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Research article

Intelligent decision framework for optimal spacecraft shielding material against cosmic radiation under IV-q-RPF settings

Loredana Ciurdariu[1], Najam Ul Sahar[2,*] and Ahmed M. Zidan[3]

[1] Department of Mathematics, Politehnica University of Timisoara, 300006 Timisoara, Romania
[2] Department of Mathematics, Government College University Faisalabad, Pakistan
[3] Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia

* **Correspondence:** Email: 6704najam@gmail.com.

Abstract: This research presents two new Yager's ordered weighted aggregation operators using interval-valued q-rung picture fuzzy (IV-q-RPF) knowledge, namely the interval-valued q-rung picture fuzzy Yager ordered weighted averaging operator (IV-q-RPFYOWAO) and the interval-valued q-rung picture fuzzy Yager ordered weighted geometric operator (IV-q-RPFYOWGO). A pair of novel score and accuracy functions for interval-valued q-rung picture fuzzy numbers (IV-q-RPFNs) is formulated. A step-by-step process is designed to solve multi-attribute decision-making (MADM) problems using the proposed methods in IV-q-RPF settings. In addition, these methods are efficiently applied to solve the MADM problem of identifying an optimal spacecraft shielding material against cosmic radiation. A detailed comparative study is presented to illustrate the validity of the suggested techniques in comparison with the existing knowledge.

Keywords: interval-valued q-rung picture fuzzy sets; Yager operational laws; interval-valued q-rung picture fuzzy Yager ordered weighted averaging/geometric operators; multi-attribute decision-making
Mathematics Subject Classification: 03E72, 94D05

1. Introduction

Table 1 describes the list of abbreviations used in this work.

Table 1. List of abbreviations.

Description	Abbreviation	Description	Abbreviation	Description	Abbreviation
Aggregation operators	AOs	Interval-valued fuzzy set	IV-FS	Picture fuzzy set	PFS
Multi-attribute decision-making	MADM	Intuitionistic fuzzy set	IFS	Interval-valued picture fuzzy set	IV-PFS
Interval-valued q-rung picture fuzzy sets	IV-q-RPFSs	Interval-valued intuitionistic fuzzy set	IV-IFS	Interval-valued picture fuzzy Aczel-Alsina weighted aggregation operators	IV-PFAAWAOs
Interval-valued q-rung picture fuzzy Yager weighted averaging operator	IV-q-RPFYWAO	Pythagorean fuzzy set	PyFS	Spherical fuzzy sets	SFS
Interval-valued q-rung picture fuzzy Yager weighted geometric operator	IV-q-RPFYWGO	q-rung ortho-pair fuzzy set	q-ROFS	Interval-valued spherical fuzzy set	IV-SFS
Interval-valued q-rung picture fuzzy numbers	IV-q-RPFNs	Interval-valued q-rung ortho-pair fuzzy set	IV-q-ROFS	Interval-valued spherical fuzzy Dombi weighted aggregation operators	IV-SFDWAOs
Decision-making	DM	p,q-quasirung ortho-pair fuzzy hybrid aggregation	p,q-QOFHA	q-rung picture fuzzy set	q-RPFS
Fuzzy set	FS	q-rung ortho-pair fuzzy hyper soft set	q-ROFHS	(p,q,r)-spherical fuzzy sets	(p,q,r)-SFSs

Table 2 describes the list of symbols used in this study and their meanings.

Table 2. List of symbols.

Description	Symbols
Membership	α
Neutral	β
Non-membership	ε
Operational parameter	τ
q-rung	q

1.1. Background and literature review

Decision-making (DM) has gained prominence in the past few decades as an essential aspect of solving complex challenges. Multi-attribute decision-making (MADM) techniques help organizations evaluate multiple factors to choose the optimal solutions. These methods require clear trade-offs and evaluations of both internal and external attributes. Traditional DM methods using crisp set theory struggled with vague data, prompting researchers to create mathematical models that would be applicable in engineering, medical diagnostics, and technology. In 1965, Zadeh [1] developed fuzzy sets (FSs) to represent human judgment through membership functions. Yager [2] proposed aggregation operators (AOs) based upon the basic concepts of Zadeh's seminal work to enable better integration of diverse fuzzy variables. Turksen [3] introduced interval-valued fuzzy sets (IV-FSs). Atanassov [4] proposed the idea of intuitionistic fuzzy sets (IFSs). Xu [5] investigated intuitionistic fuzzy aggregation operators (IFAOs). Zhao et al. [6] formulated generalized AOs for IFSs. Tan and colleagues [7,8] developed generalized geometric AOs within the framework of IFS knowledge. Chen [9] described a MADM approach employing interval-valued intuitionistic fuzzy sets (IV-IFSs). Chen [10] considered the subject of a MADM technique with IV-IFSs. Yager [11] introduced an improved framework of Pythagorean fuzzy sets (PyFSs). The PyFS model has been widely applied in numerous fields [12−14]. Yager [15] put forward the notion of q-rung ortho-pair fuzzy sets (q-ROFSs), whereby the sum of the qth powers of the MD (membership degree) and NMD (non-membership degree) is less than or equal to 1. Liu and Wang [16] explored q-ROF weighted AOs (WAOs). Liu and Liu [17] extended the concept of Bonferroni mean WAOs to q-ROF information. A multi-attribute group decision-making (MAGDM) approach with q-ROF power Maclaurin AOs was devised by [18]. The Dombi AOs for q-ROFS were described by Jana et al. [19]. A fuzzy interpretation of MAGDM build upon neutrality operators of q-ROFSs was introduced in [20]. Garg et al. [21] introduced power AOs and Vlse Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR) methodologies for intricate q-ROFSs. A complex DM structure based on q-ROFSs for optimization of evaluative approaches was investigated in [22]. Ali [23] proposed a norm-based distance metric for q-ROFSs. Joshi et al. [24] investigated the key properties of interval-valued q-ROFSs (IV-q-ROFSs). A stratified improved interval-valued q-rung ortho-pair picture bipolar fuzzy (IV-q-ROPBF) DM technique to select the method of solid waste disposal, was proposed in a study by Parthasarathy et al. [25].

Cuong [26] introduced the picture fuzzy set (PFS) framework which incorporates three elements–yes, abstain, and no–to better represent uncertainty. Cuong and Kreinovich [27] further explored PFS applications in computational intelligence. Garg [28] contributed by developing various AOs within the PFS context. In [29], novel similarity and distance metrics for PFSs were proposed. The Einstein AOs in PFS environments was introduced in [30]. Notably, Jana et al. [31] designed Dombi AOs using PFSs. In [32], new operations on IV-PFS and IV picture fuzzy soft sets (IV-PFSSs) were proposed. The interval-valued picture fuzzy Aczel-Alsina (IV-PFAA) AOs were investigated in [33]. Enhanced artificial intelligence models with IV-PFSs and Sugeno-Weber triangular norms were investigated in [34]. Gundogdu and Kahraman [35] presented the concept of spherical fuzzy sets (SFSs). The successful applications of SFSs across various fields can be found in [36–39]. The idea of interval-valued SFSs (IV-SFSs) was developed in [40]. The efficiency of MADM methodologies based on interval-valued T-spherical fuzzy aggregation operators (IV-TSFAOs) for investment policy assessments was demonstrated in [41]. The importance of IV-SF Dombi strategies was discussed in [42].

The introduction of the q-rung picture fuzzy set (q-RPFS), in which the q-th powers of MD and NMD can reach a maximum of one, was presented by Li et al. [43]. A novel q-RPF methodology was devised for group DM scenarios within q-RPF contexts in [44]. Analysis of solid waste segregation based on artificial intelligence technologies through MADM and complex q-RPF frank aggregation operators were addressed in [45]. Garg et al. [46] explored the process of integrating industry technologies into logistic management within the industrial sector, employing hybrid q-RPF DM methods. Khan and Ahmed [47] introduced MCGDM based on 2-tuple linguistic q-rung picture fuzzy sets. The DM situations were addressed with intricacy under ambiguity using several types of q-rung picture fuzzy Yager aggregation operators (q-RPFYAOs) in [48]. In addition, Cuong [49] presented the concept of a Pythagorean picture fuzzy set (PyPFS).

Jiang et al. [50] introduced a comprehensive DM framework for large groups, utilizing a rough integrated asymmetric cloud model inside a multi-granularity linguistic context to accurately capture diverse expert evaluations. Liu et al. [51] employed a case-based reasoning methodology for the categorization and detection of medical insurance fraud, underscoring the increasing application of intelligent DM approaches in intricate real-world issues. Enhancing these advancements, our research introduces IV-q-RPF Yager AOs, which offer a versatile and comprehensible framework for intricate MADM in uncertainty.

The interval-valued q-RPFS (IV-q-RPFS) was established in [52]. Jia and Jia [53] suggested a novel method for the estimation of the dependability of a ship's equipment using symbolic information integrated with IV-q-RPF projection methods. Yang et al. [54] proposed an innovative cognitive information-based DM system using IV-q-RPFSs and Heronian mean operators. Shahzadi et al. [55] studied the industrial risks under the environment of IV-q-RPFSs.

1.2. Literature gaps and motivations of the current study

In everyday DM, data often lacks clarity and uniformity. Conventional models like IFSs, PyFSs, or q-ROFSs cannot be applied because they depend on single-valued inputs. Even advanced models such as q-RPFSs fail to fully capture the uncertainty in expert judgment. On the other hand, IV-q-RPFSs address this issue by allowing the use of interval-based degrees. To illustrate this limitation, consider a simple decision problem of evaluating cybersecurity. Suppose that the experts must assess a new security system. One expert states that the system is mostly reliable but with some uncertainty, another gives a neutral response, while a third is hesitant because of missing data. If represented using single values, this uncertainty is lost. By contrast, IV-q-RPFSs can express each degree (MD, Neutral membership degree (NeD), and NMD) as intervals, e.g., MD $\in [0.55,0.75]$, NeD $\in [0.15,0.25]$, NMD $\in [0.05,0.15]$, while the q-parameter adjusts the strictness of evaluation. This example illustrates the superior flexibility of IV-q-RPFSs in modelling both the expert's uncertainty and varying attitudes toward uncertainty. They present a more adaptable framework that improves both the accuracy and flexibility of judgments. It is especially helpful in complex and ambiguous situations.

IV-q-RPFSs offer a robust framework for modelling DM scenarios under significant uncertainty by representing MD, NeD, and NMD as intervals rather than exact values. This framework accurately captures both the expert's uncertainty and measurement imprecision, providing a more realistic representation of real-world scenarios. The q-parameter increases flexibility by controlling the feasible range of degrees, where larger q values allow higher flexibility for uncertainty and smaller q values impose stricter evaluations. This flexibility allows IV-q-RPFSs to handle a wide range of decision-maker attitudes toward risk. When compared with classical fuzzy, intuitionistic fuzzy (IF), or Pythagorean fuzzy (PF) models, the use of interval representation and the q-rung structure make the model more expressive and robust. These characteristics make the IV-q-RPFSs very appropriate in solving complex MADM tasks in fields like medicine, cybersecurity, engineering design, and environmental management.

AOs play a vital role in DM by combining varied information and enabling evaluations under uncertainty. Numerous AOs, including Dombi, Einstein, Hamacher, and Heronian AOs, have been suggested; however, their dependence on inflexible numerical frameworks frequently constrains their interpretability and adaptability in qualitative or unpredictable scenarios. Yager AOs are preferable, as they provide an intuitive parameter that incorporates conjunctive and disjunctive behaviors, providing decision-makers with enhanced flexibility. They also have advantageous theoretical attributes, like commutativity, associativity, idempotency, and monotonicity, while ensuring consistency in both extreme and intermediate scenarios. Moreover, Yager operators adeptly handle fuzzy and interval-valued information, overcoming the over- or underestimating seen in various alternative methods, and their alignment with ordered weighted aggregation (OWA)-type approaches renders them especially potent in MADM under uncertainty. Consequently, we expand the IV-q-RPF framework to incorporate novel Yager-based operators, as they offer the most comprehensible and versatile solution for tackling intricate DM challenges in this study.

The novelty of the proposed interval-valued q-rung picture fuzzy Yager ordered weighted averaging (IV-q-RPFYOWA) operators stems from their ability to combine both the expressive power of IV-q-RPFSs and the mechanism of OWA as proposed by Yager. This combination effectively addresses the limitations of existing IV-q-RPF operators that are mainly based on arithmetic or geometric aggregation. Unlike conventional operators, the parameter-based aggregation functionality of interval-valued q-rung picture fuzzy Yager ordered weighted aggregation operators (IV-q-RPFYOWAOs) enables decision-makers to regulate the balance between optimism and pessimism, which gives them fine-grained control over the integration of information. This dynamic adjustment makes it adaptable in varying DM situations. The incorporation of the Yager concept into the IV-q-RPF framework creates a powerful trade-off between accuracy and flexibility, allowing better treatment of interval uncertainty and multi-dimensional assessments. This innovation greatly improves current practices by addressing the shortcomings and enhancing the modelling of uncertainty, thus making a significant contribution to decision science. The above discussion prompts us to present two new Yager ordered weighted aggregation operators (YOWAOs) in the IV-q-RPF framework and the formulation of a new mathematical process to solve MADM problems with the help of these operators in this article.

Cosmic radiation poses a serious threat to human missions beyond Earth. Astronauts' health can suffer, and spacecraft equipment can be damaged. In the absence of the shielding effect of Earth's atmosphere, space travellers will be subjected to high-energy protons, heavy ions, and subatomic particles from cosmic rays and solar events. This kind of exposure may cause cancer, neurological disorders, and acute radiation syndrome. It may also destroy electronic systems, exposing a mission to failure. Conventional materials used in spacecraft, e.g. aluminium alloys, provide limited protection. They are not very useful against heavy ions, which are the most harmful particles. In addition, their dense nature makes them too heavy and expensive to carry on long-term missions.

Researchers are investigating new lightweight materials that exhibit good mechanical strength, thermal resistance, and good radiation shielding. Polyethylene contains a high proportion of hydrogen and has been demonstrated to be more effective than aluminium in the absorption of cosmic rays, and is therefore useful in weight-sensitive missions. Nevertheless, it cannot be used in primary spacecraft construction because of its structural weakness. To alleviate this shortcoming new composite materials, particularly those reinforced with boron nitride nanotubes, are more durable and radiation resistant. Furthermore, multi-layer shielding systems contribute to the overall safety. Such developments are essential to secure and sustained human presence on the Moon and Mars.

The key objectives of this research are as follows:

(a) To create a novel IV-q-RPFN ranking system that efficiently handles MADM issues;

(b) To delineate the essential Yager operational principles pertinent to IV-q-RPFNs;

(c) To present two innovative Yager OWAOs in the context of the IV-q-RPFs settings, tailored to handle complex and ambiguous decision data and analyze their structural features;

(d) To develop a step-by-step mathematical procedure for MADM by utilizing the proposed techniques executed in the IV-q-RPF context;

(e) To ensure the applicability and efficiency of the proposed methods by addressing a real-world MADM problem of selecting an optimal spacecraft shielding material against cosmic radiation;

(f) To perform an in-depth comparison study, particularly highlighting the effectiveness of recently suggested techniques relative to the existing ones.

The remaining portion of the paper is structured into several sections as follows. Section 2 discusses the essential concepts and rules of IV-q-RPFSs. Section 3 formulates innovative scoring and accuracy functions for addressing MADM issues in IV-q-RPF scenarios. Section 4 introduces two new YOWAOs within the IV-q-RPF framework and presents an analysis of their structural characteristics. Section 5 develops a mathematical framework to tackle MADM challenges using the newly proposed techniques. It includes the solution of the MADM problem of selecting an optimal spacecraft shielding material against cosmic radiation using IV-q-RPFYOWAOs. It also examines the effectiveness of these new methodologies with the existing knowledge. Section 6 concludes the study by outlining the impact that the research could have and summarizing the most important findings.

2. Preliminaries

This section explores the foundational facets of the subject presented in this article. We provide a concise overview of the basic attributes, operations, and methodologies pertaining to IV-q-RPFSs defined on a non-empty universal set.

Definition 2.1. *[43] A q-RPFS $\mathcal{L}$ of $\mathfrak{G}$ is expressed as*

$$\mathcal{L} = \{(g, \alpha_{\mathcal{L}}(g), \beta_{\mathcal{L}}(g), \varepsilon_{\mathcal{L}}(g)) | g \in \mathfrak{G}\}, \tag{2.1}$$

where $\alpha_{\mathcal{L}}: \mathfrak{G} \to [0,1]$, $\beta_{\mathcal{L}}: \mathfrak{G} \to [0,1]$, and $\varepsilon_{\mathcal{L}}: \mathfrak{G} \to [0,1]$ represent the membership, neutral, and non-membership functions, respectively, such that $0 \leq (\alpha_{\mathcal{L}}(g))^q + (\beta_{\mathcal{L}}(g))^q + (\varepsilon_{\mathcal{L}}(g))^q \leq 1$, $\forall g \in \mathfrak{G}$, and q is a positive integer.

Definition 2.2. *[54] Assume that $\mathfrak{G}$ represents the universe and $C([0,1])$ describes the collection of all subintervals of $[0,1]$. An IV-q-RPFS $\mathfrak{T}$ is defined as*

$$\mathfrak{T} = \{(g, \alpha_{\mathfrak{T}}(g), \beta_{\mathfrak{T}}(g), \varepsilon_{\mathfrak{T}}(g)) | g \in \mathfrak{G}\}, \tag{2.2}$$

where $\alpha_{\mathfrak{T}}(g) = [\alpha_{\mathfrak{T}}^L(g), \alpha_{\mathfrak{T}}^U(g)]$, $\beta_{\mathfrak{T}}(g) = [\beta_{\mathfrak{T}}^L(g), \beta_{\mathfrak{T}}^U(g)]$, and $\varepsilon_{\mathfrak{T}}(g) = [\varepsilon_{\mathfrak{T}}^L(g), \varepsilon_{\mathfrak{T}}^U(g)]$, respectively, represent the membership, neutral, and non-membership degrees of the element g to $\mathfrak{T}$

such that $0 \le \alpha_{\mathfrak{T}}{}^{L}(\mathcal{g}) < \alpha_{\mathfrak{T}}{}^{U}(\mathcal{g}) \le 1$, $0 \le \beta_{\mathfrak{T}}{}^{L}(\mathcal{g}) < \beta_{\mathfrak{T}}{}^{U}(\mathcal{g}) \le 1$, *and* $0 \le \varepsilon_{\mathfrak{T}}{}^{L}(\mathcal{g}) < \varepsilon_{\mathfrak{T}}{}^{U}(\mathcal{g}) \le 1$. *Moreover,* $\alpha_{\mathfrak{T}}: \mathfrak{G} \to \mathcal{C}([0,1])$, $\beta_{\mathfrak{T}}: \mathfrak{G} \to \mathcal{C}([0,1])$, *and* $\varepsilon_{\mathfrak{T}}: \mathfrak{G} \to \mathcal{C}([0,1])$ *are, respectively, the membership, neutral, and non-membership functions that satisfy the conditions* $0 \le (\alpha_{\mathfrak{T}}{}^{L}(\mathcal{g}))^q + (\beta_{\mathfrak{T}}{}^{L}(\mathcal{g}))^q + (\varepsilon_{\mathfrak{T}}{}^{L}(\mathcal{g}))^q \le 1$ *and* $0 \le (\alpha_{\mathfrak{T}}{}^{U}(\mathcal{g}))^q + (\beta_{\mathfrak{T}}{}^{U}(\mathcal{g}))^q + (\varepsilon_{\mathfrak{T}}{}^{U}(\mathcal{g}))^q \le 1, \forall \mathcal{g} \in \mathfrak{G}$, *and* q *is a positive integer.*

In the subsequent portion of the study, the membership, neutral, and non-membership degrees of $\mathcal{g} \in \mathfrak{G}$ *are represented by the symbol* $\mathfrak{T} = ([\alpha^L, \alpha^U], [\beta^L, \beta^U], [\varepsilon^L, \varepsilon^U])$. *This specific representation of the element* $\mathfrak{T}$ *is termed as an IV-q-RPFN, where* $0 \le \alpha^L, \beta^L, \varepsilon^L, (\alpha^L)^q + (\beta^L)^q + (\varepsilon^L)^q \le 1$, *and* $0 \le \alpha^U, \beta^U, \varepsilon^U, (\alpha^U)^q + (\beta^U)^q + (\varepsilon^U)^q \le 1$.

Definition 2.3. *[27] Consider any two IV-PFNs* $\mathfrak{K}_1 = ([\alpha_1{}^L, \alpha_1{}^U], [\beta_1{}^L, \beta_1{}^U], [\varepsilon_1{}^L, \varepsilon_1{}^U])$ *and* $\mathfrak{K}_2 = ([\alpha_2{}^L, \alpha_2{}^U], [\beta_2{}^L, \beta_2{}^U], [\varepsilon_2{}^L, \varepsilon_2{}^U])$. *The fundamental operations of* $\mathfrak{K}_1$ *and* $\mathfrak{K}_2$ *are described as follows:*

I. $\mathfrak{K}_1 < \mathfrak{K}_2$ *iff* $\alpha_1{}^L < \alpha_2{}^L$, $\beta_1{}^L > \beta_2{}^L$, $\varepsilon_1{}^L > \varepsilon_2{}^L$, $\alpha_1{}^U < \alpha_2{}^U$, $\beta_1{}^U > \beta_2{}^U$ and $\varepsilon_1{}^U > \varepsilon_2{}^U$;

II. $\mathfrak{K}_1 = \mathfrak{K}_2$ *iff* $\alpha_1{}^L = \alpha_2{}^L$, $\beta_1{}^L = \beta_2{}^L$, $\varepsilon_1{}^L = \varepsilon_2{}^L$, $\alpha_1{}^U = \alpha_2{}^U$, $\beta_1{}^U = \beta_2{}^U$ and $\varepsilon_1{}^U = \varepsilon_2{}^U$;

III. $\mathfrak{K}_1^c = ([\varepsilon_1{}^L, \varepsilon_1{}^U], [\beta_1{}^L, \beta_1{}^U], [\alpha_1{}^L, \alpha_1{}^U])$.

Definition 2.4. *[2] Yager's t-conorm and t-norm on any* $(m, n) \in [0,1]^2$ *and for any* $\tau \epsilon (0, \infty)$ *are given by*

I. $\mathcal{S}(m, n) = min\left\{1, (m^\tau + n^\tau)^{\frac{1}{\tau}}\right\}$.

II. $\mathcal{T}(m, n) = 1 - min\left\{1, ((1-m)^\tau + (1-n)^\tau)^{\frac{1}{\tau}}\right\}$.

Definition 2.5. *[48] Let* $\mathfrak{A}$ *be a set of q-RPFNs,* $\mathfrak{T}_i = \langle \alpha_i, \beta_i, \varepsilon_i \rangle$, $i = 1, 2, \dots, \hbar$, *and* $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_\hbar)^T$ *be an associated weight vector of these q-RPFNs* $\mathfrak{T}_i$ *with* $0 \le \vartheta_i \le 1$ *such that* $\sum_{i=1}^{\hbar} \vartheta_i = 1$. *The q-RPFYWA operator is a mapping:* $\mathfrak{A}^\hbar \to \mathfrak{A}$ *specified by the expression below:*

$$q - RPFYWA_\vartheta(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_\hbar) = \oplus_{i=1}^{\hbar} \vartheta_i \mathfrak{T}_i$$

$$= \left(\begin{matrix} \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i(\alpha_i)^{q\tau})\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(1-(\beta_i)^q)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(1-(\varepsilon_i)^q)^\tau\right)\right)^{\frac{1}{\tau}}\right)} \end{matrix} \right). \quad (2.3)$$

Moreover, the q-RPFYWG operator is a mapping: $\mathfrak{A}^\hbar \to \mathfrak{A}$ *specified by the expression below:*

$$q - RPFYWG_\vartheta(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_\hbar) = \otimes_{i=1}^{\hbar} \mathfrak{T}_i^{\vartheta_i}$$

$$= \left(\begin{matrix} \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(1-(\alpha_i)^q)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i(\beta_i)^{q\tau})\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i((\varepsilon_i)^{q\tau})\right)^{\frac{1}{\tau}}\right)} \end{matrix} \right) \quad (2.4)$$

The overall methodology workflow is depicted in Figure 1.

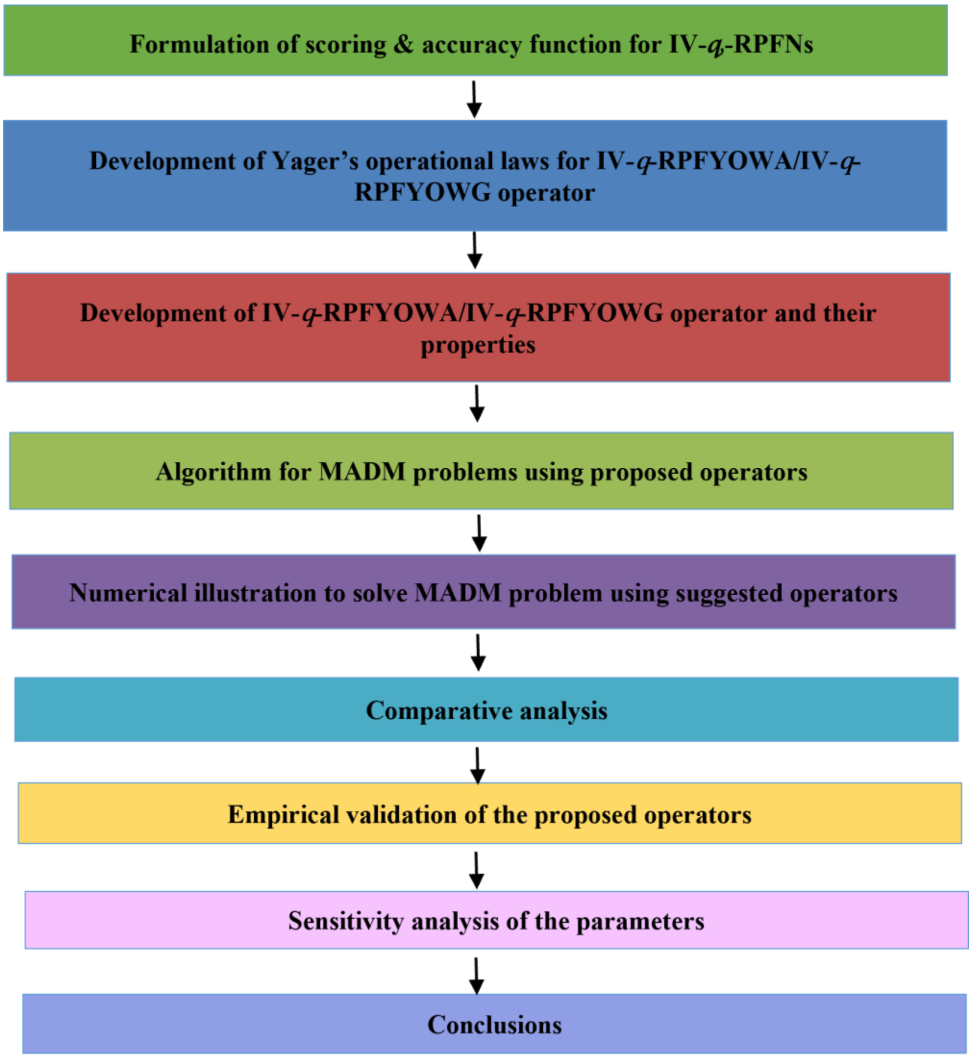

Figure 1. Overall methodology workflow.

3. Formulation of a novel ranking mechanism for IV-q-RPFNs

In this section, a pair of novel score and accuracy functions for IV-q-RPFNs is developed for MADM problems.

Definition 3.1. *Consider an IV-q-RPFN as* $\mathfrak{T} = ([\alpha^L, \alpha^U], [\beta^L, \beta^U], [\varepsilon^L, \varepsilon^U])$. *The score function* $\mathfrak{H}$ *of* $\mathfrak{T}$ *is formulated as*

$$\mathfrak{H}(\mathfrak{T}) = \frac{(\alpha^L)^q - (\beta^L)^q - (\varepsilon^L)^q + (\alpha^U)^q - (\beta^U)^q - (\varepsilon^U)^q}{2}, \text{ where } \mathfrak{H}(\mathfrak{T}) \in [-1,1].$$

And accuracy function is defined as

$$A(\mathfrak{T}) = \frac{(\alpha^L)^q + (\beta^L)^q + (\varepsilon^L)^q + (\alpha^U)^q + (\beta^U)^q + (\varepsilon^U)^q}{2}, \text{ where } A(\mathfrak{T}) \in [0,1].$$

This definition delineates the ranking criteria to any two IV-q-RPFNs $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *as follows:*

I. $\mathfrak{H}(\mathfrak{T}_1) > \mathfrak{H}(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 > \mathfrak{T}_2$, *which means that* $\mathfrak{T}_1$ *is stronger then* $\mathfrak{T}_2$*;*

II. $\mathfrak{H}(\mathfrak{T}_1) < \mathfrak{H}(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 < \mathfrak{T}_2$, *which means that* $\mathfrak{T}_1$ *is weaker then* $\mathfrak{T}_2$*;*

III. $\mathfrak{H}(\mathfrak{T}_1) = \mathfrak{H}(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 \sim \mathfrak{T}_2$, *which means that* $\mathfrak{T}_1$ *and* $\mathfrak{T}_2$ *are equivalent.*

Then if

(a) $A(\mathfrak{T}_1) > A(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 > \mathfrak{T}_2$*;*

(b) $A(\mathfrak{T}_1) < A(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 < \mathfrak{T}_2$*;*

(c) $A(\mathfrak{T}_1) = A(\mathfrak{T}_2) \Rightarrow \mathfrak{T}_1 \sim \mathfrak{T}_2$.

To substantiate the efficacy of our proposed scoring function for IV-q-RPFNs, we delineate the subsequent illustrative example.

Example 3.1. Consider any two IV-q-RPFNs $\mathfrak{T}_1 = ([0.5,0.8],[0.2,0.4],[0.1,0.3])$ and $\mathfrak{T}_2 = ([0.6,0.7],[0.2,0.5],[0.2,0.4])$, where $q = 2$. In view of Definition 3.1, we have $\mathfrak{H}(\mathfrak{T}_1) = 0.295$ and $\mathfrak{H}(\mathfrak{T}_2) = 0.180$.

Consequently, by Definition 3.1(I), we ascertain that $\mathfrak{T}_1$ is superior to $\mathfrak{T}_2$. This indicates that $\mathfrak{T}_1$ is better than $\mathfrak{T}_2$.

4. Structural characteristics of IV-q-RPFYOWAOs

In this section, we introduce the Yager operations within the framework of the IV-q-RPF environment. We introduce two innovative YOWAOs, namely the IV-q-RPFYOWA operator and the IV-q-RPFYOWG operator, and analyze the essential characteristics inherent to these operators.

Definition 4.1. *For any two IV-q-RPFNs* $\mathfrak{T}_1 = ([{\alpha_1}^L,{\alpha_1}^U],[{\beta_1}^L,{\beta_1}^U],[{\varepsilon_1}^L,{\varepsilon_1}^U])$ *and* $\mathfrak{T}_2 = ([{\alpha_2}^L,{\alpha_2}^U],[{\beta_2}^L,{\beta_2}^U],[{\varepsilon_2}^L,{\varepsilon_2}^U])$, $\tau > 0$, *and* $w > 0$. *The operational laws for IV-q-RPFNs based on Yager's t-conorm and t-norm are expressed as*

I.
$$\mathfrak{T}_1 \oplus \mathfrak{T}_2 = \left(\begin{array}{c} \left[\sqrt[q]{min\left(1, \left(({\alpha_1}^L)^{q\tau} + ({\alpha_2}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(({\alpha_1}^U)^{q\tau} + ({\alpha_2}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\left(1-({\beta_1}^L)^q\right)^\tau + \left(1-({\beta_2}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\left(1-({\beta_1}^U)^q\right)^\tau + \left(1-({\beta_2}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\left(1-({\varepsilon_1}^L)^q\right)^\tau + \left(1-({\varepsilon_2}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\left(1-({\varepsilon_1}^U)^q\right)^\tau + \left(1-({\varepsilon_2}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right] \end{array} \right)$$

II. $$\mathfrak{T}_1 \otimes \mathfrak{T}_2 = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\left(1-({\alpha_1}^L)^q\right)^\tau + \left(1-({\alpha_2}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\left(1-({\alpha_1}^U)^q\right)^\tau + \left(1-({\alpha_2}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\sqrt[q]{min\left(1, \left(({\beta_1}^L)^{q\tau} + ({\beta_2}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(({\beta_1}^U)^{q\tau} + ({\beta_2}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(({\varepsilon_1}^L)^{q\tau} + ({\varepsilon_2}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(({\varepsilon_1}^U)^{q\tau} + ({\varepsilon_2}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right)$$

III. $$w\mathfrak{T}_1 = \left(\begin{array}{c} \left[\sqrt[q]{min\left(1, \left(w({\alpha_1}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(w({\alpha_1}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(w\left(1-({\beta_1}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(w\left(1-({\beta_1}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(w\left(1-({\varepsilon_1}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(w\left(1-({\varepsilon_1}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right] \end{array} \right)$$

IV. $${\mathfrak{T}_1}^w = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\left(1-w({\alpha_1}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\left(1-w({\alpha_1}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\sqrt[q]{min\left(1, \left(w({\beta_1}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(w({\beta_1}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(w({\varepsilon_1}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(w({\varepsilon_1}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right).$$

4.1. Fundamental characteristics of the IV-q-RPFYOWA operator

In the subsequent discussion, we propose the concept of the IV-q-RPFYOWAO and examine its essential properties.

Definition 4.2. *Let* $\mathfrak{A}$ *be a set of IV-q-RPFNs,* $\mathfrak{T}_i = ([{\alpha_i}^L, {\alpha_i}^U], [{\beta_i}^L, {\beta_i}^U], [{\varepsilon_i}^L, {\varepsilon_i}^U])$, $i = 1,2,\dots,\hbar$, *and* $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_\hbar)^T$ *be an associated weight vector of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0 \le \vartheta_i \le 1$ *such that* $\sum_{i=1}^{\hbar} \vartheta_i = 1$. *Additionally,* $\left(\rho(1), \rho(2), \rho(3), \dots, \rho(\hbar)\right)$ *is a permutation of* $\{1,2,\dots,\hbar\}$

such that $\mathfrak{T}_{\rho(i-1)} \geq \mathfrak{T}_{\rho(i)}$, $\forall\, i$. *The IV-q-RPFYOWAO is a mapping:* $\mathfrak{A}^{\hbar} \to \mathfrak{A}$ *and is defined by the following rule:*

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_{\hbar})$$

$$= \oplus_{i=1}^{\hbar} \vartheta_i \mathfrak{T}_{\rho(i)} \tag{4.1}$$

$$= \left(\begin{array}{c} \left[\sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar} \left(\vartheta_i (\alpha_{\rho(i)}{}^{L})^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar} \vartheta_i (\alpha_{\rho(i)}{}^{U})^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar} \left(\vartheta_i \left(1 - (\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar} \left(\vartheta_i \left(1 - (\beta_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar} \left(\vartheta_i \left(1 - (\varepsilon_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar} \left(\vartheta_i \left(1 - (\varepsilon_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \end{array} \right] \end{array} \right).$$

Physical interpretation of the IV-q-RPFYOWAO

The IV-q-RPFYOWAO offers significant physical interpretations in DM scenarios. The IV-q-RPFYOWAO functions as a balancing mechanism, aggregating uncertain attribute values through a weighted average procedure that fairly and stably reflects the overall performance of all attributes. In addition, IV-q-RPFYOWAO prioritises consensus and seamless compromise among the attributes.

The following result shows that the aggregated value of any finite number of IV-q-RPFNs under the IV-q-RPFYOWAO is itself an IV-q-RPFN.

Theorem 4.1. *Consider* $\hbar$ *to be number of IV-q-RPFNs,* $\mathfrak{T}_i = ([\alpha_i{}^L, \alpha_i{}^U], [\beta_i{}^L, \beta_i{}^U], [\varepsilon_i{}^L, \varepsilon_i{}^U])$, $i = 1,2, \dots, \hbar$, *and* $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_{\hbar})^T$ *be an associated weight vector of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0 \leq \vartheta_i \leq 1$ *such that* $\sum_{i=1}^{\hbar} \vartheta_i = 1$ *and* $\tau > 0$. *Additionally,* $(\rho(1), \rho(2), \rho(3), \dots, \rho(\hbar))$ *is a permutation of* $\{1,2, \dots, \hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)} \geq \mathfrak{T}_{\rho(i)}$, $\forall\, i$. *Then, the aggregated value of these IV-q-RPFNs in the framework of an IV-q-RPFYOWA operator is an IV-q-RPFN and is formulated as follows:*

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_{\hbar})=\oplus_{i=1}^{\hbar}\vartheta_i\mathfrak{T}_{\rho(i)}$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\vartheta_i\left(\alpha_{\rho(i)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{U})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{L})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{U})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right).$$

Proof. The validity of this assertion is demonstrated by the application of mathematical induction on $\hbar$. Consider the base case when $\hbar=2$. Here, we have $\mathfrak{T}_1=([\alpha_1{}^L,\alpha_1{}^U],[\beta_1{}^L,\beta_1{}^U],[\varepsilon_1{}^L,\varepsilon_1{}^U])$ and $\mathfrak{T}_2=([\alpha_2{}^L,\alpha_2{}^U],[\beta_2{}^L,\beta_2{}^U],[\varepsilon_2{}^L,\varepsilon_2{}^U])$. Utilizing the formulated Yager operational laws for IV-q-RPFNs as delineated in Definition 4.1, we obtain the following expressions:

$$\vartheta_1\mathfrak{T}_{\rho(1)}=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{L})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{L})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right),$$

and

$$\vartheta_2\mathfrak{T}_{\rho(2)}=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\vartheta_2\left(\alpha_{\rho(2)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_2\left(\alpha_{\rho(2)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\beta_{\rho(2)}{}^{L})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\beta_{\rho(2)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{L})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right).$$

The aggregated value of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ in the setting of Definition 4.2 is calculated as follows:

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2)=\vartheta_1\mathfrak{T}_{\rho(1)}\oplus\vartheta_2\mathfrak{T}_{\rho(2)}$$

$$=\begin{pmatrix}\left[\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{pmatrix}\oplus\begin{pmatrix}\left[\sqrt[q]{min\left(1,\left(\vartheta_2\left(\alpha_{\rho(2)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_2\left(\alpha_{\rho(2)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\beta_{\rho(2)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\beta_{\rho(2)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{pmatrix}$$

$$=\begin{pmatrix}\left[\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{L}\right)^{q\tau}+\vartheta_2(\alpha_{\rho(2)}{}^{L})^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_1\left(\alpha_{\rho(1)}{}^{U}\right)^{q\lambda}+\vartheta_2(\alpha_{\rho(2)}{}^{U})^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{L})^{q}\right)^{\tau}+\vartheta_2\left(1-(\beta_{\rho(2)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\beta_{\rho(1)}{}^{U})^{q}\right)^{\tau}+\vartheta_2\left(1-(\beta_{\rho(2)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{L})^{q}\right)^{\tau}+\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_1\left(1-(\varepsilon_{\rho(1)}{}^{U})^{q}\right)^{\tau}+\vartheta_2\left(1-(\varepsilon_{\rho(2)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{pmatrix}.$$

Consequently,

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2)$$

$$=\begin{pmatrix}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\vartheta_i(\alpha_{\rho(i)}{}^{U})^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{pmatrix}.$$

Hence, the statement is valid for $\hbar=2$.

Suppose that the result holds for $\hbar=s$.

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_s)=\oplus_{i=1}^{s}\vartheta_i\mathfrak{T}_{\rho(i)}$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\vartheta_i(\alpha_{\rho(i)}{}^{U})^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right)$$

Now, for $\hbar=s+1$, we have

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_s,\mathfrak{T}_{s+1})$$

$$=\oplus_{i=1}^{s}\vartheta_i\mathfrak{T}_{\rho(i)}\oplus\vartheta_s\mathfrak{T}_{\rho(s+1)}$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\vartheta_i(\alpha_{\rho(i)}{}^{U})^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\varepsilon_{\rho(i)}{}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left(\alpha_{\rho(s+1)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left(\alpha_{\rho(s+1)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-(\beta_{\rho(s+1)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-(\beta_{\rho(s+1)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-(\varepsilon_{\rho(s+1)}{}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-(\varepsilon_{\rho(s+1)}{}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right).$$

Consequently,

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_{s+1})=\begin{pmatrix}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\vartheta_i\left(\alpha_{\rho(i)}{}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-\left(\beta_{\rho(i)}{}^{L}\right)^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-\left(\beta_{\rho(i)}{}^{U}\right)^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-\left(\varepsilon_{\rho(i)}{}^{L}\right)^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-\left(\varepsilon_{\rho(i)}{}^{U}\right)^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{pmatrix}.$$

This shows that the result is therefore valid for $\hbar = s+1$. Thus, the aforementioned technique demonstrates the fact that the result is valid for all positive integral values of $\hbar$.

The following example illustrates the fact stated in Theorem 4.1.

Example 4.1. Suppose four clients assess a company's customer service on the basis of their recent experiences. The opinions of these clients are represented using IV-q-RPFNs,

$$\mathfrak{T}_1=\begin{pmatrix}[0.6,0.7],\\ [0.2,0.4],\\ [0.2,0.5]\end{pmatrix},\ \mathfrak{T}_2=\begin{pmatrix}[0.5,0.8],\\ [0.3,0.5],\\ [0.3,0.4]\end{pmatrix},\ \mathfrak{T}_3=\begin{pmatrix}[0.4,0.5],\\ [0.2,0.3],\\ [0.1,0.4]\end{pmatrix}\ and\ \mathfrak{T}_4=\begin{pmatrix}[0.5,0.9],\\ [0.1,0.3],\\ [0.1,0.5]\end{pmatrix}$$

with $\vartheta=(0.1,0.2,0.3,0.4)^T$ as an associated weighted vector (WV) of these IV-q-RPFNs.

In order to aggregate these IV-q-RPFNs using Definition 4.2, we proceed as follows.

Initially, compute the score values of these IV-q-RPFNs $\mathfrak{T}_i$, $i=1,2,3,4$ using Definition 3.1 for $q=4$ as follows:

$$\mathfrak{H}(\mathfrak{T}_1)=0.139,\ \mathfrak{H}(\mathfrak{T}_2)=0.184,\ \mathfrak{H}(\mathfrak{T}_3)=0.026\ and\ \mathfrak{H}(\mathfrak{T}_4)=0.324.$$

In view of the information above, these IV-q-RPFNs are arranged in descending order as $\mathfrak{T}_{\rho(1)}=\mathfrak{T}_4,\mathfrak{T}_{\rho(2)}=\mathfrak{T}_2,\mathfrak{T}_{\rho(3)}=\mathfrak{T}_1,$ and $\mathfrak{T}_{\rho(4)}=\mathfrak{T}_3$. Then, these IV-q-RPFNs are aggregated using Definition 4.2 for the operational parameter $\tau=2$ as follows:

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\mathfrak{T}_3,\mathfrak{T}_4)=\oplus_{i=1}^{4}\vartheta_i\mathfrak{T}_{\rho(i)}$$

$$=\begin{pmatrix}\left[\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{(4)(2)}\right)^{\frac{1}{2}}\right)},\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{U}\right)^{(4)(2)}\right)^{\frac{1}{2}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[4]{1-min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(1-\left(\beta_{\rho(i)}{}^{L}\right)^{4}\right)^{2}\right)\right)^{\frac{1}{2}}\right)},\\ \sqrt[4]{1-min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(1-\left(\beta_{\rho(i)}{}^{U}\right)^{4}\right)^{2}\right)\right)^{\frac{1}{2}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[4]{1-min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(1-\left(\varepsilon_{\rho(i)}{}^{U}\right)^{4}\right)^{2}\right)\right)^{\frac{1}{2}}\right)},\\ \sqrt[4]{1-min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(1-\left(\varepsilon_{\rho(i)}{}^{U}\right)^{4}\right)^{2}\right)\right)^{\frac{1}{2}}\right)}\end{array}\right]\end{pmatrix}.$$

By substituting the values of IV-q-RPFNs and the associated WV ϑ_i in the relation above, we obtain

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\mathfrak{T}_3,\mathfrak{T}_4) = ([0.492,0.807],[0.243,0.383],[0.243,0.451]).$$

Thus, the abovementioned discussion establishes the validity of Theorem 4.1.

The following result establishes that when the IV-q-RPFYOWAO is applied to any finite number of identical IV-q-RPFNs, it yields the same value. This property is known as the idempotency property of the IV-q-RPFYOWAO.

Theorem 4.2. *(Idempotency) Consider* $\hbar$ *to be number of IV-q-RPFNs,* $\mathfrak{T}_i = ([\alpha_i{}^L,\alpha_i{}^U],[\beta_i{}^L,\beta_i{}^U],[\varepsilon_i{}^L,\varepsilon_i{}^U])$, $i=1,2,\dots,\hbar$, *and* $\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_\hbar)^T$ *be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0\le\vartheta_i\le 1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i=1$ *and* $\tau>0$. *Additionally,* $(\rho(1),\rho(2),\rho(3),\dots,\rho(\hbar))$ *is a permutation of* $\{1,2,\dots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)}\ge\mathfrak{T}_{\rho(i)}$, $\forall\, i$. *If* $\mathfrak{T}_{\rho(i)}=\mathfrak{T}_{\rho(\circ)}$, $\forall\, i$, *where* $\mathfrak{T}=([\alpha_{\rho(\circ)}{}^L,\alpha_{\rho(\circ)}{}^U],[\beta_{\rho(\circ)}{}^L,\beta_{\rho(\circ)}{}^U],[\varepsilon_{\rho(\circ)}{}^L,\varepsilon_{\rho(\circ)}{}^U])$, *then,*

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar)=\mathfrak{T}_{\rho(\circ)}.$$

Proof. Given that $\mathfrak{T}_i=([\alpha_i{}^L,\alpha_i{}^U],[\beta_i{}^L,\beta_i{}^U],[\varepsilon_i{}^L,\varepsilon_i{}^U])=\mathfrak{T}_{\rho(\circ)}, i=1,2,\dots,\hbar$. In view of Definition 2.3, the relations above give $\alpha_{\rho(i)}{}^L=\alpha_{\rho(\circ)}{}^L$, $\alpha_{\rho(i)}{}^U=\alpha_{\rho(\circ)}{}^U$, $\beta_{\rho(i)}{}^L=\beta_{\rho(\circ)}{}^L$, $\beta_{\rho(i)}{}^U=\beta_{\rho(\circ)}{}^U$, $\varepsilon_{\rho(i)}{}^L=\varepsilon_{\rho(\circ)}{}^L$, and $\varepsilon_{\rho(i)}{}^U=\varepsilon_{\rho(\circ)}{}^U$. By substituting the values of $\alpha_i{}^L,\alpha_i{}^U,\beta_i{}^L,\beta_i{}^U,\varepsilon_i{}^L$, and $\varepsilon_i{}^U$ in Eq (4.1), we get

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar)$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i(\alpha_{\rho(\circ)})^{q\tau}\right)^{\frac{1}{\tau}}\right)\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\vartheta_i(\alpha_{\rho(\circ)}{}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(\circ)}{}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(\circ)}{}^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\varepsilon_{\rho(\circ)}{}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\varepsilon_{\rho(\circ)}{}^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,\left(\left({\alpha_{\rho(\circ)}}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\left({\alpha_{\rho(\circ)}}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\left(1-({\beta_{\rho(\circ)}}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\left(1-({\beta_{\rho(\circ)}}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(\left(1-({\varepsilon_{\rho(\circ)}}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\left(\left(1-({\varepsilon_{\rho(\circ)}}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{min\left(1,({\alpha_{\rho(\circ)}}^{L})^{q}\right)},\sqrt[q]{min\left(1,({\alpha_{\rho(\circ)}}^{U})^{q}\right)}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(1-({\beta_{\rho(\circ)}}^{L})^{q}\right)\right)},\\ \sqrt[q]{1-min\left(1,\left(1-({\beta_{\rho(\circ)}}^{U})^{q}\right)\right)}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-min\left(1,\left(1-({\varepsilon_{\rho(\circ)}}^{L})^{q}\right)\right)},\\ \sqrt[q]{1-min\left(1,\left(1-({\varepsilon_{\rho(\circ)}}^{U})^{q}\right)\right)}\end{array}\right]\end{array}\right).$$

It follows that

$$IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\ldots,\mathfrak{T}_{\hbar})=\left(\left[{\alpha_{\rho(\circ)}}^{L},{\alpha_{\rho(\circ)}}^{U}\right],\left[{\beta_{\rho(\circ)}}^{L},{\beta_{\rho(\circ)}}^{U}\right],\left[{\varepsilon_{\rho(\circ)}}^{L},{\varepsilon_{\rho(\circ)}}^{U}\right]\right)=\mathfrak{T}_{\rho(\circ)}.$$

The following result describes that the aggregated value of any finite number of IV-q-RPFNs under the IV-q-RPFYOWAO, which lies between the minimum and maximum bounds of the given IV-q-RPFN. This is known as the boundedness property of an IV-q-RPFYOWAO.

Theorem 4.3. *(Boundedness) Consider* $\hbar$ *to be number of IV-q-RPFNs,* $\mathfrak{T}_i=([{\alpha_i}^L,{\alpha_i}^U],[{\beta_i}^L,{\beta_i}^U],[{\varepsilon_i}^L,{\varepsilon_i}^U])$, $i=1,2,\ldots,\hbar$, *and* $\vartheta=(\vartheta_1,\vartheta_2,\ldots,\vartheta_{\hbar})^T$ *to be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0\le\vartheta_i\le1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i=1$ *and* $\tau>0$. *Additionally,* $\left(\rho(1),\rho(2),\rho(3),\ldots,\rho(\hbar)\right)$ *is a permutation of* $\{1,2,\ldots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)}\ge\mathfrak{T}_{\rho(i)}$, $\forall\, i$. *If*

$\mathfrak{T}^-=\left(\left[min_i({\alpha_{\rho(i)}}^L),min_i\,({\alpha_{\rho(i)}}^U)\right],[max_i\,({\beta_{\rho(i)}}^L),max_i\,({\beta_{\rho(i)}}^U)],[max_i\,({\varepsilon_{\rho(i)}}^L),max_i\,({\varepsilon_{\rho(i)}}^U)]\right)$ *and*

$\mathfrak{T}^+=\left(\left[max_i\,({\alpha_{\rho(i)}}^L),max_i\,({\alpha_{\rho(i)}}^U)\right],[min_i\,({\beta_{\rho(i)}}^L),min_i\,({\beta_{\rho(i)}}^U)],[min_i\,({\varepsilon_{\rho(i)}}^L),min_i\,({\varepsilon_{\rho(i)}}^U)]\right)$, *then,*

$$\mathfrak{T}^-\le IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\ldots,\mathfrak{T}_{\hbar})\le\mathfrak{T}^+.$$

Proof. Consider the result obtained by using the IV-q-RPFYOWA operator with the set of IV-q-RPFNs, represented as $IV-q-RPFYOWA_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\ldots,\mathfrak{T}_{\hbar})=([\alpha^L,\alpha^U],[\beta^L,\beta^U],[\varepsilon^L,\varepsilon^U])$.

Suppose that $\mathfrak{T}^-=([(\alpha^L)^-,(\alpha^U)^-],[(\beta^L)^-,(\beta^U)^-],[(\varepsilon^L)^-,(\varepsilon^U)^-])$ and $\mathfrak{T}^+=([(\alpha^L)^+,(\alpha^U)^+],[(\beta^L)^+,(\beta^U)^+],[(\varepsilon^L)^+,(\varepsilon^U)^+])$, where $(\alpha^L)^-=min_i\left({\alpha_{\rho(i)}}^L\right)$, $(\alpha^U)^-=min_i\left({\alpha_{\rho(i)}}^U\right)$, $(\beta^L)^-=max_i\left({\beta_{\rho(i)}}^L\right)$, $(\beta^U)^-=max_i\left({\beta_{\rho(i)}}^U\right)$, $(\varepsilon^L)^-=max_i\left({\varepsilon_{\rho(i)}}^L\right)$, $(\varepsilon^U)^-=$

$max_i(\varepsilon_{\rho(i)}{}^U)$, $(\alpha^L)^+ = max_i(\alpha_{\rho(i)}{}^L)$, $(\alpha^U)^+ = max_i(\alpha_{\rho(i)}{}^U)$, $(\beta^L)^+ = min_i\ (\beta_{\rho(i)}{}^L)$, $(\beta^U)^+ = min_i\ (\beta_{\rho(i)}{}^U)$, $(\varepsilon^L)^+ = min_i\ (\varepsilon_{\rho(i)}{}^L)$, *and* $(\varepsilon^U)^+ = min_i(\varepsilon_{\rho(i)}{}^U)$. Thus, for each IV-$q$-RPFN $\mathfrak{T}_i$, we have

$$min_i(\alpha_{\rho(i)}{}^L) \leq \alpha_{\rho(i)}{}^L \leq max_i\ (\alpha_i{}^L)$$

$$\Rightarrow (min_i(\alpha_{\rho(i)}{}^L))^{q\tau} \leq (\alpha_{\rho(i)}{}^L)^{q\tau} \leq (max_i(\alpha_{\rho(i)}{}^L))^{q\tau}$$

$$\Rightarrow \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(min_i(\alpha_{\rho(i)}{}^L))^{q\tau}\right)\right)^{\frac{1}{\tau}} \leq \left(\sum_{i=1}^{\hbar}\left(\left(\vartheta_i(\alpha_{\rho(i)}{}^L)^{q\tau}\right)\right)\right)^{\frac{1}{\tau}} \leq \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(max_i(\alpha_{\rho(i)}{}^L))^{q\tau}\right)\right)^{\frac{1}{\tau}}$$

$$\Rightarrow \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(min_i(\alpha_{\rho(i)}{}^L))^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\left(\vartheta_i(\alpha_{\rho(i)}{}^L)^{q\tau}\right)\right)\right)^{\frac{1}{\tau}}\right)}$$

$$\leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i(max_i(\alpha_{\rho(i)}{}^L))^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}$$

$$\Rightarrow \sqrt[q]{min\ \left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i((\alpha^L)^-)^{q\tau})\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}(\left(\vartheta_i(\alpha_{\rho(i)}{}^L)^{q\tau}\right))\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i((\alpha^L)^+)^{q\tau})\right)^{\frac{1}{\tau}}\right)}.$$

It follows that

$$(\alpha^L)^- \leq (\alpha^L) \leq (\alpha^L)^+. \tag{4.2}$$

Similarly, by following the mathematical steps above, we can obtain the following expression for the relation $min_i(\alpha_{\rho(i)}{}^U) \leq \alpha_{\rho(i)}{}^U \leq max_i\ (\alpha_{\rho(i)}{}^U)$:

$$(\alpha^U)^- \leq (\alpha^U) \leq (\alpha^U)^+. \tag{4.3}$$

Now, consider

$$max_i(\beta_{\rho(i)}{}^L) \leq \beta_{\rho(i)}{}^L \leq min_i(\beta_{\rho(i)}{}^L) \Rightarrow (max_i(\beta_{\rho(i)}{}^L))^q \leq (\beta_{\rho(i)}{}^L)^q \leq (min_i(\beta_{\rho(i)}{}^L))^q$$

$$\Rightarrow \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(max_i(\beta_{\rho(i)}{}^L))^q\right)^\tau\right)\right)^{\frac{1}{\tau}} \geq \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}} \geq \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(min_i(\beta_{\rho(i)}{}^L))^q\right)^\tau\right)\right)^{\frac{1}{\tau}}$$

$$\Rightarrow \sqrt[q]{1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(max_i(\beta_{\rho(i)}{}^L))^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}$$

$$\leq \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1 - \left(min_i\left(\beta_{\rho(i)}{}^L\right)\right)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}$$

$$\Rightarrow \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i(1 - ((\beta^L)^-)^q)^\tau)\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1 - \left(\beta_{\rho(i)}{}^L\right)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}$$

$$\leq \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}(\vartheta_i(1 - ((\beta^L)^+)^q)^\tau)\right)^{\frac{1}{\tau}}\right)}.$$

It follows that

$$(\beta^L)^- \leq (\beta^L) \leq (\beta^L)^+. \tag{4.4}$$

By adopting the mathematical process above for the relations $max_i\left(\beta_{\rho(i)}{}^U\right) \leq \beta_{\rho(i)}{}^U \leq min_i\left(\beta_{\rho(i)}{}^U\right)$, $\max_i\left(\varepsilon_{\rho(i)}{}^L\right) \leq \varepsilon_{\rho(i)}{}^L \leq \min_i\left(\varepsilon_{\rho(i)}{}^L\right)$, and $\max_i\left(\varepsilon_{\rho(i)}{}^U\right) \leq \varepsilon_{\rho(i)}{}^U \leq \min_i\left(\varepsilon_{\rho(i)}{}^U\right)$, we obtain their respective outcomes as follows:

$$(\beta^U)^- \leq (\beta^U) \leq (\beta^U)^+, \tag{4.5}$$

$$(\varepsilon^L)^- \leq (\varepsilon^L) \leq (\varepsilon^L)^+, \tag{4.6}$$

and

$$(\varepsilon^U)^- \leq (\varepsilon^U) \leq (\varepsilon^U)^+. \tag{4.7}$$

Hence, from the comparison of the relations (4.2)–(4.7), we get

$$\mathfrak{T}^- \leq IV - q - RPFYOWA_\vartheta(\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_\hbar) \leq \mathfrak{T}^+.$$

The following result establishes that if a particular set containing a finite number of IV-q-RPFNs exhibits improvement under the IV-q-RPFYOWAO with respect to another collection of finite number of IV-q-RPFNs, then the overall outcome will not diminish. This is known as the monotonicity property of an IV-q-RPFYOWAO.

Theorem 4.4. *(Monotonicity) Let* $\mathfrak{T}_i = ([\alpha_i{}^L, \alpha_i{}^U], [\beta_i{}^L, \beta_i{}^U], [\varepsilon_i{}^L, \varepsilon_i{}^U])$ *and* $\mathfrak{T}_i{}' = ([\alpha_i{}'^L, \alpha_i{}'^U], [\beta_i{}'^L, \beta_i{}'^U], [\varepsilon_i{}'^L, \varepsilon_i{}'^U])$, $i = 1,2, \ldots, \hbar$, *be any two sets of IV-q-RPFNs, and let* $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_\hbar)^T$ *be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *and* $\mathfrak{T}_i{}'$ *with* $0 \leq \vartheta_i \leq 1$ *such that* $\sum_{i=1}^{\hbar} \vartheta_i = 1$ *and* $\tau > 0$. *Additionally,* $(\rho(1), \rho(2), \rho(3), \ldots, \rho(\hbar))$ *is a permutation of* $\{1,2, \ldots, \hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)} \geq \mathfrak{T}_{\rho(i)}$, $\forall\, i$. *If* $\alpha_{\rho(i)}{}^L \leq \alpha_{\rho(i)}{}'^L$, $\alpha_{\rho(i)}{}^U \leq \alpha_{\rho(i)}{}'^U$, $\beta_{\rho(i)}{}^L \geq \beta_{\rho(i)}{}'^L$, $\beta_{\rho(i)}{}^U \geq \beta_{\rho(i)}{}'^U$, $\varepsilon_{\rho(i)}{}^L \geq \varepsilon_{\rho(i)}{}'^L$, *and* $\varepsilon_{\rho(i)}{}^U \geq \varepsilon_{\rho(i)}{}'^U$, $\forall\, i$. *Then,*

$$IV - q - RPFYOWA_\vartheta(\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_\hbar) \leq IV - q - RPFYOWA_\vartheta(\mathfrak{T}_1{}', \mathfrak{T}_2{}', \ldots, \mathfrak{T}_\hbar{}').$$

Proof. Consider

$$\alpha_{\rho(\mathrm{i})}{}^{L} \leq \alpha_{\rho(\mathrm{i})}{}'^{L} \Longrightarrow (\alpha_{\rho(\mathrm{i})}{}^{L})^{q} \leq \left(\alpha_{\rho(\mathrm{i})}{}'^{L}\right)^{q}$$

$$\Rightarrow \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(\mathrm{i})}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}} \leq \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(\mathrm{i})}{}'^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}$$

$$\Rightarrow min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right) \leq min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}'^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right).$$

It follows that

$$\sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}'^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}. \tag{4.8}$$

Similarly, by following the aforementioned mathematical steps, we can establish the following expression for the relation $\alpha_{\rho(\mathrm{i})}{}^{U} \leq \alpha_{\rho(\mathrm{i})}{}'^{U}$, we have

$$\sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \leq \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\alpha_{\rho(i)}{}'^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}. \tag{4.9}$$

Now, consider

$$\beta_{\rho(i)}{}^{L} \geq \beta_{\rho(i)}{}'^{L} \Longrightarrow (\beta_{\rho(i)}{}^{L})^{q} \geq \left(\beta_{\rho(i)}{}'^{L}\right)^{q}$$

$$\Rightarrow \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}} \leq \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}'^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}$$

$$\Rightarrow 1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right) \geq 1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}'^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right).$$

It follows that

$$\sqrt[q]{1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \geq \sqrt[q]{1-min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-(\beta_{\rho(i)}{}'^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}. \tag{4.10}$$

By following the abovementioned mathematical procedure for the relations $\beta_{\rho(\mathrm{i})}{}^{U} \geq \beta_{\rho(\mathrm{i})}{}'^{U}$, $\varepsilon_{\rho(\mathrm{i})}{}^{L} \geq \varepsilon_{\rho(\mathrm{i})}{}'^{L}$ and $\varepsilon_{\rho(\mathrm{i})}{}^{U} \geq \varepsilon_{\rho(\mathrm{i})}{}'^{U}$, we obtain their respective outcomes as follows:

$$\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\beta_{\rho(i)}}^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)} \geq \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\beta_{\rho(i)}}'^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \tag{4.11}$$

$$\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\varepsilon_{\rho(i)}}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)} \geq \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\varepsilon_{\rho(i)}}'^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \tag{4.12}$$

and

$$\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\varepsilon_{\rho(i)}}^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)} \geq \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\varepsilon_{\rho(i)}}'^U)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}. \tag{4.13}$$

Comparing the relations from (4.8)–(4.13) and making use of Definition 2.3, we get

$$IV-q-RPFYOWA_\vartheta(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar) \leq IV-q-RPFYOWA_\vartheta({\mathfrak{T}_1}',{\mathfrak{T}_2}',\dots,{\mathfrak{T}_\hbar}').$$

4.2. Fundamental characteristics of the interval-valued q-rung picture fuzzy Yager ordered weighted geometric operator (IV-q-RPFYOWGO)

This subsection introduces the notion of the interval-valued q-rung picture fuzzy Yager ordered weighted geometric operator (IV-q-RPFYOWGO) and analyzes its essential properties.

Definition 4.3. *Let* $\mathfrak{A}$ *be a collection of IV-q-RPFNs,* $\mathfrak{T}_i = ([{\alpha_i}^L,{\alpha_i}^U],[{\beta_i}^L,{\beta_i}^U],[{\varepsilon_i}^L,{\varepsilon_i}^U])$*,* $i = 1,2,\dots,\hbar$*, and* $\vartheta = (\vartheta_1,\vartheta_2,\dots,\vartheta_\hbar)^T$ *be an associated weight vector of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0 \leq \vartheta_i \leq 1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i = 1$*. Additionally,* $\left(\rho(1),\rho(2),\rho(3),\dots,\rho(\hbar)\right)$ *is a permutation of* $\{1,2,\dots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)} \geq \mathfrak{T}_{\rho(i)}$*,* $\forall\, i$*. The IV-q-RPFYOWGO is a mapping:* $\mathfrak{A}^\hbar \to \mathfrak{A}$ *and is formulated by the following rule:*

$$IV-q-RPFYOWG_\vartheta(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar)$$

$$= \begin{pmatrix} \begin{bmatrix} \sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1-min\left(1,\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{bmatrix}, \\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left({\beta_{\rho(i)}}^L\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left({\beta_{\rho(i)}}^U\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right] \\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^L\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1,\left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^U\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right] \end{pmatrix}. \tag{4.14}$$

Physical interpretation of the IV-q-RPFYOWGO

The IV-q-RPFYOWGO offers significant physical interpretations in DM scenarios. The IV-q-RPFYOWGO encapsulates a synergistic interaction among attributes via multiplicative aggregation, whereby a less potent attribute can substantially influence the total result. Moreover, IV-q-RPFYOWGO underscores sensitivity to the weakest link, rendering both operators suitable for modelling real-world MADM contexts.

The following result shows that the aggregated value of any finite number of IV-q-RPFNs under the IV-q-RPFYOWGO, is itself an IV-q-RPFN.

Theorem 4.5. *Consider* $\hbar$ *to be number of IV-q-RPFNs,* $\mathfrak{T}_i = ([\alpha_i{}^L, \alpha_i{}^U], [\beta_i{}^L, \beta_i{}^U], [\varepsilon_i{}^L, \varepsilon_i{}^U])$, $i = 1,2,\dots,\hbar$, *and let* $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_\hbar)^T$ *be an associated weight vector of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0 \le \vartheta_i \le 1$ *such that* $\sum_{i=1}^{\hbar} \vartheta_i = 1$ *and* $\tau > 0$. *Additionally,* $(\rho(1), \rho(2), \rho(3), \dots, \rho(\hbar))$ *is a permutation of* $\{1,2,\dots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)} \ge \mathfrak{T}_{\rho(i)}$, $\forall\, i$. *Then, the aggregated value of these IV-q-RPFNs in the framework of the IV-q-RPFYOWGO is an IV-q-RPFN and is formulated as follows:*

$$IV - q - RPFYOWG_\vartheta(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_\hbar) = \otimes_{i=1}^{\hbar} \mathfrak{T}_{\rho(i)}{}^{\vartheta_i}$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(1 - (\alpha_{\rho(i)}{}^L)^q\right)^\tau\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \sum_{i=1}^{\hbar}\left(\vartheta_i\left(1 - (\alpha_{\rho(i)}{}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\beta_{\rho(i)}{}^L\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\beta_{\rho(i)}{}^U\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^L\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{i=1}^{\hbar}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^U\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right).$$

Proof. The validity of this assertion is established by mathematical induction on $\hbar$. Consider the base case when $\hbar = 2$. Here, we have $\mathfrak{T}_1 = ([\alpha_1{}^L, \alpha_1{}^U], [\beta_1{}^L, \beta_1{}^U], [\varepsilon_1{}^L, \varepsilon_1{}^U])$ and $\mathfrak{T}_2 = ([\alpha_2{}^L, \alpha_2{}^U], [\beta_2{}^L, \beta_2{}^U], [\varepsilon_2{}^L, \varepsilon_2{}^U])$. Utilizing the formulated Yager operational laws for IV-q-RPFNs as delineated in Definition 4.1, we obtain the following expressions:

$$\mathfrak{T}_{\rho(1)}{}^{\vartheta_1} = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - (\alpha_{\rho(1)}{}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - (\alpha_{\rho(1)}{}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left(\beta_{\rho(1)}{}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left(\beta_{\rho(1)}{}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left(\varepsilon_{\rho(1)}{}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left(\varepsilon_{\rho(1)}{}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right),$$

and

$$\mathfrak{T}_{\rho(2)}{}^{\vartheta_2} = \left(\begin{array}{c} \left[\sqrt[q]{1 - min\left(1, \left(\vartheta_2\left(1 - ({\alpha_{\rho(2)}}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \atop \sqrt[q]{1 - min\left(1, \left(\vartheta_2\left(1 - ({\alpha_{\rho(2)}}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_2\left({\beta_{\rho(2)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_2\left({\beta_{\rho(2)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_2\left({\varepsilon_{\rho(2)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_2\left({\varepsilon_{\rho(2)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right).$$

The aggregated value of $\mathfrak{T}_{\rho(1)}$ and $\mathfrak{T}_{\rho(2)}$ in the setting of Definition 4.3 is calculated as follows:

$$IV - q - RPFYOWG_{\vartheta}(\mathfrak{T}_1, \mathfrak{T}_2) = \mathfrak{T}_{\rho(1)}{}^{\vartheta_1} \otimes \mathfrak{T}_{\rho(2)}{}^{\vartheta_2}$$

$$= \left(\begin{array}{c} \left[\sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - ({\alpha_{\rho(1)}}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \atop \sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - ({\alpha_{\rho(1)}}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left({\beta_{\rho(1)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left({\beta_{\rho(1)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left({\varepsilon_{\rho(1)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left({\varepsilon_{\rho(1)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \left[\sqrt[q]{1 - min\left(1, \left(\vartheta_2\left(1 - ({\alpha_{\rho(2)}}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \atop \sqrt[q]{1 - min\left(1, \left(\vartheta_2\left(1 - ({\alpha_{\rho(2)}}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_2\left({\beta_{\rho(2)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_2\left({\beta_{\rho(2)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_2\left({\varepsilon_{\rho(2)}}^L\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_2\left({\varepsilon_{\rho(2)}}^U\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - ({\alpha_{\rho(1)}}^L)^q\right)^\tau + \vartheta_2\left(1 - ({\alpha_{\rho(2)}}^L)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)}, \atop \sqrt[q]{1 - min\left(1, \left(\vartheta_1\left(1 - ({\alpha_{\rho(1)}}^U)^q\right)^\tau + \vartheta_2\left(1 - ({\alpha_{\rho(2)}}^U)^q\right)^\tau\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left({\beta_{\rho(1)}}^L\right)^{q\tau} + \vartheta_2({\beta_{\rho(2)}}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left({\beta_{\rho(1)}}^U\right)^{q\tau} + \vartheta_2({\beta_{\rho(2)}}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\sqrt[q]{min\left(1, \left(\vartheta_1\left({\varepsilon_{\rho(1)}}^L\right)^{q\tau} + \vartheta_2({\varepsilon_{\rho(2)}}^L)^{q\tau}\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\vartheta_1\left({\varepsilon_{\rho(1)}}^U\right)^{q\tau} + \vartheta_2({\varepsilon_{\rho(2)}}^U)^{q\tau}\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right).$$

It follows that

$$V-q-RPFYOWG_{\omega}(\mathfrak{T}_1,\mathfrak{T}_2)=\begin{pmatrix}\begin{bmatrix}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\alpha_{\rho(i)}{}^{L})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\sum_{i=1}^{2}\left(\vartheta_i\left(1-(\alpha_{\rho(i)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{bmatrix},\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{2}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\end{pmatrix}.$$

Hence, the statement is valid for $\hbar=2$.

Suppose that the result holds for $\hbar=s$.

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_s)=\bigotimes_{i=1}^{s}\mathfrak{T}_{\rho(i)}{}^{\vartheta_i}$$

$$=\begin{pmatrix}\begin{bmatrix}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\alpha_{\rho(i)}{}^{L})^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\alpha_{\rho(i)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{bmatrix},\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\end{pmatrix}.$$

Now, for $\hbar=s+1$, we have

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_s,\mathfrak{T}_{s+1})$$

$$=\bigotimes_{i=1}^{s}\mathfrak{T}_{\rho(i)}{}^{\vartheta_i}\otimes\mathfrak{T}_{\rho(s+1)}{}^{\vartheta_{s+1}}$$

$$=\begin{pmatrix}\begin{bmatrix}\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(1-\left(\alpha_{\rho(i)}{}^{L}\right)^q\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\\ \sqrt[q]{1-min\left(1,\sum_{i=1}^{s}\left(\vartheta_i\left(1-(\alpha_{\rho(i)}{}^{U})^q\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\end{bmatrix},\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\beta_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s}\left(\vartheta_i\left(\varepsilon_{\rho(i)}{}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\end{pmatrix}$$

$$\otimes\left(\begin{array}{c}\left[\sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-({\alpha_{\rho(s+1)}}^{L})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{1-min\left(1,\left(\vartheta_{s+1}\left(1-({\alpha_{\rho(s+1)}}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left({\beta_{\rho(s+1)}}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left({\beta_{\rho(s+1)}}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right]\\ \left[\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left({\varepsilon_{\rho(s+1)}}^{L}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\vartheta_{s+1}\left({\varepsilon_{\rho(s+1)}}^{U}\right)^{q\tau}\right)^{\frac{1}{\tau}}\right)}\right]\end{array}\right).$$

It follows that

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\ldots,\mathfrak{T}_{s+1})$$

$$=\left(\begin{array}{c}\left[\sqrt[q]{1-min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{1-min\left(1,\sum_{i=1}^{s+1}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)}\right],\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left({\beta_{\rho(i)}}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left({\beta_{\rho(i)}}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\\ \left[\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)},\sqrt[q]{min\left(1,\left(\sum_{i=1}^{s+1}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}\right]\end{array}\right).$$

This shows that the result is therefore valid for $\hbar = s+1$. Thus, the aforementioned technique demonstrates the fact that the result is valid for all positive integral values of $\hbar$.

The following example illustrates the fact stated in Theorem 4.5.

Example 4.2. In Example 4.1, we aggregated IV-q-RPFNs using an IV-q-RPFYOWAO. In the following discussion, we aggregate the same IV-q-RPFNs within the scope of an IV-q-RPFYOWGO for $q=4$ and $\tau=2$.

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\mathfrak{T}_3,\mathfrak{T}_4)=\otimes_{i=1}^{4}{\mathfrak{T}_{\rho(i)}}^{\vartheta_i}$$

$$=\left(\begin{array}{c}\left[\sqrt[4]{1-min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^{L})^{4}\right)^{2}\right)\right)^{\frac{1}{2}}\right)},\sqrt[4]{1-min\left(1,\sum_{i=1}^{4}\left(\vartheta_i\left(1-({\alpha_{\rho(i)}}^{U})^{4}\right)^{2}\right)^{\frac{1}{2}}\right)}\right],\\ \left[\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left({\beta_{\rho(i)}}^{L}\right)^{(4)(2)}\right)\right)^{\frac{1}{2}}\right)},\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left({\beta_{\rho(i)}}^{U}\right)^{(4)(2)}\right)\right)^{\frac{1}{2}}\right)}\right]\\ \left[\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^{L}\right)^{(4)(2)}\right)\right)^{\frac{1}{2}}\right)},\sqrt[4]{min\left(1,\left(\sum_{i=1}^{4}\left(\vartheta_i\left({\varepsilon_{\rho(i)}}^{U}\right)^{(4)(2)}\right)\right)^{\frac{1}{2}}\right)}\right]\end{array}\right).$$

By substituting the values of the IV-q-RPFNs and the associated WV ϑ_i in the relation above, we obtain

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\mathfrak{T}_3,\mathfrak{T}_4)=([0.493,0.763],[0.248,0.415],[0.246,0.467]).$$

The following result establishes that when the IV-q-RPFYOWGO is applied to any finite number of identical IV-q-RPFNs, it yields the same value. This property is known as the idempotency property of an IV-q-RPFYOWGO.

Theorem 4.6. *(Idempotency) Consider* $\hbar$ *to be number of IV-q-RPFNs,* $\mathfrak{T}_i=([\alpha_i{}^L,\alpha_i{}^U],[\beta_i{}^L,\beta_i{}^U],[\varepsilon_i{}^L,\varepsilon_i{}^U])$, $i=1,2,\dots,\hbar$, *and let* $\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_\hbar)^T$ *be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0\le\vartheta_i\le1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i=1$ *and* $\tau>0$. *Additionally,* $(\rho(1),\rho(2),\rho(3),\dots,\rho(\hbar))$ *is a permutation of* $\{1,2,\dots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)}\ge\mathfrak{T}_{\rho(i)}$, $\forall\, i$ *.If* $\mathfrak{T}_{\rho(i)}=\mathfrak{T}_{\rho(\circ)}$, $\forall\, i$, *where* $\mathfrak{T}=\left([\alpha_{\rho(\circ)}{}^L,\alpha_{\rho(\circ)}{}^U],[\beta_{\rho(\circ)}{}^L,\beta_{\rho(\circ)}{}^U],[\varepsilon_{\rho(\circ)}{}^L,\varepsilon_{\rho(\circ)}{}^U]\right)$*, then,*

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar)=\mathfrak{T}_{\rho(\circ)}.$$

Proof. The proof of Theorem 4.2 and this theorem is analogous.

The following result describes the aggregated value of any finite number of IV-q-RPFNs under the IV-q-RPFYOWGO, which lies between the minimum and maximum bounds of the given IV-q-RPFN. This is known as the boundedness property of the IV-q-RPFYOWGO.

Theorem 4.7. *(Boundedness) Consider* $\hbar$ *to be a number of IV-q-RPFNs,* $\mathfrak{T}_i=([\alpha_i{}^L,\alpha_i{}^U],[\beta_i{}^L,\beta_i{}^U],[\varepsilon_i{}^L,\varepsilon_i{}^U])$, $i=1,2,\dots,\hbar$, *and* $\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_\hbar)^T$ *to be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *with* $0\le\vartheta_i\le1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i=1$ *and* $\tau>0$. *Additionally,* $(\rho(1),\rho(2),\rho(3),\dots,\rho(\hbar))$ *is a permutation of* $\{1,2,\dots,\hbar\}$ *such that* $\mathfrak{T}_{\rho(i-1)}\ge\mathfrak{T}_{\rho(i)}$, $\forall\, i$*. If* $\mathfrak{T}^-=\left([min_i(\alpha_{\rho(i)}{}^L),min_i\,(\alpha_{\rho(i)}{}^U)],[max_i\,(\beta_{\rho(i)}{}^L),max_i\,(\beta_{\rho(i)}{}^U)],[max_i\,(\varepsilon_{\rho(i)}{}^L),max_i\,(\varepsilon_{\rho(i)}{}^U)]\right)$ *and* $\mathfrak{T}^+=\left([max_i\,(\alpha_{\rho(i)}{}^L),max_i\,(\alpha_{\rho(i)}{}^U)],[min_i\,(\beta_{\rho(i)}{}^L),min_i\,(\beta_{\rho(i)}{}^U)],[min_i\,(\varepsilon_{\rho(i)}{}^L),min_i\,(\varepsilon_{\rho(i)}{}^U)]\right)$.
Then,

$$\mathfrak{T}^-\le IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1,\mathfrak{T}_2,\dots,\mathfrak{T}_\hbar)\le\mathfrak{T}^+.$$

Proof. The proof of Theorem 4.3 and this theorem are analogous. Therefore, we omit the repetition here.

The following result establishes that if a particular set containing a finite number of IV-q-RPFNs exhibits improvement under the IV-q-RPFYOWGO with respect to another collection of a finite number of IV-q-RPFNs, then the overall outcome will not diminish. This is known as the monotonicity property of the IV-q-RPFYOWGO.

Theorem 4.8. *(Monotonicity) Let* $\mathfrak{T}_i=([\alpha_i{}^L,\alpha_i{}^U],[\beta_i{}^L,\beta_i{}^U],[\varepsilon_i{}^L,\varepsilon_i{}^U])$ *and* $\mathfrak{T}_i'=([\alpha_i{}'^L,\alpha_i{}'^U],[\beta_i{}'^L,\beta_i{}'^U],[\varepsilon_i{}'^L,\varepsilon_i{}'^U])$, $i=1,2,\dots,\hbar$, *be any two collections of IV-q-RPFNs, and let* $\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_\hbar)^T$ *be an associated WV of these IV-q-RPFNs* $\mathfrak{T}_i$ *and* $\mathfrak{T}_i'$ *with* $0\le\vartheta_i\le1$ *such that* $\sum_{i=1}^{\hbar}\vartheta_i=1$ *and* $\tau>0$. *Additionally,* $(\rho(1),\rho(2),\rho(3),\dots,\rho(\hbar))$ *is a permutation of* $\{1,2,\dots,\hbar\}$

such that $\mathfrak{T}_{\rho(i-1)} \geq \mathfrak{T}_{\rho(i)}$, $\forall i$. *If* ${\alpha_{\rho(i)}}^L \leq {\alpha_{\rho(i)}}'^L$, ${\alpha_{\rho(i)}}^U \leq {\alpha_{\rho(i)}}'^U$, ${\beta_{\rho(i)}}^L \geq {\beta_{\rho(i)}}'^L$, ${\beta_{\rho(i)}}^U \geq {\beta_{\rho(i)}}'^U$, ${\varepsilon_{\rho(i)}}^L \geq {\varepsilon_{\rho(i)}}'^L$, *and* ${\varepsilon_{\rho(i)}}^U \geq {\varepsilon_{\rho(i)}}'^U$, $\forall i$. *Then,*

$$IV-q-RPFYOWG_{\vartheta}(\mathfrak{T}_1, \mathfrak{T}_2, \dots, \mathfrak{T}_{\hbar}) \leq IV-q-RPFYOWG_{\vartheta}({\mathfrak{T}_1}', {\mathfrak{T}_2}', \dots, {\mathfrak{T}_{\hbar}}').$$

Proof. Since the proof is analogous to that of Theorem 4.4, we omit the details.

5. Application of proposed strategies operators in MADM

In this section, we present a DM approach employing the recently proposed IV-q-RPFYOWAOs for MADM situations, where the weights of the attributes are real numbers and the values of the attributes are IV-q-RPFNs. Suppose that $M = \{M_1, M_2, \dots, M_r\}$ is a collection of alternatives, and let $N = \{N_1, N_2, \dots, N_s\}$ be the set of attributes. In addition, $\vartheta=(\vartheta_1, \vartheta_2, \dots, \vartheta_s)^T$ denotes the corresponding WV of attributes, where $0 \leq \vartheta_l \leq 1$ for all l=1,2,…,s such that $\sum_{l=1}^{s} \vartheta_l = 1$. Additionally, $(\rho(1), \rho(2), \rho(3), \dots, \rho(s))$ is a permutation of $\{1,2,\dots,s\}$ such that $\mathfrak{T}_{\rho(l-1)} \geq \mathfrak{T}_{\rho(l)}$, $\forall l$. Assume that the decision-maker assesses the available alternatives on the basis of many attributes and articulates his/her preferred values using IV-q-RPFNs, which are expressed as $\mathfrak{T}_{kl} = ([\alpha_{kl}^L, \alpha_{kl}^U], [\beta_{kl}^L, \beta_{kl}^U], [\varepsilon_{kl}^L, \varepsilon_{kl}^U])$, where $k = 1,2,3, \dots, r$; $l = 1,2,3, \dots, s$. The information supplied by the decision-maker is encapsulated in an IV-q-RPF decision matrix $\mathfrak{D}$=$[\mathfrak{T}_{kl}]_{r\times s}$.

The suggested methodology, using the IV-q-RPFYOWAOs for resolving MADM issues, primarily comprises the following steps.

Step 1. Create an IV-q-RPF decision matrix $\mathfrak{D}$=$[\mathfrak{T}_{kl}]_{r\times s}$ using the data obtained from the decision-maker as follows:

$$\mathfrak{D}=\begin{bmatrix} \begin{pmatrix} [\alpha_{11}^L, \alpha_{11}^U], \\ [\beta_{11}^L, \beta_{11}^U], \\ [\varepsilon_{11}^L, \varepsilon_{11}^U] \end{pmatrix} & \begin{pmatrix} [\alpha_{12}^L, \alpha_{12}^U], \\ [\beta_{12}^L, \beta_{12}^U], \\ [\varepsilon_{12}^L, \varepsilon_{12}^U] \end{pmatrix} & \cdots & \begin{pmatrix} [\alpha_{1s}^L, \alpha_{1s}^U], \\ [\beta_{1s}^L, \beta_{1s}^U], \\ [\varepsilon_{1s}^L, \varepsilon_{1s}^U] \end{pmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{pmatrix} [\alpha_{r1}^L, \alpha_{r1}^U], \\ [\beta_{r1}^L, \beta_{r1}^U], \\ [\varepsilon_{r1}^L, \varepsilon_{r1}^U] \end{pmatrix} & \begin{pmatrix} [\alpha_{r2}^L, \alpha_{r2}^U], \\ [\beta_{r2}^L, \beta_{r2}^U], \\ [\varepsilon_{r2}^L, \varepsilon_{r2}^U] \end{pmatrix} & \cdots & \begin{pmatrix} [\alpha_{rs}^L, \alpha_{rs}^U], \\ [\beta_{rs}^L, \beta_{rs}^U], \\ [\varepsilon_{rs}^L, \varepsilon_{rs}^U] \end{pmatrix} \end{bmatrix}.$$

Step 2. To obtain the IV-q-RPF permuted decision matrix $\mathfrak{D}_{\rho(kl)} = \left[\mathfrak{T}_{\rho(kl)}\right]_{r\times s}$, we follow the subsequent two stages.

(1) Calculate the score values of all attributes N_l, corresponding to each alternative M_k of the IV-q-RPF decision matrix $\mathfrak{D}$, using Definition 3.1.

(2) Arrange the calculated values from the previous stage in descending order to obtain the IV-q-RPF permuted decision matrix $\mathfrak{D}_{\rho(kl)}$.

Step 3. (a) Calculate the aggregated values $\mathfrak{T}_k = ([\alpha_k^L, \alpha_k^U], [\beta_k^L, \beta_k^U], [\varepsilon_k^L, \varepsilon_k^U])$ of each alternatives M_k, corresponding to all attributes N_l using the IV-q-RPFYOWA operator in the following way:

$$\mathfrak{T}_k = IV - q - RPFYOWA(\mathfrak{T}_{k1}, \mathfrak{T}_{k2}, \dots, \mathfrak{T}_{ks})$$

$$= \left(\begin{array}{c} \left[\sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\alpha_{\rho(kl)}}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\alpha_{\rho(kl)}}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\beta_{\rho(kl)}}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\beta_{\rho(kl)}}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\varepsilon_{\rho(kl)}}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\varepsilon_{\rho(kl)}}^{U})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)} \end{array} \right] \end{array} \right), \quad k = 1,2, \dots, r.$$

(b) Calculate the aggregated values $\mathfrak{T}_k = ([\alpha_k^L, \alpha_k^U], [\beta_k^L, \beta_k^U], [\varepsilon_k^L, \varepsilon_k^U])$ of each alternatives M_k, corresponding to all attributes N_l using the IV-q-RPFYOWG operator in the following way:

$$\mathfrak{T}_k = IV - q - RPFYOWG(\mathfrak{T}_{k1}, \mathfrak{T}_{k2}, \dots, \mathfrak{T}_{ks})$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{1 - min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\alpha_{\rho(kl)}}^{L})^{q}\right)^{\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \\ \sqrt[q]{1 - min\left(1, \sum_{l=1}^{s}\left(\vartheta_l\left(1 - ({\alpha_{\rho(kl)}}^{U})^{q}\right)^{\tau}\right)^{\frac{1}{\tau}}\right)} \end{array} \right], \\ \left[\sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\beta_{\rho(kl)}}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\beta_{\rho(kl)}}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right] \\ \left[\sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\varepsilon_{\rho(kl)}}^{L}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)}, \sqrt[q]{min\left(1, \left(\sum_{l=1}^{s}\left(\vartheta_l\left({\varepsilon_{\rho(kl)}}^{U}\right)^{q\tau}\right)\right)^{\frac{1}{\tau}}\right)} \right] \end{array} \right), k = 1,2, \dots, r.$$

Step 4. Determine the score values of $\mathfrak{T}_k$, $k = 1,2, \dots, r$ using Definition 3.1.
Step 5. Rank all the alternatives utilizing the information obtained from the preceding step and select the most optimal choice.

A pictorial representation of the algorithm above is depicted in Figure 2.

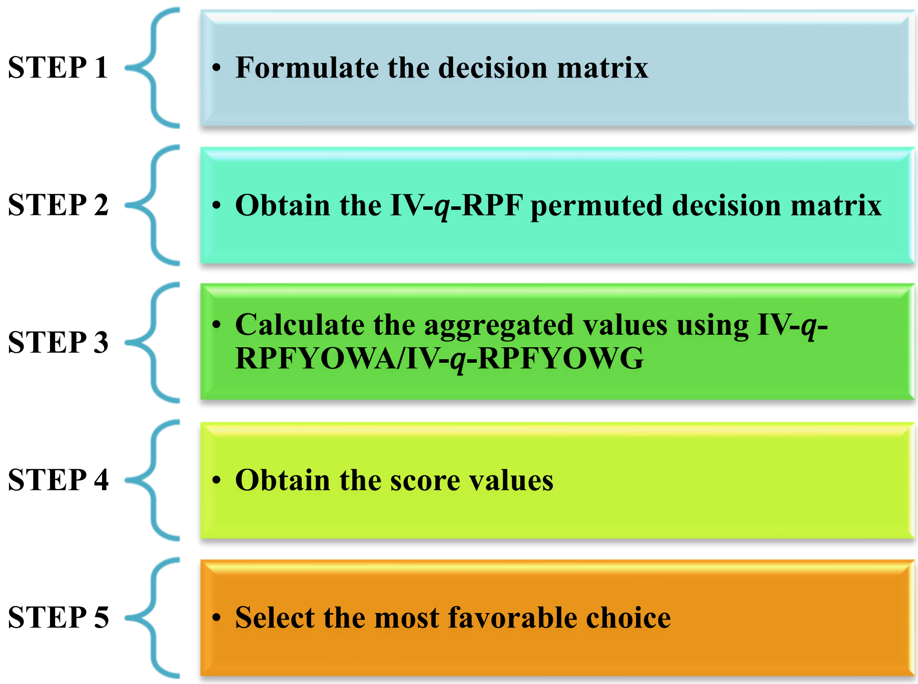

Figure 2. Schematic workflow of the algorithm based on the IV-q-RPFYOWA/IV-q-RPFYOWG operators.

5.1. Case study: selection of an optimal spacecraft shielding materials against cosmic radiation

Human missions beyond Earth's atmosphere face significant challenges from cosmic radiation, which poses serious risks to astronauts and equipment. High-energy cosmic rays and solar particle events can damage spacecraft and human tissue, leading to increased cancer risks, nervous system harm, and acute radiation sickness. Unlike Earth, where atmospheric protection exists, space missions rely solely on engineered materials for shielding. The selection of appropriate materials is crucial for the safety and success of lunar and Martian constructions. Space radiation differs from terrestrial radiation, consisting of rapid-moving protons, heavy ions, and subatomic particles that can have destructive cellular effects. Spacecraft electronics are at risk from high-energy particles that cause memory bit flips, degrade systems, and lead to mission failures. This challenge intensifies for extended Moon and Mars missions, where astronauts may exceed radiation limits during travel. Protecting astronauts from space radiation is crucial for their survival. Current spacecraft predominantly utilize aluminum alloys to provide structural integrity and some level of radiation shielding, but these materials are insufficient to counter the most dangerous cosmic rays—heavy ions—posing significant risks to space missions.

Aluminum's high density makes it unsuitable for deep-space missions, making them costly and complex. Scientists are exploring other materials that can be lightweight and yet flexible with greater radiation protection. The ideal shielding materials should have four main characteristics: High radiation attenuation, lightweight, strong mechanical properties, and thermal tolerance. The combination of polymers, hydrogen technology, and nanotechnology is being explored, and various mission-specific shielding proposals require independent assessments.

Polyethylene has a high hydrogen content that makes it an ideal candidate to use as radiation shielding in space because it is able to scatter and absorb cosmic rays. Laboratory experiments show that polyethylene shields are more protective than aluminium, especially when weight is taken into account, as observed on the International Space Station. Pure polyethylene is not suitable as a primary spacecraft material. Scientists are working on creating composite materials that are structurally sound yet protective against radiation, with boron nitride nanotubes being a leading candidate due to their high strength and ability to block radiation. The most effective shielding systems are those that use well-spaced multilayered materials to optimize the stopping power. These developments are essential for establishing human settlements on other celestial bodies during future missions. The feasibility of shielding solutions will be determined by their practicality, which includes the cost and compatibility with current technologies, and these factors will determine how they are used in space. Radiation protection is a critical consideration in safe human exploration, and the materials selected today will determine the future of human space exploration over the next few generations.

5.2. Illustrated example

This study offers a structured framework to test spacecraft shielding materials to protect against cosmic radiation through application of the IV-q-RPFS model. The space research agency is concerned by space exploration missions extending beyond short durations because of the damaging effects of cosmic radiation on the astronauts and equipment. To secure the mission's goals and the astronaut's safety, an aerospace materials engineer examines shielding materials by considering important properties, intending to design a structurally robust spacecraft to protect against cosmic radiations.

The engineer selects four shielding materials (alternatives) $\{M_1, M_2, M_3, M_4\}$.

M_1**:** Polyethylene,
M_2: Aluminium alloy,
M_3: Boron nitride nanotubes,
M_4: A Multi-layered composite.

Furthermore, the engineer specifies four key attributes $\{N_1, N_2, N_3, N_4\}$ that affect the efficiency and appropriateness of these alternatives:

N_1: Radiation shielding effectiveness,
N_2: Structural stability,
N_3: Weight efficiency, and
N_4: Thermal resilience.

The engineer assigns $\vartheta = (0.3,0.2,0.2,0.3)^T$ as an associated WV to these attributes such that $\sum_{l=1}^{4} \vartheta_l = 1$.

This MADM problem is solved within the framework of IV-q-RPFYOWAOs as follows.

Step 1. Table 3 specifies the IV-q-RPF decision matrix representing the researcher's estimation for each alternative $M_k,\ k = 1,2,3,4$ relative to each attribute N_l in the form of an IV-q-RPFN.

Table 3. IV-q-RPF decision matrix representing the evaluation of alternatives across multiple attributes under IV-q-RPF information.

	N_1	N_2	N_3	N_4
M_1	$\begin{pmatrix}[0.6,0.8],\\ [0.3,0.5],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.4,0.6],\\ [0.2,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.9],\\ [0.2,0.6],\\ [0.4,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\ [0.3,0.6],\\ [0.3,0.5]\end{pmatrix}$
M_2	$\begin{pmatrix}[0.5,0.9],\\ [0.1,0.5],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.2,0.5],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\ [0.3,0.5],\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.3,0.4],\\ [0.3,0.5]\end{pmatrix}$
M_3	$\begin{pmatrix}[0.4,0.7],\\ [0.3,0.5],\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\ [0.3,0.6],\\ [0.1,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],\\ [0.2,0.6],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.7],\\ [0.2,0.4],\\ [0.3,0.4]\end{pmatrix}$
M_4	$\begin{pmatrix}[0.5,0.6],\\ [0.2,0.5],\\ [0.2,0.5],\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],\\ [0.4,0.5],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.2,0.3],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],\\ [0.2,0.6],\\ [0.4,0.5]\end{pmatrix}$

Step 2. In order to obtain the IV-q-RPF permuted decision matrix, we proceed to the subsequent two stages.

(1) Calculate the score values of all attributes N_l, relative to each alternative M_k of the IV-q-RPF decision matrix above using Definition 3.1 for a particular value of $q = 3$.

For M_1, $\mathfrak{H}(\mathfrak{T}_{11}) = 0.116$, $\mathfrak{H}(\mathfrak{T}_{12}) = 0.073$, $\mathfrak{H}(\mathfrak{T}_{13}) = 0.112$, $\mathfrak{H}(\mathfrak{T}_{14}) = 0.091$.
For M_2, $\mathfrak{H}(\mathfrak{T}_{21}) = 0.224$, $\mathfrak{H}(\mathfrak{T}_{22}) = 0.131$, $\mathfrak{H}(\mathfrak{T}_{23}) = 0.028$, $\mathfrak{H}(\mathfrak{T}_{24}) = 0.197$.
For M_3, $\mathfrak{H}(\mathfrak{T}_{31}) = 0.092$, $\mathfrak{H}(\mathfrak{T}_{32}) = 0.068$, $\mathfrak{H}(\mathfrak{T}_{33}) = 0.221$, $\mathfrak{H}(\mathfrak{T}_{34}) = 0.152$.
For M_4, $\mathfrak{H}(\mathfrak{T}_{41}) = 0.038$, $\mathfrak{H}(\mathfrak{T}_{42}) = 0.238$, $\mathfrak{H}(\mathfrak{T}_{43}) = 0.179$, $\mathfrak{H}(\mathfrak{T}_{44}) = 0.266$.

(2) Arrange the calculated values in descending order to obtain the IV-q-RPF permuted decision matrix.

The outcomes of this mathematical procedure are listed in Table 4.

Table 4. IV-q-RPF permuted decision matrix after arranging the attributes in descending order.

	N_1	N_2	N_3	N_4
M_1	$\begin{pmatrix}[0.6,0.8],\\ [0.3,0.5],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.9],\\ [0.2,0.6],\\ [0.4,0.7]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.8],\\ [0.3,0.6],\\ [0.3,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.4,0.6],\\ [0.2,0.5]\end{pmatrix}$
M_2	$\begin{pmatrix}[0.5,0.9],\\ [0.1,0.5],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.3,0.4],\\ [0.3,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.2,0.5],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\ [0.3,0.5],\\ [0.2,0.4]\end{pmatrix}$
M_3	$\begin{pmatrix}[0.6,0.9],\\ [0.2,0.6],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.7],\\ [0.2,0.4],\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\ [0.3,0.5],\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\ [0.3,0.6],\\ [0.1,0.3]\end{pmatrix}$
M_4	$\begin{pmatrix}[0.6,0.9],\\ [0.2,0.6],\\ [0.4,0.5]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.9],\\ [0.4,0.5],\\ [0.4,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.8],\\ [0.2,0.3],\\ [0.3,0.6]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.2,0.5],\\ [0.2,0.5]\end{pmatrix}$

Part A: Solution of the MADM problem using an IV-q-RPFYOWA operator

Step 3. Obtain the aggregated values $\mathfrak{T}_k$, $k = 1,2,3,4$ of each alternative M_k by applying an IV-q-RPFYOWA to the IV-q-RPFNs listed in Table 4 for particular value of $q = 3$ and an operational parameter $\tau = 2$. The outcomes of this process are tabulated in Table 5.

Table 5. Aggregated values of alternatives under the IV-q-RPFYOWA operator.

Alternatives	$\mathfrak{T}_k$
M_1	([0.563,0.804], [0.325,0.573], [0.307,0.577])
M_2	([0.463,0.807], [0.372,0.484], [0.319,0.532])
M_3	([0.517,0.788], [0.260,0.549], [0.297,0.532])
M_4	([0.586,0.840], [0.268,0.451], [0.343,0.544])

Step 4. Evaluate the score values of all IV-q-RPF numbers obtained in Table 5 using Definition 3.1 as follows:

$$\mathfrak{H}(\mathfrak{T}_1) = 0.129,\ \mathfrak{H}(\mathfrak{T}_2) = 0.164,\ \mathfrak{H}(\mathfrak{T}_3) = 0.163, \text{ and } \mathfrak{H}(\mathfrak{T}_4) = 0.223.$$

Step 5. Since $\mathfrak{H}(\mathfrak{T}_4) > \mathfrak{H}(\mathfrak{T}_2) > \mathfrak{H}(\mathfrak{T}_3) > \mathfrak{H}(\mathfrak{T}_1)$, the ranking order of the alternatives is $M_4 > M_2 > M_3 > M_1$.

Consequently, the multi-layered composite is the most optimal shielding material against cosmic radiation according to the IV-q-RPFYOWA model.

A graphical representation of the selection of the most suitable alternative using the IV-q-RPFYOWA is depicted in Figure 3.

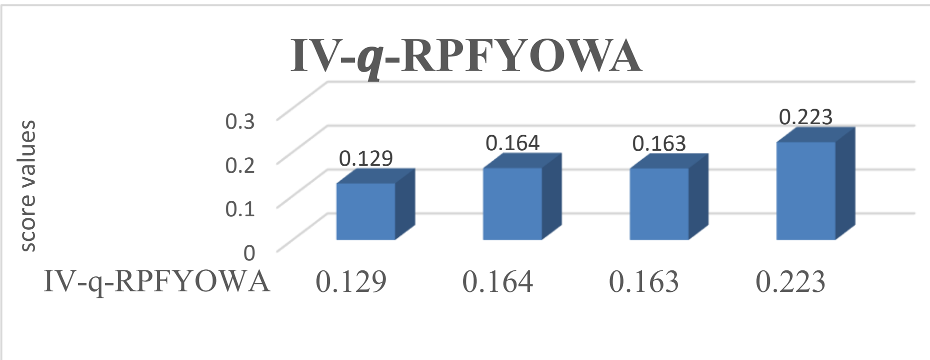

Figure 3. Ranking of alternatives using the IV-q-RPFYOWA operator.

Part B: Solution of the MADM problem using the IV-q-RPFYOWG operator

Step 3. Obtain the aggregated values $\mathfrak{T}_k$, $k = 1,2,3,4$ of each alternative M_k by applying the IV-q-RPFYOWG to IV-q-RPFNs listed in Table 4 for a particular value of $q = 3$ and an operational parameter $\tau = 2$. The outcomes of this process are tabulated in Table 6.

Table 6. Aggregated values of alternatives under the IV-q-RPFYOWG operator.

Alternatives	$\mathfrak{x}_k$
M_1	([0.549,0.787], [0.342,0.577], [0.326,0.578])
M_2	([0.474,0.767], [0.269,0.487], [0.334,0.506])
M_3	([0.493,0.756], [0.271,0.562], [0.339,0.547])
M_4	([0.582,0.813], [0.225,0.508], [0.306,0.576])

Step 4. Evaluate the score values of all IV-q-RPF numbers obtained in Table 5 using Definition 3.1 as follows:

$$\mathfrak{H}(\mathfrak{T}_1) = 0.086,\ \mathfrak{H}(\mathfrak{T}_2) = 0.108,\ \mathfrak{H}(\mathfrak{T}_3) = 0.094, \text{ and } \mathfrak{H}(\mathfrak{T}_4) = 0.171.$$

Step 5. Since $\mathfrak{H}(\mathfrak{T}_4) > \mathfrak{H}(\mathfrak{T}_2) > \mathfrak{H}(\mathfrak{T}_3) > \mathfrak{H}(\mathfrak{T}_1)$, the ranking order of alternatives is $M_4 > M_2 > M_3 > M_1$.

Consequently, the multi-layered composite is the most optimal shielding material against cosmic radiation according to the IV-q-RPFYOWG model.

A graphical representation of the selection of the most suitable alternative using the IV-q-RPFYOWG is depicted in Figure 4.

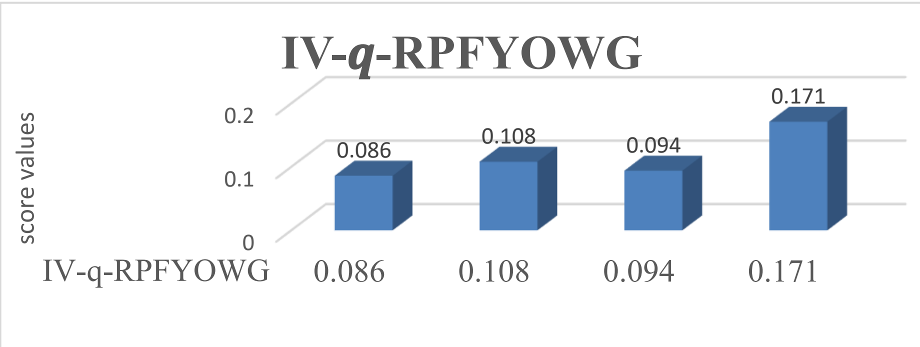

Figure 4. Ranking of alternatives using the IV-q-RPFYOWG operator.

5.3. Comparative analysis

This comparison study intends to demonstrate the effectiveness and robustness of our suggested techniques by evaluating several existing methods, including T-spherical fuzzy ordered weighted averaging (IV-TSFOWA) [41], T-spherical fuzzy ordered weighted geometric (IV-TSFOWG) [41], spherical fuzzy Dombi ordered weighted averaging (IV-SFDOWA) [42], spherical fuzzy Dombi ordered weighted geometric (IV-SFDOWG) [42], rung picture fuzzy Yager ordered weighted averaging (q-RPFYOWA) [48] and rung picture fuzzy Yager ordered weighted geometric (q-RPFYOWG) [48]. Table 7 summarizes the aggregated values of the alternatives derived by various techniques, while Table 8 shows their corresponding rankings.

Table 7. Aggregated values of alternatives obtained with different existing operators.

	IV-TSFOWA [41]	IV-TSFOWG [41]	IV-SFDOWA [42]	IV-SFDOWG [42]
M_1	([0.556,0.807], [0.300,0.568], [0.280,0.564])	([0.534,0.788], [0.300,0.280], [0.303,0.586])	([0.564,0.837], [0.265,0.557], [0.248,0.543])	([0.497,0.766], [0.336,0.578], [0.318,0.605])
M_2	([0.475,0.809], [0.198,0.478], [0.289,0.512])	([0.468,0.759], [0.198,0.478], [0.317,0.538])	([0.478,0.850], [0.133,0.468], [0.251,0.476])	([0.455,0.699], [0.260,0.485], [0.332,0.551])
M_3	([0.502,0.793], [0.245,0.534], [0.216,0.414])	([0.472,0.753], [0.245,0.534], [0.303,0.468])	([0.518,0.843], [0.227,0.498], [0.133,0.364])	([0.444,0.729], [0.265,0.564], [0.321,0.501])
M_4	([0.584,0.845], [0.229,0.477], [0.306,0.538])	([0.579,0.815], [0.229,0.477], [0.341,0.546])	([0.587,0.872], [0.210,0.405], [0.255,0.528])	([0.569,0.779], [0.291,0.529], [0.355,0.552])

Table 8. Score values and ranking of alternatives using different existing techniques.

Operators	$\mathfrak{H}(\mathfrak{T}_1)$	$\mathfrak{H}(\mathfrak{T}_2)$	$\mathfrak{H}(\mathfrak{T}_3)$	$\mathfrak{H}(\mathfrak{T}_4)$	Ranking
IV-TSFOWA [41]	0.143	0.181	0.188	0.248	$M_4 > M_3 > M_2 > M_1$
IV-TSFOWG [41]	0.181	0.118	0.117	0.206	$M_4 > M_1 > M_2 > M_3$
IV-SFDOWA [41]	0.199	0.247	0.276	0.313	$M_4 > M_3 > M_2 > M_1$
IV-SFDOWG [41]	0.044	0.050	0.059	0.136	$M_4 > M_3 > M_2 > M_1$
IV-q-RPFYOWA	0.129	0.164	0.163	0.223	$M_4 > M_2 > M_3 > M_1$
IV-q-RPFYOWG	0.086	0.108	0.094	0.171	$M_4 > M_2 > M_3 > M_1$

- The methodologies presented in this article offer more advanced and flexible aggregation methods compared with the techniques presented in [41], particularly in contexts characterized by high uncertainty and hesitation. By leveraging the q-rung picture structure, IV-q-RPFYOWAOs allow a more nuanced expression of membership, non-membership, and abstention degrees, thereby enabling finer discrimination among alternatives. Unlike IV-TSFOWAOs, which are constrained by the spherical fuzzy model, IV-q-RPFYOWAOs effectively encapsulate acceptance, rejection, and indeterminacy within an interval-valued framework, enhancing both robustness and the decision's precision. By enabling finer discrimination among competing alternatives and enhancing aggregation robustness, IV-q-RPFYOWAOs significantly improve the accuracy and reliability of MADM processes. Therefore, they are particularly well-suited for tackling intricate real-world DM problems involving multiple conflicting criteria and imprecise information.
- The methodologies proposed in this article offer a more adaptable and resilient approach to OWA in MADM compared with the strategies outlined in [42], specifically in complex and uncertain environments. The techniques in [42] rely on fixed exponential-like functions, which

may constrain adaptability when handling high levels of uncertainty and hesitation. In contrast, the recently developed IV-q-RPFYOWAOs introduce a configurable parameter structure within the q-rung picture fuzzy framework, enabling dynamic adjustment of MD, NeD, and NMD. This structural flexibility empowers decision-makers to better accommodate varying levels of hesitation, ambiguity, and risk preferences, aspects that are often oversimplified in traditional IV-SF Dombi-based models. Furthermore, by integrating interval-valued information with the q-rung picture fuzzy environment, IV-q-RPFYOWAOs enhance the accuracy, robustness, and contextual relevance of the aggregation process, leading to more reliable and decision-sensitive outcomes in MADM tasks marked by deep uncertainty.

- Numerous drawbacks affect the approaches introduced in [48]. The q-RPFS framework has excellent uncertain information management but its single-digit membership degrees create potential information loss during DM, whereas the proposed strategies improve existing the models by incorporating intervals for membership, neutral, and non-membership degrees. This upgraded method extends uncertainty clusters enabling decision-makers to consider a wider range of possible values while delivering more accurate representations of real-world ambiguity. The newly proposed modelling methods establish sophisticated and dependable foundations for complex DM applications which generate more efficient and precise results.

The proposed IV-q-RPFYOWAOs offer a more flexible and robust alternative to the existing methods by incorporating Yager's ordered weighted averaging and tunable q-parameterization within an interval-valued q-rung picture framework. Unlike earlier models that rely on fixed exponential functions or single-valued memberships, these approaches better handle ambiguity and hesitation in complex decision-making scenarios. As a result, they provide more accurate, adaptable, and contextually relevant outcomes in MADM under profound uncertainty.

A graphical view of the information presented in Tables 7 and 8 is depicted in Figure 5.

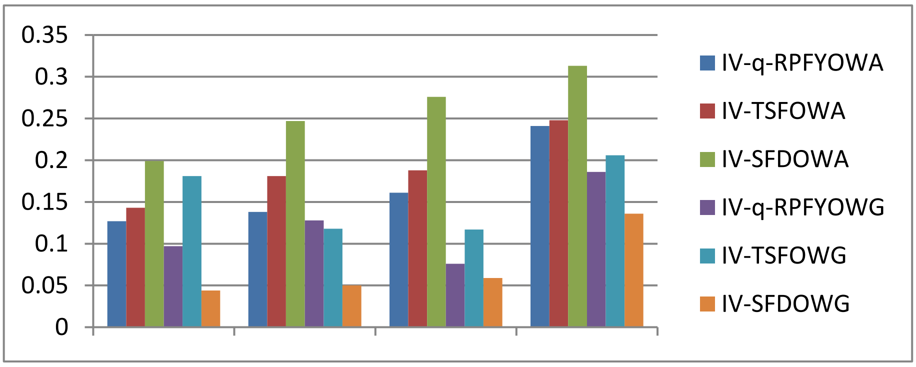

Figure 5. Graphical illustration of the ranking of alternatives using the recently proposed and existing methodologies.

5.4. *Empirical analysis*

Table 9 outlines an empirical evaluation of the proposed strategies in comparison with the existing models like fuzzy technique for order of preference by similarity to ideal solution (TOPSIS), fuzzy multi-objective optimaztion on the basis of ratio analysis (MOORA) and fuzzy analytic hierarchy process (AHP).

Table 9. Empirical analysis of the proposed operators in comparison with the existing models.

Criteria	IV-q-RPFYWAOs (proposed)	Fuzzy TOPSIS	Fuzzy MORA (e.g., VIKOR/PROMETHEE)	Fuzzy AHP
Representation of uncertainty	Highest. Uses interval-valued q-rung picture fuzzy sets, capturing membership, non membership, and hesitation as intervals with tunable parameter q for flexibility	Moderate. Uses fuzzy numbers; no direct modelling of hesitation	Moderate. Handles fuzzy numbers, but treatment of hesitation is limited	Low to moderate. Relies on crisp or fuzzy pairwise comparisons; hesitation is often lost in defuzzification
Discriminatory power	High. The q parameter adjusts the emphasis to separate close alternatives; avoids excessive ties	Moderate. Close alternatives often have similar closeness scores	Moderate to high. VIKOR produces compromize rankings; PROMETHEE may leave incomparable	Low to moderate. Small differences in pairwise scores may be hidden
Parameter sensitivity	Controlled q tuning allows targeted adaptability without great instability	High. The distance metric choice affects the results	Medium. The compromise coefficient influences the rankings	High. Consistency ratio and scale affect weights
Interpretability	High. Experts can understand the effect of q and Yager weights; intervals explain hesitation	High. Geometric closeness is intuitive	Medium. Outranking logic is less intuitive for non-technical users	Medium. hierarchy is clear but fuzziness in the judgments is harder to explain
Weight flexibility	Very high. Works with equal, expert, entropy, or hybrid weights	High. Any weights are applicable	High. Supports different weighting schemes	Medium. Weights must come from pairwise judgments
Robustness to noisy/conflicting data	Excellent	Good	Good	Low

5.5. Sensitivity analysis for parameter q

Tables 10 and 11 encapsulate the score values and ranking of alternatives utilizing IV-q-RPFYOWAO and IV-q-RPFYOWGO, respectively, for different values of the parameter q.

For the IV-q-RPFYOWAO, the ranking at $q = 1$ is $M_4 > M_3 > M_2 > M_1$; however, from $q \geq 2$ onwards, it stabilizes to $M_4 > M_2 > M_3 > M_1$. The operator's sensitivity indicates a preference change between M_2 and M_3, while consistently favouring M_4 as the most preferred and M_1 as the least favored. By contrast, the IV-q-RPFYOWGO has increased variability: When $q = 1$, the ranking is $M_4 > M_3 > M_2 > M_1$. At $q = 2,3,$ it transitions to $M_4 > M_2 > M_3 > M_1$. For $q \geq 4$, it stabilizes at $M_4 > M_2 > M_1 > M_3$. Overall, M_4 consistently, emerges as the best choice among both operators, while the intermediate rankings fluctuate, depending on the value of q, underscoring the impact of parametric modifications on aggregation sensitivity and the decision's results.

Table 10. Score values and ranking of alternatives using the IV-q-RPFYOWAO for different values of q.

q	$\mathfrak{H}(\mathfrak{T}_1)$	$\mathfrak{H}(\mathfrak{T}_2)$	$\mathfrak{H}(\mathfrak{T}_3)$	$\mathfrak{H}(\mathfrak{T}_4)$	Ranking of the alternatives
1	-0.192	-0.125	-0.093	-0.080	$M_4 > M_3 > M_2 > M_1$
2	0.055	0.102	0.116	0.163	$M_4 > M_3 > M_2 > M_1$
3	0.129	0.164	0.163	0.223	$M_4 > M_2 > M_3 > M_1$
4	0.146	0.174	0.166	0.227	$M_4 > M_2 > M_3 > M_1$
5	0.141	0.167	0.155	0.213	$M_4 > M_2 > M_3 > M_1$
6	0.130	0.153	0.142	0.194	$M_4 > M_2 > M_3 > M_1$
7	0.117	0.139	0.129	0.175	$M_4 > M_2 > M_3 > M_1$
8	0.104	0.125	0.117	0.157	$M_4 > M_2 > M_3 > M_1$
9	0.093	0.112	0.105	0.140	$M_4 > M_2 > M_3 > M_1$
10	0.083	0.099	0.095	0.126	$M_4 > M_2 > M_3 > M_1$

Table 11. Score values and ranking of alternatives using the IV-q-RPFYOWGO for different values of q.

q	$\mathfrak{H}(\mathfrak{T}_1)$	$\mathfrak{H}(\mathfrak{T}_2)$	$\mathfrak{H}(\mathfrak{T}_3)$	$\mathfrak{H}(\mathfrak{T}_4)$	Ranking of the alternatives
1	-0.226	-0.173	-0.154	-0.125	$M_4 > M_3 > M_2 > M_1$
2	0.012	0.046	0.045	0.109	$M_4 > M_2 > M_3 > M_1$
3	0.086	0.108	0.094	0.171	$M_4 > M_2 > M_3 > M_1$
4	0.104	0.121	0.101	0.179	$M_4 > M_2 > M_1 > M_3$
5	0.102	0.117	0.095	0.168	$M_4 > M_2 > M_1 > M_3$
6	0.093	0.108	0.086	0.152	$M_4 > M_2 > M_1 > M_3$
7	0.083	0.097	0.077	0.135	$M_4 > M_2 > M_1 > M_3$
8	0.073	0.086	0.068	0.119	$M_4 > M_2 > M_1 > M_3$
9	0.063	0.076	0.059	0.106	$M_4 > M_2 > M_1 > M_3$
10	0.055	0.067	0.053	0.094	$M_4 > M_2 > M_1 > M_3$

5.6. Sensitivity analysis for parameter τ

Tables 12 and 13 present the score values and rankings of alternatives based on varying values of the operational parameter τ, utilizing the IV-q-RPFYOWAO and IV-q-RPFYOWGO, respectively. For the IV-q-RPFYOWAO (Table 12), the ranking at $\tau = 1,2$ is $M_4 > M_2 > M_3 > M_1$; however, from $\tau \geq 3$ onwards, the ranking alters and stabilizes as $M_4 > M_3 > M_2 > M_1$. This suggests that elevated values of τ enhance the superiority of M_3 relative to M_2, although M_4 constantly remains the most favoured and M_1 the least favored. On the other hand, the IV-q-RPFYOWGO (Table 13) exhibits greater variability: At $\tau = 1$, the ranking is $M_4 > M_3 > M_2 > M_1$; at $\tau = 2,3$, it transitions to $M_4 > M_2 > M_3 > M_1$; and from $\tau \geq 4$, it stabilizes as $M_4 > M_2 > M_1 > M_3$. Consequently, in view of the discussion above, M_4 emerges as the most robust alternative across both operators regardless of the value of τ, whereas the shifts among M_2 and M_2 highlight the sensitivity of middle-ranked alternatives to variations in the operational parameter.

Table 12. Score values and ranking of alternatives using the IV-q-RPFYOWAO for different values of the operational parameter τ.

τ	$\mathfrak{H}(\mathfrak{T}_1)$	$\mathfrak{H}(\mathfrak{T}_2)$	$\mathfrak{H}(\mathfrak{T}_3)$	$\mathfrak{H}(\mathfrak{T}_4)$	Ranking of the alternatives
1	0.112	0.141	0.135	0.207	$M_4 > M_2 > M_3 > M_1$
2	0.129	0.164	0.163	0.223	$M_4 > M_2 > M_3 > M_1$
3	0.145	0.182	0.189	0.237	$M_4 > M_3 > M_2 > M_1$
4	0.158	0.196	0.213	0.248	$M_4 > M_3 > M_2 > M_1$
5	0.169	0.207	0.232	0.258	$M_4 > M_3 > M_2 > M_1$
6	0.181	0.217	0.248	0.266	$M_4 > M_3 > M_2 > M_1$
7	0.190	0.225	0.261	0.273	$M_4 > M_3 > M_2 > M_1$
8	0.198	0.232	0.272	0.278	$M_4 > M_3 > M_2 > M_1$
9	0.206	0.238	0.281	0.283	$M_4 > M_3 > M_2 > M_1$
10	0.213	0.243	0.286	0.288	$M_4 > M_3 > M_2 > M_1$

Table 13. Score values and ranking of alternatives using the IV-q-RPFYOWGO for different values of the operational parameter τ.

τ	$\mathfrak{H}(\mathfrak{T}_1)$	$\mathfrak{H}(\mathfrak{T}_2)$	$\mathfrak{H}(\mathfrak{T}_3)$	$\mathfrak{H}(\mathfrak{T}_4)$	Ranking of the alternatives
1	0.112	0.141	0.135	0.207	$M_4 > M_3 > M_2 > M_1$
2	0.086	0.108	0.094	0.171	$M_4 > M_2 > M_3 > M_1$
3	0.062	0.082	0.065	0.142	$M_4 > M_2 > M_3 > M_1$
4	0.042	0.062	0.045	0.119	$M_4 > M_2 > M_1 > M_3$
5	0.025	0.045	0.030	0.101	$M_4 > M_2 > M_1 > M_3$
6	0.011	0.032	0.019	0.087	$M_4 > M_2 > M_1 > M_3$
7	-0.001	0.022	0.011	0.075	$M_4 > M_2 > M_1 > M_3$
8	-0.011	0.013	0.005	0.066	$M_4 > M_2 > M_1 > M_3$
9	-0.019	0.006	-0.001	0.058	$M_4 > M_2 > M_1 > M_3$
10	-0.027	0.001	-0.005	0.051	$M_4 > M_2 > M_1 > M_3$

Computational complexity of the proposed methods

The computational complexity of the proposed IV-q-RPFYOWAOs is delineated by three principal steps: Computing the score and accuracy functions for each alternative, necessitating $O(mn)$ operations for m alternatives and n attributes; weighting and ordering the interval-valued q-rung picture fuzzy numbers, which entails a sorting step of $O(n\ log\ n)$; and executing the final aggregation in $O(n)$. Consequently, the total complexity is $O\ (mn + n\ log\ n)$, meaning that the operators exhibit computational efficiency and scalability for DM challenges involving larger datasets.

Real-world feasibility and constraints of the suggested techniques

The real-world feasibility of IV-q-RPFYOWAOs in MADM problems is attributed to their superior capacity to model uncertainty, hesitation, and incomplete information compared with traditional fuzzy methods, rendering them appropriate for domains such as financial risk management, engineering design, medical diagnosis, and cybersecurity assessments. Their interval-valued and q-rung frameworks afford adaptability in encapsulating varied expert perspectives, whilst Yager's OWA presents a balanced aggregation methodology. Nevertheless, practical constraints of the suggested operators encompass heightened computational complexity for extensive problems; the difficulty of selecting suitable q-parameters and weight vectors, which can profoundly influence the results; and the necessity for decision-makers to possess adequate knowledge to deliver consistent and meaningful input data.

More practical implications of the suggested techniques

This study's results have numerous implications for daily life. In real-world scenarios including financial risk assessments, engineering material selection, medical diagnosis, and evaluations of cybersecurity systems, decision-makers often encounter uncertainty, hesitancy, and inadequate knowledge. By using IV-q-RPFYOWAOs, they may better manage these issues. The adaptability of the q-rung and interval-valued structures enables experts to articulate viewpoints with enhanced precision, while Yager's OWAOs consolidates varied preferences, resulting in more resilient and comprehensible outcomes. These attributes augment the dependability of intricate DM procedures and provide organizations with a systematic, flexible framework that is applicable to real-world challenges necessitating both accuracy and versatility.

The proposed methodologies assist managers, mission planners, and materials engineers in assessing shielding materials by incorporating essential factors such as radiation protection, structural integrity, weight efficiency, and cost under unknown conditions. These approaches integrate expert assessments into a clear ranking system, ensuring that the selected materials adequately protect against cosmic radiation while adhering to the mission's requirements, thus improving safety and cost-effectiveness in space exploration. Furthermore, the IV-q-RPFYOWAOs equip managers with systematic tools for resolving trade-offs among competing criteria, thus diminishing dependence on subjective evaluations. By explicitly addressing the uncertainty inherent in expert opinions, they enhance the robustness and reliability of material selection decisions, increase confidence in the selected shielding technologies, and facilitate long-term strategic planning for sustainable and reliable space operations.

6. Conclusions

In this study, we have introduced two new Yager's OWAOs, namely the IV-q-RPFYOWAO and IV-q-RPFYOWGO, and have analyzed their structural features. We have also designed a novel ranking mechanism for IV-q-RPFNs and have presented a step-by-step mathematical approach to handle MADM situations with the newly developed techniques. Furthermore, we have implemented these methodologies to resolve the MADM challenge of selecting an optimal spacecraft shielding material against cosmic radiation. Finally, we have thoroughly compared our technical approaches against existing knowledge to verify their effectiveness.

6.1. Limitations of the current study

Despite the contributions of proposed strategies, it is important to acknowledge their limitations.

(1) The computational complexity of the proposed operators escalates with elevated values of q and longer interval datasets, potentially impacting their efficiency in extensive applications.

(2) The procedure necessitates precise identification of the weight vectors; any erroneous allocation may affect the dependability of outcomes.

6.2. Future research recommendations

We will extend the study of YAOs within the scope of complex interval-valued q-rung picture fuzzy sets, along with linguistic and probabilistic variants, to capture deeper uncertainty in our future studies. In addition, the suggested models will be further adapted for diverse decision-making scenarios, particularly in healthcare diagnostics, cybersecurity protection, Internet of Things enabled networks, disaster management, and sustainable environmental systems.

Author contributions

L. Ciurdariu, N. Ul Sahar and A. M. Zidan: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review and editing; N. Ul Sahar: Supervision. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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