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Research article

Investigating positive solutions in p-Laplacian fractional systems with infinite-point boundaries

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Abstract: This study explored the theoretical characteristics of a p-Laplacian fractional-order differential equation. This equation was subject to both infinite-point boundary value requirements and nonlocal integral value constraints. We began by examining the associated Green's function, deriving its explicit expression and unique properties. These properties were then utilized to establish the existence and uniqueness of positive solutions through the application of the Banach fixed-point technique. Furthermore, we employed the nonlinear alternative of Leray-Schauder type, specifically Guo-Krasnoselskii's fixed-point theorem on cones, to demonstrate existence results for cases where the nonlinearity exhibits singularity with respect to the time variable. The practical relevance and applicability of our findings were illustrated through compelling examples. This research significantly contributed to the field of fractional differential equations, particularly within the domain of p-Laplacian Hadamard fractional differential equations.

Keywords: nonlinear equation; p-Laplacian fractional operator; existence result; fixed point methodology; singular nonlinear term; fractional derivative
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1. Introduction

The p-Laplacian differential equation (DE) is a nonlinear partial DE that expands upon the classical Laplace equation. It's incredibly versatile, showing up in fields like fluid mechanics, nonlinear elasticity, and image processing. Originally, Leibenson introduced it to model turbulent flow in porous media [1]. Since then, its usefulness has been proven across various disciplines, including cosmic physics, elasticity theory, plasma problems, and non-Newtonian mechanics [2–4].

The p-Laplacian operator, symbolized as L_p, is defined as follows:

$$L_p(\ell) = |\ell|^{p-2}\,\ell,\ \ p, h > 1 \text{ with } \frac{1}{p} + \frac{1}{h} > 1.$$

For recent advancements, the existence of solutions for fractional DEs under p-Laplacian operators and boundary conditions is presented in [5–7] and references [8–10].

Beyond the theoretical realm, fractional operators offer a rich tapestry of practical applications across diverse scientific and engineering disciplines. For instance, they prove invaluable in modeling time-fractional damage hyperelasticity, providing a more accurate description of material behavior under stress over time. Their utility extends to generalized Birkhoffian systems, offering novel perspectives in classical mechanics, and even to the nonlinear Schrödinger equation. These examples merely scratch the surface of the growing influence of fractional calculus in solving real-world problems. For more applications, see [11, 13, 14]

Recent research has focused on the existence, multiplicity, and numerical solutions of fractional DEs, as well as their applications to nonlinear deformation, flexural wave propagation, and coupled problems [15–17]. A variety of methods have been employed to investigate fractional DEs, including the comparison principle [18], the substitute approach proposed by Leray-Schauder [19–21], methodology of Hussein-Jassim [22], Banach's fixed point (FP) theorem [23–25], and others. In recent years, there has been significant interest in Hadamard fractional DEs and related nonlinear dynamical systems [26–28].

When a continuous function $\Phi : (0, \infty) \to \mathbb{R}_+^1$ has order $\zeta > 0$, its Hadamard fractional (HF) derivative is described as

$${}^{H}D_{1^+}^{\zeta}\Phi(\gimel) = \frac{1}{\Gamma(l-\zeta)}\left(\gimel\frac{d}{d\gimel}\right)^{l}\int_{1}^{\gimel}\left(\ln\frac{\gimel}{\ell}\right)^{l-\zeta-1}\frac{\Phi(\ell)}{\ell}d\ell,$$

where $\mathbb{R}_+^1 = [0, +\infty)$, $l = [\zeta] + 1$, $[\zeta]$ represents the integer part of ζ, whenever, the righthand integral is described point-wise on $(0, \infty)$. The HF integral of order $\zeta > 0$ of the function $\Phi : (0, \infty) \to \mathbb{R}_+^1$ is obtained by

$${}^{H}I_{1^+}^{\zeta}\Phi(\gimel) = \frac{1}{\Gamma(\zeta)}\int_{1}^{\gimel}\left(\ln\frac{\gimel}{\ell}\right)^{\zeta-1}\frac{\Phi(\ell)}{\ell}d\ell.$$

Many people concentrate on the fundamental theoretical study of fractional DEs, including the structure of HF DEs, in an effort to better direct practice. Some fractional DE structures have been observed to be derived from HF DEs. Authors in [29] investigated the following HF equations:

$$\begin{aligned} {}^{H}D_{1^+}^{h}v(\gimel) + \mathfrak{R}\left(\gimel, v(\gimel), w(\gimel)\right) &= 0,\ \gimel \in (1, e), \\ {}^{H}D_{1^+}^{h}w(\gimel) + \widetilde{\mathfrak{R}}\left(\gimel, v(\gimel), w(\gimel)\right) &= 0,\ \gimel \in (1, e), \end{aligned}$$

under conditions

$$v(1) \;=\; \delta v(1) = 0,\ v(e) = \sum_{k=1}^{m-1}\ell_k v(\rho_k),$$

$$w(1) = \delta w(1)=0,\ w(e)=\sum_{k=1}^{m-1}\widetilde{\ell}_k w(\widetilde{\rho}_k),$$

where ${}^{H}D^{h}_{1^+}$ is the standard HF derivative with order $h\in(2,3]$, $\ell_k,\widetilde{\ell}_k\geq 0$, $\rho_k,\widetilde{\rho}_k\in(1,e)$ such that $\sum\limits_{k=1}^{m-1}\ell_k(\log\rho_k)^{h-1}, \sum\limits_{k=1}^{m-1}\widetilde{\ell}_k(\log\widetilde{\rho}_k)^{h-1}\in[0,1)$, and $\mathfrak{R},\widetilde{\mathfrak{R}}\in C\left([1,e]\times\mathbb{R}^1_+\times\mathbb{R}^1_+,\mathbb{R}^1_+\right)$. By applying the FP technique and exploiting the relationship between nonlinear and linear operators, the authors were able to establish the existence of a triple positive solution and a nontrivial solution to the following HF DEs, which were investigated by Ardjouni in [30]:

$${}^{H}D^{\varepsilon}_{1^+}v(\gimel)+\psi(\gimel,v(\gimel))={}^{H}D^{\mu}_{1^+}v(\gimel)+\varphi(\gimel,v(\gimel)),\ \gimel\in(1,e),$$

under stipulations

$$v(1)=0,\ v(e)=\frac{1}{\Gamma(\varepsilon-\mu)}\int_1^e\left(\log\frac{e}{\ell}\right)^{\varepsilon-\mu-1}q\left(\ell,\varkappa(\ell)\right)\frac{d\ell}{\ell},$$

where the functions $\psi,\varphi:[1,e]\times[0,\infty)\to[0,\infty)$ are continuous, $\varepsilon\in(1,2]$ and $\mu\in(0,\varepsilon-1]$. Utilizing the approach of both the higher and lower solutions along with FP techniques, the authors were able to use spectral analysis to demonstrate the existence and uniqueness of positive solutions.

Inspired by these findings, we examine the HF differential system that follows:

$$\begin{cases} L_{p_1}\left({}^{H}D^{\zeta}_{1^+}v(\gimel)\right)+\mathfrak{R}\left(\gimel,w(\gimel),{}^{H}D^{\alpha}_{1^+}w(\gimel)\right)=0,\ \gimel\in(1,e),\\ L_{p_2}\left({}^{H}D^{\gamma}_{1^+}w(\gimel)\right)+\widetilde{\mathfrak{R}}\left(\gimel,z(\gimel),{}^{H}D^{\alpha}_{1^+}z(\gimel)\right)=0,\ \gimel\in(1,e),\\ L_{p_3}\left({}^{H}D^{\beta}_{1^+}z(\gimel)\right)+\widehat{\mathfrak{R}}\left(\gimel,v(\gimel),{}^{H}D^{\alpha}_{1^+}v(\gimel)\right)=0,\ \gimel\in(1,e),\end{cases}\tag{1.1}$$

under infinite boundary conditions:

$$\begin{cases} v^{(a)}(1)=0,\ a=0,1,2,\cdots,l-2,\ v(e)=\sum\limits_{k=1}^{\infty}\lambda_k v\left(\rho_k\right)+\int\limits_1^e g(\gimel)v\left(\gimel\right)dG(\gimel),\\ w^{(b)}(1)=0,\ b=0,1,2,\cdots,m-2,\ w(e)=\sum\limits_{k=1}^{\infty}\widetilde{\lambda}_k w\left(\widetilde{\rho}_k\right)+\int\limits_1^e \widetilde{g}(\gimel)w\left(\gimel\right)d\widetilde{G}(\gimel),\\ z^{(c)}(1)=0,\ c=0,1,2,\cdots,n-2,\ z(e)=\sum\limits_{k=1}^{\infty}\widehat{\lambda}_k z\left(\widehat{\rho}_k\right)+\int\limits_1^e \widehat{g}(\gimel)z\left(\gimel\right)d\widehat{G}(\gimel),\end{cases}\tag{1.2}$$

where ${}^{H}D^{\zeta}_{1^+},{}^{H}D^{\gamma}_{1^+},{}^{H}D^{\beta}_{1^+}$ and ${}^{H}D^{\alpha}_{1^+}$ are HF derivatives with orders $\zeta\in(l-1,l]$, $\gamma\in(m-1,m]$, $\beta\in(n-1,n]$, and $0<\alpha<\min\{\zeta-1,\gamma-1,\beta-1\}$, respectively. Also, L_{p_i} refers to the $p-$Laplacian operator and is defined as $L_{p_i}(\ell)=|\ell|^{p_i-2}\,\ell$, $p_i,h_i>1$, $\frac{1}{p_i}+\frac{1}{h_i}=1$, $(i=1,2,3)$. Moreover, $\lambda_k,\widetilde{\lambda}_k,\widetilde{\lambda}_k\geq0$, $\rho_k,\widetilde{\rho}_k,\widehat{\rho}_k\in(1,e)$ are parameters, $G,\widetilde{G},\widehat{G}$ are functions of bounded variation, $g(\gimel),\widetilde{g}(\gimel),\widehat{g}(\gimel)\in L^1(1,e)$, and $\int\limits_1^e g(\gimel)v\left(\gimel\right)dG(\gimel)$, $\int\limits_1^e \widetilde{g}(\gimel)w\left(\gimel\right)d\widetilde{G}(\gimel)$, $\int\limits_1^e \widehat{g}(\gimel)z\left(\gimel\right)d\widehat{G}(\gimel)$ represent the Riemann-Stieltjes (RS) integral on $G(\gimel)$, $\widetilde{G}(\gimel)$, $\widehat{G}(\gimel)$, respectively. Additionally, $\mathfrak{R},\widetilde{\mathfrak{R}},\widehat{\mathfrak{R}}\in C\left((1,e)\times\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}^1_+\right)$, where $\mathbb{R}_+=(0,+\infty)$, and $\mathfrak{R}\left(\gimel,\varkappa_1,\varkappa_2\right)$, $\widetilde{\mathfrak{R}}\left(\gimel,\varkappa_1,\varkappa_2\right)$, $\widehat{\mathfrak{R}}\left(\gimel,\varkappa_1,\varkappa_2\right)$ have singularity at $\gimel=1$ and $\gimel=e$.

In this paper, we conduct an in-depth investigation into the existence of positive solutions for a complex, singular infinite-point tripled $p-$Laplacian boundary value system (BVS). This study significantly distinguishes itself from prior research, particularly studies [29, 31], by introducing a singular nonlinear term that exhibits dependence on the time variable. Furthermore, we integrate fractional derivatives into the nonlinear terms and impose infinite-point boundary conditions upon the HF differential system (1.1), subject to the stipulations outlined in (1.2). While earlier works [29, 31] addressed BVSs with continuous nonlinear terms devoid of derivative terms, our present work extends the analysis to a substantially more intricate $p-$Laplacian BVSs. This newly formulated system allows for the modeling of more complex phenomena characterized by singular nonlinearities, and through rigorous analysis, we conclusively demonstrate the existence and uniqueness of solutions to these equations.

Our manuscript is organized as follows: Section 2 introduces the fundamental fractional calculus principles relevant to our work. In Section 3, we derive and present the integral representation for the solution to problem (1.1), subject to the boundary conditions and constraints defined in (1.2). Section 4 develops the necessary theoretical framework by establishing auxiliary lemmas critical for transforming the original problem into an FP problem. This transformation is the fundamental step required to apply FP theory to prove the existence and uniqueness of solutions. Section 5 is devoted to discussing the existence of positive solutions for the considered system. In Section 6, we present practical examples to demonstrate and confirm the applicability of our theoretical results. Section 7 employs abbreviated terminology for frequently used phrases to enhance the efficiency of the discussion. Section 7 provides a list of abbreviations used for simplicity throughout the paper. Finally, Section 8 contains the conclusion which summarizes our work and suggests several open problems for future research.

2. Preliminaries

This section introduces some fundamental fractional calculus principles that are pertinent to our work. For a deeper understanding of the topic, readers are encouraged to consult the recent bibliographies in [32, 33].

Lemma 2.1. *[34] For $\zeta, \gamma > 0$, we have*

$$^{H}I_{1^+}^{\zeta}\left(\ln(\varkappa)\right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\zeta)}\left(\ln(\varkappa)\right)^{\gamma+\zeta-1}, \ \textit{and}\ ^{H}D_{1^+}^{\zeta}\left(\ln(\varkappa)\right)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\zeta)}\left(\ln(\varkappa)\right)^{\gamma-\zeta-1}.$$

Lemma 2.2. *[34] Assume that $\Phi \in C[0,\infty) \cap L^1[0,\infty)$ and $\zeta > 0$. Then, the solution to the equation $^{H}D_{1^+}^{\zeta}\Phi(\beth) = 0$ takes the form*

$$\Phi(\beth) = \tau_1(\ln\beth)^{\zeta-1} + \tau_2(\ln\beth)^{\zeta-2} + \cdots + \tau_l(\ln\beth)^{\zeta-l},$$

where $\tau_j \in \mathbb{R}$ $(j = 1, 2, ..., l)$ and $l = [l] + 1$.

Lemma 2.3. *[34] Let $\zeta \in (0, +\infty)$, $\Phi, f \in C[0,\infty) \cap L^1[0,\infty)$, then*

$$\Phi(\beth) =^{H} I_{1^+}^{\zeta}\, {}^{H}D_{1^+}^{\zeta} f(\beth) + \sum_{i=1}^{l}\tau_i(\ln\beth)^{\zeta-i}, \ \textit{for}\ \beth \in (1, e],$$

where $\tau_j \in \mathbb{R}$ $(j = 1, 2, ..., l)$ and $l = [\zeta] + 1$.

Lemma 2.4. *[35] Let Q be a cone subset in a real Banach space (BS) $\mho$. Assume that ϕ_1, ϕ_2 are two bounded open subsets in $\mho$ such that $0 \in \phi_1$ and $\overline{\phi}_1 \subset \phi_2$. Suppose that $\mathfrak{I} : \left(\overline{\phi}_2 \backslash \phi_1\right) \cap Q \to Q$ is a completely continuous operator. If one of the following hypotheses is true: for all $\varkappa \in Q \cap \partial\phi_1$,*

(i) if $\left\|\mathfrak{I}\varkappa\right\| \leq \|\varkappa\|$, then $\left\|\mathfrak{I}\varkappa\right\| \geq \|\varkappa\|$ for all $\varkappa \in Q \cap \partial\phi_2$,
(ii) if $\left\|\mathfrak{I}\varkappa\right\| \geq \|\varkappa\|$, then $\left\|\mathfrak{I}\varkappa\right\| \leq \|\varkappa\|$ for all $\varkappa \in Q \cap \partial\phi_2$.

Then, $\mathfrak{I}$ has an FP in $\left(\overline{\phi}_2 \backslash \phi_1\right) \cap Q$.

Lemma 2.5. *[36] Let $\mho$ be a BS, $\phi \subset \mho$ be a closed convex subset, and $V \subset \phi$ be a relatively open subset with $0 \in \phi$. Assume that the continuous compact map $\mathfrak{I} : V \to \phi$ exists. In that case, one of the following options is accurate:*

(i) $\mathfrak{I}$ owns an FP in V,
(ii) there is $v \in \partial V$ and $\iota \in (0,1)$ such that $v = \iota\mathfrak{I}v$.

Lemma 2.6. *([37], Theorem 1.2.7) Let $\Upsilon \subset C^1[J, \mho]$, then Υ is a relatively compact set if and only if the following are true:*

(i) Υ' is equicontinuous, and $\Upsilon'(\gimel)$ is a relatively compact set in $\mho$.
(ii) There exists $\gimel_0 \in J$ such that $\Upsilon(\gimel_0)$ is a relatively compact set in $\mho$.

3. Integral representation of the considered problem

In this section, we present the integral representation to the problem (1.1) with stipulations (1.2).

Lemma 3.1. *Assume that $\psi, \widetilde{\psi}, \widehat{\psi} \in L^1(1,e) \cap C^1(1,e)$, then the linear systems*

$$\begin{cases} L_{p_1}\left({}^H D_{1^+}^{\zeta} v(\gimel)\right) + \psi(\gimel) = 0, \ \gimel \in (1,e), \\ v^{(a)}(1) = 0, \ a = 0,1,2,\cdots,l-2, \ v(e) = \sum\limits_{k=1}^{\infty} \lambda_k v(\rho_k) + \int\limits_1^e g(\gimel) v(\gimel)\, dG(\gimel), \end{cases} \tag{3.1}$$

$$\begin{cases} L_{p_2}\left({}^H D_{1^+}^{\gamma} w(\gimel)\right) + \widetilde{\psi}(\gimel) = 0, \ \gimel \in (1,e), \\ w^{(b)}(1) = 0, \ b = 0,1,2,\cdots,m-2, \ w(e) = \sum\limits_{k=1}^{\infty} \widetilde{\lambda}_k w(\widetilde{\rho}_k) + \int\limits_1^e \widetilde{g}(\gimel) w(\gimel)\, d\widetilde{G}(\gimel), \end{cases} \tag{3.2}$$

and

$$\begin{cases} L_{p_3}\left({}^H D_{1^+}^{\beta} z(\gimel)\right) + \widehat{\psi}(\gimel) = 0, \ \gimel \in (1,e), \\ z^{(c)}(1) = 0, \ c = 0,1,2,\cdots,n-2, \ z(e) = \sum\limits_{k=1}^{\infty} \widehat{\lambda}_k z(\widehat{\rho}_k) + \int\limits_1^e \widehat{g}(\gimel) z(\gimel)\, d\widehat{G}(\gimel), \end{cases} \tag{3.3}$$

have integral representation

$$\begin{cases} v(\gimel) = \int\limits_1^e \supset(\gimel,\ell) \frac{\varphi_{h_1}(\psi(\ell))}{\ell} d\ell, \\ w(\gimel) = \int\limits_1^e \widetilde{\supset}(\gimel,\ell) \frac{\varphi_{h_2}(\widetilde{\psi}(\ell))}{\ell} d\ell, \\ z(\gimel) = \int\limits_1^e \widehat{\supset}(\gimel,\ell) \frac{\varphi_{h_2}(\widehat{\psi}(\ell))}{\ell} d\ell, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \Game(\gimel,\ell) = \Game_1(\gimel,\ell) + \Game_2(\gimel,\ell), \\ \widetilde{\Game}(\gimel,\ell) = \widetilde{\Game}_1(\gimel,\ell) + \widetilde{\Game}_2(\gimel,\ell), \\ \widehat{\Game}(\gimel,\ell) = \widehat{\Game}_1(\gimel,\ell) + \widehat{\Game}_2(\gimel,\ell), \end{cases}$$

in which

$$\begin{aligned} \Game_1(\gimel,\ell) &= \frac{1}{\Gamma(\zeta)} \begin{cases} A(\ell)\Gamma(\zeta)(\ln\gimel)^{\zeta-1}\left(\ln\frac{e}{\ell}\right)^{\zeta-1} - \nabla\left(\ln\frac{\gimel}{\ell}\right)^{\zeta-1}, & 1 \le \ell \le \gimel \le e, \\ A(\ell)\Gamma(\zeta)(\ln\gimel)^{\zeta-1}\left(\ln\frac{e}{\ell}\right)^{\zeta-1}, & 1 \le \gimel \le \ell \le e, \end{cases} \\ \Game_2(\gimel,\ell) &= \frac{(\ln\gimel)^{\zeta-1}}{\nabla_1}\int_1^e g(\gimel)\Game_1(\gimel,\ell)\,dG(\gimel), \\ \nabla_1 &= \nabla - \int_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\,dG(\gimel),\ \nabla = 1 - \sum_{k=1}^{\infty}\lambda_k\ln(\rho_k)^{\zeta-1}, \\ A(\ell) &= \frac{1}{\Gamma(\zeta)} - \frac{1}{\Gamma(\zeta)}\sum_{k=1}^{\infty}\lambda_k\left(\frac{\ln\rho_k - \ln\ell}{\ln e - \ln\ell}\right)^{\zeta-1}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \widetilde{\Game}_1(\gimel,\ell) &= \frac{1}{\Gamma(\gamma)} \begin{cases} \widetilde{A}(\ell)\Gamma(\gamma)(\ln\gimel)^{\gamma-1}\left(\ln\frac{e}{\ell}\right)^{\gamma-1} - \widetilde{\nabla}\left(\ln\frac{\gimel}{\ell}\right)^{\gamma-1}, & 1 \le \ell \le \gimel \le e, \\ \widetilde{A}(\ell)\Gamma(\gamma)(\ln\gimel)^{\gamma-1}\left(\ln\frac{e}{\ell}\right)^{\gamma-1}, & 1 \le \gimel \le \ell \le e, \end{cases} \\ \widetilde{\Game}_2(\gimel,\ell) &= \frac{(\ln\gimel)^{\gamma-1}}{\widetilde{\nabla}_1}\int_1^e \widetilde{g}(\gimel)\widetilde{\Game}_1(\gimel,\ell)\,d\widetilde{G}(\gimel), \\ \widetilde{\nabla}_1 &= \widetilde{\nabla} - \int_1^e \widetilde{g}(\gimel)(\ln\gimel)^{\gamma-1}\,d\widetilde{G}(r),\ \widetilde{\nabla} = 1 - \sum_{k=1}^{\infty}\widetilde{\lambda}_k\ln(\widetilde{\rho}_k)^{\gamma-1}, \\ \widetilde{A}(t) &= \frac{1}{\Gamma(\gamma)} - \frac{1}{\Gamma(\gamma)}\sum_{k=1}^{\infty}\widetilde{\lambda}_k\left(\frac{\ln\widetilde{\rho}_k - \ln\ell}{\ln e - \ln\ell}\right)^{\gamma-1}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \widehat{\Game}_1(\gimel,\ell) &= \frac{1}{\Gamma(\beta)} \begin{cases} \widehat{A}(\ell)\Gamma(\beta)(\ln\gimel)^{\beta-1}\left(\ln\frac{e}{\gimel}\right)^{\beta-1} - \widehat{\nabla}\left(\ln\frac{\gimel}{\ell}\right)^{\beta-1}, & 1 \le \ell \le \gimel \le e, \\ \widehat{A}(\ell)\Gamma(\beta)(\ln\gimel)^{\beta-1}\left(\ln\frac{e}{\gimel}\right)^{\beta-1}, & 1 \le r \le s \le e, \end{cases} \\ \widehat{\Game}_2(\gimel,\ell) &= \frac{(\ln\gimel)^{\beta-1}}{\widehat{\nabla}_1}\int_1^e \widehat{g}(\gimel)\widehat{\Game}_1(\gimel,\ell)\,d\widehat{G}(\gimel), \\ \widehat{\nabla}_1 &= \widehat{\nabla} - \int_1^e \widehat{g}(\gimel)(\ln\gimel)^{\beta-1}\,d\widehat{G}(\gimel),\ \widehat{\nabla} = 1 - \sum_{k=1}^{\infty}\widehat{\lambda}_k\ln(\widehat{\rho}_k)^{\beta-1}, \\ \widehat{A}(\ell) &= \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta)}\sum_{k=1}^{\infty}\widehat{\lambda}_k\left(\frac{\ln\widehat{\rho}_k - \ln\ell}{\ln e - \ln\ell}\right)^{\beta-1}, \end{aligned} \tag{3.7}$$

provided that $\nabla_1 \neq 0$, $\widetilde{\nabla}_1 \neq 0$, *and* $\widehat{\nabla}_1 \neq 0$.

Proof. Adding L_{h_1} and selecting ${}^{H}I^{\zeta}_{1^+}$ to the first equation of (3.1), and using Lemma 2.3, we have

$$v(\gimel) = -{}^{H}I^{\zeta}_{1^+}L_{h_1}\left(\psi\left(\gimel\right)\right) + \tau_1\left(\ln\gimel\right)^{\zeta-1} + \tau_2\left(\ln\gimel\right)^{\zeta-2} + \cdots + \tau_l\left(\ln\gimel\right)^{\zeta-l},$$

for some $\tau_1, \tau_2, \cdots, \tau_l \in \mathbb{R}^{+1}$. From conditions (1.2), since $v(1) = 0$, we have $\tau_l = 0$, then

$$\begin{aligned} v'(\gimel) &= -{}^{H}I^{\zeta-1}_{1^+}L_{h_1}\left(\psi\left(\gimel\right)\right) + \tau_1\left(\zeta-1\right)\left(\ln\gimel\right)^{\zeta-2}\left(\frac{1}{\gimel}\right) + \tau_2\left(\zeta-2\right)\left(\ln\gimel\right)^{\zeta-3}\left(\frac{1}{\gimel}\right) \\ &\quad + \cdots + \tau_{l-1}\left(\zeta-(l-1)\right)\left(\ln\gimel\right)^{\zeta-l}\left(\frac{1}{\gimel}\right). \end{aligned}$$

Since $v'(1) = 0$, we get $\tau_{l-1} = 0$. Continue in this process, we obtain that $\tau_2 = \tau_3 = \cdots = \tau_l = 0$, and hence

$$v(\gimel) = -\int_1^e \frac{\left(\ln\frac{e}{\ell}\right)^{\zeta-1}}{\Gamma(\zeta)}\frac{L_{h_1}\left(\psi\left(\ell\right)\right)}{\ell}d\ell + \tau_1\left(\ln\gimel\right)^{\zeta-1}. \tag{3.8}$$

By routine calculation, one gets

$$v(e) = -{}^{H}I^{\zeta}_{1^+}L_{h_1}\left(\psi\left(e\right)\right) + \tau_1\left(\ln e = 1\right). \tag{3.9}$$

Applying (3.9) in the condition $v(e) = \sum_{k=1}^{\infty}\lambda_k v\left(\rho_k\right) + \int_1^e g(\gimel)v\left(\gimel\right)dG(\gimel)$, and using (3.8), we have

$$\tau_1 -^{H} I^{\zeta}_{1^+}L_{h_1}\left(\psi\left(e\right)\right) = \sum_{k=1}^{\infty}\lambda_k\left[\tau_1\left(\ln\rho_k\right)^{\zeta-1} -^{H} I^{\zeta}_{1^+}L_{p_1}\left(\psi\left(\rho_k\right)\right)\right] + \int_1^e g(\gimel)v\left(\gimel\right)dG(\gimel),$$

which implies that

$$\begin{aligned} \tau_1 &= \int_1^e \frac{\left(\ln e - \ln\ell\right)^{\zeta-1}}{\Gamma(\zeta)\nabla}L_{h_1}\left(\psi\left(\ell\right)\right)\frac{d\ell}{\ell} - \sum_{k=1}^{\infty}\lambda_k\int_1^{\rho_k}\frac{\left(\ln\rho_k - \ln\ell\right)^{\zeta-1}}{\Gamma(\zeta)\nabla}L_{h_1}\left(\psi\left(\ell\right)\right)\frac{d\ell}{\ell} \\ &\quad + \frac{1}{\nabla}\int_1^e g(\gimel)v\left(\gimel\right)dG(\gimel) \\ &= \int_1^e \frac{\left(\ln e - \ln\ell\right)^{\zeta-1}A\left(\ell\right)}{\nabla}L_{h_1}\left(\psi\left(\ell\right)\right)\frac{d\ell}{\ell} + \frac{1}{\nabla}\int_1^e g(\gimel)v\left(\gimel\right)dG(\gimel). \end{aligned} \tag{3.10}$$

From (3.10) in (3.8), one can write

$$v(\gimel) = -\frac{1}{\Gamma(\zeta)}\int_1^e\left(\ln\frac{e}{\ell}\right)^{\zeta-1}\frac{L_{h_1}\left(\psi\left(\ell\right)\right)}{\ell}d\ell$$

$$
\begin{aligned}
& +(\ln \gimel)^{\zeta-1} \frac{\frac{1}{\Gamma(\zeta)} \int_1^e \left(\ln \frac{e}{\ell}\right)^{\zeta-1} \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell - \sum_{k=1}^{\infty} \frac{\lambda_k}{\Gamma(\zeta)} \int_1^{\rho_k} \left(\ln \frac{\rho_k}{\ell}\right)^{\zeta-1} \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell}{\nabla} \\
& + \frac{(\ln \gimel)^{\zeta-1}}{\nabla} \int_1^e g(\gimel) v(\gimel)\, dG(\gimel) \\
= \; & \int_1^e \Game(\gimel, \ell) \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell + \frac{(\ln \gimel)^{\zeta-1}}{\nabla} \int_1^e g(\gimel) v(\gimel)\, dG(\gimel).
\end{aligned}
$$

Multiply the above equation by $g(\gimel)$ and take the integral of RS from 1 to e, we have

$$
\begin{aligned}
\int_1^e g(\gimel) v(\gimel)\, dG(\gimel) &= \int_1^e g(\gimel) \left[\int_1^e \Game(\gimel, \ell) \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell \right] dG(\gimel) \\
&\quad + \frac{1}{\nabla} \left(\int_1^e g(\gimel) (\ln \gimel)^{\zeta-1}\, dG(\gimel) \right) \int_1^e g(\gimel) v(\gimel)\, dG(\gimel),
\end{aligned}
$$

which yields

$$
\int_1^e g(\gimel) v(\gimel)\, dG(\gimel) = \frac{\nabla}{\nabla_1} \int_1^e g(\gimel) \left[\int_1^e \Game(\gimel, \ell) \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell \right] dG(\gimel).
$$

Hence,

$$
\begin{aligned}
v(\gimel) &= \int_1^e \Game_1(\gimel, \ell) L_{h_1}(\psi(\ell)) \frac{d\ell}{\ell} + \frac{(\ln \gimel)^{\zeta-1}}{\nabla} \frac{\nabla}{\nabla_1} \int_1^e g(\gimel) \left[\int_1^e \Game(\gimel, \ell) \frac{L_{h_1}(\psi(\ell))}{\ell} d\ell \right] dG(\gimel) \\
&= \int_1^e \Game_1(\gimel, \ell) L_{h_1}(\psi(\ell)) \frac{d\ell}{\ell} + \int_1^e \Game_2(\gimel, \ell) L_{h_1}(\psi(\ell)) \frac{d\ell}{\ell} \\
&= \int_1^e \Game(\gimel, \ell) L_{h_1}(\psi(\ell)) \frac{d\ell}{\ell},
\end{aligned}
$$

where $\Game(\gimel, \ell)$, $\Game_1(\gimel, \ell)$, $\Game_2(\gimel, \ell)$, ∇, and ∇_1 are defined in (3.5). Further, we have

$$
{}^{H}D_{1^+}^{\alpha} \Game(\gimel, \ell) = {}^{H}D_{1^+}^{\alpha} \Game_1(\gimel, \ell) + \frac{\Gamma(\zeta)}{\Gamma(\zeta - \alpha)} (\ln \gimel)^{\zeta-\alpha-1} \int_1^e v(\gimel) \Game_1(\gimel, \ell)\, dG(\gimel),
$$

and

$$
{}^{H}D_{1^+}^{\alpha} \Game_1(\gimel, \ell) = \frac{1}{\nabla \Gamma(\zeta - \alpha)} \begin{cases} A(\ell) \Gamma(\zeta) (\ln \gimel)^{\zeta-\alpha-1} \left(\ln \frac{e}{\ell}\right)^{\zeta-\alpha-1} - \nabla \left(\ln \frac{\gimel}{\ell}\right)^{\zeta-\alpha-1}, & 1 \le \ell \le \gimel \le e, \\ A(\ell) \Gamma(\zeta) (\ln \gimel)^{\zeta-\alpha-1} \left(\ln \frac{e}{\ell}\right)^{\zeta-\alpha-1}, & 1 \le \gimel \le \ell \le e, \end{cases}
$$

where $A(\ell)$ is described as in (3.5). Similarly, by (3.2) and (3.3), we can obtain the results (3.6) and (3.7), respectively, so we omitted it. Moreover, we have

$$^{H}D^{\alpha}_{1^{+}}\widetilde{Ↄ}(\gimel,\ell) = {^{H}D^{\alpha}_{1^{+}}}\widetilde{Ↄ}_{1}(\gimel,\ell) + \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(\ln\gimel)^{\gamma-\alpha-1}\int_{1}^{e} w(\gimel)\widetilde{Ↄ}_{1}(\gimel,\ell)\,d\widetilde{G}(\gimel),$$

and

$$^{H}D^{\alpha}_{1^{+}}\widetilde{Ↄ}_{1}(\gimel,\ell) = \frac{1}{\widetilde{\nabla}\Gamma(\gamma-\alpha)}\begin{cases}\widetilde{A}(\ell)\Gamma(\gamma)(\ln\gimel)^{\gamma-\alpha-1}\left(\ln\frac{e}{\ell}\right)^{\gamma-\alpha-1} - \nabla\left(\ln\frac{\gimel}{\ell}\right)^{\gamma-\alpha-1}, & 1\le\ell\le\gimel\le e,\\ \widetilde{A}(\ell)\Gamma(\gamma)(\ln\gimel)^{\gamma-\alpha-1}\left(\ln\frac{e}{\ell}\right)^{\gamma-\alpha-1}, & 1\le\gimel\le\ell\le e.\end{cases}$$

Also,

$$^{H}D^{\alpha}_{1^{+}}\widehat{Ↄ}(\gimel,\ell) = {^{H}D^{\alpha}_{1^{+}}}\widehat{Ↄ}_{1}(\gimel,\ell) + \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\gimel)^{\beta-\alpha-1}\int_{1}^{e} z(\gimel)\widehat{Ↄ}_{1}(\gimel,\ell)\,d\widehat{G}(\gimel),$$

and

$$^{H}D^{\alpha}_{1^{+}}\widehat{Ↄ}_{1}(\gimel,\ell) = \frac{1}{\widehat{\nabla}\Gamma(\beta-\alpha)}\begin{cases}\widehat{A}(\ell)\Gamma(\beta)(\ln\gimel)^{\beta-\alpha-1}\left(\ln\frac{e}{\ell}\right)^{\beta-\alpha-1} - \nabla\left(\ln\frac{\gimel}{\ell}\right)^{\beta-\alpha-1}, & 1\le\ell\le\gimel\le e,\\ \widehat{A}(\ell)\Gamma(\beta)(\ln\gimel)^{\beta-\alpha-1}\left(\ln\frac{e}{\ell}\right)^{\beta-\alpha-1}, & 1\le\gimel\le\ell\le e.\end{cases}$$

□

4. Auxiliary lemmas

This section focuses on developing the necessary theoretical framework by establishing auxiliary lemmas. These lemmas will be instrumental in transforming the original problem into an FP problem, a fundamental step in utilizing FP theory to prove the existence and uniqueness of solutions.

Lemma 4.1. *The functions* $Ↄ(\gimel,\ell)$, ${^{H}D^{\alpha}_{1^{+}}}Ↄ(\gimel,\ell)$, $\widetilde{Ↄ}(\gimel,\ell)$, ${^{H}D^{\alpha}_{1^{+}}}\widetilde{Ↄ}(\gimel,\ell)$, $\widehat{Ↄ}(\gimel,\ell)$, *and* ${^{H}D^{\alpha}_{1^{+}}}\widehat{Ↄ}(\gimel,\ell)$ *described by (3.4) have the axioms below:*

(i) On the product $[1,e]\times[1,e]$, *the functions* $Ↄ(\gimel,\ell)$, ${^{H}D^{\alpha}_{1^{+}}}Ↄ(\gimel,\ell)$, $\widetilde{Ↄ}(\gimel,\ell)$, ${^{H}D^{\alpha}_{1^{+}}}\widetilde{Ↄ}(\gimel,\ell)$, $\widehat{Ↄ}(\gimel,\ell)$, *and* ${^{H}D^{\alpha}_{1^{+}}}\widehat{Ↄ}(\gimel,\ell)$ *are uniformly continuous,*

(ii) for $\gimel,\ell\in[1,e]$, *we have*

$$\begin{cases} Ↄ(\gimel,\ell)\le Ↄ_{1}(e,\ell)Z,\\ Ↄ(\gimel,\ell)\ge(\ln\gimel)^{\zeta-1}Ↄ_{1}(e,\ell)Z_{1},\\ {^{H}D^{\alpha}_{1^{+}}}Ↄ(\gimel,\ell)\le {^{H}D^{\alpha}_{1^{+}}}Ↄ_{1}(e,\ell)Z,\\ {^{H}D^{\alpha}_{1^{+}}}Ↄ(\gimel,\ell)\ge(\ln\gimel)^{\zeta-\alpha-1}\,{^{H}D^{\alpha}_{1^{+}}}Ↄ(e,\ell)Z_{1},\end{cases}$$

where

$$Z = 1+\frac{1}{\nabla_{1}}\int_{1}^{e} g(\gimel)dG(\gimel),\; Z_{1} = 1+\frac{1}{\nabla_{1}}\int_{1}^{e} g(\gimel)(\ln\gimel)^{\zeta-1}dG(\gimel), \tag{4.1}$$

(iii) for $\gimel, \ell \in [1,e]$, *we have*

$$\begin{cases} \widetilde{\Game}(\gimel,\ell) \leq \widetilde{\Game}_1(e,\ell)\widetilde{Z}, \\ \widetilde{\Game}(\gimel,\ell) \geq (\ln\gimel)^{\gamma-1}\widetilde{\Game}_1(e,\ell)\widetilde{Z}_1, \\ {}^{H}D^{\alpha}_{1^+}\widetilde{\Game}(\gimel,\ell) \leq^{H} D^{\alpha}_{1^+}\widetilde{\Game}_1(e,\ell)\widetilde{Z}, \\ {}^{H}D^{\alpha}_{1^+}\widetilde{\Game}(\gimel,\ell) \geq (\ln\gimel)^{\gamma-\alpha-1}\ {}^{H}D^{\alpha}_{1^+}\widetilde{\Game}(e,\ell)\widetilde{Z}_1, \end{cases}$$

where

$$\widetilde{Z} = 1 + \frac{1}{\widetilde{\nabla}_1}\int_1^e \widetilde{g}(\gimel)d\widetilde{G}(\gimel),\ \widetilde{Z}_1 = 1 + \frac{1}{\widetilde{\nabla}_1}\int_1^e \widetilde{g}(\gimel)(\ln\gimel)^{\gamma-1}d\widetilde{G}(\gimel), \tag{4.2}$$

(iv) for $\gimel, \ell \in [1,e]$, *we have*

$$\begin{cases} \widehat{\Game}(\gimel,\ell) \leq \widehat{\Game}_1(e,\ell)\widehat{Z}, \\ \widehat{\Game}(\gimel,\ell) \geq (\ln\gimel)^{\beta-1}\widehat{\Game}_1(e,\ell)\widehat{Z}_1, \\ {}^{H}D^{\alpha}_{1^+}\widehat{\Game}(\gimel,\ell) \leq^{H} D^{\alpha}_{1^+}\widehat{\Game}_1(e,\ell)\widehat{Z}, \\ {}^{H}D^{\alpha}_{1^+}\widehat{\Game}(\gimel,\ell) \geq (\ln\gimel)^{\beta-\alpha-1}\ {}^{H}D^{\alpha}_{1^+}\widehat{\Game}(e,\ell)\widehat{Z}_1, \end{cases}$$

where

$$\widehat{Z} = 1 + \frac{1}{\widehat{\nabla}_1}\int_1^e \widehat{g}(\gimel)d\widehat{G}(\gimel),\ \widehat{Z}_1 = 1 + \frac{1}{\widehat{\nabla}_1}\int_1^e \widehat{g}(\gimel)(\ln\gimel)^{\beta-1}d\widehat{G}(\gimel). \tag{4.3}$$

Proof. (i) By definitions of the functions $\Game(\gimel,\ell)$, $\widetilde{\Game}(\gimel,\ell)$, and $\widehat{\Game}(\gimel,\ell)$, it is easy to check that $\Game(\gimel,\ell)$, $\Game(\gimel,\ell)$, ${}^{H}D^{\alpha}_{1^+}\Game(\gimel,\ell)$, $\widetilde{\Game}(\gimel,\ell)$, ${}^{H}D^{\alpha}_{1^+}\widetilde{\Game}(\gimel,\ell)$, $\widehat{\Game}(\gimel,\ell)$, and ${}^{H}D^{\alpha}_{1^+}\widehat{\Game}(\gimel,\ell)$ are uniformly continuous on $[1,e]\times[1,e]$.

(ii) With the aid of [38], for $\gimel, \ell \in [1,e]$, we have

$$(\ln\gimel)^{\zeta-1}\Game_1(e,\ell) \leq \Game_1(\gimel,\ell) \leq \Game_1(e,\ell),$$

hence, we can write

$$\begin{aligned} \Game(\gimel,\ell) &= \Game_1(\gimel,\ell) + \Game_2(\gimel,\ell) \\ &\leq \Game_1(e,\ell) + \frac{(\ln\gimel)^{\zeta-1}}{\nabla_1}\int_1^e g(\gimel)\Game_1(e,\ell)\,dG(\gimel) \\ &= \Game_1(e,\gimel)\left[1 + \frac{(\ln\gimel)^{\zeta-1}}{\nabla_1}\int_1^e g(\gimel)dG(\gimel)\right] = \Game_1(e,\ell)Z. \end{aligned}$$

On the other hand

$$\begin{aligned} \Game(\gimel,\ell) &= \Game_1(\gimel,\ell) + \Game_2(\gimel,\ell) \\ &= \Game_1(\gimel,\ell) + \frac{(\ln\gimel)^{\zeta-1}}{\nabla_1}\int_1^e g(\gimel)\Game_1(\gimel,\ell)\,dG(\gimel) \end{aligned}$$

$$\geq \quad (\ln \daleth)^{\zeta-1} \eth_1(e,\ell) + \frac{(\ln \daleth)^{\zeta-1}}{\nabla_1} \int_1^e g(\daleth)(\ln \daleth)^{\zeta-1} \eth_1(e,\ell)\, dG(\daleth)$$

$$\geq \quad (\ln \daleth)^{\zeta-1} \eth_1(e,\ell) \left[1 + \frac{1}{\nabla_1} \int_1^e g(\daleth)(\ln \daleth)^{\zeta-1}\, dG(\daleth) \right] = (\ln \daleth)^{\zeta-1} \eth_1(e,\ell)\, Z_1.$$

Similarly, we can obtain

$${}^H D_{1^+}^{\alpha} \eth(\daleth,\ell) \leq {}^H D_{1^+}^{\alpha} \eth_1(e,\ell)\, Z, \text{ and } {}^H D_{1^+}^{\alpha} \eth(\daleth,\ell) \geq (\ln \daleth)^{\zeta-\alpha-1}\ {}^H D_{1^+}^{\alpha} \eth(e,\ell)\, Z_1.$$

The proof of axioms (iii) and (iv) follows by the same method in (ii). □

Now, assume that $\mho = C^{\alpha}[1,e]$, and

$$\begin{aligned} \|v\| &= \max\left\{ \|v\|_0, \left\| {}^H D_{1^+}^{\alpha} v \right\|_0 \right\}, \\ \|w\| &= \max\left\{ \|w\|_0, \left\| {}^H D_{1^+}^{\alpha} w \right\|_0 \right\}, \\ \|z\| &= \max\left\{ \|z\|_0, \left\| {}^H D_{1^+}^{\alpha} z \right\|_0 \right\}, \end{aligned}$$

where

$$\begin{aligned} \|v\|_0 &= \max_{\daleth\in[1,e]} |v(\daleth)|, \ \left\| {}^H D_{1^+}^{\alpha} v \right\|_0 = \max_{\daleth\in[1,e]} \left| {}^H D_{1^+}^{\alpha} v(\daleth) \right|, \\ \|w\|_0 &= \max_{\daleth\in[1,e]} |w(\daleth)|, \ \left\| {}^H D_{1^+}^{\alpha} w \right\|_0 = \max_{\daleth\in[1,e]} \left| {}^H D_{1^+}^{\alpha} w(\daleth) \right|, \\ \|z\|_0 &= \max_{\daleth\in[1,e]} |z(\daleth)|, \ \left\| {}^H D_{1^+}^{\alpha} z \right\|_0 = \max_{\daleth\in[1,e]} \left| {}^H D_{1^+}^{\alpha} z(\daleth) \right|. \end{aligned}$$

Then, $(\mho, \|.\|)$ is a real Banach space and $\mho^3 = \mho \times \mho \times \mho$ is a Banach space with the norm $\|(v,w,z)\| = \max\{\|v\|, \|w\|, \|z\|\}$.

Furthermore, define

$$Q_0 = \left\{ v \in \mho : v(\daleth) \geq 0,\ {}^H D_{1^+}^{\alpha} v(\daleth) \geq 0, \text{ for all } \daleth \in [1,e] \right\},$$

and assume the cone Q on $\mho^3$ $\left(\mho^3 = \mho \times \mho \times \mho\right)$ is defined by

$$Q = \left\{ (v,w,z) \in \mho^3 : v \in Q_0,\ w \in Q_0,\ z \in Q_0, \daleth \in [1,e] \right\}.$$

The HF differential systems (1.1) and (1.2) can be solved by a trio $(v,w,z) \in \mho^3$ if it can also solve the following system of integral equations:

$$\left\{ \begin{array}{c} v(\daleth) \\ w(\daleth) \\ z(\daleth) \end{array} \right\} = \left\{ \begin{array}{c} \int_1^e \eth(\daleth,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell} \\ \int_1^e \widetilde{\eth}(\daleth,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right) \frac{d\ell}{\ell} \\ \int_1^e \widehat{\eth}(\daleth,\ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right) \frac{d\ell}{\ell} \end{array} \right\} = \left\{ \begin{array}{c} B_1(w)(\daleth) \\ B_2(z)(\daleth) \\ B_3(v)(\daleth) \end{array} \right\}, \tag{4.4}$$

for all $v, w, z \in Q$, $\gimel \in (1, e)$. Also,

$$\left\{\begin{array}{c} {}^{H}D_{1^+}^{\zeta}v(\gimel) \\ {}^{H}D_{1^+}^{\gamma}w(\gimel) \\ {}^{H}D_{1^+}^{\beta}z(\gimel) \end{array}\right\} = \left\{\begin{array}{c} \int\limits_1^e {}^{H}D_{1^+}^{\zeta}\backepsilon(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha}w(\ell)\right)\right)\frac{d\ell}{\ell} \\ \int\limits_1^e {}^{H}D_{1^+}^{\gamma}\widetilde{\backepsilon}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^{H}D_{1^+}^{\alpha}z(\ell)\right)\right)\frac{d\ell}{\ell} \\ \int\limits_1^e {}^{H}D_{1^+}^{\beta}\widehat{\backepsilon}(\gimel,\ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^{H}D_{1^+}^{\alpha}v(\ell)\right)\right)\frac{d\ell}{\ell} \end{array}\right\} = \left\{\begin{array}{c} {}^{H}D_{1^+}^{\zeta}B_1(w)(\gimel) \\ {}^{H}D_{1^+}^{\gamma}B_2(z)(\gimel) \\ {}^{H}D_{1^+}^{\beta}B_3(v)(\gimel) \end{array}\right\},$$

where the mapping $B : Q \to Q$ is described as

$$B(v, w, z)(\gimel) = \left(B_1(w)(\gimel), B_2(z)(\gimel), B_3(v)(\gimel)\right),\ v, w, z \in Q,\ \gimel \in (1, e).$$

First, we prove that the mappings B, $B_j : Q \to Q$ $(j = 1, 2, 3)$ are continuous. For this, we assume the following hypotheses:

(A_0) The functions $\mathfrak{R}, \widetilde{\mathfrak{R}}, \widehat{\mathfrak{R}} : (1, e) \times (\mathbb{R}_+)^2 \to \mathbb{R}_+$ are continuous and functions $\sigma(\gimel), \widetilde{\sigma}(\gimel), \widehat{\sigma}(\gimel) : (1, e) \to \mathbb{R}_+$ are such that for all $\varkappa_0, \varkappa_1 \in \mho$,

$$\mathfrak{R}(\gimel, \varkappa_0, \varkappa_1) \leq \sigma(r),\ \widetilde{\mathfrak{R}}(\gimel, \varkappa_0, \varkappa_1) \leq \widetilde{\sigma}(\gimel),\ \widehat{\mathfrak{R}}(\gimel, \varkappa_0, \varkappa_1) \leq \widehat{\sigma}(\gimel),$$

and

$$\int\limits_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell} < +\infty,\ \int\limits_1^e L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell} < +\infty,\ \int\limits_1^e L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell} < +\infty;$$

(A_1) For all $\gimel \in (1, e)$, we have

$$\int\limits_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\, dG(\gimel) < +\infty,\ \int\limits_1^e \widetilde{g}(\gimel)(\ln\gimel)^{\gamma-1}\, d\widetilde{G}(\gimel) < +\infty,\ \int\limits_1^e \widehat{g}(\gimel)(\ln\gimel)^{\beta-1}\, d\widehat{G}(\gimel) < +\infty.$$

Lemma 4.2. *Under the hypotheses (A_0) and (A_1), the mappings B, $B_j : Q \to Q$ $(j = 1, 2, 3)$ are continuous.*

Proof. According to definitions of $\backepsilon, \widetilde{\backepsilon}, \widehat{\backepsilon}$ and (A_0), for $v, w, z \in Q$, $\gimel, \ell \in (1, e)$, one can write

$$\begin{aligned} B_1(w)(\gimel) &= \int\limits_1^e \backepsilon(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha}w(\ell)\right)\right)\frac{d\ell}{\ell} \\ &\leq \left(\frac{1}{\Gamma(\zeta)} + \frac{1}{\Gamma(\zeta)\nabla_1}\int\limits_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\, dG(\gimel)\right)\int\limits_1^e L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha}w(\ell)\right)\right)\frac{d\ell}{\ell} \\ &\leq \left(\frac{1}{\Gamma(\zeta)} + \frac{1}{\Gamma(\zeta)\nabla_1}\int\limits_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\, dG(\gimel)\right)\int\limits_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell} \\ &< +\infty, \end{aligned}$$

$$
\begin{aligned}
B_2(z)(\gimel) &= \int_1^e \widetilde{\beth}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\Re}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\le \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma)\widetilde{\nabla}_1}\int_1^e \widetilde{g}(\gimel)(\ln \gimel)^{\gamma-1}\, d\widetilde{G}(\gimel)\right)\int_1^e L_{h_2}\left(\widetilde{\Re}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\le \left(\frac{1}{\Gamma(\gamma)} + \frac{1}{\Gamma(\gamma)\widetilde{\nabla}_1}\int_1^e \widetilde{g}(\gimel)(\ln \gimel)^{\gamma-1}\, d\widetilde{G}(\gimel)\right)\int_1^e L_{h_2}\left(\widetilde{\sigma}(\ell)\right)\frac{d\ell}{\ell} \\
&< +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
B_3(v)(\gimel) &= \int_1^e \widehat{\beth}(\gimel,\ell)\, L_{h_3}\left(\widehat{\Re}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\le \left(\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta)\widehat{\nabla}_1}\int_1^e \widehat{g}(\gimel)(\ln \gimel)^{\beta-1}\, d\widehat{G}(\gimel)\right)\int_1^e L_{h_3}\left(\widehat{\Re}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\le \left(\frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta)\widehat{\nabla}_1}\int_1^e \widehat{g}(\gimel)(\ln \gimel)^{\beta-1}\, d\widehat{G}(\gimel)\right)\int_1^e L_{h_3}\left(\widehat{\sigma}(\ell)\right)\frac{d\ell}{\ell} \\
&< +\infty.
\end{aligned}
$$

The three above inequalities prove that the mapping $B(v,w,z)(\gimel) = (B_1(w)(\gimel), B_2(z)(\gimel), B_3(v)(\gimel))$ is well-defined on Q. The uniform continuity of $\beth(\gimel,\ell)$, ${}^H D_{1^+}^{\alpha}\beth(\gimel,\ell)$, $\widetilde{\beth}(\gimel,\ell)$, ${}^H D_{1^+}^{\alpha}\widetilde{\beth}(\gimel,\ell)$, $\widehat{\beth}(\gimel,\ell)$, and ${}^H D_{1^+}^{\alpha}\widehat{\beth}(\gimel,\ell)$ implies that there exists a large $Z > 0$ such that

$$
\max\left\{\beth(\gimel,\ell), {}^H D_{1^+}^{\alpha}\beth(\gimel,\ell), \widetilde{\beth}(\gimel,\ell), {}^H D_{1^+}^{\alpha}\widetilde{\beth}(\gimel,\ell), \widehat{\beth}(\gimel,\ell), {}^H D_{1^+}^{\alpha}\widehat{\beth}(\gimel,\ell)\right\} \le Z. \tag{4.5}
$$

In order to prove that $B: Q \to Q$ is continuous, assume that

$$
\begin{cases}
(v_l, w_l, z_l) \to (v, w, z), \\
\left({}^H D_{1^+}^{\alpha} v_l, {}^H D_{1^+}^{\alpha} w_l, {}^H D_{1^+}^{\alpha} z_l\right) \to \left({}^H D_{1^+}^{\alpha} v, {}^H D_{1^+}^{\alpha} w, {}^H D_{1^+}^{\alpha} z\right)
\end{cases}
\quad \text{as } l \to +\infty,
$$

which implies that there exists a large enough $\Lambda > 0$ such that

$$
\|(v_l, w_l, z_l)\| \le \Lambda \text{ and } \left\|\left({}^H D_{1^+}^{\alpha} v_l, {}^H D_{1^+}^{\alpha} w_l, {}^H D_{1^+}^{\alpha} z_l\right)\right\| \le \Lambda,\ l \in \mathbb{N}.
$$

Then,

$$
\|(v, w, z)\| \le \Lambda \text{ and } \left\|\left({}^H D_{1^+}^{\alpha} v, {}^H D_{1^+}^{\alpha} w, {}^H D_{1^+}^{\alpha} z\right)\right\| \le \Lambda,
$$

that is,

$$
\begin{cases}
\|v\| \le \Lambda, & \left\|{}^H D_{1^+}^{\alpha} v\right\| \le \Lambda, \\
\|w\| \le \Lambda, & \left\|{}^H D_{1^+}^{\alpha} w\right\| \le \Lambda, \\
\|z\| \le \Lambda, & \left\|{}^H D_{1^+}^{\alpha} z\right\| \le \Lambda.
\end{cases}
$$

From (A_0) and (4.5), we have

$$\begin{cases} \left|\int\limits_1^e \Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w_l(\ell), {}^H D_{1^+}^{\alpha} w_l(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\right| < +\infty, \\ \left|\int\limits_1^e {}^H D_{1^+}^{\zeta}\Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w_l(\ell), {}^H D_{1^+}^{\alpha} w_l(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\right| < +\infty, \\ \left|\int\limits_1^e \widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell}\right| < +\infty, \\ \left|\int\limits_1^e {}^H D_{1^+}^{\gamma}\widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell}\right| < +\infty, \\ \left|\int\limits_1^e \widehat{\Game}(\gimel,\ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell}\right| < +\infty, \\ \left|\int\limits_1^e {}^H D_{1^+}^{\beta}\widehat{\Game}(\gimel,\ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right)\frac{d\ell}{\ell}\right| < Z\left|\int\limits_1^e L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell}\right| < +\infty. \end{cases} \tag{4.6}$$

The condition (A_0) implies that $L_{h_1}(\sigma(\ell))$, $L_{h_2}(\widetilde{\sigma}(\ell))$, and $L_{h_3}(\widehat{\sigma}(\ell))$ are integrable. Therefore, for $\gimel \in [1,e]$, $l > \mathbb{N}$, using (4.5) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \left|(B_1 w_l)(\gimel) - (B_1 w)(\gimel)\right| &= \left|\int\limits_1^e \Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w_l(\ell), {}^H D_{1^+}^{\alpha} w_l(\ell)\right)\right)\frac{d\ell}{\ell}\right. \\ &\quad \left. - \int\limits_1^e \Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right)\frac{d\ell}{\ell}\right| \\ &\leq \epsilon, \\ \left|{}^H D_{1^+}^{\zeta}(B_1 w_l)(\gimel) - {}^H D_{1^+}^{\zeta}(B_1 w)(\gimel)\right| &= \left|\int\limits_1^e {}^H D_{1^+}^{\zeta}\Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w_l(\ell), {}^H D_{1^+}^{\alpha} w_l(\ell)\right)\right)\frac{d\ell}{\ell}\right. \\ &\quad \left. - \int\limits_1^e {}^H D_{1^+}^{\zeta}\Game(\gimel,\ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right)\frac{d\ell}{\ell}\right| \\ &\leq \epsilon, \\ \left|(B_2 z_l)(\gimel) - (B_2 z)(\gimel)\right| &= \left|\int\limits_1^e \widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z_l(\ell), {}^H D_{1^+}^{\alpha} z_l(\ell)\right)\right)\frac{d\ell}{\ell}\right. \\ &\quad \left. - \int\limits_1^e \widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell}\right| \\ &\leq \epsilon, \\ \left|{}^H D_{1^+}^{\gamma}(B_2 z_l)(\gimel) - {}^H D_{1^+}^{\gamma}(B_2 z)(\gimel)\right| &= \left|\int\limits_1^e {}^H D_{1^+}^{\gamma}\widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z_l(\ell), {}^H D_{1^+}^{\alpha} z_l(\ell)\right)\right)\frac{d\ell}{\ell}\right. \\ &\quad \left. - \int\limits_1^e {}^H D_{1^+}^{\gamma}\widetilde{\Game}(\gimel,\ell)\, L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right)\frac{d\ell}{\ell}\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon, \\
\left|(B_3 v_l)(\gimel) - (B_3 v)(\gimel)\right| &= \left| \int_1^e \widehat{\text{Ↄ}}(\gimel, \ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(t, v_l(t), {}^H D_{1^+}^{\alpha} v_l(t)\right)\right) \frac{d\ell}{\ell} \right. \\
&\quad \left. - \int_1^e \widehat{\text{Ↄ}}(\gimel, \ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(t, v(t), {}^H D_{1^+}^{\alpha} v(t)\right)\right) \frac{d\ell}{\ell} \right| \\
&\leq \epsilon, \\
\left|{}^H D_{1^+}^{\beta}(B_3 v_l)(\gimel) - {}^H D_{1^+}^{\beta}(B_3 v)(\gimel)\right| &= \left| \int_1^e {}^H D_{1^+}^{\beta} \widehat{\text{Ↄ}}(\gimel, \ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v_l(\ell), {}^H D_{1^+}^{\alpha} v_l(\ell)\right)\right) \frac{d\ell}{\ell} \right. \\
&\quad \left. - \int_1^e {}^H D_{1^+}^{\beta} \widehat{\text{Ↄ}}(\gimel, \ell)\, L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right) \frac{d\ell}{\ell} \right| \\
&\leq \epsilon.
\end{aligned}
$$

Hence, when $l \to \infty$, we get

$$
\begin{cases}
\|(B_1 w_l)(\gimel) - (B_1 w)(\gimel)\|_0 \to 0, & \left\|{}^H D_{1^+}^{\zeta}(B_1 w_l)(\gimel) - {}^H D_{1^+}^{\zeta}(B_1 w)(\gimel)\right\|_0 \to 0, \\
\|(B_2 z_l)(\gimel) - (B_2 z)(\gimel)\|_0 \to 0, & \left\|{}^H D_{1^+}^{\gamma}(B_2 z_l)(\gimel) - {}^H D_{1^+}^{\gamma}(B_2 z)(\gimel)\right\|_0 \to 0, \\
\|(B_3 v_l)(\gimel) - (B_3 v)(\gimel)\|_0 \to 0, & \left\|{}^H D_{1^+}^{\beta}(B_3 v_l)(\gimel) - {}^H D_{1^+}^{\beta}(B_3 v)(\gimel)\right\|_0 \to 0.
\end{cases}
$$

This proves that

$$
\begin{cases}
\|(B_1 w_l)(\gimel) - (B_1 w)(\gimel)\|_0 \to 0, \\
\|(B_2 z_l)(\gimel) - (B_2 z)(\gimel)\|_0 \to 0, \qquad \text{as } l \to \infty. \\
\|(B_3 v_l)(\gimel) - (B_3 v)(\gimel)\|_0 \to 0,
\end{cases}
$$

Therefore, B_j $(j = 1, 2, 3)$ is continuous, and, hence, B is continuous, too, in the space $\mho$. □

Next, we show that the mapping B_j $(j = 1, 2, 3)$ and B are completely continuous (CPC).

Lemma 4.3. *Via the hypotheses* (A_0) *and* (A_1), *the mappings* $B, B_j : Q \to Q$ $(j = 1, 2, 3)$ *are CPC.*

Proof. Thanks to Lemma 4.1,

$$
(B_1 w)(\gimel), {}^H D_{1^+}^{\alpha}(B_1 w)(\gimel), (B_2 z)(\gimel), {}^H D_{1^+}^{\alpha}(B_2 z)(\gimel), (B_3 v)(\gimel), {}^H D_{1^+}^{\alpha}(B_3 v)(\gimel) \geq 0,
$$

for all $\gimel \in [1, e]$. Thus, $B(Q) \subset Q$. Suppose that U is a bounded subset of Q. We claim that $B(U)$ is relatively compact. The boundedness of U implies that there exists $H > 0$ such that $\|(w, z, v)\| \leq H$, for any $(w, z, v) \in U$.

Now, for $(w, z, v) \in U$, and $\gimel, \ell \in [1, e]$, one gets

$$
\begin{aligned}
B_1(w)(\gimel) &= \int_1^e \text{Ↄ}(\gimel, \ell)\, L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell} \\
&\leq \left(\frac{1}{\Gamma(\zeta)} + \frac{1}{\Gamma(\zeta)\nabla_1} \int_1^e g(\gimel)(\ln \gimel)^{\zeta - 1}\, dG(\gimel) \right) \int_1^e L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{1}{\Gamma(\zeta)}+\frac{1}{\Gamma(\zeta)\nabla_1}\int_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\,dG(\gimel)\right)\int_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\\
&= \left(\frac{1}{\Gamma(\zeta)}+\frac{1}{\Gamma(\zeta)\nabla_1}\Xi\right)F<+\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{H}D^{\zeta}_{1^+}B_1(w)(\gimel) &= \int_1^e {}^{H}D^{\zeta}_{1^+}\eth(\gimel,\ell)\,L_{h_1}\left(\Re\left(\ell,w(\ell),{}^{H}D^{\alpha}_{1^+}w(\ell)\right)\right)\frac{d\ell}{\ell}\\
&\leq \left(\frac{1}{\Gamma(\zeta-\alpha)}+\frac{1}{\Gamma(\zeta-\alpha)\nabla_1}\int_1^e g(\gimel)(\ln\gimel)^{\zeta-\alpha-1}\,dG(\gimel)\right)\int_1^e L_{h_1}\left(\Re\left(\ell,w(\ell),{}^{H}D^{\alpha}_{1^+}w(\ell)\right)\right)\frac{d\ell}{\ell}\\
&\leq \left(\frac{1}{\Gamma(\zeta-\alpha)}+\frac{1}{\Gamma(\zeta-\alpha)\nabla_1}\int_1^e g(\gimel)(\ln\gimel)^{\zeta-\alpha-1}\,dG(\gimel)\right)\int_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\\
&\leq \left(\frac{1}{\Gamma(\zeta-\alpha)}+\frac{1}{\Gamma(\zeta-\alpha)\nabla_1}\int_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\,dG(\gimel)\right)\int_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\\
&= \left(\frac{1}{\Gamma(\zeta-\alpha)}+\frac{1}{\Gamma(\zeta-\alpha)\nabla_1}\Xi\right)F<+\infty,
\end{aligned}
$$

where $\Xi=\int_1^e g(\gimel)(\ln\gimel)^{\zeta-1}\,dG(\gimel)$ and $F=\int_1^e L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}$. Analogously, for $\gimel,\ell\in[1,e]$, and $(w,z,v)\in U$, one can write

$$
\begin{aligned}
B_2(z)(\gimel) &= \left(\frac{1}{\Gamma(\gamma)}+\frac{1}{\Gamma(\gamma)\widetilde{\nabla}_1}\widetilde{\Xi}\right)\widetilde{F}<+\infty,\\
{}^{H}D^{\gamma}_{1^+}B_2(z)(\gimel) &= \left(\frac{1}{\Gamma(\gamma-\alpha)}+\frac{1}{\Gamma(\gamma-\alpha)\widetilde{\nabla}_1}\widetilde{\Xi}\right)\widetilde{F}<+\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
B_3(v)(\gimel) &= \left(\frac{1}{\Gamma(\beta)}+\frac{1}{\Gamma(\beta)\widehat{\nabla}_1}\widehat{\Xi}\right)\widehat{F}<+\infty,\\
{}^{H}D^{\beta}_{1^+}B_3(v)(\gimel) &= \left(\frac{1}{\Gamma(\beta-\alpha)}+\frac{1}{\Gamma(\beta-\alpha)\widehat{\nabla}_1}\widehat{\Xi}\right)\widehat{F}<+\infty,
\end{aligned}
$$

where $\widetilde{\Xi}=\int_1^e\widetilde{g}(\gimel)(\ln\gimel)^{\gamma-1}\,d\widetilde{G}(\gimel)$, $\widetilde{F}=\int_1^e L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell}$, $\widehat{\Xi}=\int_1^e\widehat{g}(\gimel)(\ln\gimel)^{\beta-1}\,d\widehat{G}(\gimel)$ and $\widehat{F}=\int_1^e L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell}$. Thus, $B(Q)$ is bounded. After that, we show that ${}^{H}D^{\alpha}_{1^+}B(U)$ is equicontinuous. For this regard, consider $\gimel_1,\gimel_2\in[1,e]$ with $\gimel_1<\gimel_2$. Then, for $(v,w,z)\in U$, we have

$$
\left|{}^{H}D^{\zeta}_{1^+}(B_1w)(\gimel_2)-{}^{H}D^{\zeta}_{1^+}(B_1w)(\gimel_1)\right|
$$

$$
\begin{aligned}
&= \left| \int_1^e {}^H D_{1^+}^{\zeta} \supset (\gimel_2, \ell)\, L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \right. \\
&\quad \left. - \int_1^e {}^H D_{1^+}^{\zeta} \supset (\gimel_1, \ell)\, L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \right| \\
&= \left| (\ln \gimel_2)^{\zeta-\alpha-1} - (\ln \gimel_1)^{\zeta-\alpha-1} \right| \int_1^e L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \\
&\quad + \left| (\ln \gimel_2)^{\zeta-\alpha-1} - (\ln \gimel_1)^{\zeta-\alpha-1} \right| \frac{\Gamma(\zeta) \int_1^e h(\gimel) \supset (\gimel, \ell)\, dG(\gimel)}{\nabla_1 \Gamma(\zeta-\alpha)} \int_1^e L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \\
&\quad + \frac{1}{\Gamma(\zeta-\alpha)} \int_1^{r_1} \left| (\ln \gimel_1 - \ln \ell)^{\zeta-\alpha-1} - (\ln \gimel_2 - \ln \ell)^{\zeta-\alpha-1} \right| \int_1^e L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \\
&\quad + \int_{\gimel_1}^{\gimel_2} \left| (\ln \gimel_2 - \ln \ell)^{\zeta-\alpha-1} \right| \int_1^e L_{h_1} \left(\Re \left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell) \right) \right) \frac{d\ell}{\ell} \\
&\leq \left| (\ln \gimel_2)^{\zeta-\alpha-1} - (\ln \gimel_1)^{\zeta-\alpha-1} \right| \frac{1}{\Gamma(\zeta-\alpha)} \int_1^e L_{h_1} (\sigma(\ell)) \frac{d\ell}{\ell} \\
&\quad + \frac{1}{\Gamma(\zeta-\alpha)} \int_1^{\gimel_1} \left| (\ln \gimel_1 - \ln \ell)^{\zeta-\alpha-1} - (\ln \gimel_2 - \ln \ell)^{\zeta-\alpha-1} \right| \int_1^e L_{h_1} (\sigma(\ell)) \frac{d\ell}{\ell} \\
&\quad + \int_{\gimel_1}^{\gimel_2} \left| (\ln \gimel_2 - \ln \ell)^{\zeta-\alpha-1} \right| \int_1^e L_{h_1} (\sigma(\ell)) \frac{d\ell}{\ell}.
\end{aligned}
\tag{4.7}
$$

Similarly, we can obtain

$$
\begin{aligned}
&\left| {}^H D_{1^+}^{\gamma} (B_2 z)(\gimel_2) - {}^H D_{1^+}^{\gamma} (B_2 z)(\gimel_1) \right| \\
&\leq \left| (\ln \gimel_2)^{\gamma-\alpha-1} - (\ln \gimel_1)^{\gamma-\alpha-1} \right| \frac{1}{\Gamma(\gamma-\alpha)} \int_1^e L_{h_2} (\widetilde{\sigma}(\ell)) \frac{d\ell}{\ell} \\
&\quad + \frac{1}{\Gamma(\gamma-\alpha)} \int_1^{r_1} \left| (\ln \gimel_1 - \ln \ell)^{\gamma-\alpha-1} - (\ln \gimel_2 - \ln \ell)^{\gamma-\alpha-1} \right| \int_1^e L_{h_2} (\widetilde{\sigma}(\ell)) \frac{d\ell}{\ell} \\
&\quad + \int_{r_1}^{r_2} \left| (\ln \gimel_2 - \ln \ell)^{\gamma-\alpha-1} \right| \int_1^e L_{h_2} (\widetilde{\sigma}(\ell)) \frac{d\ell}{\ell},
\end{aligned}
\tag{4.8}
$$

and

$$
\left| {}^H D_{1^+}^{\beta} (B_3 v)(\gimel_2) - {}^H D_{1^+}^{\beta} (B_3 v)(\gimel_1) \right|
$$

$$\leq \left|(\ln \daleth_2)^{\beta-\alpha-1} - (\ln \daleth_1)^{\beta-\alpha-1}\right| \frac{1}{\Gamma(\beta-\alpha)} \int_1^e L_{h_3}(\widehat{\sigma}(\ell)) \frac{d\ell}{\ell}$$

$$+\frac{1}{\Gamma(\beta-\alpha)} \int_1^{r_1} \left|(\ln \daleth_1 - \ln \ell)^{\beta-\alpha-1} - (\ln \daleth_2 - \ln \ell)^{\beta-\alpha-1}\right| \int_1^e L_{h_3}(\widehat{\sigma}(\ell)) \frac{d\ell}{\ell}$$

$$+\int_{r_1}^{r_2} \left|(\ln \daleth_2 - \ln \ell)^{\beta-\alpha-1}\right| \int_1^e L_{h_3}(\widehat{\sigma}(\ell)) \frac{d\ell}{\ell}. \tag{4.9}$$

Since $(\ln \daleth)^{\zeta-\alpha-1}$, $(\ln \daleth)^{\gamma-\alpha-1}$, $(\ln \daleth)^{\beta-\alpha-1}$, $(\ln \daleth - \ln \ell)^{\zeta-\alpha-1}$, $(\ln \daleth - \ln \ell)^{\gamma-\alpha-1}$, and $(\ln \daleth - \ln \ell)^{\beta-\alpha-1}$ are uniformly continuous, it follows from Lemma 2.6 and (4.7)–(4.9) that $B(U)$ is relatively compact in the space $\mho$. Therefore, by Lemma 4.2, we conclude that $B : Q \to Q$ is CPC. $\square$

5. Existence of positive solutions

In this section, we discuss the positive solutions to the HF differential system (1.1) with conditions (1.2). Also, we consider here the functions $\mathfrak{R}$, $\widetilde{\mathfrak{R}}$, and $\widehat{\mathfrak{R}}$ are continuous. To accomplish this task, we present the following hypotheses:

(A_2) There exist constants $\wp > 0$ and $\daleth_0 \in (0,1)$ such that for all $(\daleth, v, w, z) \in [1,e] \times [0,\wp]^3$ (where $[0,\wp]^3 = [0,\wp] \times [0,\wp] \times [0,\wp]$), we have

$$L_{h_1}\left(\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right)\right) \geq \wp \max \left\{ \begin{array}{c} \frac{1}{(\ln \daleth_0)^{\zeta-1} Z_1} \left(\int_{r_0}^{e} \eth_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1}, \\ \frac{1}{(\ln \daleth_0)^{\zeta-\alpha-1} Z_1} \left(\int_{r_0}^{e} {}^H D_{1^+}^{\alpha} \eth_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1} \end{array} \right\},$$

$$L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\ell, z(\ell), {}^H D_{1^+}^{\alpha} z(\ell)\right)\right) \geq \wp \max \left\{ \begin{array}{c} \frac{1}{(\ln \daleth_0)^{\gamma-1} \widetilde{Z}_1} \left(\int_{r_0}^{e} \widetilde{\eth}_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1}, \\ \frac{1}{(\ln \daleth_0)^{\gamma-\alpha-1} \widetilde{Z}_1} \left(\int_{r_0}^{e} {}^H D_{1^+}^{\alpha} \widetilde{\eth}_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1} \end{array} \right\},$$

$$L_{h_3}\left(\widehat{\mathfrak{R}}\left(\ell, v(\ell), {}^H D_{1^+}^{\alpha} v(\ell)\right)\right) \geq \wp \max \left\{ \begin{array}{c} \frac{1}{(\ln \daleth_0)^{\beta-1} \widehat{Z}_1} \left(\int_{r_0}^{e} \widehat{\eth}_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1}, \\ \frac{1}{(\ln \daleth_0)^{\beta-\alpha-1} \widehat{Z}_1} \left(\int_{r_0}^{e} {}^H D_{1^+}^{\alpha} \widehat{\eth}_1(\daleth_0, \ell) \frac{d\ell}{\ell} \right)^{-1} \end{array} \right\},$$

where Z_1, $\widetilde{Z}_1$, and $\widehat{Z}_1$ are defined in (4.1)–(4.3), respectively.

(A_3) There exist a positive constant $\upsilon > \wp$ and functions $\sigma(\ell), \widetilde{\sigma}(\ell), \widehat{\sigma}(\ell)$ such that for all $(\ell, v, w, z) \in [1,e] \times [0,\upsilon]^3$, we get

$$\mathfrak{R}\left(\ell, w(\ell), {}^H D_{1^+}^{\alpha} w(\ell)\right) \leq \sigma(\ell) \upsilon^{\frac{1}{h_1-1}},$$

$$
\begin{aligned}
\widetilde{\mathfrak{R}}\left(\ell, z(\ell),{}^{H}D_{1^+}^{\alpha} z(\ell)\right) &\leq \widetilde{\sigma}(\ell)\upsilon^{\frac{1}{h_2-1}},\\
\widehat{\mathfrak{R}}\left(\ell, v(\ell),{}^{H}D_{1^+}^{\alpha} v(\ell)\right) &\leq \widehat{\sigma}(\ell)\upsilon^{\frac{1}{h_3-1}},
\end{aligned}
$$

with

$$
\begin{aligned}
Z\int_1^e \max\left\{\eth(e,\ell),{}^{H}D_{1^+}^{\alpha}\eth(e,\ell)\right\} L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell} &< 1,\\
\widetilde{Z}\int_1^e \max\left\{\widetilde{\eth}(e,\ell),{}^{H}D_{1^+}^{\alpha}\widetilde{\eth}(e,\ell)\right\} L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell} &< 1,\\
\widehat{Z}\int_1^e \max\left\{\widehat{\eth}(e,\ell),{}^{H}D_{1^+}^{\alpha}\widehat{\eth}(e,\ell)\right\} L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell} &< 1,
\end{aligned}
$$

where Z, $\widetilde{Z}$, and $\widehat{Z}$ are defined in (4.1), (4.2) and (4.3), respectively.

(A_4) There exist nondecreasing functions $\vartheta, \kappa, \varrho : \mathbb{R}_+^1 \to \mathbb{R}_+$ and functions $\sigma(\ell), \widetilde{\sigma}(\ell), \widehat{\sigma}(\ell)$ such that

$$
\begin{aligned}
\mathfrak{R}\left(\ell, w(\ell),{}^{H}D_{1^+}^{\alpha} w(\ell)\right) &\leq \sigma(\ell)\vartheta^{\frac{1}{h_1-1}}(w),\\
\widetilde{\mathfrak{R}}\left(\ell, z(\ell),{}^{H}D_{1^+}^{\alpha} z(\ell)\right) &\leq \widetilde{\sigma}(\ell)\kappa^{\frac{1}{h_2-1}}(z),\\
\widehat{\mathfrak{R}}\left(\ell, v(\ell),{}^{H}D_{1^+}^{\alpha} v(\ell)\right) &\leq \widehat{\sigma}(\ell)\varrho^{\frac{1}{h_3-1}}(v),
\end{aligned}
$$

for all $(\beth, v, w, z) \in [1, e] \times \left(\mathbb{R}_+^1\right)^3$, where σ, $\widetilde{\sigma}$ and $\widehat{\sigma}$ are defined in (A_0).

(A_5) There exist a positive constant M and, the functions $\sigma(\ell), \widetilde{\sigma}(\ell), \widehat{\sigma}(\ell)$ such that

$$
\max\left\{\begin{array}{l}
\int_1^e \eth_1(e,\ell) Z L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell},\\
\int_1^e \widetilde{\eth}_1(e,\ell) \widetilde{Z} L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell},\\
\int_1^e \widehat{\eth}_1(e,\ell) \widehat{Z} L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell}
\end{array}\right\} < \frac{M}{\max\{\vartheta(M), \kappa(M), \varrho(M)\}}.
$$

(A_6) There exist functions $\sigma(\ell), \widetilde{\sigma}(\ell), \widehat{\sigma}(\ell)$, which are defined in ($A_0$) such that

$$
\begin{cases}
\left|L_{h_1}\left(\mathfrak{R}(\ell,\varkappa_1,\varpi_1)\right) - L_{h_1}\left(\mathfrak{R}(\ell,\varkappa_2,\varpi_2)\right)\right| \leq N_1\sigma(\ell)\left|\varkappa_1-\varkappa_2\right| + N_2\sigma(\ell)\left|\varpi_1-\varpi_2\right|,\\
\left|L_{h_2}\left(\widetilde{\mathfrak{R}}(\ell,\varkappa_1,\varpi_1)\right) - L_{h_2}\left(\widetilde{\mathfrak{R}}(\ell,\varkappa_2,\varpi_2)\right)\right| \leq \widetilde{N}_1\widetilde{\sigma}(\ell)\left|\varkappa_1-\varkappa_2\right| + \widetilde{N}_2\widetilde{\sigma}(\ell)\left|\varpi_1-\varpi_2\right|,\\
\left|L_{h_3}\left(\widehat{\mathfrak{R}}(\ell,\varkappa_1,\varpi_1)\right) - L_{h_3}\left(\widehat{\mathfrak{R}}(\ell,\varkappa_2,\varpi_2)\right)\right| \leq \widehat{N}_1\widehat{\sigma}(\ell)\left|\varkappa_1-\varkappa_2\right| + \widehat{N}_2\widehat{\sigma}(\ell)\left|\varpi_1-\varpi_2\right|,
\end{cases}
$$

and

$$
\begin{cases}
\frac{1}{\Gamma(\zeta)} Z\left(N_1\sigma(\ell) + N_2\sigma(\ell)\right) < 1,\\
\frac{1}{\Gamma(\gamma)} \widetilde{Z}\left(\widetilde{N}_1\widetilde{\sigma}(\ell) + \widetilde{N}_2\widetilde{\sigma}(\ell)\right) < 1,\\
\frac{1}{\Gamma(\beta)} \widehat{Z}\left(\widehat{N}_1\widehat{\sigma}(\ell) + \widehat{N}_2\widehat{\sigma}(\ell)\right) < 1,
\end{cases}
$$

for all $\varkappa_1, \varpi_1, \varkappa_2, \varpi_2 \in \mho$, where Z_1, $\widetilde{Z}_1$, and $\widehat{Z}_1$ are defined in (4.1)–(4.3), respectively.

Theorem 5.1. *There exist at least one positive solution to the problem (1.1) with conditions (1.2), provided that the hypotheses (A_0)–(A_3) hold.*

Proof. Assume that $\phi_1 = \{(w,z,v) \in Q : \|w\| \le \wp,\ \|z\| \le \wp,\ \|v\| \le \wp\}$ with $0 < w(\gimel), z(\gimel), v(\gimel) \le \wp$, for each $(w,z,v) \in Q \cap \partial\phi_1$ and all $\gimel \in [1,e]$. Based on (A_2) and Lemma 4.1, one has

$$\begin{aligned}
B_1(w)(\gimel_0) &= \int_1^e \eth(\gimel_0,\ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^H D^{\alpha}_{1^+} w(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\ge \int_{r_0}^e \eth_1(\gimel_0,\ell)\,(\ln \gimel_0)^{\zeta-1} Z_1 \frac{d\ell}{\ell} \times \frac{\wp}{(\ln r_0)^{\zeta-1} Z_1}\left(\int_{r_0}^e \eth_1(\gimel_0,\ell)\frac{d\ell}{\ell}\right)^{-1} \\
&= \wp, \\
{}^H D^{\alpha}_{1^+} B_1(w)(\gimel_0) &= \int_1^e {}^H D^{\alpha}_{1^+}\eth(\gimel_0,\ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^H D^{\alpha}_{1^+} w(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\ge \int_{r_0}^e {}^H D^{\alpha}_{1^+}\eth_1(\gimel_0,\ell)\,(\ln \gimel_0)^{\zeta-\alpha-1} Z_1 \frac{d\ell}{\ell} \times \frac{1}{(\ln \gimel_0)^{\zeta-\alpha-1} Z_1}\left(\int_{r_0}^e {}^H D^{\alpha}_{1^+}\eth_1(\gimel_0,\ell)\frac{d\ell}{\ell}\right)^{-1} \\
&= \wp.
\end{aligned}$$

Similarly, we have

$$B_2(z)(\gimel_0) \ge \wp,\ {}^H D^{\alpha}_{1^+} B_2(z)(\gimel_0) \ge \wp,\ B_3(v)(\gimel_0) \ge \wp,\ {}^H D^{\alpha}_{1^+} B_3(v)(\gimel_0) \ge \wp.$$

This implies that

$$\begin{aligned}
\|B_1 w\| &= \max\left\{\|B_1 w\|_0, \left\|{}^H D^{\alpha}_{1^+} B_1 w\right\|_0\right\} \ge \wp, \\
\|B_2 z\| &= \max\left\{\|B_2 z\|_0, \left\|{}^H D^{\alpha}_{1^+} B_2 z\right\|_0\right\} \ge \wp, \\
\|B_3 v\| &= \max\left\{\|B_3 v\|_0, \left\|{}^H D^{\alpha}_{1^+} B_3 v\right\|_0\right\} \ge \wp.
\end{aligned}$$

Hence,

$$B(w,z,v) = \max\{\|B_1 w\|, \|B_2 z\|, \|B_3 v\|\} \ge \wp = \|(w,z,v)\|. \tag{5.1}$$

Again, let $\phi_2 = \{(v,w,z) \in Q : \|v\| \le \upsilon,\ \|w\| \le \upsilon,\ \|z\| \le \upsilon\}$ with $\upsilon > \wp$. Now, for any $(v,w,z) \in Q \cap \partial\phi_2$ and for $\gimel \in [1,e]$, we have $0 < v(\gimel), w(\gimel), z(\gimel) \le \upsilon$. According to ($A_3$) and Lemma 4.1, we can write

$$\begin{aligned}
B_1(w)(\gimel_0) &= \int_1^e \eth(\gimel_0,\ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^H D^{\alpha}_{1^+} w(\ell)\right)\right)\frac{d\ell}{\ell} \\
&\le \int_1^e \eth_1(e,\ell)\, Z\upsilon L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell} \\
&= \upsilon Z \int_1^e \eth_1(e,\ell)\, L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}
\end{aligned}$$

$$
\begin{aligned}
&= \upsilon, \\
{}^{H}D_{1^+}^{\alpha} B_1(w)(\daleth_0) &= \int_1^e {}^{H}D_{1^+}^{\alpha} \eth(\daleth_0, \ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell} \\
&\le \int_{r_0}^e {}^{H}D_{1^+}^{\alpha} \eth_1(e, \ell)\, Z\upsilon L_{h_1}(\sigma(\ell)) \frac{d\ell}{\ell} = \upsilon Z \int_1^e \eth_1(e, \ell)\, L_{h_1}(\sigma(\ell)) \frac{d\ell}{\ell} \\
&= \upsilon.
\end{aligned}
$$

Analogously, we have

$$
B_2(z)(\daleth_0) \le \upsilon,\ {}^{H}D_{1^+}^{\alpha} B_2(z)(\daleth_0) \le \upsilon,\ B_3(v)(\daleth_0) \le \upsilon,\ {}^{H}D_{1^+}^{\alpha} B_3(v)(\daleth_0) \le \upsilon.
$$

Hence,

$$
\begin{aligned}
\|B_1 v\| &= \max\left\{\|B_1 v\|_0, \left\|{}^{H}D_{1^+}^{\alpha} B_1 v\right\|_0\right\} \le \upsilon, \\
\|B_2 w\| &= \max\left\{\|B_2 w\|_0, \left\|{}^{H}D_{1^+}^{\alpha} B_2 w\right\|_0\right\} \le \upsilon, \\
\|B_3 z\| &= \max\left\{\|B_3 z\|_0, \left\|{}^{H}D_{1^+}^{\alpha} B_3 z\right\|_0\right\} \le \upsilon.
\end{aligned}
$$

Therefore,

$$
B(v, w, z) = \max\{\|B_1 v\|, \|B_2 w\|, \|B_3 z\|\} \le \upsilon = \|(v, w, z)\|. \tag{5.2}
$$

Thanks to Lemma 4.3, the mapping B is CPC. Then, by (5.1), (5.2) and Lemma 2.4, the mapping B has at least one FP, which is a positive solution to the considered problem. This finishes the proof. □

Theorem 5.2. *Via hypotheses (A_0), (A_1), (A_4) and (A_5), the HF problem (1.1) with conditions (1.2) possesses a positive solution.*

Proof. Assume that $V = \{(v, w, z) \in Q : \|(v, w, z)\| < M\}$, then $V \subset Q$. Based on Lemmas 4.2 and 4.3, we have that the mapping $B : \overline{V} \to P$ is CPC. Now, if there are $(v, w, z) \in \partial V$ and $\iota \in (0, 1)$, we have $(v, w, z) = \iota B(v, w, z)$, hence, by ($A_4$) and (4.4) for $\daleth \in [1, e]$, we can write

$$
\begin{aligned}
w(\daleth) &= \iota B_1(w)(\daleth) = \iota \int_1^e \eth(\daleth, \ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell} \\
&< \int_1^e \eth(\daleth, \ell)\, L_{h_1}\left(\Re\left(\ell, w(\ell), {}^{H}D_{1^+}^{\alpha} w(\ell)\right)\right) \frac{d\ell}{\ell} \\
&\le \int_1^e \eth(\daleth, \ell)\, L_{h_1}\left(\sigma(\ell)\vartheta^{\frac{1}{h_1-1}}(w)\right) \frac{d\ell}{\ell} \\
&\le \vartheta(\|w\|) \int_1^e \eth_1(e, \ell)\, Z L_{h_1}(\sigma(\ell)) \frac{d\ell}{\ell} \\
&\le \vartheta(\|(v, w, z)\|) \int_1^e \eth_1(e, \ell)\, Z L_{h_1}(\sigma(\ell)) \frac{d\ell}{\ell},
\end{aligned}
$$

which implies that

$$\frac{\|w\|}{\vartheta(\|(v,w,z)\|)} \leq \int_1^e \eth_1(e,\ell)\, ZL_{h_1}(\sigma(\ell))\, \frac{d\ell}{\ell}. \tag{5.3}$$

Similarly, one can obtain

$$\frac{\|z\|}{\vartheta(\|(v,w,z)\|)} \leq \int_1^e \widetilde{\eth}_1(e,\ell)\, \widetilde{Z}L_{h_2}(\widetilde{\sigma}(\ell))\, \frac{d\ell}{\ell}, \tag{5.4}$$

and

$$\frac{\|v\|}{\vartheta(\|(v,w,z)\|)} \leq \int_1^e \widehat{\eth}_1(e,\ell)\, \widehat{Z}L_{h_1}(\widehat{\sigma}(\ell))\, \frac{d\ell}{\ell}. \tag{5.5}$$

It follows from (5.3)–(5.5) that

$$\begin{aligned}
&\frac{\|(v,w,z)\|}{\max\{\vartheta(\|(v,w,z)\|), \kappa(\|(v,w,z)\|), \varrho(\|(v,w,z)\|)\}} \\
\leq\ & \max\left\{\int_1^e \eth_1(e,\ell)\, ZL_{h_1}(\sigma(\ell))\, \frac{d\ell}{\ell}, \int_1^e \widetilde{\eth}_1(e,\ell)\, \widetilde{Z}L_{h_2}(\widetilde{\sigma}(\ell))\, \frac{d\ell}{\ell}, \int_1^e \widehat{\eth}_1(e,\ell)\, \widehat{Z}L_{h_3}(\widehat{\sigma}(\ell))\, \frac{d\ell}{\ell}\right\}.
\end{aligned}$$

From (A_4), $\|(v,w,z)\| \neq M$, this means $(v,w,z) \notin \partial V$. Hence, by Lemma 2.5, the mapping B has an FP $(v,w,z) \in V$, which is a positive solution to the HF problem (1.1) with stipulations (1.2). □

In the next theorem, the Banach contraction principle (BCP) will be applied to discuss the uniqueness of the solution of our purposed problem.

Theorem 5.3. *Under the assumption (A_6), the HF problem (1.1) with conditions (1.2) owns a unique positive solution, provided that $N < 1$ where*

$$N = \max\left\{\frac{\sigma(\ell)}{\Gamma(\zeta)} Z(N_1+N_2), \frac{\widetilde{\sigma}(\ell)}{\Gamma(\gamma)} \widetilde{Z}\left(\widetilde{N}_1+\widetilde{N}_2\right), \frac{\widehat{\sigma}(\ell)}{\Gamma(\beta)} \widehat{Z}\left(\widehat{N}_1+\widehat{N}_2\right)\right\} < 1.$$

Proof. For all $\daleth \in [1,e]$, $v_1, v_2, w_1, w_2, z_1, z_2 \in \mho$, we get

$$\begin{aligned}
&|B_1(w_2)(\daleth) - B_1(w_1)(\daleth)| \\
=\ & \left|\int_1^e \eth(\daleth,\ell)\, L_{h_1}\left(\Re\left(\ell, w_2(\ell), {}^H D_{1^+}^{\alpha} w_2(\ell)\right)\right) \frac{d\ell}{\ell} \right.\\
&\left. - \int_1^e \eth(\daleth,\ell)\, L_{h_1}\left(\Re\left(\ell, w_1(\ell), {}^H D_{1^+}^{\alpha} w_1(\ell)\right)\right) \frac{d\ell}{\ell}\right| \\
\leq\ & \int_1^e \eth(\daleth,\ell) \left| L_{h_1}\left(\Re\left(\ell, w_2(\ell), {}^H D_{1^+}^{\alpha} w_2(\ell)\right)\right) - L_{h_1}\left(\Re\left(\ell, w_1(\ell), {}^H D_{1^+}^{\alpha} w_1(\ell)\right)\right)\right| \frac{d\ell}{\ell}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_1^e \eth_1(\gimel,\ell)\, Z\left(N_1\sigma(\ell)\left|w_2(\ell)-w_1(\ell)\right| + N_2\sigma(\ell)\left|{}^H D_{1^+}^{\alpha} w_2(\ell) -{}^H D_{1^+}^{\alpha} w_1(\ell)\right|\right)\frac{d\ell}{\ell} \\
&\leq \frac{\sigma(\ell)}{\Gamma(\zeta)}\left\|w_2-w_1\right\| Z\left(N_1+N_2\right). \qquad (5.6)
\end{aligned}
$$

Similarly, one can write

$$
|B_2(z_2)(\gimel) - B_2(z_1)(\gimel)| \leq \frac{\widetilde{\sigma}(\ell)}{\Gamma(\gamma)}\left\|z_2-z_1\right\| \widetilde{Z}\left(\widetilde{N}_1+\widetilde{N}_2\right), \qquad (5.7)
$$

and

$$
|B_3(v_2)(\gimel) - B_3(v_1)(\gimel)| \leq \frac{\widehat{\sigma}(\ell)}{\Gamma(\beta)}\left\|v_2-v_1\right\| \widehat{Z}\left(\widehat{N}_1+\widehat{N}_2\right). \qquad (5.8)
$$

It follows from (5.6)–(5.8) that

$$
\begin{aligned}
\|B(w_2,z_2,v_2)-B(w_1,z_1,v_1)\| &= \max\left\{|B(w_2,z_2,v_2)-B(w_1,z_1,v_1)|\right\} \\
&= \|(B_2w_2, B_3z_2, B_1v_2)-(B_2w_1,B_3z_1,B_1v_1)\| \\
&= \|(B_2w_2-B_2w_1, B_3z_2-B_3z_1, B_1v_2-B_1v_1)\| \\
&\leq N\left\|(w_2-w_1, z_2-z_1, v_2-v_1)\right\|,
\end{aligned}
$$

where

$$
N = \max\left\{\frac{\sigma(\gimel)}{\Gamma(\zeta)} Z\left(N_1+N_2\right), \frac{\widetilde{\sigma}(\gimel)}{\Gamma(\gamma)}\widetilde{Z}\left(\widetilde{N}_1+\widetilde{N}_2\right), \frac{\widehat{\sigma}(\gimel)}{\Gamma(\beta)}\widehat{Z}\left(\widehat{N}_1+\widehat{N}_2\right)\right\}.
$$

Since $N<1$, this implies that B is a contraction mapping on a Banach space $\mho$. Hence, by BCP, B owns a unique FP, which is a unique solution to the HF system (1.1) with conditions (1.2). □

6. Supportive examples

The main goal of this part is to bolster and validate our theoretical findings with a few practical examples.

Example 6.1. Consider the following HF problem:

$$
\begin{cases}
L_4\left({}^H D_{1^+}^{\frac{3}{2}} v(\gimel)\right) + \mathfrak{R}\left(\gimel, w(\gimel), {}^H D_{1^+}^{\alpha} w(\gimel)\right) = 0, \ \gimel \in (1,e), \\
L_3\left({}^H D_{1^+}^{\frac{3}{2}} w(\gimel)\right) + \widetilde{\mathfrak{R}}\left(\gimel, z(\gimel), {}^H D_{1^+}^{\alpha} z(\gimel)\right) = 0, \ \gimel \in (1,e), \\
L_2\left({}^H D_{1^+}^{\frac{3}{2}} z(\gimel)\right) + \widehat{\mathfrak{R}}\left(\gimel, v(\gimel), {}^H D_{1^+}^{\alpha} v(\gimel)\right) = 0, \ \gimel \in (1,e), \\
v(1) = v'(1) = 0, \ v(e) = \sum_{k=1}^{\infty} \lambda_k v\left(e^{\frac{1}{k^3}}\right) + \int_1^e g(\gimel) v(\gimel)\, dG(\gimel), \\
w(1) = w'(1) = 0, \ w(e) = \sum_{k=1}^{\infty} \widetilde{\lambda}_k w\left(e^{\frac{1}{k^3}}\right) + \int_1^e \widetilde{g}(\gimel) w(\gimel)\, d\widetilde{G}(\gimel), \\
z(1) = z'(1) = 0, \ z(e) = \sum_{k=1}^{\infty} \widehat{\lambda}_k w\left(e^{\frac{1}{k^3}}\right) + \int_1^e \widehat{g}(\gimel) z(\gimel)\, d\widehat{G}(\gimel),
\end{cases} \qquad (6.1)
$$

where $\zeta=\gamma=\beta=\frac{3}{2}$, $\alpha=\frac{1}{2}$, $\lambda_k=\widetilde{\lambda}_k=\widehat{\lambda}_k=\frac{1}{3k^2}$, $\rho_k=\widetilde{\rho}_k=\widehat{\rho}_k=e^{\frac{1}{k^3}}$, $p_1=4$, $h_1=\frac{4}{3}$, $p_2=3$, $h_2=\frac{3}{2}$, $p_3=2$, $h_3=2$, $g(\gimel)=\widetilde{g}(\gimel)=\widehat{g}(\gimel)=(\ln\gimel)^{\frac{1}{3}}$,

$$G(\gimel)=\widetilde{G}(\gimel)=\widehat{G}(\gimel)=\begin{cases}0, & \gimel\in[0,\frac{e}{3}),\\ 3, & \gimel\in[\frac{e}{3},\frac{2e}{3}],\\ 1, & \gimel\in(\frac{2e}{3},e],\end{cases}$$

$$\begin{aligned}&\mathfrak{R}\left(\gimel, w(\gimel), {}^H D_{1^+}^{\frac{1}{2}} w(\gimel)\right)\\ = &\begin{cases}\frac{1}{3\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}}\left(w^3+\left({}^H D_{1^+}^{\frac{1}{2}} w\right)^3\right), & \left(\gimel, w(\gimel), {}^H D_{1^+}^{\frac{1}{2}} w(\gimel)\right)\in(1,e)\times[0,1]\times[0,1],\\ \frac{255^3}{3\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}}, & \left(\gimel, w(\gimel), {}^H D_{1^+}^{\frac{1}{2}} w(\gimel)\right)\in(1,e)\times(1,+\infty)\times(1,+\infty),\end{cases}\end{aligned}$$

$$\begin{aligned}&\widetilde{\mathfrak{R}}\left(\gimel, z(\gimel), {}^H D_{1^+}^{\frac{1}{2}} z(\gimel)\right)\\ = &\begin{cases}\frac{1}{3\pi(\ln\gimel)^{\frac{2}{3}}(1-\ln\gimel)^{\frac{2}{3}}}\left(z^3+\left({}^H D_{1^+}^{\frac{1}{2}} z\right)^3\right), & \left(\gimel, z(\gimel), {}^H D_{1^+}^{\frac{1}{2}} z(\gimel)\right)\in(1,e)\times[0,1]\times[0,1],\\ \frac{255^3}{3\pi(\ln\gimel)^{\frac{2}{3}}(1-\ln\gimel)^{\frac{2}{3}}}, & \left(\gimel, z(\gimel), {}^H D_{1^+}^{\frac{1}{2}} z(\gimel)\right)\in(1,e)\times(1,+\infty)\times(1,+\infty),\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\widehat{\mathfrak{R}}\left(\gimel, v(\gimel), {}^H D_{1^+}^{\frac{1}{2}} v(\gimel)\right)\\ = &\begin{cases}\frac{1}{3\pi(\ln\gimel)^{\frac{3}{4}}(1-\ln\gimel)^{\frac{3}{4}}}\left(v^3+\left({}^H D_{1^+}^{\frac{1}{2}} v\right)^3\right), & \left(\gimel, v(\gimel), {}^H D_{1^+}^{\frac{1}{2}} v(\gimel)\right)\in(1,e)\times[0,1]\times[0,1],\\ \frac{255^3}{3\pi(\ln\gimel)^{\frac{3}{4}}(1-\ln\gimel)^{\frac{3}{4}}}, & \left(\gimel, v(\gimel), {}^H D_{1^+}^{\frac{1}{2}} v(\gimel)\right)\in(1,e)\times(1,+\infty)\times(1,+\infty).\end{cases}\end{aligned}$$

Now, to check reveal the influence of $\mathfrak{R}$, we simplify the expression of $\mathfrak{R}$ as

$$\mathfrak{R}(\gimel,\varkappa,\varpi)=\frac{1}{3\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}}\left(\varkappa^3+\varpi^3\right)\ \text{for}\ (\gimel,\varkappa,\varpi)\in(1,e)\times[0,1]\times[0,1],$$

and draw the function $\mathfrak{R}$ in Figures 1 and 2.

Figures 1 and 2 visually represent the singular nonlinear terms of $\mathfrak{R}$. Given the similarity between $\mathfrak{R}$, $\widetilde{\mathfrak{R}}$ and $\widehat{\mathfrak{R}}$, we omit the influence of $\widetilde{\mathfrak{R}}$ and $\widehat{\mathfrak{R}}$ for brevity. Notably, the nonlinearity exhibits singularities at $\gimel=1$ and e. Despite these significant singularities, and in order to guarantee the stability and robustness of the equation's solutions, the nonlinear term can still be controlled by an integrable function.

By simple calculations, we have

$$\nabla=\widetilde{\nabla}=\widehat{\nabla}=1-\sum_{k=1}^{\infty}\lambda_k\ln(\rho_k)^{\zeta-1}=1-\sum_{k=1}^{\infty}\frac{1}{3k^2}\ln\left(e^{\frac{1}{k^3}}\right)^{\frac{1}{2}}=1-\sum_{k=1}^{\infty}\frac{1}{6k^2k^3}\approx 1-\frac{1}{6}\frac{\pi^5}{933}\approx 0.94533,$$

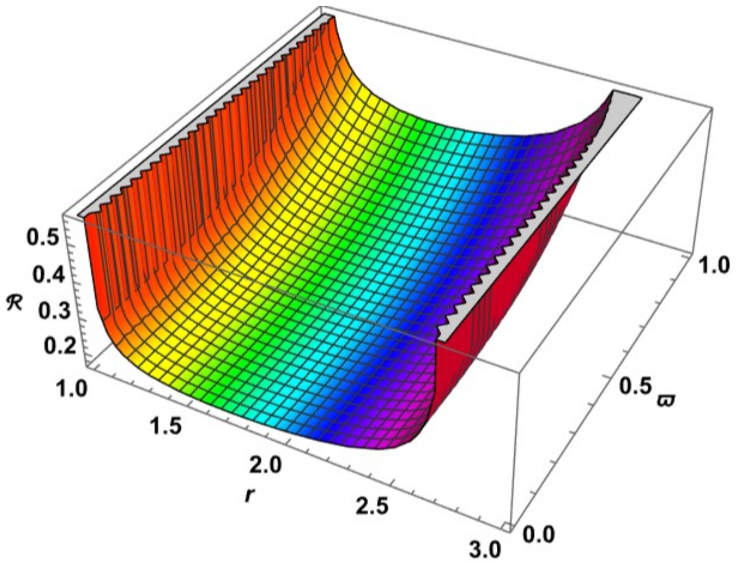

Figure 1. The singularities of $\mathfrak{R}$ at $r = 1$ and $r = e$ in the case of $\varkappa = 1$.

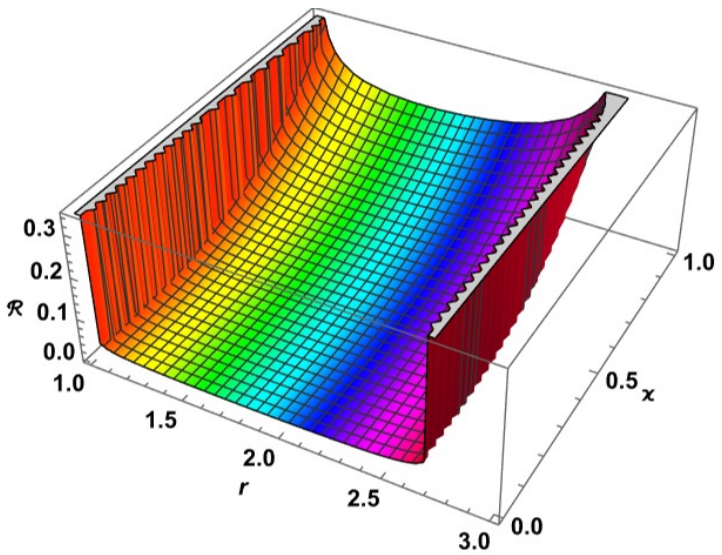

Figure 2. The singularities of $\mathfrak{R}$ at $r = 1$ and $r = e$ in the case of $\varkappa = 0.5$.

$$\begin{aligned}\nabla_1 &= \widetilde{\nabla}_1 = \widehat{\nabla}_1 = \nabla - \int_1^e g(\beth)\,(\ln \beth)^{\zeta-1}\, dG(\beth)\\ &= \nabla - \int_1^e (\ln \beth)^{\frac{5}{6}}\, dG(\beth) = 0.94533 - \left[3\left(\ln \frac{e}{3}\right)^{\frac{5}{6}} - \left(\ln \frac{2e}{3}\right)^{\frac{5}{6}}\right]\\ &\approx 2.7368,\end{aligned}$$

$$Z = \widetilde{Z} = \widehat{Z} = 1 + \frac{1}{\nabla_1}\int_1^e g(\beth)dG(\beth) = 1 + \frac{1}{2.7368}\int_1^e (\ln \beth)^{\frac{5}{6}}\, dG(\beth) \approx 0.5632,$$

and, apparently,

$$\begin{cases} \Re\left(\beth, w(\beth), {}^H D_{1^+}^{\frac{1}{2}} w(\beth)\right) \le \frac{1}{\pi(\ln \beth)^{\frac{1}{3}}(1-\ln \beth)^{\frac{1}{3}}} = \sigma(\beth),\\ \widetilde{\Re}\left(\beth, z(\beth), {}^H D_{1^+}^{\frac{1}{2}} z(\beth)\right) \le \frac{1}{\pi(\ln \beth)^{\frac{2}{3}}(1-\ln \beth)^{\frac{2}{3}}} = \widetilde{\sigma}(\beth),\\ \widehat{\Re}\left(\beth, v(\beth), {}^H D_{1^+}^{\frac{1}{2}} v(\beth)\right) \le \frac{1}{\pi(\ln \beth)^{\frac{3}{4}}(1-\ln \beth)^{\frac{3}{4}}} = \widehat{\sigma}(\beth). \end{cases}$$

Taking $\wp = \frac{2}{3}$, $\beth_0 = \frac{4}{3}$, then for $(\beth, \varkappa, \varpi) \in [1, e] \times [0, \frac{2}{3}] \times [0, \frac{2}{3}]$, we have

$$\begin{cases} L_4\left(\Re\left(\beth, w(\beth), {}^H D_{1^+}^{\alpha} w(\beth)\right)\right) = \frac{\wp}{(\ln \beth_0)^{\zeta-1} Z_1}\left(\int_{\beth_0}^e \eth_1(\beth_0, \ell)\,\frac{d\ell}{\ell}\right)^{-1} < +\infty,\\ L_3\left(\widetilde{\Re}\left(\beth, z(\beth), {}^H D_{1^+}^{\alpha} z(\beth)\right)\right) = \frac{\wp}{(\ln \beth_0)^{\gamma-1} \widetilde{Z}_1}\left(\int_{\beth_0}^e \widetilde{\eth}_1(\beth_0, \ell)\,\frac{d\ell}{\ell}\right)^{-1} < +\infty,\\ L_2\left(\widehat{\Re}\left(\beth, v(\beth), {}^H D_{1^+}^{\alpha} v(\beth)\right)\right) = \frac{\wp}{(\ln \beth_0)^{\beta-1} \widehat{Z}_1}\left(\int_{\beth_0}^e \eth_1(\beth_0, \ell)\,\frac{d\ell}{\ell}\right)^{-1} < +\infty. \end{cases}$$

Further, for $(\ell, \varkappa, \varpi) \in [1, e] \times [0, \upsilon] \times [0, \upsilon]$, we have

$$\begin{aligned} & Z\int_1^e \max\left\{\eth(e,\ell), {}^H D_{1^+}^{\alpha}\eth(e,\ell)\right\} L_{h_1}(\sigma(\ell))\frac{d\ell}{\ell}\\ =\ & Z\int_1^e \max\left\{\eth(e,\ell), {}^H D_{1^+}^{\alpha}\eth(e,\ell)\right\}\left(\frac{1}{\pi^{\frac{2}{3}}(\ln \ell)^{\frac{2}{9}}(1-\ln \ell)^{\frac{2}{9}}}\right)\frac{d\ell}{\ell}\\ \le\ & \frac{1}{\pi^{\frac{2}{3}}\Gamma(\zeta)} Z\int_1^e \frac{1}{(\ln \ell)^{\frac{2}{9}}(1-\ln \ell)^{\frac{2}{9}}}\frac{d\ell}{\ell}\\ =\ & \frac{0.5632}{\pi^{\frac{2}{3}}\Gamma(1.5)} Beta\left(\frac{11}{9}, \frac{11}{9}\right) \approx 0.1927 < 1, \end{aligned}$$

$$\widetilde{Z}\int_1^e \max\left\{\widetilde{\eth}(e,\ell), {}^H D_{1^+}^{\alpha}\widetilde{\eth}(e,\ell)\right\} L_{h_2}(\widetilde{\sigma}(\ell))\frac{d\ell}{\ell}$$

$$
\begin{aligned}
&= \widetilde{Z}\int_{1}^{e} \max\left\{\widetilde{\backepsilon}\left(e,\ell\right), {}^{H}D_{1^{+}}^{\alpha}\widetilde{\backepsilon}\left(e,\ell\right)\right\}\left(\frac{1}{\pi^{\frac{1}{2}}\left(\ln \ell\right)^{\frac{1}{3}}\left(1-\ln \ell\right)^{\frac{1}{3}}}\right)\frac{d\ell}{\ell}\\
&= \frac{1}{\pi^{\frac{1}{2}}\Gamma(\gamma)}\widetilde{Z}\int_{1}^{e}\frac{1}{\left(\ln \gimel\right)^{\frac{1}{3}}\left(1-\ln \gimel\right)^{\frac{1}{3}}}\frac{d\ell}{\ell}\\
&= \frac{0.5632}{\pi^{\frac{1}{2}}\Gamma(1.5)}Beta\left(\frac{4}{3},\frac{4}{3}\right)\approx 0.4281<1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\widehat{Z}\int_{1}^{e} \max\left\{\widehat{\backepsilon}\left(e,\ell\right), {}^{H}D_{1^{+}}^{\alpha}\widehat{\backepsilon}\left(e,\ell\right)\right\}L_{h_3}(\widehat{\sigma}(\ell))\frac{d\ell}{\ell}\\
&= \widehat{Z}\int_{1}^{e} \max\left\{\widehat{\backepsilon}\left(e,\ell\right), {}^{H}D_{1^{+}}^{\alpha}\widehat{\backepsilon}\left(e,\ell\right)\right\}\left(\frac{1}{\pi^{\frac{3}{2}}\left(\ln \gimel\right)^{\frac{9}{8}}\left(1-\ln \gimel\right)^{\frac{9}{8}}}\right)\frac{d\ell}{\ell}\\
&= \frac{1}{\pi^{\frac{3}{2}}\Gamma(\beta)}\widehat{Z}\int_{1}^{e}\frac{1}{\frac{3}{2}\left(\ln \gimel\right)^{\frac{9}{8}}\left(1-\ln \gimel\right)^{\frac{9}{8}}}\frac{d\ell}{\ell}\\
&= \frac{0.5632}{\pi^{\frac{3}{2}}\Gamma(1.5)}Beta\left(\frac{17}{8},\frac{17}{8}\right)\approx 0.0974<1.
\end{aligned}
$$

Therefore, all requirements of Theorem 5.1 are satisfied. Hence, the problem (6.1) has a positive solution.

Example 6.2. Consider the BVS (6.1) with

$$
\begin{cases}
\Re\left(\gimel, w(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}w(\gimel)\right) = \frac{1}{3\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}}\left(w^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}w\right)^{3}\right)^{3},\\
\widetilde{\Re}\left(\gimel, z(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}z(\gimel)\right) = \frac{1}{3\pi(\ln\gimel)^{\frac{2}{3}}(1-\ln\gimel)^{\frac{2}{3}}}\left(z^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}z\right)^{3}\right)^{2},\\
\widehat{\Re}\left(\gimel, v(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}v(\gimel)\right) = \frac{1}{3\pi(\ln\gimel)^{\frac{3}{4}}(1-\ln\gimel)^{\frac{3}{4}}}\left(v^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}v\right)^{3}\right).
\end{cases}
$$

Hence, for all $(\gimel, v, w, z)\in[1,e]\times\mathbb{R}_{+}^{1}\times\mathbb{R}_{+}^{1}\times\mathbb{R}_{+}^{1}$, we conclude that

$$
\begin{cases}
\Re\left(\gimel, w(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}w(\gimel)\right)\leq \sigma(\gimel)\vartheta^{\frac{1}{h_1-1}},\\
\widetilde{\Re}\left(\gimel, z(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}z(\gimel)\right)\leq \widetilde{\sigma}(\gimel)\kappa^{\frac{1}{h_2-1}},\\
\widehat{\Re}\left(\gimel, v(\gimel), {}^{H}D_{1^{+}}^{\frac{1}{2}}v(\gimel)\right)\leq \widehat{\sigma}(\gimel)\varrho^{\frac{1}{h_3-1}},
\end{cases}
$$

where

$$
\begin{cases}
\sigma(\gimel) = \frac{1}{3\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}},\ \vartheta(w) = w^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}w\right)^{3},\\
\widetilde{\sigma}(\gimel) = \frac{1}{3\pi(\ln\gimel)^{\frac{2}{3}}(1-\ln\gimel)^{\frac{2}{3}}},\ \kappa(z) = z^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}z\right)^{3},\\
\widehat{\sigma}(\gimel) = \frac{1}{3\pi(\ln\gimel)^{\frac{3}{4}}(1-\ln\gimel)^{\frac{3}{4}}},\ \varrho(v) = v^{3}+\left({}^{H}D_{1^{+}}^{\frac{1}{2}}v\right)^{3}.
\end{cases}
$$

Therefore, the condition (A_4) is true. Taking $M = 1 > 0$,

$$\max\left\{\begin{array}{l}\int\limits_1^e \eth_1(e,t)\, ZL_{h_1}(\sigma(t))\,\frac{dt}{t},\\ \int\limits_1^e \widetilde{\eth}_1(e,t)\,\widetilde{Z}L_{h_2}(\widetilde{\sigma}(t))\,\frac{dt}{t},\\ \int\limits_1^e \widehat{\eth}_1(e,t)\,\widehat{Z}L_{h_3}(\widehat{\sigma}(t))\,\frac{dt}{t}\end{array}\right\} = 0.4281 < 1 = \frac{M}{\max\{\vartheta(M),\kappa(M),\varrho(M)\}}.$$

Hence, the condition (A_5) holds. Therefore, all hypotheses of Theorem 5.2 are fulfilled. Thus, the problem (6.1) has at least one positive solution.

Example 6.3. Consider the BVS (6.1) with

$$\begin{cases} \mathfrak{R}\left(\gimel, w(\gimel),^{H} D^{\frac{1}{2}}_{1^+} w(\gimel)\right) = \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left(w + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}w\right)\right)^3,\\ \widetilde{\mathfrak{R}}\left(\gimel, z(\gimel),^{H} D^{\frac{1}{2}}_{1^+} z(\gimel)\right) = \pi(\ln\gimel)^{\frac{2}{3}}(1-\ln\gimel)^{\frac{2}{3}}\left(z + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}z\right)\right)^2,\\ \widehat{\mathfrak{R}}\left(\gimel, v(\gimel),^{H} D^{\frac{1}{2}}_{1^+} v(\gimel)\right) = \pi(\ln\gimel)^{\frac{3}{4}}(1-\ln\gimel)^{\frac{3}{4}}\left(v + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}v\right)\right). \end{cases}$$

By simple calculations, we can write

$$\begin{aligned} &\left|L_{h_1}\left(\mathfrak{R}\left(\gimel, w_1(\gimel),^{H} D^{\frac{1}{2}}_{1^+} w_2(\gimel)\right)\right) - L_{h_1}\left(\mathfrak{R}\left(\gimel, w_2(\gimel),^{H} D^{\frac{1}{2}}_{1^+} w_2(\gimel)\right)\right)\right|\\ = &\left|\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left(w_1 + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}w_1\right)\right) - \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left(w_2 + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}w_2\right)\right)\right|\\ \le &\ \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left[|w_1 - w_2| + 2\left|{}^{H}D^{\frac{1}{2}}_{1^+}w_1 -^{H} D^{\frac{1}{2}}_{1^+}w_2\right|\right], \end{aligned}$$

where $N_1 = 1$, $N_1 = 2$, $\sigma(\gimel) = \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}$. Also,

$$\begin{aligned} &\left|L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\gimel, z_1(\gimel),^{H} D^{\frac{1}{2}}_{1^+} z_1(\gimel)\right)\right) - L_{h_2}\left(\widetilde{\mathfrak{R}}\left(\gimel, z_2(\gimel),^{H} D^{\frac{1}{2}}_{1^+} z_2(\gimel)\right)\right)\right|\\ = &\left|\pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left(z_1 + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}z_1\right)\right) - \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left(z_2 + 2\left({}^{H}D^{\frac{1}{2}}_{1^+}z_2\right)\right)\right|\\ \le &\ \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}\left[|z_1 - z_2| + 2\left|{}^{H}D^{\frac{1}{2}}_{1^+}z_1 -^{H} D^{\frac{1}{2}}_{1^+}z_2\right|\right], \end{aligned}$$

where $\widetilde{N}_1 = 1$, $\widetilde{N}_2 = 2$, $\widetilde{\sigma}(r) = \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}$. Similarly, we have $\widehat{N}_1 = 1$, $\widehat{N}_2 = 2$, $\widehat{\sigma}(r) = \pi(\ln\gimel)^{\frac{1}{3}}(1-\ln\gimel)^{\frac{1}{3}}$.

Furthermore,

$$\begin{cases} \frac{1}{\Gamma(\zeta)}Z\left(N_1\sigma(\ell) + N_2\sigma(\ell)\right) \approx 0.7365 < 1,\\ \frac{1}{\Gamma(\gamma)}\widetilde{Z}\left(\widetilde{N}_1\widetilde{\sigma}(\ell) + \widetilde{N}_2\widetilde{\sigma}(\ell)\right) \approx 0.7365 < 1,\\ \frac{1}{\Gamma(\beta)}\widehat{Z}\left(\widehat{N}_1\widehat{\sigma}(\ell) + \widehat{N}_2\widehat{\sigma}(\ell)\right) \approx 0.7365 < 1, \end{cases}$$

which implies that

$$N = \max\left\{\frac{\sigma(\ell)}{\Gamma(\zeta)}Z(N_1 + N_2), \frac{\widetilde{\sigma}(\ell)}{\Gamma(\gamma)}\widetilde{Z}\left(\widetilde{N}_1 + \widetilde{N}_2\right), \frac{\widehat{\sigma}(\ell)}{\Gamma(\beta)}\widehat{Z}\left(\widehat{N}_1 + \widehat{N}_2\right)\right\} = 0.7365 < 1.$$

Therefore, all hypotheses of Theorem 5.3 are fulfilled. Hence, the BVS (6.1) has a unique solution.

7. Abbreviations

DE⟹differential equation.
HF⟹Hadamard fractional.
FP⟹fixed point.
RS⟹Riemann-Stieltjes.
BVS⟹boundary value system.
CPC⟹completely continuous.
BCP⟹Banach contraction principle.

8. Conclusions and future work

This paper rigorously investigated the existence and uniqueness of positive solutions for a singular p-Laplacian hybrid fractional DE. The study considered this equation under infinite-point boundary conditions and nonlocal integrals. Utilizing the Leray-Schauder-type Guo-Krasnoselskii FP technique and the BCP in the framework of cones, the study first derived and analyzed the properties of the Green's function. A key contribution was the successful demonstration of solution existence for singular nonlinearities, which represented a significant generalization over prior works. The uniqueness of these positive solutions was then conclusively established. This explicit inclusion of singular nonlinearities, despite inherent analytical challenges, revealed that solutions remained stable and dependable under appropriate integrability conditions.

Future research could explore the impact of individual factors on engineering applications, focusing on solution durability and stability. Additionally, researchers could investigate the existence and multiplicity of solutions with various nonlinearities (e.g., critical growth or sign-changing behavior), analyze the influence of varying fractional orders and infinite-point boundary conditions, and extend these findings to more general operators like those with variable exponents or nonlocal terms beyond the fractional Laplacian. Developing numerical methods and computational simulations to approximate and visualize these positive solutions would also offer practical insights for scientific and engineering applications.

Author contributions

Each author made a substantial and equal contribution to the writing of this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors confirm that they have no conflict of interest.

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