)$, thus, we find that the positivity of $I(v)$ depends on $I(0^+)$.

Thus we get that $S(v) \geq 0, I(v) \geq 0$ for all $v \geq 0$. This completes the proof. □

Lemma 3.3. *(Comparison theorem, [40]) Let $V : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be a function belonging to the class V_0. Suppose that the following inequalities hold:*

$$\begin{cases} D^+V(v, x(v)) \leq g(v, V(v, x(v))), & v \neq n\omega, \\ V(v, x(v^+)) \leq \Psi_n(V(v, x(v))), & v = n\omega, \end{cases}$$

where the mapping $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is continuous on each domain* $(n\omega, (n+1)\omega] \times \mathbb{R}_+$, *and for all* $z \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$, *the limit condition*

$$\lim_{(v,y)\to(n\omega^+,z)} g(v,y) = g(n\omega^+, z)$$

is satisfied. Furthermore $\Psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ *are monotonic non-decreasing.*

Now, consider the following impulsive differential system:

$$\begin{cases} \frac{du}{dv} = g(v, u(v)), & v \neq n\omega, \\ u^+(v) = \Psi_n(u(v, x(v))), & v = n\omega, \\ u(0^+) = u(0), \end{cases}$$

and consider its maximal solution $R(v)$ *to be defined on the interval* $[0, \infty)$. *It then follows that*

$$V(v, x(v)) \leq R(v), \quad \forall\, v \geq 0,$$

provided that the initial condition $V(0^+, x(0)) \leq v(0)$ *holds, where* $x(v)$ *denotes an arbitrary solution to the system* (2.1).

Lemma 3.4. *A positive constant* $G > 0$ *exists such that for every solution* $(S(v), I(v))$ *of the system* (2.1), *the inequalities* $S(v) \leq G$ *and* $I(v) \leq G$ *hold for all sufficiently large values of* v.

Proof. Let $V(v) = S(v) + I(v)$, and define the constant $d_\lambda = \min\{d_1, h_1, d_2 + h_2\}$. Under this notation, the following relation holds:

$$\begin{cases} D^+V(v) + d_\lambda V(v) = r_1 S(v)(1 - \dfrac{S(v) + \theta_1 I(v)}{K_1}) + d_\lambda S(v) - (d_1 - d_\lambda)I(v) \\ \quad \leq (r_1 + d_\lambda)S(v) - \dfrac{r_1 S(v)^2}{K_1} \leq L_0, v \in (n\omega, (n+\gamma)\omega], \\ V(v^+) \leq V(v), v = (n+\gamma)\omega, \\ D^+V(v) + d_\lambda V(v) = r_2 S(v)(1 - \dfrac{S(v) + \theta_2 I(v)}{K_2}) - (h_1 - d_\lambda)S(v) - (d_2 + h_2 - d_\lambda)I(v) \\ \quad \leq r_2 S(v)(1 - \frac{S(v)}{K_2}) \leq L_1, v \in ((n+\gamma)\omega, (n+1)\omega], \\ V(v^+) \leq V(v) + \mu_{max}, v = (n+1)\omega, \end{cases}$$

where

$$L_0 = \frac{K_1(r_1 + d_\lambda)^2}{4r_1}, \quad L_1 = \frac{K_2 r_2}{4}.$$

Take $L = \max\{L_0, L_1\}$, when $v \neq (n+\gamma)\omega$, and $v \neq (n+1)\omega$, and obtain

$$\begin{cases} D^+V(v) + d_\lambda V(v) \leq L, v \in (n\omega, (n+1)\omega], \\ V(v^+) \leq V(v) + \mu_{max}, v = (n+1)\omega. \end{cases}$$

In light of Lemma 3.1, it follows that

$$\begin{aligned} V(v) \leq & V(0)\exp(-d_\lambda v) + \int_0^v L\exp(-d_\lambda(v-\mu))\,d\mu + \sum_{0<n\omega<v} \mu_{max}\exp(-d_\lambda(v - n\omega)) \\ \leq & V(0)\exp(-d_\lambda v) + \frac{L}{d_\lambda}(1 - \exp(-d_\lambda v)) + \frac{\mu_{max}\exp(-d_\lambda(\mu - \omega))}{1 - \exp(d_\lambda \omega)} + \frac{\mu_{max}\exp(d_\lambda\omega)}{\exp(d_\lambda\omega) - 1} \\ \to & \frac{L}{d_\lambda} + \frac{\mu_{max}\exp(d_\lambda\omega)}{\exp(d_\lambda\omega) - 1} \quad \text{as } v \to \infty. \end{aligned}$$

Therefore, the function $V(v)$ has been shown to be uniformly ultimately bounded. Recalling the definition of $V(v)$, one concludes that a constant $G \geq 0$ exists such that the inequalities $S(v) \leq G$ and $I(v) \leq G$ hold for all sufficiently large values of v. The proof is thus completed. □

If the condition $S(v) = 0$ holds, then the system (2.1) simplifies to the corresponding reduced subsystem as follows:

$$\begin{cases} \frac{dI(v)}{dv} = -d_1 I(v), & v \in (n\omega, (n+\gamma)\omega], \\ \Delta I(v) = -\mu_2 I(v), & v = (n+\gamma)\omega, \\ \frac{dI(v)}{dv} = -d_2 I(v) - h_2 I(v), & v \in ((n+\gamma)\omega, (n+1)\omega], \\ \Delta I(v) = \frac{\mu_{max}}{1+\alpha I(v)}, & v = (n+1)\omega. \end{cases} \tag{3.1}$$

The analytical expression for the solution of the system (3.1) in the interval between successive impulses can be derived as follows:

$$I(v) = \begin{cases} I(n\omega^+)e^{-d_1(v-n\omega)}, & v \in (n\omega, (n+\gamma)\omega], \\ I((n+\gamma)\omega^+)e^{-(d_2+h_2)(v-(n+\gamma)\omega)}, & v \in ((n+\gamma)\omega, (n+1)\omega], \end{cases} \tag{3.2}$$

and the stroboscopic map of the system (3.1) is given by:

$$\begin{aligned} I((n+1)\omega^+) =& I(n\omega^+)(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega} \\ &+ \frac{\mu_{max}}{1+\alpha I(n\omega^+)(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega}}. \end{aligned} \tag{3.3}$$

It is straightforward to demonstrate that Eq (3.3) admits a unique positive fixed point

$$I^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{3.4}$$

where

$$\begin{aligned} A_0 =& (1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega} < 1, \\ A =& \alpha A_0(1-A_0), \\ B =& 1 - A_0, \\ C =& -\mu_{max}. \end{aligned}$$

Lemma 3.5. *The positive equilibrium point I^*, as specified in Eq* (3.4)*, exhibits GAS.*

Proof. If we take $I_n = I(n\omega^+)$, then the system (3.3) can be rewritten as

$$\begin{aligned} I_{n+1} = F(I_n) = & I_n(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega} \\ &+ \frac{\mu_{max}}{1+\alpha I_n(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega}}, \end{aligned} \tag{3.5}$$

since

$$F^{'}(I_n) = \frac{\partial F(I_n)}{\partial I_n} = A_0 - \frac{\mu_{max}\alpha A_0}{(1+\alpha A_0 I_n)^2}, \tag{3.6}$$

we can obtain

$$\left.\frac{\partial F(I_n)}{\partial I_n}\right|_{I_n=I^*} = A_0 - \frac{\mu_{max}\alpha A_0}{(1+\alpha A_0 I^*)^2} < A_0 < 1,$$

and

$$\left.\frac{\partial F(I_n)}{\partial I_n}\right|_{I_n=I^*} > \frac{-\alpha A_0 I^*(1-A_0)}{(1+\alpha A_0 I^*)} > \frac{-\alpha A_0 I^*}{(1+\alpha A_0 I^*)} > -1,$$

Therefore, the preceding analysis verifies that

$$\left|\frac{\partial F(I_n)}{\partial I_n}\right|_{I_n=I^*} < 1,$$

which ensures the local asymptotic stability of the positive fixed point I^*.

Subsequently, we proceed to establish the global attractivity of I^*. According to [32], setting $F'(I) = 0$, we solve for

$$I_1 = \frac{-\sqrt{\mu_{max}\alpha} - 1}{\alpha A_0}, \quad I_2 = \frac{\sqrt{\mu_{max}\alpha} - 1}{\alpha A_0},$$

where I_2 denotes the point at which the function $F(I)$ reaches its minimum, and it holds that

$$F(I_2) = \frac{2\sqrt{\mu_{\max}\alpha} - 1}{\alpha}.$$

By calculation, we have

$$F''(I) = \frac{2\mu_{max}\alpha^2 A_0^2}{(1+\alpha A_0 I)^3}.$$

Therefore, when $I > 0$, $F(I)$ is a convex function, and $F(0) = \mu_{max}$.

Below, we prove the global attractivity of I^* by discussing three cases.

(i) $I_2 < 0 < I^*$

From $I_2 < 0$, we know that $0 < \mu_{max}\alpha < 1$. When $I > 0$, $F(I)$ is a convex function and a monotonically increasing function. Also, since $F'(I) < 1$, the equation $F(I) = I$ admits a unique positive fixed point, denoted I^*. Define the sequence $F^n(I)$ recursively by $F^n(I) = F(F^{n-1}(I))$ for $n = 2, 3, \ldots$.

If $0 < I_0 < I^*$, because $F(I)$ is a convex function, $F(I) > I$, so the sequence $F^n(I_0)$ is monotonically increasing with respect to n. In a similar manner, it can be shown that $\lim\limits_{n\to\infty} F^n(I_0) = I^*$.

If $I^* < I_0$, then $F(I) < I$. The sequence $F^n(I_0)$ a exhibits monotonic decrease as n increases. Similarly, we can obtain $\lim_{n\to\infty} F^n(I_0) = I^*$.

(ii) $0 < I_2 < I^*$

From $I_2 > 0$, we know that $\mu_{max}\alpha > 1$. If $I_0 > I_2$, the proof is the same as in case (i). Therefore, we can get $\lim_{n\to\infty} F^n(I_0) = I^*$. If $0 < I_0 < I_2$, there must be a smallest positive integer n such that $F^n(I_0) > I_2$. In summary, $\lim\limits_{n\to\infty} F^{n+m}(I_0) = I^*$.

(iii) $0 < I^* < I_2$

On the interval $(0, I_2]$, $F(I)$ is a monotonically decreasing convex function, and on $(I_2, +\infty)$, $F(I)$ is a convex function that exhibits monotonic growth with respect to I. Due to the properties of $F(I)$ mentioned above, for any $I_0 > 0$, let $m \in \mathbb{Z}_+$ be such that $I' \triangleq F^m(I_0) \in [I^*, I_2]$. Therefore, it suffices to prove that

$$\lim_{n\to\infty} F^n(I') = \lim_{n\to\infty} F^{m+n}(I_0) = I^*, I' \in [I^*, I_2].$$

Let $G(I) = \frac{F(F(I))}{I}$, and calculate to obtain

$$G'(I) = \frac{F'(F(I))F'(I)I - F(F(I))}{I^2}.$$

Since $F(I)$ is a monotonically decreasing convex function on the interval $(I^*, I_2]$, and $F''(I) > 0$ holds for all $I > 0$, $F'(I)$ is monotonically increasing on $(I^*, I_2]$, and thus we have

$$\begin{aligned} F'(I)I + F(I) &= 2IA_0 + \frac{\mu_{max}}{(1+\alpha A_0 I)^2} \\ &> 2IA_0 > 0 \\ &\Rightarrow -F(I) < F'(I)I \leq 0. \end{aligned} \tag{3.7}$$

Furthermore,

$$F'(F(I))F(I) + F(F(I)) > 2F(I)A_0 > 0.$$

Since $0 < F(I) < F(I^*) = I^*$, and for any $I \in (I^*, I_2]$, $F'(I) < 0$, it follows that

$$-\frac{F(F(I))}{F(I)} < F'(F(I)) < 0. \tag{3.8}$$

From the inequalities (3.7) and (3.8), it is known that when $I \in (I^*, I_2]$, we have

$$F'(F(I)F'(I)I < F(I)\frac{F(F(I))}{F(I)} = F(F(I)).$$

Therefore, when $I \in (I^*, I_2]$, $G'(I) < 0$, $G(I)$ decreases on the interval $(I^*, I_2]$. Since

$$G(I^*) = \frac{F(F(I^*))}{I^*} = 1,$$

it follows that

$$G(I) < G(I^*) = 1 \Leftrightarrow F(F(I)) < I.$$

Let $F^2(I) \triangleq F(F(I))$, thus for any $I_0 \in (I^*, I_2]$, the function $F^2(I_0)$ decreases monotonically as n increases, and $F^{2n}(I_0) > I^*$. By the monotone convergence theorem, $\lim_{n\to\infty} F^{2n}(I_0) = I^*$. Similarly, it can be proven that $\lim_{n\to\infty} F^{2n+1}(I_0) = I^*$, and thus for any $I > 0$, we have $\lim_{n\to\infty} F^{n+m}(I) = I^*$.

In summary, it is proven that the difference equation (3.1) possesses a unique equilibrium point I^*, which is GAS. □

Similar to [26], the following lemma is obtained:

Lemma 3.6. *The positive periodic solution of the system* (3.1) *is GAS, where*

$$\widetilde{I(v)} = \begin{cases} I^* e^{-d_1(v-n\omega)}, & v \in (n\omega, (n+\gamma)\omega], \\ I^{**} e^{-(d_2+h_2)(v-(n+\gamma)\omega)}, & v \in ((n+\gamma)\omega, (n+1)\omega]. \end{cases} \tag{3.9}$$

I^* *is defined as given in* (3.4)*, and*

$$I^{**} = I^*(1-\mu_2)e^{-d_1\gamma\omega}. \tag{3.10}$$

4. The dynamics

From the above discussion, our attention is now directed toward analyzing the GAS of the pest extinction periodic solution $(0, \widetilde{I(v)})$, while also demonstrating the permanence of the system (2.1).

4.1. Extinction

Theorem 4.1. *If the following inequality holds:*

$$\begin{aligned} & r_1\gamma\omega + r_2(1-\gamma)\omega + \frac{I^*(r_1\theta_1+\beta_1K_1)}{d_1K_1}(e^{-d_1\gamma\omega}-1) - h_1(1-\gamma)\omega \\ & + \frac{I^*(1-\mu_2)e^{-d_1\gamma\omega}(r_2\theta_2+\beta_2K_2)}{K_2(d_2+h_2)}(e^{-(d_2+h_2)(1-\gamma)\omega}-1) < \ln\frac{1}{1-\mu_1}, \end{aligned} \tag{4.1}$$

then, the periodic solution corresponding to the pest eradication state of the system (2.1) *is GAS, where* I^* *is specified by* (3.4).

Proof. We begin by establishing the local stability. Let $(0,\widetilde{I(v)}$ be defined, and set $S_1(v) = S(v)$, $I_1(v) = I(v) - \widetilde{I(v)})$. Under this transformation, the system can be rewritten in the following form:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(v)}{dv} &= r_1S_1(v)(1-\frac{S_1(v)+\theta_1(I_1(v)+\widetilde{I(v)})}{K_1}) - \beta_1S_1(v)(I_1(v)+\widetilde{I(v)}), \\ \frac{dI_1(v)}{dv} &= \beta_1S_1(v)(I_1(v)+\widetilde{I(v)}) - d_1I_1(v), \end{aligned}\right\} v \in (n\omega,(n+\gamma)\omega], \\ \left.\begin{aligned} \Delta S_1(v) &= -\mu_1S_1(v), \\ \Delta I_1(v) &= -\mu_2I_1(v), \end{aligned}\right\} v = (n+\gamma)\omega, \\ \left.\begin{aligned} \frac{dS_1(v)}{dv} &= r_2S_1(v)(1-\frac{S_1(v)+\theta_2(I_1(v)+\widetilde{I(v)})}{K_2}) - \beta_2S_1(I_1(v)+\widetilde{I(v)}) - h_1S_1(v), \\ \frac{dI_1(v)}{dv} &= \beta_2S_1(v)(I_1(v)+\widetilde{I(v)}) - d_2I_1(v) - h_2I_1(v), \end{aligned}\right\} v \in ((n+\gamma)\omega,(n+1)\omega], \\ \left.\begin{aligned} \Delta S_1(v) &= 0 \\ \Delta I_1(v) &= \frac{-\mu_{\max}\alpha I_1(v)}{(1+\alpha(I_1(v)+\widetilde{I(v)}))(1+\alpha\widetilde{I(v)})}, \end{aligned}\right\} v = (n+1)\omega. \end{cases}$$

Furthermore, the following linearly similar system is obtained by using the Taylor expansion:

$$\begin{cases} \left.\begin{aligned} \frac{dS_1(v)}{dv} &= (r_1 - \frac{r_1\theta_1\widetilde{I(v)}}{K_1} - \beta_1\widetilde{I(v)})S_1(v), \\ \frac{dI_1(v)}{dv} &= \beta_1\widetilde{I(v)}S_1(v) - d_1I_1(v), \end{aligned}\right\} v \in (n\omega,(n+\gamma)\omega], \\ \left.\begin{aligned} \Delta S_1(v) &= -\mu_1S_1(v), \\ \Delta I_1(v) &= -\mu_2I_1(v), \end{aligned}\right\} v = (n+\gamma)\omega, \\ \left.\begin{aligned} \frac{dS_1(v)}{dv} &= (r_2 - \frac{r_2\theta_2\widetilde{I(v)}}{K_2} - \beta_2\widetilde{I(v)} - h_1)S_1(v), \\ \frac{dI_1(v)}{dv} &= \beta_2\widetilde{I(v)}S_1(v) - d_2I_1(v) - h_2I_1(v), \end{aligned}\right\} v \in ((n+\gamma)\omega,(n+1)\omega], \\ \left.\begin{aligned} \Delta S_1(v) &= 0 \\ \Delta I_1(v) &= \frac{-\mu_{\max}\alpha I_1(v)}{(1+\alpha\widetilde{I(v)})^2}, \end{aligned}\right\} v = (n+1)\omega. \end{cases}$$

Therefore, we obtain:

$$\begin{pmatrix} \frac{dS_1(v)}{dv} \\ \frac{dI_1(v)}{dv} \end{pmatrix} = \begin{pmatrix} r_1 - \frac{r_1\theta_1\widetilde{I(v)}}{K_1} - \beta_1\widetilde{I(v)} & 0 \\ \beta_1\widetilde{I(v)} & -d_1 \end{pmatrix} \begin{pmatrix} S_1(v) \\ I_1(v) \end{pmatrix}, v \in (n\omega,(n+\gamma)\omega],$$

and

$$\begin{pmatrix} \frac{dS_1(v)}{dv} \\ \frac{dI_1(v)}{dv} \end{pmatrix} = \begin{pmatrix} r_2 - \frac{r_2\theta_2\widetilde{I(v)}}{K_2} - \beta_2\widetilde{I(v)} - h_1 & 0 \\ \beta_2\widetilde{I(v)} & -d_2-h_2 \end{pmatrix} \begin{pmatrix} S_1(v) \\ I_1(v) \end{pmatrix}, v \in ((n+\gamma)\omega,(n+1)\omega].$$

The fundamental solution matrix can be derived for $v \in (n\omega, (n+\gamma)\omega]$,

$$\Phi_1(v) = \begin{pmatrix} \exp\left(\int_{n\omega}^{v}(r_1 - \frac{r_1\theta_1\widetilde{I(\mu)}}{K_1} - \beta_1\widetilde{I(\mu)})d\mu\right) & 0 \\ *_1 & \exp\left(\int_{n\omega}^{v}(-d_1)\,d\mu\right) \end{pmatrix}.$$

For $v \in ((n+\gamma)\omega, (n+1)\omega]$, we have

$$\Phi_2(v) = \begin{pmatrix} \exp\left(\int_{(n+\gamma)\omega}^{v}(r_2 - \frac{r_2\theta_2\widetilde{I(\mu)}}{K_2} - \beta_2\widetilde{I(\mu)} - h_1)d\mu\right) & 0 \\ *_2 & \exp\left(\int_{(n+\gamma)\omega}^{v}(-d_2 - h_2)\,d\mu\right) \end{pmatrix}.$$

The terms $*_1$ and $*_2$ are not needed in the subsequent analysis, so their exact forms will not be provided. The linearization of the third and fourth equations of the system (2.1) is:

$$\begin{pmatrix} S_1((n+\gamma)\omega^+) \\ I_1((n+\gamma)\omega^+) \end{pmatrix} = \begin{pmatrix} 1-\mu_1 & 0 \\ 0 & 1-\mu_2 \end{pmatrix}\begin{pmatrix} S_1((n+\gamma)\omega) \\ I_1((n+\gamma)\omega) \end{pmatrix}.$$

The linearization of the seventh, and eighth equations of the system (2.1) is

$$\begin{pmatrix} S_1((n+1)\omega^+) \\ I_1((n+1)\omega^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2} \end{pmatrix}\begin{pmatrix} S_1((n+1)\omega) \\ I_1((n+1)\omega) \end{pmatrix}.$$

The stability of $(0, \widetilde{I(\omega)})$ is determined by the eigenvalues of

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2} \end{pmatrix}\Phi_2(\omega)\begin{pmatrix} 1-\mu_1 & 0 \\ 0 & 1-\mu_2 \end{pmatrix}\Phi_1(\gamma\omega),$$

which are

$$\begin{aligned} \lambda_1 =&(1-\mu_1)\exp\left(\int_0^{\gamma\omega}(r_1 - \frac{r_1\theta_1\widetilde{I(\mu)}}{K_1} - \beta_1\widetilde{I(\mu)})d\mu + \int_{\gamma\omega}^{\omega}(r_2 - \frac{r_2\theta_2\widetilde{I(\mu)}}{K_2} - \beta_2\widetilde{I(\mu)} - h_1)d\mu\right), \\ \lambda_2 =&(1-\mu_2)(1 - \frac{\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2})\exp\left(\int_0^{\gamma\omega}(-d_1)\,d\mu + \int_{\gamma\omega}^{\omega}(-d_2 - h_2)d\mu\right). \end{aligned}$$

For the conditions of this theorem, we have $|\lambda_1| < 1$,

$$\lambda_2 = (1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-l)\omega}(1 - \frac{\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2}) < 1,$$

and from (3.4), we can get

$$A_0 = (1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega}.$$

Because

$$\begin{aligned} \frac{(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega}\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2} &= \frac{4\alpha\mu_{max}A_0}{\left(1+\sqrt{1+4\alpha\mu_{max}A_0(1-A_0)^{-1}}\right)^2} \\ &< \frac{4\alpha\mu_{max}A_0}{\left(1+\sqrt{4\alpha\mu_{max}A_0}\right)^2} \\ &< 1, \end{aligned}$$

we obtain

$$\lambda_2 > -(1-\mu_2)e^{-d_1\gamma\omega}e^{-(d_2+h_2)(1-\gamma)\omega}\frac{\alpha\mu_{max}}{(1+\alpha\widetilde{I(\omega)})^2} > -1.$$

Thus, $|\lambda_2| < 1$. In light of the Floquet theory for impulsive differential equations [41], the periodic solution corresponding to pest eradication in the system (2.1) is locally asymptotically stable. We now proceed to establish its global attractivity. Let $\varepsilon > 0$ be sufficiently small such that

$$\begin{aligned}\rho =&(1-\mu_1)\exp\Bigg(\int_0^{\gamma\omega}\Bigg(r_1-(\frac{r_1\theta_1}{K_1}+\beta_1)(\widetilde{I(\mu)}-\varepsilon)\Bigg)d\mu\\ &+\int_{\gamma\omega}^{\omega}\Bigg(r_2-h_1-(\frac{r_2\theta_2}{K_2}+\beta_2)(\widetilde{I(\mu)}-\varepsilon)\Bigg)d\mu\Bigg) < 1.\end{aligned}$$

It is observed from the system (2.1) that the following inequality holds:

$$\begin{cases}\dfrac{dI(v)}{dv} \geq -d_1I(v), v \in (n\omega,(n+\gamma)\omega],\\ \Delta I(v) = -\mu_2 I(v), v = (n+\gamma)\omega,\\ \dfrac{dI(v)}{dv} \geq -(d_2+h_2)I(v), v \in ((n+\gamma)\omega,(n+1)\omega],\\ \Delta I(v) = \dfrac{\mu_{max}}{1+\alpha I(v)}, v = (n+1)\omega.\end{cases} \tag{4.2}$$

The impulsive differential equation under investigation is given by

$$\begin{cases}\dfrac{dI_0(v)}{dv} = -d_1I_0(v), v \in (n\omega,(n+\gamma)\omega],\\ \triangle I_0(v) = -\mu_2 I_0(v), v = (n+\gamma)\omega,\\ \dfrac{dI_0(v)}{dv} = -(d_2+h_2)I_0(v), v \in ((n+\gamma)\omega,(n+1)\omega],\\ \triangle I_0(v) = \dfrac{\mu_{max}}{1+\alpha I_0(v)}, v = (n+1)\omega.\end{cases} \tag{4.3}$$

By applying Lemma 3.6 together with the comparison theorem for impulsive differential equations [39], it follows that $I(v) \geq I_0(v)$, where $I_0(v)$ converges to $\widetilde{I(v)}$ as $v \to \infty$. Consequently, for sufficiently large values of v, the following inequality holds:

$$I(v) \geq I_0(v) \geq \widetilde{I(v)} - \varepsilon. \tag{4.4}$$

Analogously, presuming that the aforementioned inequality is satisfied for every $v \geq 0$ in the context of the systems (2.1) and (4.4), it follows that

$$\begin{cases}\dfrac{dS(v)}{dv} \leq S(v)(r_1-(\dfrac{r_1\theta_1}{K_1}+\beta_1)(\widetilde{I(v)}-\varepsilon)), v \in (n\omega,(n+\gamma)\omega],\\ \triangle S(v) = -\mu_1 S(v), v = (n+\gamma)\omega,\\ \dfrac{dS(v)}{dv} \leq S(v)(r_2-h_1-(\dfrac{r_2\theta_2}{K_2}+\beta_2)(\widetilde{I(v)}-\varepsilon)), v \in ((n+\gamma)\omega,(n+1)\omega],\\ \triangle S(v) = 0, v = (n+1)\omega.\end{cases} \tag{4.5}$$

This leads to

$$\begin{aligned} S((n+1)\omega) \leq &(1-\mu_1)S(n\omega^+)\exp\Bigg(\int_{n\omega}^{(n+\gamma)\omega}\Bigg(r_1-(\frac{r_1\theta_1}{K_1}+\beta_1)(\widetilde{I(\mu)}-\varepsilon)\Bigg)d\mu \\ &+\int_{(n+\gamma)\omega}^{(n+1)\omega}\Bigg(r_2-h_1-(\frac{r_2\theta_2}{K_2}+\beta_2)(\widetilde{I(\mu)}-\varepsilon)\Bigg)d\mu\Bigg). \end{aligned}$$

So $S((n+1)\omega) \leq S(0^+)\rho^{(n+1)}$, and $S((n+1)\omega) \to 0$, as $n \to \infty$. That is, $S(v) \to 0$, as $v \to \infty$.

Next, it will be demonstrated that $I(v)$ converges to $\widetilde{I}(v)$ as $v \to \infty$. For any sufficiently small $0 < \varepsilon_1 < \min\left\{\frac{d_1}{\beta_1}, \frac{d_1+h_2}{\beta_2}\right\}$, a time $v_0 > 0$ exists such that $0 < S(v) < \varepsilon_1$ holds for all $v \geq v_0$. Without loss of generality, we suppose that $0 < S(v) < \varepsilon_1$ for all $v \geq 0$. Under this assumption, the system (2.1) satisfies the following:

$$\begin{cases} \dfrac{dI(v)}{dv} \leq \beta_1\varepsilon_1 I(v) - d_1 I(v), v \in (n\omega, (n+\gamma)\omega], \\ \triangle I(v) = -\mu_2 I(v), v = (n+\gamma)\omega, \\ \dfrac{dI(v)}{dv} \leq \beta_2\varepsilon_1 I(v) - d_2 I(v) - h_2 I(v), v \in ((n+\gamma)\omega, (n+1)\omega]. \\ \triangle I(v) = \dfrac{\mu_{max}}{1+\alpha I(v)}, v = (n+1)\omega. \end{cases} \tag{4.6}$$

Considering the following system of comparison:

$$\begin{cases} \dfrac{dI_1'(v)}{dv} = (\beta_1\varepsilon_1 - d_1)I_1'(v), v \in (n\omega, (n+\gamma)\omega], \\ \triangle I_1'(v) = -\mu_2 I_1'(v), v = (n+\gamma)\omega, \\ \dfrac{dI_1'(v)}{dv} = (\beta_2\varepsilon_1 - d_2 - h_2)I_1'(v), v \in ((n+\gamma)\omega, (n+1)\omega], \\ \triangle I_1'(v) = \dfrac{\mu_{max}}{1+\alpha I_1'(v)}, v = (n+1)\omega. \end{cases} \tag{4.7}$$

The GAS periodic solution of (4.7) can be readily derived, leading to the determination of the periodic solution for (4.7)

$$\widetilde{I_1'(v)} = \begin{cases} I_1'^{*} e^{(\beta_1\varepsilon_1-d_1)(v-n\tau)}, v \in (n\omega, (n+\gamma)\omega], \\ I_1'^{**} e^{(\beta_1\varepsilon_1-d_2-h_2)(v-(n+\gamma)\omega)}, v \in ((n+\gamma)\omega, (n+1)\omega], \end{cases} \tag{4.8}$$

where

$$\begin{aligned} I_1'^{*} =& \frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1}, I_1'^{**} = (1-\mu_2)e^{(\beta_1\varepsilon_1-d_1)\omega}I_1^{*}, \\ A_{10} =&(1-\mu_2)e^{(\beta_1\varepsilon_1-d_1)\gamma\omega}(e^{(\beta_2\varepsilon_1-d_2-h_2)(1-\gamma)\omega}) > 0, \\ A_1 =&\alpha A_{10}(1-A_{10}), \\ B_1 =&1-A_{10}, \\ C_1 =&-\mu_{max}. \end{aligned}$$

Hence, given any $\varepsilon_2 > 0$, we can find a time v_1 such that for all $v > v_1$, the following holds

$$\widetilde{I(v)} - \varepsilon_2 < I(v) < \widetilde{I_1'(v)} + \varepsilon_2.$$

As $\varepsilon_1 \to 0$, it follows that

$$\widetilde{I(v)} - \varepsilon_2 < I(v) < \widetilde{I(v)} + \varepsilon_2. \tag{4.9}$$

Hence, given any $\varepsilon_2 > 0$, for a sufficiently large v, the inequality $\widetilde{I}(v) - \varepsilon < I(v) < \widetilde{I}(v) + \varepsilon$ holds, indicating that $I(v)$ converges to $\widetilde{I}(v)$ as $v \to \infty$. This concludes the proof. □

The subsequent step involves examining the permanence of the system (2.1). Prior to undertaking this analysis, the following definition is presented.

4.2. Permanence

Definition 4.1. *System* (2.1) *is defined to be permanent if there are positive constants g and G, that are independent of the initial conditions, along with a finite time V_0, such that for every solution $(S(v), I(v))$ corresponding to the initial values $S(0^+) > 0$ and $I(0^+) > 0$, the inequalities*

$$g \le S(v) \le G, \quad g \le I(v) \le G$$

hold for all $v \ge V_0$. Note that V_0 may depend on the initial values $(S(0^+), I(0^+))$.

Theorem 4.2. *If*

$$\begin{aligned} r_1\gamma\omega + r_2(1-\gamma)\omega + \frac{I^*(r_1\theta_1 + \beta_1 K_1)}{d_1 K_1}(e^{-d_1\gamma\omega} - 1) - h_1(1-\gamma)\omega \\ + \frac{I^*(1-\mu_2)e^{-d_1\gamma\omega}(r_2\theta_2 + \beta_2 K_2)}{K_2(d_1 + d_2)}(e^{-(d_2+h_2)(1-\gamma)\omega} - 1) > \ln\frac{1}{1-\mu_1} \end{aligned}$$

holds, then the system (2.1) *is permanent, where I^* is given by* (3.4).

Proof. By Lemma 3.4, it has been shown that $S(v) \le G$ and $I(v) \le G$ hold for sufficiently large values of v. We further assume that $S(v) \le G$ and $I(v) \le G$ for all $v \ge 0$. From the proof of Theorem 4.1, it follows that $\varepsilon > 0$ exists such that for all sufficiently large values of v. Consequently, for a large enough v, the inequality

$$I(v) \ge I^* e^{-d_1\gamma\omega} + I^{**} e^{-(d_2+h_2)((1-\gamma)\omega)} - \varepsilon =: g_2$$

is satisfied, where I^* and I^{**} are defined in Eqs (3.4) and (3.10), respectively. Therefore, it suffices to identify a positive constant g_1 such that $S(v) \ge g_1$ holds for sufficiently large v. This will be demonstrated in the following two steps.

Step 1. According to the assumptions of Theorem 4.2, it is possible to choose sufficiently small positive constants g_3 and ε_3 such that

$$0 < g_3 < \min\left\{\frac{d_1}{\beta_1}, \frac{d_2 + h_2}{\beta_2}\right\}$$

and

$$\begin{aligned} \delta =& (1-\mu_1)\exp\left(\int_{n\omega}^{(n+\gamma)\omega}\left(r_1 - \frac{r_1}{K_1}g_3 - (\frac{r_1\theta_1}{K_1} + \beta_1)(\widetilde{I_2(\mu)} + \varepsilon_3)\right)d\mu\right. \\ &\left. + \int_{(n+\gamma)\omega}^{(n+1)\omega}\left(r_2 - \frac{r_2}{K_2}g_3 - h_1 - (\frac{r_2\theta_2}{K_2} + \beta_2)((\widetilde{I_2(\mu)} + \varepsilon_3)\right)d\mu\right) > 1. \end{aligned}$$

We will prove that $S(v) < g_3$ can not hold for $v \geq 0$. Otherwise,

$$\begin{cases} \dfrac{dI(v)}{dv} \leq (\beta_1 g_3 - d_1)I(v), v \in (n\omega, (n+\gamma)\omega], \\ \triangle I(v) = -\mu_2 I(v), v = (n+\gamma)\omega, \\ \dfrac{dI(v)}{dv} \leq (\beta_2 g_3 - d_2 - h_2)I(v), v \in ((n+\gamma)\omega, (n+1)\omega], \\ \triangle I(v) = \dfrac{\mu_{max}}{1+\alpha I(v)}, v = (n+\gamma)\omega. \end{cases} \tag{4.10}$$

By invoking the comparison principle for impulsive differential equations [39] in conjunction with Lemma 3.5, it follows that $I(v) \leq I_2(v)$, moreover, $I_2(v) \to \widetilde{I_2(v)}$ as $v \to \infty$. Here

$$\begin{cases} \dfrac{dI_2(v)}{dv} = (\beta_1 g_3 - d_1)I_2(v), v \in (n\omega, (n+\gamma)\omega], \\ \triangle I_2(v) = -\mu_2 I_2(v), v = (n+\gamma)\omega, \\ \dfrac{dI_2(v)}{dv} = (\beta_2 g_3 - d_2 - h_2)I_2(v), v \in ((n+\gamma)\omega, (n+1)\omega], \\ \triangle I_2(v) = \dfrac{\mu_{max}}{1+\alpha I_2(v)}, v = (n+\gamma)\omega. \end{cases} \tag{4.11}$$

Analyzing (4.11) in a similar way to (4.7), we get

$$\widetilde{I_2(v)} = \begin{cases} I_2^* e^{(\beta_1 g_3 - d_1)(v - n\omega)}, v \in (n\omega, (n+\gamma)\omega], \\ I_2^{**} e^{(\beta_1 g_3 - d_2 - h_2)(v-(n+\gamma)\omega)}, v \in ((n+\gamma)\omega, (n+1)\omega], \end{cases} \tag{4.12}$$

where

$$\begin{aligned} I_2{}^* =& \frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}, I_1^{**} = (1-\mu_2)e^{(\beta_1 m_3 - d_1)\omega} I_2^*, \\ A_{11} =& (1-\mu_2)e^{(\beta_1 m_3 - d_1)\gamma\omega}(e^{(\beta_2 m_3 - d_2 - h_2)(1-\gamma)\omega} > 0, \\ A_2 =& \alpha A_{11}(1 - A_{11}), \\ B_2 =& 1 - A_{11}, \\ C_2 =& -\mu_{max}. \end{aligned}$$

Consequently, we can identify a time $V_1 > 0$ and a constant ε_3 such that

$$I(v) \leq I_2(v) \leq \widetilde{I_2(v)} + \varepsilon_3$$

and

$$\begin{cases} \dfrac{dS(v)}{dv} \geq S(v)(r_1 - \dfrac{r_1}{K_1} g_3 - (\dfrac{r_1\theta_1}{K_1} + \beta_1)(\widetilde{I_2(v)} + \varepsilon_3)), v \in (n\omega, (n+\gamma)\omega], \\ \triangle S(v) = -\mu_1 S(v), v = (n+\gamma)\omega, \\ \dfrac{dS(v)}{dv} \geq S(v)(r_2 - \dfrac{r_2}{K_2} g_3 - h_1 - (\dfrac{r_2\theta_2}{K_2} + \beta_2)(\widetilde{I_2(v)} + \varepsilon_3)), v \in ((n+\gamma)\omega, (n+1)\omega], \\ \triangle S(v) = 0, v = (n+\gamma)\omega. \end{cases} \tag{4.13}$$

For all $v \geq V_1$, consider a $N_1 \in \mathbb{Z}_+$ satisfying $N_1\omega > V_1$. Integrating (4.13) over the interval $(n\omega, (n+\gamma)\omega]$ for $n \geq N_1$, we obtain

$$\begin{aligned} S((n+1)\omega) \geq &(1-\mu_1)S(n\omega^+)\exp\Bigg(\int_{n\omega}^{(n+\gamma)\omega}\Big(r_1 - \frac{r_1}{K_1}g_3 - (\frac{r_1\theta_1}{K_1} + \beta_1)(\widetilde{I_2(\mu)} + \varepsilon_3)\Big)d\mu \\ &+ \int_{(n+\gamma)\omega}^{(n+1)\omega}\Big(r_2 - \frac{r_2}{K_2}g_3 - h_1 - (\frac{r_2\theta_2}{K_2} + \beta_2)(\widetilde{I_2(\mu)} + \varepsilon_3)\Big)d\mu\Bigg). \end{aligned}$$

Consequently, we have $S((N_1+k)\omega) \geq S(N_1\omega^+)\delta^k \to \infty$ as $k \to \infty$, which contradicts the assumed boundedness of $S(v)$. Therefore, one can conclude that some $v_1 > 0$ exists such that $S(v_1) \geq g_3$.

Step 2. The analysis of $S(v)$ is conducted by considering two cases.

Case (i) If $S(v) \geq g_3$, which holds for all $v \geq v_1$, then we are done. Otherwise, we discuss the following case.

Case (ii) $S(v)$ is oscillatory around g_3 for all $v \geq v_1$. Let $v^* = \inf_{v \geq v_1}\{v : S(v) < g_3\}$, we also have $S(v) \geq g_3$ for $v \in [v_1, v^*)$, where $v^* \in (n_1\omega, (n_1+1)\omega]$, $n_1 \in \mathbb{Z}_+$.

Case 1. If $v^* = (n_1+\gamma)\omega$, then $S(v) \geq g_3$ for $v \in [v_1, v^*)$ and $S(v^*) = g_3$, and $(1-\mu_1)g_3 \leq S(v^{*+}) = (1-\mu_1)S(v^*) < g_3$. Choose integers $n_2, n_3 \in \mathbb{Z}_+$ such that

$$(1-\mu_1)^{n_2}e^{n_2\rho_1\omega}\delta^{n_3} > (1-\mu_1)^{n_2}e^{(n_2+1)\rho_1\omega}\delta^{n_3} > 1,$$

where

$$\rho_1 = \min\{r_1 - \frac{r_1}{K_1}(g_3 + \theta_1 G) - \beta_1 G, r_2 - \frac{r_2}{K_2}(g_3 + \theta_2 G) - \beta_2 G - h_1\} < 0.$$

By setting $V = (n_2+n_3)\omega$, we assert that there exists a time $v_2 \in (v^*, v^*+V]$ such that $S(v_2) \geq g_3$. Otherwise, if $S(v) < g_3$ for all $v \in (v^*, v^*+V]$. Therefore, from Eq (4.11) and the identity $I_2(v^{*+}) = I(v^{*+})$, it follows that

$$I_2(v) = \begin{cases} A_{11}(1 - \frac{\mu_{max}\alpha}{(1+\alpha A_{11}I_2(n-1)\omega^+))(1+\alpha A_{11}I_2^*)})(I_2(n-1)\omega^+) - I_2^*)e^{(\beta_1 g_3 - d_1)(v-n\omega)} + \widetilde{I_2(v)}, \quad v \in (n\omega, (n+\gamma)\omega], \\ (1-\mu_2)A_{11}(1 - \frac{\mu_{max}\alpha}{(1+\alpha A_{11}I_2(n-1)\omega^+))(1+\alpha A_{11}I_2^*)})(I_2(n-1)\omega^+) - I_2^*)e^{(\beta_1 g_3 - d_1)\gamma\omega}e^{(\beta_2 g_3 - d_2 - h_2)(v-(n+\gamma)\omega)} \\ +\widetilde{I_2(v)}, \quad v \in ((n+\gamma)\omega, (n+1)\omega], \end{cases}$$

$n_1 + 1 \leq n \leq n_1 + 1 + n_2 + n_3$. Furthermore, according to Lemma 3.5 and (4.12), we can acquire that

$$\left|I_2(v) - \widetilde{I_2(v)}\right| < \varepsilon_4. \tag{4.14}$$

Furthermore, for any v within the interval $[v^* + n_2\tau,\ v^* + V]$, the inequalities $I(v) \leq I_2(v) \leq \widetilde{I_2}(v) + \varepsilon_4$ are satisfied.

According to (4.13), we obtian

$$S(v^* + V) \geq S(v^* + n_2\omega)\delta^{n_3}, \tag{4.15}$$

It can be deduced from the system (2.1) that

$$\begin{cases} \dfrac{dS(v)}{dv} \geq \rho_1 S(v), v \neq (n+\gamma)\omega, \\ \Delta S(v) = -\mu_1 S(v), v = (n+\gamma)\omega. \end{cases} \tag{4.16}$$

By integrating (4.16) over the interval $(v^*, (v^* + n_2)\omega]$, we obtain

$$S(v^* + n_2\tau)) \geq g_3(1 - \mu_1)^{n_2} e^{n_2\rho_1\omega}. \tag{4.17}$$

Then by (4.15) and (4.17), we obtain

$$S(v^* + V) \geq g_3(1 - \mu_1)^{n_2} e^{n_2\rho_1\omega}\delta^{n_3} > g_3. \tag{4.18}$$

This result stands in contradiction to the initial assumption that $S(v) < g_3$. Let

$$\overline{v} = \inf_{v^*<v\leq t^*+V}\{S(v) \geq g_3\}.$$

Consequently, we have $S(\bar{t}) \geq m_3$, for all $v \in (v^*, \overline{v})$, integrating Eq (4.16) over the interval $(v^*, \overline{v})$ yields the inequality

$$S(v) \geq S(v^{*+})e^{\sigma_1(v-v^*)} \geq g_3(1 - \mu_1)^{n_2+n_3} e^{(n_2+n_3)\rho_1\omega} > g_3(1 - \mu_1)^{n_2+n_3} e^{(n_2+n_3+1)\rho_1\omega} \triangleq \widehat{g},$$

which holds for $v \in (v^*, \overline{v})$. The same reasoning can be extended for $v > \overline{v}$: therefore, we conclude that $S(v) \geq \widehat{g}$ for all $v \geq v_1$.

Case 2. $v^* \neq (n_1 + \gamma)\omega$, then it holds that $S(v) \geq g_3$ for all $v \in [v_1, v^*)$, and $S(v^*) = g_3$. Assume further that $v^* \in ((n_1^{'} + \gamma)\omega, (n_1^{'} + \gamma + 1)\omega]$. For any $v \in (v^*, (n_1^{'} + \gamma + 1)\omega)$, where $n_1^{'} \in \mathbb{Z}_+$, two distinct scenarios may arise.

Case 2a: Assume that $S(v) < g_3$ holds for all $v \in (v^*, (n_1' + \gamma + 1)\omega]$. Following an approach analogous to Case 1, it can be demonstrated that there is a moment $v_2' \in ((n_1' + \gamma)\omega, (n_1' + \gamma + 1)\omega + V)$ at which $S(v_2') > g_3$ is satisfied. Let $\widetilde{v} = \inf_{v>v^*}\{S(v) \geq g_3\}$. Consequently, one obtains $S(v) < g_3$ for all $v \in (v^*, \widetilde{v})$ and $S(\widetilde{v}) = g_3$. By integrating Eq (4.16) over the interval $(v^*, \widetilde{v})$, we obtain

$$S(v) \geq S(v^*)e^{\rho_1(v-v^*)} \geq g_3(1 - \mu_1)^{n_2+n_3} e^{(n_2+n_3+1)\rho_1\omega} \triangleq \widehat{g}.$$

Consequently, $S(\widetilde{v}) \geq g_3$ for $v \in [v^*, \widetilde{v})$, the same reasoning can be extended for $v > \widetilde{v}$. Therefore, we conclude that $S(v) \geq \widehat{g}$ for all $v \geq v_1$.

Case 2b: A time point $v \in (v^*, (n_1' + \gamma + 1)\omega]$ exists such that $S(v) \geq g_3$. Let $\hat{v} = \inf_{v>v^*}\{S(v) \geq g_3\}$. It follows that $S(v) < g_3$ for all $v \in [v^*, \hat{v})$, and $S(\hat{v}) = g_3$. The differential equation (4.16) remains valid throughout the interval $[v^*, \hat{v})$. Upon integrating both sides over this interval, we obtain the following result

$$S(v) \geq S(v^*)e^{\rho_1(v-v^*)} \geq g_3 e^{\rho_1\omega} > \widehat{g}.$$

This procedure remains valid as long as the condition $S(\hat{v}) \geq \widehat{g}$ holds, and thus we can deduce that $S(v) \geq \widehat{g}$ for all $v \geq \hat{v}$. Consequently, by Cases 1 and 2, the inequality $S(v) \geq \widehat{g}$ holds for all $v \geq v_1$. This completes the proof. □

Corollary 4.1. *Denote $R_0 = f_1/f_2$. If $R_0 < 1$, then the periodic solution corresponding to the pest-eradication state of the system* (2.1) *is GAS. If $R_0 > 1$, then the system* (2.1) *is permanent.*

Here,

$$\begin{aligned} f_1 &= r_1\gamma\omega + r_2(1 - \gamma)\omega + \frac{I^*(r_1\theta_1 + \beta_1 K_1)}{d_1 K_1}(e^{-d_1\gamma\omega} - 1) - h_1(1 - \gamma)\omega \\ &\quad + \frac{I^*(1 - \mu_2)e^{-d_1\gamma\omega}(r_2\theta_2 + \beta_2 K_2)}{K_2(d_2 + h_2)}(e^{-(d_2+h_2)(1-\gamma)\omega} - 1), \\ f_2 &= \ln\frac{1}{1 - \mu_1}. \end{aligned}$$

5. Numerical simulation and discussions

5.1. Extinction and permanence

In order to verify our theoretical mathematical results, we will analyze the influence of impulsive effects on the dynamics of the system (2.1) through numerical simulations and a brief discussion. Moreover, the parameter values employed in the simulations are, to a great extent, derived from scholarly publications and have biological plausibility, which is illustrated in Table 1.

Table 1. Values of parameters of the system (2.1).

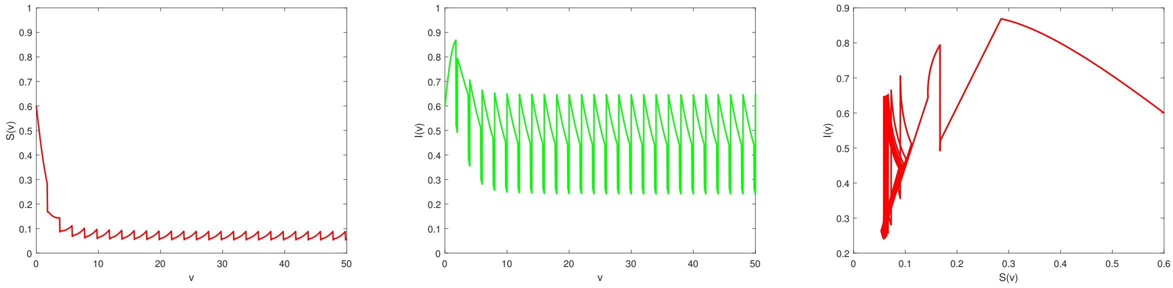
Parameter	Descriptions	Values range	Source
r_1	The intrinsic growth rate of pests	$(0, 3.5]$	[32]
K_1	The environment's carrying capacity	$(0, 50]$	[32]
θ_1	The competitive ability of infected pests	$(0, 1)$	The definition of the competitive ability
β_1	The transmission rate of I on pests	$(0, 4]$	[18]
u_1	The instantaneous impulsive mortality rates of pesticides on pests	$(0, 1)$	The definition of mortality rate
u_2	The instantaneous impulsive mortality rates of I	$(0, 1)$	The definition of mortality rate
r_2	The intrinsic growth rate of pests	$(0, 3.5]$	[32]
K_2	The environment's carrying capacity	$(0, 50]$	[32]
θ_2	The competitive ability of infected pests	$(0, 1)$	The definition of the competitive ability
β_2	The transmission rate of I on pests	$(0, 4]$	[18]
h_1	The non-instantaneous impulsive killing rates of pesticides acting on pests	$(0, 1)$	The definition of mortality rate
h_2	The non-instantaneous impulsive killing rates of pesticides acting on I	$(0, 1)$	The definition of mortality rate
α	Shape parameter	–	Assumed
$\mu_{\max}$	The maximum amount of I released	$(0, 4]$	[37]
γ	The interval between two different impulsive controls	$(0, 1)$	The definition of the interval between two different impulsive controls
ω	The impulsive control period	$(1, 10]$	[39]

Here, we assume that the parameters are set as follows: $S(v) = 0.6$, $I(v) = 0.6$, $r_1 = 1$, $K_1 = 1$, $\theta_1 = 0.1, \beta_1 = 1.2$, $d_1 = 0.3$, $\mu_2 = 0.4$, $r_2 = 1$, $K_2 = 1$, $\theta_2 = 0.1, \beta_2 = 1.2$, $h_1 = 0.2$, $h_2 = 0.2$, $d_2 = 0.2$, $\alpha = 2$, $\gamma = 0.9$, and $\omega = 2$.

- (1) When $\mu_1 = 0.4$, $\mu_{\max} = 0.6$, all conditions of Theorem 4.2 are satisfied, and the system (2.1) is permanent (see Figure 1).
- (2) When $\mu_1 = 0.7$, $\mu_{\max} = 0.6$, the pest extinction periodic solution $(0, \widetilde{I(v)})$ of the system (2.1) is GAS (see Figure 2).
- (3) When $\mu_1 = 0.3$, $\mu_{\max} = 0.6$, all conditions of Theorem 4.2 are again satisfied, and the system (2.1) remains permanent (see Figure 3).
- (4) When $\mu_1 = 0.4$, $\mu_{\max} = 0.9$, the pest extinction periodic solution $(0, \widetilde{I(v)})$ is GAS (see Figure 4).

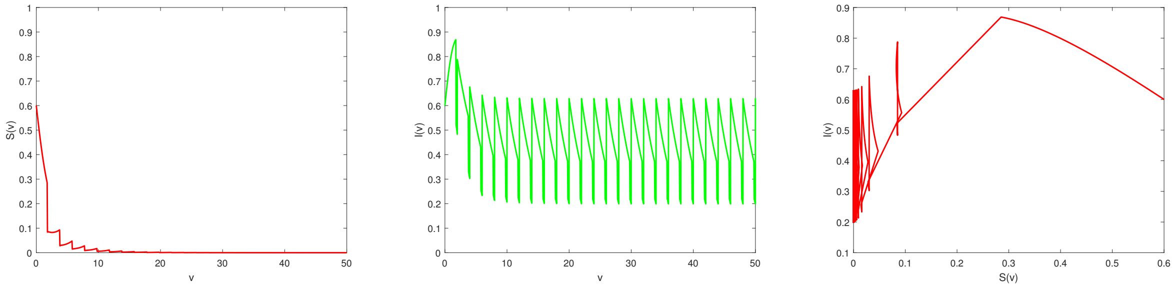
- (5) When $\mu_1 = 0.4$, $\mu_{\max} = 0.2$, all conditions of Theorem 4.2 are satisfied, and the system (2.1) is permanent (see Figure 5).

These simulation results further confirm the theoretical analysis and highlight how variations in μ_1 and $\mu_{\max}$ influence the long-term behavior of the system.



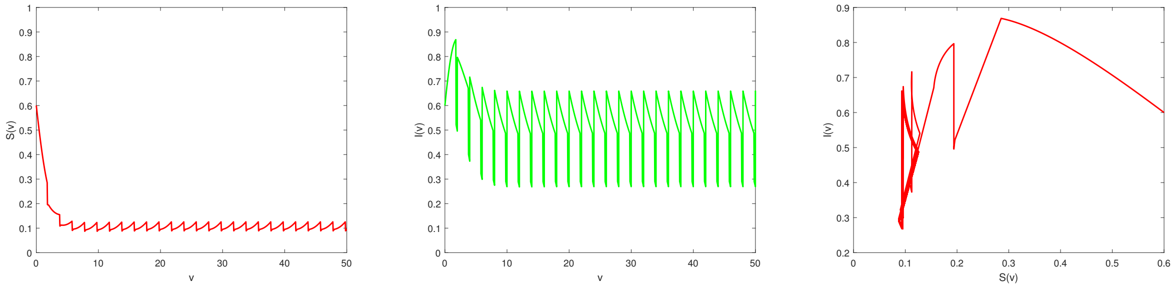
(a) (b) (c)

Figure 1. The permanence of the system (2.1) with $\mu_1 = 0.4, \mu_{\max} = 0.6$. (a) The time series of $S(v)$. (b) The time series of $I(v)$. (c) The system's phase portrait.



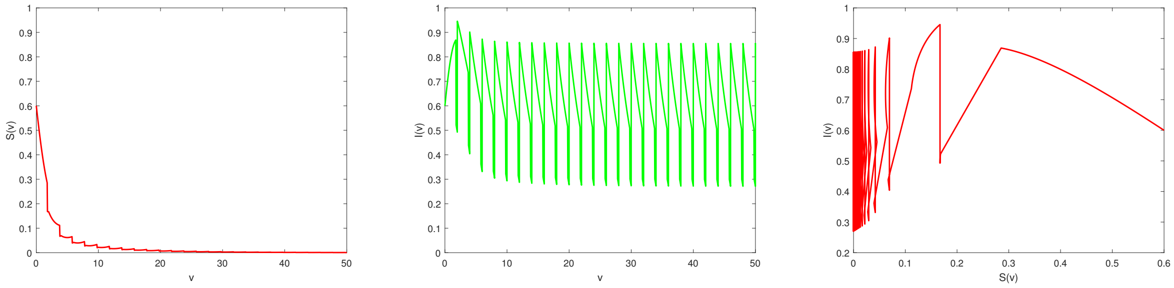
(a) (b) (c)

Figure 2. The periodic solution of system (3.1) for GAS of the susceptible pest eradication with $\mu_1 = 0.7, \mu_{\max} = 0.6$. (a) The time series of $S(v)$. (b)The time series of $I(v)$. (c)The phase portrait of portrait of the system (2.1).



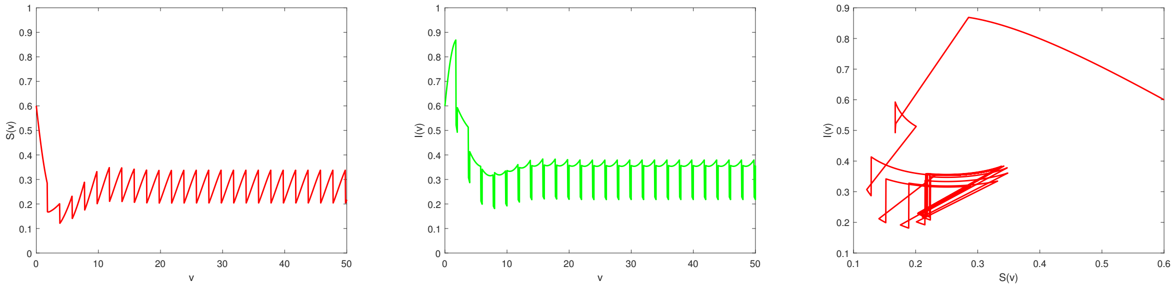
(a) (b) (c)

Figure 3. The permanence of system (2.1) with $\mu_1 = 0.3, \mu_{\max} = 0.6$. (a) The time series of $S(v)$. (b) The time series of $I(v)$. (c) The phase portrait of the system (2.1).



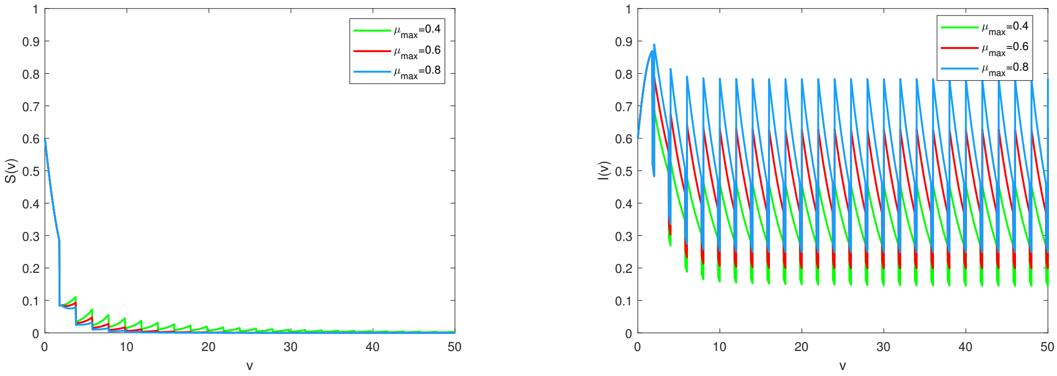
(a) (b) (c)

Figure 4. The periodic solution of system (3.1) for GAS of the susceptible pest eradication with (3.1) with $\mu_1 = 0.4, \mu_{\max} = 0.9$. (a) The time series of $S(v)$, (b)The time series of $I(v)$. (c) The phase portrait of the system (2.1).



(a) (b) (c)

Figure 5. The permanence of system (2.1) with $\mu_1 = 0.4, \mu_{\max} = 0.2$. (a) The time series of $S(v)$. (b) The time series of $I(v)$. (c)The Phase portrait of the system (2.1).



(a) (b)

Figure 6. The effect of different parameter values of $\mu_{\max}$ on the pest extinction periodic solution $(0, \widetilde{I(v)})$ of the system (3.1). (a) The time series of $S(v)$. (b)The time series of $I(v)$.

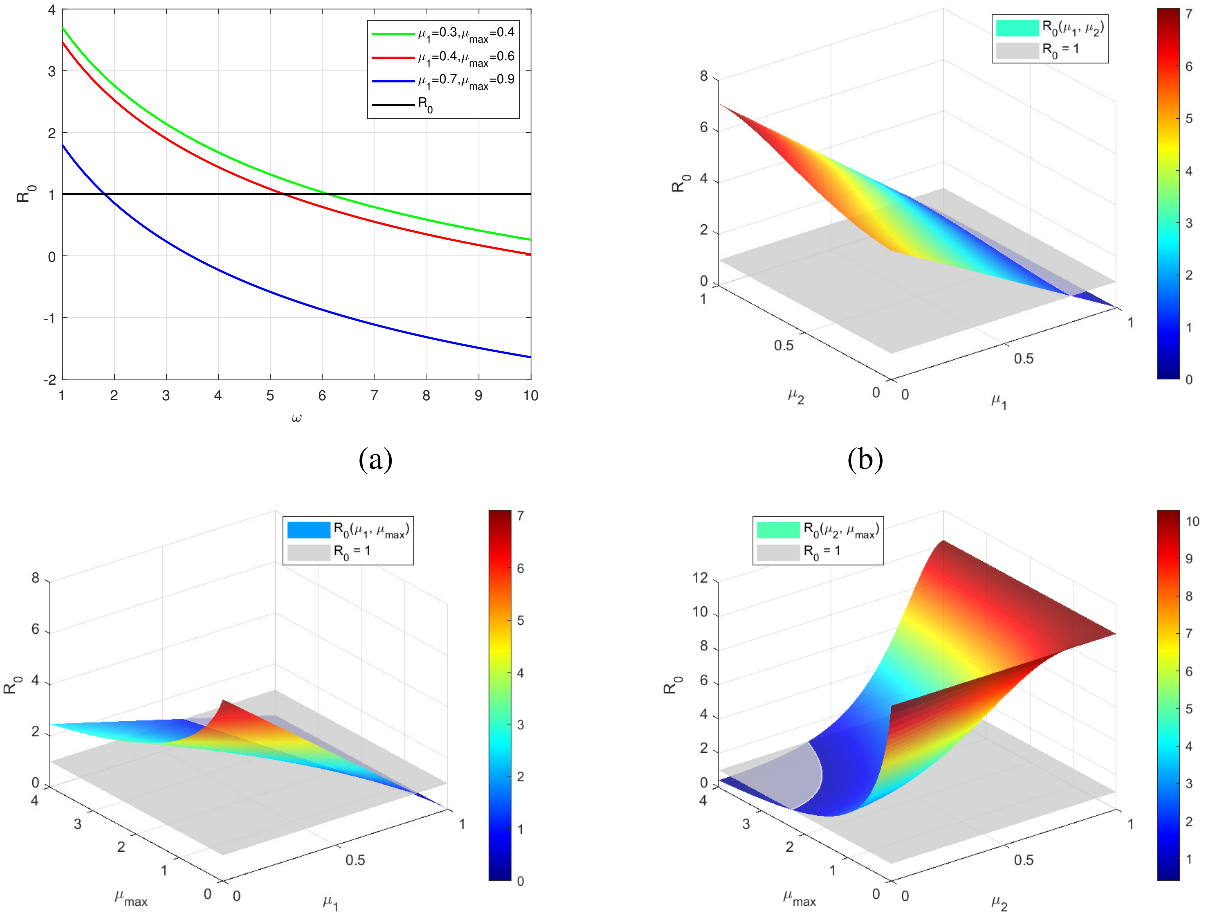
The simulation results in Figures 1–3 demonstrate that the parameter μ_1 significantly influences the dynamic behavior of the system (2.1). When all other parameters of the system (2.1) are fixed, Theorems 4.1 and 4.2 suggest the existence of a threshold value μ_1^*. Specifically, if $\mu_1 > \mu_1^*$, the pest extinction periodic solution $(0, \widetilde{I(v)})$ of the system (2.1) is GAS. Conversely, if $\mu_1 < \mu_1^*$, the system (2.1) is permanent. These findings indicate that the instantaneous impulsive killing rate of the pesticide on pests plays a crucial role in the system's dynamics and can effectively suppress pest populations.

The simulation results presented in Figures 1, 4, and 5 clearly demonstrate that the parameter $\mu_{\max}$ has a significant impact on the dynamical behavior of the system (2.1). With all other parameters held constant, Theorems 4.1 and 4.2 suggest the existence of a critical threshold $\mu_{\max}^*$. Specifically, when $\mu_{\max} > \mu_{\max}^*$, the pest extinction periodic solution $(0, \widetilde{I(v)})$ of the system (2.1) is GAS. In contrast, if $\mu_{\max} < \mu_{\max}^*$, the system remains permanent. These findings highlight the pivotal role of the pulse release of natural enemies in shaping the system's dynamics, effectively suppressing pest populations.

Assuming that the pest extinction periodic solution $(0, \widetilde{I(v)})$ of the system (2.1) is GAS, we assume that the parameters are set as follows: $S(v) = 0.6$, $I(v) = 0.6$, $r_1 = 1$, $K_1 = 1$, $\theta_1 = 0.1$, $\beta_1 = 1.2$, $d_1 = 0.3$, $\mu_1 = 0.7$, $\mu_2 = 0.4$, $r_2 = 1$, $K_2 = 1$, $\theta_2 = 0.1$, $\beta_2 = 1.2$, $h_1 = 0.2$, $h_2 = 0.2$, $d_2 = 0.2$, $\alpha = 2$, $\gamma = 0.9$, and $\omega = 2$. The simulation results in Figure 6 indicate that the population density of $I(v)$ increases with the rise in the transient pulse release level $\mu_{\max}$. Meanwhile, the population density of pests decreases accordingly, which, in turn, accelerates the extinction process of $I(v)$ as $\mu_{\max}$ increases from 0.4 to 0.8.

5.2. *Threshold condition*

As established in Corollary 4.1, the pest periodic solution $(0, S(v))$ achieves GAS when $R_0 < 1$; in contrast, the system (2.1) maintains permanence when $R_0 > 1$. Thus $R_0 = 1$ functions as the critical threshold governing the GAS of pest extinction. In the subsequent analysis, numerical methods are employed to examine how key parameters within the system (2.1) impact this critical threshold R_0. First, R_0 exhibits a monotonic decreasing trend with respect to μ_1, μ_{max}, γ, ω, and h_1 individually. Second, a set of parameters is specified as follows: $r_1 = 1, r_2 = 1, K_1 = 1, K_2 = 1, \theta_1 = 0.1, \theta_2 = 0.1, \beta_1 = 1.2, \beta_2 = 1.2, d_1 = 0.3, d_2 = 0.3, h_1 = 0.2, h_2 = 0.2, \mu_1 = 0.7, \mu_2 = 0.4, \mu_{max} = 0.9, \alpha = 2, \gamma = 0.9$. Using these parameters, the change in R_0 corresponding to variations in the impulsive period v is calculated under different instantaneous lethal rates of natural predators and the number of natural predators released; these results presented (see Figure 7(a)). Thereafter, three distinct parameter groups are defined: (C1) $r_1 = 1, r_2 = 1, K_1 = 1, K_2 = 1, \theta_1 = 0.1, \theta_2 = 0.1, \beta_1 = 1.2, \beta_2 = 1.2, d_1 = 0.3, d_2 = 0.3, h_1 = 0.2, h_2 = 0.2, \mu_{max} = 0.6, \alpha = 2, \gamma = 0.9$, and $\omega = 2$; (C2) $r_1 = 1.5, r_2 = 1, K_1 = 1, K_2 = 1, \theta_1 = 0.1, \theta_2 = 0.1, \beta_1 = 3.5, \beta_2 = 3, d_1 = 0.3, d_2 = 0.2, h_1 = 0.3, h_2 = 0.3, \mu_1 = 0.4, \alpha = 2, \gamma = 0.9$, and $\omega = 2$; (C3) $r_1 = 1, r_2 = 1, K_1 = 1, K_2 = 1, \theta_1 = 0.1, \theta_2 = 0.1, \beta_1 = 1.2, \beta_2 = 1.2, d_1 = 0.3, d_2 = 0.3, h_1 = 0.2, h_2 = 0.2, \mu_2 = 0.2, \alpha = 2, \gamma = 0.9$, and $\omega = 2$. The two-parameter trend plots corresponding to these groups are illustrated in Figure 7(b–d). As can be deduced from Figure 7(a), with an increase in the impulsive period ω, the threshold R_0 undergoes a corresponding change; more effective management of pest populations can be achieved by appropriately raising both the number of natural enemies released and the instantaneous lethal rate that pesticides exert on pests. As can be deduced from Figure 7(b), the threshold R_0 exhibits a decreasing trend with increase in both μ_1 and μ_2, and pest populations can be effectively managed through the appropriate augmentation of pesticide-induced lethal rates targeting both pests and their natural enemies. As can be deduced from Figure 7(c), the threshold R_0 demonstrates a decreasing tendency with an increase in μ_1 and $\mu_{\max}$, and effective pest control can be realized by appropriately raising the instantaneous lethal rate of pesticides on pests and increasing the number of natural predators released, which better reflects the IPM control strategy. As deduced from Figure 7(d), the threshold R_0 exhibits an increasing trend as $\mu_{\max}$ and μ_2 rise, and thus in cases where the number of natural predators released is insufficient, reducing the instantaneous lethal rate of pesticides on natural enemies can achieve the goal of pest management.



(a) (b)

(c) (d)

Figure 7. Variations of R_0 with certain parameters. (a) Change chart of R_0 with respect to ω under different cooperation coefficients. (b) Change chart of R_0 with respect to μ_1 and μ_2 under parameters (C1). (c) Change chart of R_0 with respect to μ_1 and μ_{max} under parameters (C2). (d) Change chart of R_0 with respect to μ_2 and μ_{max} under parameters (C3).

Following the approach outlined in the literature [39, 42], a sensitivity analysis is performed on the threshold parameter R_0, along with several associated key parameters. The values employed in this analysis are specified as follows: $r_1 = 1.2, r_2 = 1, K_1 = 1, K_2 = 1, \theta_1 = 0.15, \theta_2 = 0.2, b_1 = 1, b_2 = 1, d_1 = 0.2, d_2 = 0.2, h_1 = 0.2, h_2 = 0.3, \mu_1 = 0.6, \mu_2 = 0.3, \mu_{max} = 0.8, \alpha = 2, \gamma = 0.95$, and $\omega = 4$. These settings yield the results illustrated (see in Figure 8).

Figure 8 consists of two visual components: A bar plot presenting the partial rank correlation coefficient (PRCC) values of R_0 with respect to each parameter, and a scatter plot that depicts the variation of R_0 in response to changes in the parameters' values. As observed from the bar plot, the parameters h_2 and μ_2 exhibit positive PRCC values, implying that an increase in any of these parameters is likely to facilitate pest population outbreaks.

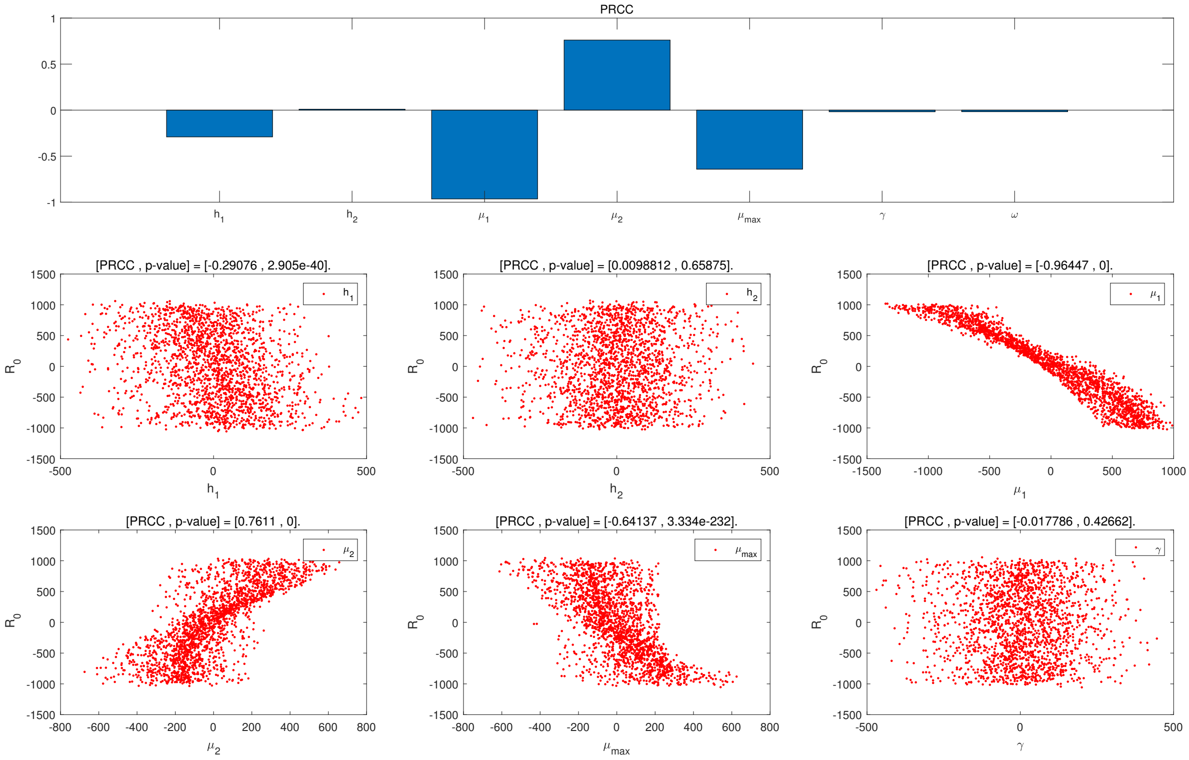


Figure 8. Scatter plots illustrating the partial rank correlation coefficient (PRCC) values of the threshold R_0 with respect to key parameters (h_1, μ_1, μ_2, μ_{max}). A sample size of 2000 is used, and all parameters are varied concurrently.

In contrast, parameters such as h_1, μ_2, μ_{max}, γ, and ω yield negative PRCC values, suggesting that increasing these values may contribute to effective pest suppression. Notably, μ_1, μ_2, and μ_{max} exhibit PRCC values exceeding 0.4, highlighting their pivotal role in pest population control. Additionally, the PRCC value of h_1 falls within the range of 0.3 to 0.4, indicating that the non-instantaneous lethal rate significantly influences pest management dynamics.

6. Conclusions

In this paper, we considered a pest management SI epidemic model with instantaneous and non-instantaneous impulsive effects. At the instantaneous pulse moments, the nonlinear release of infected pests and the reduction in the populations of both susceptible and infected pests occur from spraying pesticide. At non-instantaneous pulse moments, the mortality rates of susceptible and infected pests following pesticide application are considered. These factors can significantly influence the IPM system. On the basis of the practical aspects of IPM, we constructed a SI epidemic model for pest control that incorporated the nonlinear release of natural predators at specific impulsive times as well as instantaneous and noninstantaneous kill rates following the pesticide treatment. We demonstrated that all of the system's (2.1) solutions are uniformly eventually bounded. Meanwhile, from Corollary 4.1, the pest-periodic solution $(0, S(v))$ achieves GAS when $R_0 < 1$(see Figures 2 and 4); in contrast, the system (2.1) maintains permanence when $R_0 > 1$ (see Figures 1, 3, and 5). $R_0 = 1$ thus functions as the critical threshold governing the GAS of pest extinction. These results are confirmed by numerical simulations; we find that μ_1, μ_2, and μ_{max} play a role in pest control (see Figures 6, 7(c), and 8).

According to the analysis above, it is evident that increasing the proportion of pests killed instantaneously and enhancing the release of natural enemies can significantly improve pest control outcomes. The results of our study offer a reliable reference for informed decision-making in practical pest management strategies. In our current model, the residual lethality of sprayed pesticides is assumed to be constant. However, in reality, it is better represented by a time-dependent function. Furthermore, our study only considers applying pesticide at fixed time pulses, whereas in practice, spraying often depends on the current state of the pest population—referred to as state-dependent pulses. Concurrently, fractional differential equations find extensive application in the development of mathematical modeling [43–45], in the event that the model (2.1) is modified to take the form of a fractional mathematical model. We plan to address these aspects in future research.

Author contributions

Hu Pu: constructed the model and writes the original draft. Jianjun Jiao: conducted numerical simulations of the results, and coordinated the project. Sergey Meleshko: validated the results of the study. Eckart Schulz: manuscript editing and critical revision of the written content.All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that there are no conflicts of interest related to the content of this work.

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