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Research article

New developments in convex analysis: Harmonically trigonometric p-coordinated convex functions and related inequalities

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Abstract: In this paper, we introduce two novel classes of functions termed the harmonically trigonometric p-convex functions on $\Delta = [\tau, \upsilon] \times [\varphi, \chi]$ and harmonically trigonometric p-coordinated convex functions. We discuss relations between these two newly introduced classes of convex functions in two variables and validate the theoretical findings with visual 3D graphs that illustrate the relation landscapes and transition regimes between the function classes. We then study several special or limiting cases (e.g. $p = 1$, $p = -1$, pure harmonic trigonometric and pure trigonometric), showing that our general formulations leads to new novel convexities. Hermite-Hadamard, Fejér-Hermite-Hadamard, and related type integral inequalities, with bounds including hypergeometric functions, are presented via a novel coordinated class of convex functions.

Keywords: harmonically trigonometric p-coordinated convex functions; Fejér-Hermite-Hadamard type inequalities; Hermite-Hadamard type inequalities; Hölder's inequality
Mathematics Subject Classification: 26A51, 26D15

1. Introduction

The theory of convexity has been a key instrument in advancing the field of inequalities theory with many well-known results leveraging the convexity property of functions [1]. Convex functions and inequalities are intensely interconnected in diverse mathematical fields providing new horizons of applications [2, 3]. Convexity inherently tends to useful inequalities that express their behavior and are often used to develop bounds or optimality conditions [4, 5]. Among these, the Hermite-Hadamard double inequality stands out as an excessively studied outcome involving convex functions [6]. This result offers a comprehensive condition for identifying new types of convex functions and their extensive study.

In recent years, there has been a proliferation of novel approaches extending and generalizing classical concepts of convex functions [7, 8]. Convex functions, along with their generalizations and refinements, form a central part of optimization theory and have extensive applications in both theoretical and applied disciplines [9]. One such contribution by İmdat İşcan [10] introduced the category of harmonic convex functions and developed some Hermite-Hadamard type inequalities for this class of functions. Zhang et al. [11] introduced p-convex functions. Kadakal et al. [12] defined trigonometrically convex functions, established their properties, and used them to derive Hermite-Hadamard type inequalities. The authors [13] elegantly developed the notion of HT p-covex functions and the harmonically inverse sine trigonometric convex function [14]. Moreover, Dragomir [15] introduced convex functions on $[\tau, \upsilon] \times [\varphi, \chi]$ as well as convex functions on coordinates, and used them to provide Hermite-Hadamard inequality and related results. Aslam et al. [17] gave the concept of coordinated harmonically convex functions and derived Hermite-Hadamard type inequalities. Set and İşcan also discussed the same notion, and provided a refinement of the Hermite-Hadamard inequality presented by Aslam et al. [18]. Almutairi and Kiliçman provided refinement of the Hadamard inequality for coordinated convex functions [16]. In this work, we introduce a novel class of coordinated convex functions that can extend the coordinated convexity presented in [15]. We provide such functions along with graphs which satisfy novel convexity. First, we recall the following definitions used in the construction of our main results.

The extensive utility of convex functions and their extensions in fractional inequality theory stems from their diverse features, making them invaluable in various applications such as numerical integration, convex programming, and specialized means [19]. Researchers frequently employ different classes of convex functions to drive innovations in both academic literature and real-world problem solving [20]. The concept of convexity is becoming increasingly prominent with researchers regularly introducing new classes of functions that extend the principles of generalized convexity.

Convex functions are a widely utilized function by the researchers to make innovations in literature and in real-world problems [21]. Convexity has been prevailing day by day, and researchers are frequently introducing new classes of functions due to their importance and utilizations to solve complex real life problems.

Definition 1. *[1] Suppose ϑ is a subset of $\mathbb{R}^{\varkappa}$. Then, ϑ is convex set satisfying the property*

$$(1-\hbar)\,\alpha + \hbar\beta \in \vartheta, \qquad \forall\, \alpha, \beta \in \vartheta,\ \hbar \in [0,1].$$

Definition 2. *[1] If ϑ is a convex set and $\Omega : \vartheta \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function, then for $\hbar \in [0,1]$, $\forall\, \tau, \upsilon \in \vartheta$, the function Ω is said to be convex if*

$$\Omega\left(\hbar\tau + (1-\hbar)\upsilon\right) \le \hbar\Omega(\tau) + (1-\hbar)\Omega(\upsilon)$$

holds.

Noor et al. [22] defined p-harmonic convex sets and corresponding p-harmonic convex functions as follows.

Definition 3. *Let H_p be a subset of $\mathbb{R}^{\varkappa}/\{0\}$. Then, for $\hbar \in [0,1]$, $p \neq 0$, H_p is p-harmonic set if*

$$\left(\frac{\tau^p \upsilon^p}{\hbar\tau^p + (1-\hbar)\upsilon^p}\right)^{\frac{1}{p}} \in H_p$$

holds for all $\tau, \upsilon \in H_p$.

Definition 4. *Let* $\Omega : H_p \to \mathbb{R}$ *be a real-valued function. Then,* $\forall\, \tau, \upsilon \in H_p$ *and* $\hbar \in [0,1]$, Ω *is a p-harmonic convex function if*

$$\Omega\left(\left(\frac{\tau^p \upsilon^p}{\hbar\tau^p + (1-\hbar)\upsilon^p}\right)^{\frac{1}{p}}\right) \leq (1-\hbar)\Omega(\tau) + \hbar\Omega(\upsilon)$$

holds.

Kadakal et al. defined trigonometrically convex functions [12] as follows.

Definition 5. *Let* $\Omega : \vartheta \to \mathbb{R}$ *be a real valued non negative function, then* $\forall\, \tau, \upsilon \in \vartheta$ *and* $\hbar \in [0,1]$, Ω *is trigonometrically convex function if*

$$\Omega\left(\hbar\tau + (1-\hbar)\upsilon\right) \leq \sin\frac{\pi\hbar}{2}\Omega(\tau) + \cos\frac{\pi\hbar}{2}\Omega(\upsilon)$$

holds.

Dragomir gives the idea of convexity on Δ as well as coordinated class of convex functions in the following ways.

Definition 6. *[15] Let* $\Delta =: [\tau, \upsilon] \times [\varphi, \chi]$ *be a bidimensional interval in* $\mathbb{R}^2$ *with* $\tau < \upsilon$, $\varphi < \chi$, *and* $\Omega : \Delta \to \mathbb{R}$, *is a real valued function. Then, for* $\hbar \in [0,1]$, *the function* Ω *is said to be convex on* Δ *if*

$$\Omega\left(\hbar\eta + (1-\hbar)\mu, \hbar\nu + (1-\hbar)\rho\right) \leq \hbar\Omega(\eta, \nu) + (1-\hbar)\Omega(\mu, \rho)$$

holds for all (η, ν), $(\mu, \rho) \in \Delta$.

Definition 7. *[15] A function* $\Omega : \Delta \to \mathbb{R}$ *is called convex on coordinates of* Δ *if the partial correspondences* $\Omega_y : [\tau, \upsilon] \to \mathbb{R}$, $\Omega_y(\rho) = \Omega(\rho, y)$, *and* $\Omega_x : [\varphi, \chi] \to \mathbb{R}$, $\Omega_x(\psi) = \Omega(x, \psi)$, *are convex for all* $x \in [\tau, \upsilon]$ *and* $y \in [\varphi, \chi]$.

Erhan Set and Imdat İşcan presented the notion of harmonically convex functions on coordinates as follows.

Definition 8. *[18] Let* $\Delta =: [\tau, \upsilon] \times [\varphi, \chi]$ *in* $(0,\infty) \times (0,\infty)$ *be a bidimensional interval in* $\mathbb{R}^2$ *with* $\tau < \upsilon$, $\varphi < \chi$, *and* $\Omega : \Delta \to \mathbb{R}$, *is a real valued function. Then, for* $\hbar \in [0,1]$, *the function* Ω *is said to be harmonically convex on* Δ *if*

$$\Omega\left(\frac{\eta\mu}{\hbar\mu + (1-\hbar)\eta}, \frac{\nu\rho}{\hbar\rho + (1-\hbar)\nu}\right) \leq \hbar\Omega(\eta, \nu) + (1-\hbar)\Omega(\mu, \rho)$$

holds for all (η, ν), $(\mu, \rho) \in \Delta$.

Definition 9. *[18] A function* $\Omega : \Delta \to \mathbb{R}$ *is called harmonically convex on coordinates of* Δ *if the partial correspondences* $\Omega_y : [\tau, \upsilon] \to \mathbb{R}$, $\Omega_y(\rho) = \Omega(\rho, y)$, *and* $\Omega_x : [\varphi, \chi] \to \mathbb{R}$, $\Omega_x(\psi) = \Omega(x, \psi)$ *are harmonically convex for all* $x \in [\tau, \upsilon]$ *and* $y \in [\varphi, \chi]$.

The relationship between integral inequalities and convex functions is fundamental, as evidenced by numerous studies [23]. Researchers have shown substantial interest in exploring various inequalities, with particular emphasis on the Hermite-Hadamard integral inequality, which holds considerable significance due to its extensive applications in academic literature [24–26]. It is a fundamental result in convex analysis that provides bounds for integrals of convex functions [27]. Due to its foundational properties, many researchers have formulated its types for differnet convexities [28, 29]. The formulation of the Hermite-Hadamard integral inequality as presented in [6] can be expressed through the following theorems.

Theorem 1. *[6] If $\Omega : \vartheta \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, then $\forall\, \tau, \upsilon \in \vartheta$ and $\tau < \upsilon$, the Hermite-Hadamard inequality is defined as*

$$\Omega\left(\frac{\tau+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\tau}\int_{\tau}^{\upsilon}\Omega(\hbar)d\hbar \leq \frac{\Omega(\tau)+\Omega(\upsilon)}{2}.$$

Another important inequality established by Fejér [30] is a weighted version of the Hermite-Hadamard inequality, as stated by the following theorem.

Theorem 2. *[30] Let $\Omega : \vartheta \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then for $\tau, \upsilon \in \vartheta$, and $\tau < \upsilon$, the Fejér Hermite-Hadamard inequality is defined as*

$$\Omega\left(\frac{\tau+\upsilon}{2}\right)\int_{\tau}^{\upsilon}\Phi(u)du \leq \int_{\tau}^{\upsilon}\Omega(u)\Phi(u)du \leq \frac{\Omega(\tau)+\Omega(\upsilon)}{2}\int_{\tau}^{\upsilon}\Phi(u)du,$$

where $\Phi : [\tau, \upsilon] \to \mathbb{R}$ is a nonnegative, integrable, and symmetric function with respect to $u = \frac{\tau+\upsilon}{2}$.

This study aims to introduce the concept of harmonically trigonometric (HT) p-coordinated convex functions on Δ, along with harmonically trigonometric (HT) p-coordinated convex functions, elucidating their relationship. It is shown that this novel class encompasses various generalizations of coordinated convex functions. Furthermore, this paper formulates several refinements of Hermite-Hadamard like inequalities involving this function.

The paper is structured as follows: Section 2 introduces a new class of coordinated convex functions, termed harmonically trigonometric (HT) p-coordinated convex functions. Section 3 presents the Hermite-Hadamard inequality, Fejèr Hermite-Hadamard inequality, and other related types of inequalities. Finally, the concluding section summarizes the paper's findings.

2. Main results

In this section, we present novel convexities, namely HT p-convexity on $\Delta =: [\tau, \upsilon] \times [\varphi, \chi]$ and HT p-coordinated convexity. The introduction of new classes of convex functions extends the scope of mathematical problems, providing a broader framework for understanding convexity. It allows for the generalization of existing literature, potentially including a wider range of functions. This novel idea enriches mathematical theory, provides powerful analytical tools, and contributes to the solution of mathematical problems, particularly in the realm of inequalities and special means.

Definition 10. *A function $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ is an HT p-convex function on Δ if for $\tau, \upsilon, \varphi, \chi \in H_p$, $\hbar \in [0,1]$, $p \neq 0$, the following inequality is true:*

$$\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p + (1-\hbar)\tau^p}\right)^{\frac{1}{p}}, \left(\frac{\varphi^p\chi^p}{\hbar\chi^p + (1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right) \leq \sin\frac{\pi\hbar}{2}\Omega(\tau,\varphi) + \cos\frac{\pi\hbar}{2}\Omega(\upsilon,\chi), \tag{2.1}$$

for all $(\tau,\varphi), (\upsilon,\chi) \in \Delta$.
It is obvious that if the inequality is reversed, then $\Omega : \Delta \to \mathbb{R}$ is HT p-concave on Δ.

Now, we will discuss some special cases of Definition 10 as follows.
If we take $p = 1$, we get a new convexity on Δ, namely the HT convex function on Δ.

Definition 11. *A function $\Omega : \Delta = [\tau,\upsilon]\times[\varphi,\chi] \to \mathbb{R}$ is an HT convex function on Δ if for $\tau, \upsilon, \varphi, \chi \in \mathbb{R}$, $\hbar \in [0,1]$, the inequality*

$$\Omega\left(\frac{\tau\upsilon}{\hbar\upsilon + (1-\hbar)\tau}, \frac{\varphi\chi}{\hbar\chi + (1-\hbar)\varphi}\right) \leq \sin\frac{\pi\hbar}{2}\Omega(\tau,\varphi) + \cos\frac{\pi\hbar}{2}\Omega(\upsilon,\chi), \tag{2.2}$$

is true for all $(\tau,\varphi), (\upsilon,\chi) \in \Delta$.
If the inequality is reversed, then $\Omega : \Delta \to \mathbb{R}$ is HT concave on Δ.

If we take $p = -1$, we get a new convexity on Δ, namely a trigonometric convex function on Δ.

Definition 12. *A function $\Omega : \Delta = [\tau,\upsilon] \times [\varphi,\chi] \to \mathbb{R}$ is a trigonometric convex function on Δ if for $\tau, \upsilon, \varphi, \chi \in \mathbb{R}$, $\hbar \in [0,1]$, the inequality*

$$\Omega\left(\hbar\upsilon + (1-\hbar)\tau, \hbar\chi + (1-\hbar)\varphi\right) \leq \cos\frac{\pi\hbar}{2}\Omega(\tau,\varphi) + \sin\frac{\pi\hbar}{2}\Omega(\upsilon,\chi), \tag{2.3}$$

is true for all $(\tau,\varphi), (\upsilon,\chi) \in \Delta$.
If the inequality is reversed, then $\Omega : \Delta \to \mathbb{R}$ is a trigonometric concave function on Δ.

Definition 13. *A function $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ is an HT p-coordinated convex function if the partial mappings $\Omega_\upsilon : H_p \to \mathbb{R}$ defined as $\Omega_\upsilon(\varphi) = \Omega(\varphi,\upsilon)$ and $\Omega_\tau : H_p \to \mathbb{R}$ defined as $\Omega_\tau(\chi) = \Omega(\tau,\chi)$ are HT p-convex functions.*

A formal definition for HT p-coordinated convex functions can be given as follows.

Definition 14. *A function $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ is an HT p-coordinated convex function if for $\tau, \upsilon, \varphi, \chi \in H_p$, $\hbar, \ell \in [0,1]$, $p \neq 0$, the inequality*

$$\begin{aligned}
&\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p + (1-\hbar)\tau^p}\right)^{\frac{1}{p}}, \left(\frac{\varphi^p\chi^p}{\ell\chi^p + (1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right) \\
&\leq \frac{1}{2}\Bigg(\cos\frac{\pi(\hbar+\ell)}{2}\left(\Omega(\tau,\varphi) - \Omega(\upsilon,\chi)\right) + \sin\frac{\pi(\hbar+\ell)}{2}\left(\Omega(\upsilon,\varphi) + \Omega(\tau,\chi)\right) \\
&+ \cos\frac{\pi(\hbar-\ell)}{2}\left(\Omega(\tau,\varphi) + \Omega(\upsilon,\chi)\right) + \sin\frac{\pi(\hbar-\ell)}{2}\left(\Omega(\upsilon,\varphi) - \Omega(\tau,\chi)\right)\Bigg),
\end{aligned}$$

is true for all $(\tau,\varphi), (\upsilon,\chi) \in \Delta$.
It is obvious that if the inequality is reversed, then $\Omega : \Delta \to \mathbb{R}$ is an HT p-coordinated concave.

Here, one can raise the question that many researchers are generalizing convex functions, but what are the benefits of generalizing the concept of convexity? One of the answers to this question is that there exist many functions which are not convex but follow some type of generalized convexity. For example, the function $\Omega : \Delta = [-\pi, \pi] \times [-\pi, \pi] \to \mathbb{R}$ defined by $\Omega(\tau, \varphi) = \frac{1}{\tau^p \varphi^p}$ is not convex, but it is an HT p-convex function for $p = -2$. The graphs of Figure 1 with choice of the parameters $\tau \in [-\pi, 0]$, $\upsilon = \pi, \varphi = -\pi, \chi = \pi$, and $0 \leq \hbar \leq 1$ show that $\Omega(\tau, \varphi) = \frac{1}{\tau^p \varphi^p}$ is not convex while the graphs of Figure 2 show that it is an HT p-convex function.

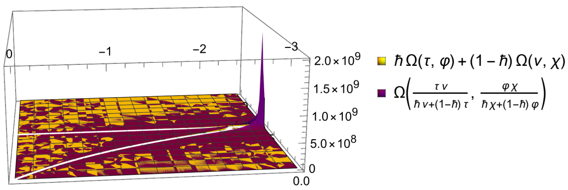

Figure 1. The graph illustrates that $\Omega(\tau, \varphi) = \frac{1}{\tau^p \varphi^p}$ is not convex.

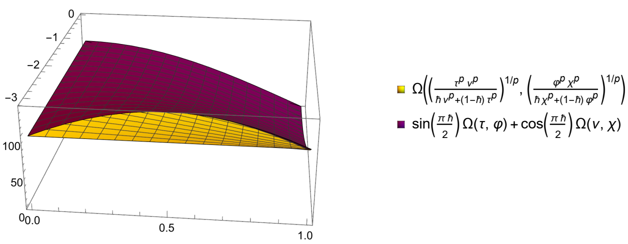

Figure 2. The graph illustrates that $\Omega(\tau, \varphi) = \frac{1}{\tau^p \varphi^p}$ is an HT p-convex function.

So, the key benefit of generalizations of numerous convexities is that they are useful in applying the potent characteristics of convex functions to a broader class of non-convex problems, which we encounter in various situations of our everyday life. Generalizations greatly broaden the use of mathematical programming by providing effective analysis and solution techniques for unsolvable non-convex optimization problems, and its applications can be found in applied mathematics, economics, engineering, and medicine. In order to identify global solutions or comprehend the behavior of local minima in intricate, non-convex environments, these generalizations offer analytical tools and a framework.

Lemma 1. *Every HT p-convex function $\Omega : \Delta \to \mathbb{R}$ on Δ is HT p-convex on coordinates.*

Proof. Suppose that $\Omega : \Delta \to \mathbb{R}$ is HT p-convex on Δ. Consider $\Omega_\tau : H_p \to \mathbb{R}$ defined as $\Omega_\tau(\chi) =$

$\Omega(\tau,\chi)$. Then for all $\hbar \in [0,1]$ and $\varphi,\chi \in H_p$, using the HT p-convexity on Δ, we have

$$\begin{aligned}
&\Omega_\tau\left(\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=\Omega\left(\tau,\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=\Omega\left(\left(\frac{\tau^p\tau^p}{\hbar\tau^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&\le \sin\frac{\pi\hbar}{2}\Omega(\tau,\varphi)+\cos\frac{\pi\hbar}{2}\Omega(\tau,\chi)\\
&=\sin\frac{\pi\hbar}{2}\Omega_\tau(\varphi)+\cos\frac{\pi\hbar}{2}\Omega_\tau(\chi).
\end{aligned}$$

This implies that Ω_τ is HT p-convex. Similarly, it can be proved that Ω_υ is also HT p-convex. □

To show that the converse of Lemma 1 may or may not hold, we provide two examples.

Example 1. *Let $\Omega(x,y) = k$, $k > 0$. Then Ω is an HT p-convex function on Δ, and Ω is HT p-coordinated convex function.*

Proof. By using the fact $\sin\frac{\pi\hbar}{2}+\cos\frac{\pi\hbar}{2}\ge 1$, $\forall\ell,\hbar\in[0,1]$, $k>0$, we can write

$$\begin{aligned}
&\Omega_\tau\left(\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=\Omega\left(\tau,\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=k\le k\left(\sin\frac{\pi\hbar}{2}+\cos\frac{\pi\hbar}{2}\right)\\
&=\sin\frac{\pi\hbar}{2}\Omega(\tau,\chi)+\cos\frac{\pi\hbar}{2}\Omega(\tau,\varphi)\\
&=\sin\frac{\pi\hbar}{2}\Omega_\tau(\chi)+\cos\frac{\pi\hbar}{2}\Omega_\tau(\varphi),
\end{aligned}$$

which shows that Ω_τ is HT p-convex. Similarly, it can easily be seen that Ω_υ is also HT p-convex. Hence, Ω is an HT p-coordinated convex function. To show that it is also HT p-convex on Δ, we proceed as follows:

$$\begin{aligned}
&\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=k\le k(\sin\frac{\pi\hbar}{2}+\cos\frac{\pi\hbar}{2})\\
&=\cos\frac{\pi\hbar}{2}\Omega(\upsilon,\chi)+\sin\frac{\pi\hbar}{2}\Omega(\tau,\varphi), \qquad (2.4)
\end{aligned}$$

which leads to the fact that Ω is HT p-convex on Δ. The validation of the result can be confirmed by the following graphs of Figure 3 via the choice of parameters $k = 10, \tau \in [-1, 0]$, $\upsilon = 2, \varphi = -2, \chi = 4, p = 5$, and $0 \leq \hbar \leq 1$.

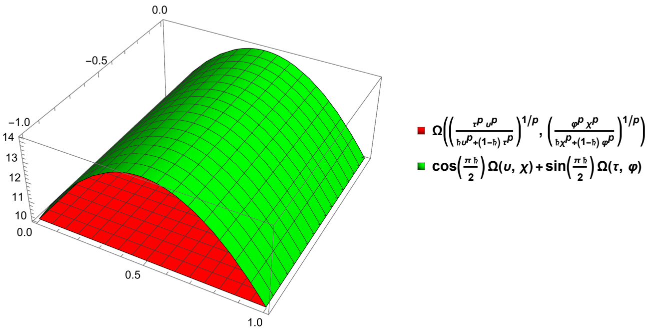

Figure 3. The graph illustrates the validity of (2.4).

□

Example 2. *Let $\Omega(\tau, \upsilon) = \frac{1}{\tau^p \upsilon^p}$, $\tau, \upsilon \in H_p$ be positive valued function. Then Ω is an HT p-coordinated convex function [13], but it is not HT p-convex on Δ.*

Proof. Consider

$$\begin{aligned}
&\Omega_\tau\left(\left(\frac{\varphi^p \chi^p}{\hbar \chi^p + (1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right) \\
&= \Omega\left(\tau, \left(\frac{\varphi^p \chi^p}{\hbar \chi^p + (1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right) \\
&= \frac{1}{\tau^p \left(\left(\frac{\varphi^p \chi^p}{\hbar \chi^p + (1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)^p} \\
&= \frac{1}{\tau^p} \frac{\hbar \chi^p + (1-\hbar)\varphi^p}{\varphi^p \chi^p} \\
&= \frac{1}{\tau^p}\left(\frac{\hbar}{\varphi^p} + \frac{1-\hbar}{\chi^p}\right) \\
&= \hbar\Omega(\tau, \varphi) + (1-\hbar)\Omega(\tau, \chi).
\end{aligned}$$

Now using the fact $\hbar \leq \sin\frac{\pi\hbar}{2}$ and $1-\hbar \leq \cos\frac{\pi\hbar}{2}$, for all $\hbar \in [0, 1]$, we can write

$$\begin{aligned}
&\Omega_\tau\left(\left(\frac{\varphi^p \chi^p}{\hbar \chi^p + (1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right) f \\
&\leq \sin\frac{\pi\hbar}{2}\Omega(\tau, \varphi) + \cos\frac{\pi\hbar}{2}\Omega(\tau, \chi) \\
&= \sin\frac{\pi\hbar}{2}\Omega_\tau(\varphi) + \cos\frac{\pi\hbar}{2}\Omega_\tau(\chi). \qquad (2.5)
\end{aligned}$$

On the same lines, the HT p-convexity of Ω_υ can be seen. But, this function fails to be HT p-convex on Δ, as

$$\begin{aligned}
&\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&=\frac{1}{\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)\left(\frac{\varphi^p\chi^p}{\hbar\chi^p+(1-\hbar)\varphi^p}\right)}\\
&=\frac{\hbar\upsilon^p+(1-\hbar)\tau^p}{\tau^p\upsilon^p}\times\frac{\hbar\chi^p+(1-\hbar)\varphi^p}{\varphi^p\chi^p}\\
&=\left(\frac{\hbar}{\tau^p}+\frac{1-\hbar}{\upsilon^p}\right)\left(\frac{\hbar}{\varphi^p}+\frac{1-\hbar}{\chi^p}\right)\\
&\le\hbar^2\frac{1}{\tau^p\varphi^p}+\frac{\hbar(1-\hbar)}{\tau^p\chi^p}+\frac{\hbar(1-\hbar)}{\upsilon^p\varphi^p}+\frac{(1-\hbar)^2}{\upsilon^p\chi^p}\\
&\le\sin^2\frac{\pi\hbar}{2}\Omega(\tau,\varphi)+\cos^2\frac{\pi\hbar}{2}\Omega(\upsilon,\chi)+\frac{1}{2}\sin\pi\hbar\left(\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)\right)\\
&\le\sin\frac{\pi\hbar}{2}\Omega(\tau,\varphi)+\cos\frac{\pi\hbar}{2}\Omega(\upsilon,\chi)+\frac{1}{2}\left(\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)\right).
\end{aligned}$$

Hence, Ω is not HT p-convex on Δ as $\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)>0$.
The validation of the result can be confirmed by the following graph of Figure 4 via the choice of parameters $\Omega(u,v)=\frac{1}{u^pv^p}$, $\tau\in[1,1.5]$, $\upsilon=2$, $\varphi=2$, $\chi=4$, $p=2$, and $0\le\hbar\le 1$.

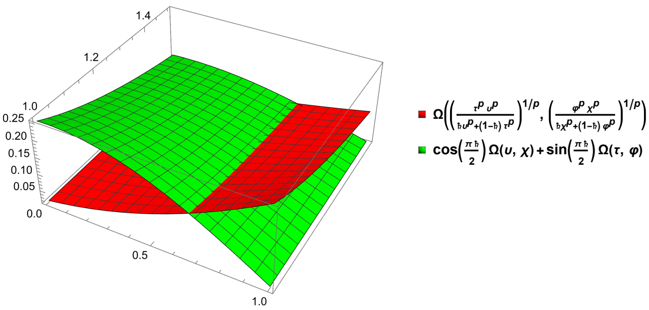

Figure 4. The graph illustrates that Ω is not HT p-convex on Δ.

□

3. Hermite-Hadamard, Fejér-Hermite-Hadamard and related type inequalities

In this section, we present Hermite-Hadamard, Fejér-Hermite-Hadamard, and related inequalities for HT p-coordinated convex functions. To validate the obtained results, 3D graphs were sketched using Mathematica.

Theorem 3. *Let $\Omega:\Delta=H_p\times H_p\to\mathbb{R}$ be an HT-p coordinated convex function and $\tau,\upsilon,\varphi,\chi\in H_p$,*

$\tau < \upsilon$, $\varphi < \chi$. If $\Omega \in L(\Delta)$, then the following Hermite-Hadamard inequality holds:

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\\
&\leq \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx\\
&\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\upsilon,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\chi)}{2}\right), \qquad p\neq 0. \qquad (3.1)
\end{aligned}$$

Proof. Choosing $X^p = \frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}$, $Y^p = \frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}$, $Z^p = \frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}$, and $W^p = \frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}$, we have

$$\frac{\tau^p\upsilon^p}{\tau^p+\upsilon^p} = \frac{X^pY^p}{X^p+Y^p},$$

and

$$\frac{\varphi^p\chi^p}{\varphi^p+\chi^p} = \frac{Z^pW^p}{Z^p+W^p}.$$

Applying the definition of HT p-coordinated convexity with $\hbar = \ell = \frac{1}{2}$, we have

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\\
&= \Omega\left(\left(\frac{2X^pY^p}{X^p+Y^p}\right)^{\frac{1}{p}},\left(\frac{2Z^pW^p}{Z^p+W^p}\right)^{\frac{1}{p}}\right)\\
&\leq \frac{1}{2}\left(\Omega(X,Z)+\Omega(X,W)+\Omega(Y,Z)+\Omega(Y,W)\right)\\
&= \frac{1}{2}\left(\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right.\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\\
&\left.+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\right). \qquad (3.2)
\end{aligned}$$

Integrating inequality (3.2) on $[0,1]\times[0,1]$, we get

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)\\
&\leq \int_0^1\int_0^1 \Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)d\hbar d\ell
\end{aligned}$$

$$+\int_0^1\int_0^1 \Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)d\hbar d\ell. \tag{3.3}$$

By employing suitable substitutions in (3.3), we have

$$\begin{aligned}&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)\\&\le \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx.\end{aligned} \tag{3.4}$$

By the definition of HT p-coordinated convexity, we have

$$\begin{aligned}&\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\\&\le \frac{1}{2}\Big((\Omega(\tau,\varphi)-\Omega(\upsilon,\chi))\cos\frac{\pi}{2}(\hbar+\ell)+(\Omega(\upsilon,\varphi)+\Omega(\tau,\chi))\sin\frac{\pi}{2}(\hbar+\ell)\\&+(\Omega(\upsilon,\chi)+\Omega(\tau,\varphi))\cos\frac{\pi}{2}(\hbar-\ell)+(\Omega(\upsilon,\varphi)-\Omega(\tau,\chi))\sin\frac{\pi}{2}(\hbar-\ell)\Big).\end{aligned} \tag{3.5}$$

Integrating inequality (3.5) over $[0,1]\times[0,1]$, we have

$$\begin{aligned}&\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx\\&\le \frac{1}{2}\Big((\Omega(\tau,\varphi)-\Omega(\upsilon,\chi))\int_0^1\int_0^1\cos\frac{\pi}{2}(\hbar+\ell)d\hbar d\ell\\&+(\Omega(\upsilon,\varphi)+\Omega(\tau,\chi))\int_0^1\int_0^1\sin\frac{\pi}{2}(\hbar+\ell)d\hbar d\ell\\&+(\Omega(\upsilon,\chi)+\Omega(\tau,\varphi))\int_0^1\int_0^1\cos\frac{\pi}{2}(\hbar-\ell)d\hbar d\ell\\&+(\Omega(\upsilon,\varphi)-\Omega(\tau,\chi))\int_0^1\int_0^1\sin\frac{\pi}{2}(\hbar-\ell)d\hbar d\ell\Big).\end{aligned}$$

This can be written as

$$\begin{aligned}&\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx\\&\le \frac{8}{\pi^2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)\\&\le 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right),\end{aligned} \tag{3.6}$$

where

$$\int_0^1\int_0^1\cos\frac{\pi}{2}(\hbar+\ell)d\hbar d\ell=\int_0^1\int_0^1\sin\frac{\pi}{2}(\hbar-\ell)d\hbar d\ell=0$$

and

$$\int_0^1\int_0^1 \cos\frac{\pi}{2}(\hbar-\ell)d\hbar d\ell=\int_0^1\int_0^1 \sin\frac{\pi}{2}(\hbar+\ell)d\hbar d\ell=\frac{8}{\pi^2}$$

and combining (3.4) and (3.6), we get the desired result. □

Corollary 1. *Choosing* $p = 1$ *in* (3.1)*, we get the Hermite-Hadamard inequality for HT coordinated convexity.*

$$\begin{aligned}&\Omega\left(\frac{2\tau\upsilon}{\tau+\upsilon},\frac{2\varphi\chi}{\varphi+\chi}\right)\\ &\leq \frac{2\tau\upsilon\varphi\chi}{(\upsilon-\tau)(\chi-\varphi)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^2}dydx\\ &\leq \frac{8}{\pi^2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)\\ &\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right).\end{aligned}$$

Corollary 2. *Choosing* $p = -1$ *in* (3.1)*, we get the Hermite-Hadamard inequality for trigonometric coordinated convexity:*

$$\begin{aligned}\Omega\left(\frac{\tau+\upsilon}{2},\frac{\varphi+\chi}{2}\right) &\leq \frac{2}{(\upsilon-\tau)(\chi-\varphi)}\int_\tau^\upsilon\int_\varphi^\chi \Omega(x,y)dydx\\ &\leq \frac{8}{\pi^2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)\\ &\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right).\end{aligned}$$

Example 3. *To authenticate the result obtained in Theorem 3, we used Mathematica to get the following graph in Figure 5 by choosing* $\Omega(x,y) = \frac{1}{x^p y^p}$*,* $p = 2$*, and* $\upsilon = 2$*,* $\tau \in [1, 1.5]$*,* $\varphi = 2$*,* $\chi \in [2.5, 3]$*, for Hermite-Hadamard inequality* (3.1)*.*

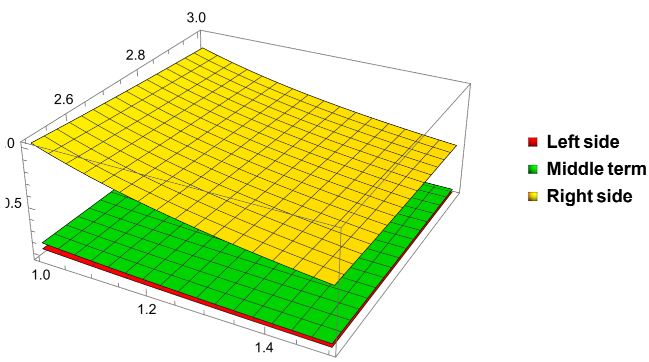

Figure 5. The graphs of functions validate the Hermite-Hadamard inequality (3.1).

By using Definition 13, we provide a refined form of the Hermite-Hadamard inequality given under Theorem 3.

Theorem 4. *Let $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ be an HT p-coordinated convex function and $\tau, \upsilon, \varphi, \chi \in H_p$, $\tau < \upsilon$, $\varphi < \chi$. If $\Omega \in L(\Delta)$, then the following Hermite-Hadamard inequality holds:*

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\\
&\leq \frac{p}{\sqrt{2}}\left(\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_{\varphi}^{\chi}\frac{\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}}, y\right)}{y^{p+1}}dy\right.\\
&\left.+\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_{\tau}^{\upsilon}\frac{\Omega\left(x, \left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)}{x^{p+1}}dx\right)\\
&\leq \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Omega(x,y)}{(xy)^{p+1}}dydx\\
&\leq p\left(\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_{\tau}^{\upsilon}\frac{\Omega(x,\varphi)}{x^{p+1}}dx+\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_{\tau}^{\upsilon}\frac{\Omega(x,\chi)}{x^{p+1}}dx\right.\\
&\left.+\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_{\varphi}^{\chi}\frac{\Omega(\tau,y)}{y^{p+1}}dy+\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_{\varphi}^{\chi}\frac{\Omega(\upsilon,y)}{y^{p+1}}dy\right)\\
&\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\upsilon,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\chi)}{2}\right), \qquad p\neq 0. \qquad (3.7)
\end{aligned}$$

Proof. Since Ω is HT-p convex on coordinates, the function $\Omega_x : [\varphi, \chi] \to \mathbb{R}$, $\Omega_x(y) = \Omega(x, y)$ is HT-p convex on $[\varphi, \chi]$ by definition. So, the following Hermite-Hadamard inequality [13] for Ω_x holds:

$$\begin{aligned}
&\Omega\left(x, \left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)\\
&\leq \frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p-\varphi^p}\int_{\varphi}^{\chi}\frac{\Omega(x,y)}{y^{p+1}}dy\\
&\leq 2\sqrt{2}\left(\frac{\Omega(x,\varphi)+\Omega(x,\chi)}{2}\right). \qquad (3.8)
\end{aligned}$$

Multiplying inequality (3.8) by $\frac{\sqrt{2}p\tau^p\upsilon^p}{(\upsilon^p-\tau^p)x^{p+1}}$ and then integrating on the interval $[\tau, \upsilon]$ with respect to x leads to

$$\begin{aligned}
&\frac{\sqrt{2}p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_{\tau}^{\upsilon}\frac{\Omega\left(x, \left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)}{x^{p+1}}dx\\
&\leq \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Omega(x,y)}{(xy)^{p+1}}dydx
\end{aligned}$$

$$\leq \frac{2p\tau^p\upsilon^p}{\upsilon^p - \tau^p}\int_\tau^\upsilon \frac{\Omega(x,\varphi) + \Omega(x,\chi)}{x^{p+1}}dx. \tag{3.9}$$

Following the same argument for Ω_y gives

$$\begin{aligned} &\frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}}, y\right)}{y^{p+1}}dy \\ &\leq \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p - \tau^p)(\chi^p - \varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx \\ &\leq \frac{2p\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega(\tau,y) + \Omega(\upsilon,y)}{y^{p+1}}dy. \end{aligned} \tag{3.10}$$

Adding (3.9) and (3.10), we have

$$\begin{aligned} &\frac{p}{\sqrt{2}}\left(\frac{\tau^p\upsilon^p}{\upsilon^p - \tau^p}\int_\tau^\upsilon \frac{\Omega\left(x, \left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)}{x^{p+1}}dx + \frac{\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}}, y\right)}{y^{p+1}}dy\right) \\ &\leq \frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p - \tau^p)(\chi^p - \varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(x,y)}{(xy)^{p+1}}dydx \\ &\leq p\left(\frac{\tau^p\upsilon^p}{\upsilon^p - \tau^p}\int_\tau^\upsilon \frac{\Omega(x,\varphi)}{x^{p+1}}dx + \frac{\tau^p\upsilon^p}{\upsilon^p - \tau^p}\int_\tau^\upsilon \frac{\Omega(x,\chi)}{x^{p+1}}dx \right. \\ &\left. + \frac{\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega(\tau,y)}{y^{p+1}}dy + \frac{\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega(\upsilon,y)}{y^{p+1}}dy\right). \end{aligned} \tag{3.11}$$

By the Hermite-Hadamard inequality for Ω_x with $x = \left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}}$,

$$\begin{aligned} &\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p + \tau^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\chi^p + \varphi^p}\right)^{\frac{1}{p}}\right) \\ &\quad \leq \frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p - \varphi^p}\int_\varphi^\chi \frac{\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}}, y\right)}{y^{p+1}}dy. \end{aligned} \tag{3.12}$$

Similarly, by the Hermite-Hadamard inequality for Ω_y with $y = \left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}$,

$$\begin{aligned} &\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p + \tau^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\chi^p + \varphi^p}\right)^{\frac{1}{p}}\right) \\ &\quad \leq \frac{\sqrt{2}p\tau^p\upsilon^p}{\upsilon^p - \tau^p}\int_\tau^\upsilon \frac{\Omega\left(\left(x, \frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)}{x^{p+1}}dx. \end{aligned} \tag{3.13}$$

Addition of inequalities (3.12) and (3.13) gives

$$\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p + \tau^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\chi^p + \varphi^p}\right)^{\frac{1}{p}}\right)$$

$$\leq \frac{p}{\sqrt{2}}\left[\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega\left(x,\left(\frac{2\varphi^p\chi^p}{\chi^p+\varphi^p}\right)^{\frac{1}{p}}\right)}{x^{p+1}}dx+\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p+\tau^p}\right)^{\frac{1}{p}},y\right)}{y^{p+1}}dy\right]. \tag{3.14}$$

For the rightmost inequality of (3.1), again taking into account the Hermite-Hadamard inequality for Ω_x, we have

$$\frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega_x(y)}{y^{p+1}}dy \leq 2\sqrt{2}\frac{\Omega_x(\varphi)+\Omega_x(\chi)}{2}. \tag{3.15}$$

For $x=\tau$ and $x=\upsilon$ in (3.15), we have

$$\frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega(\tau,y)}{y^{p+1}}dy \leq 2\sqrt{2}\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)}{2}. \tag{3.16}$$

$$\frac{\sqrt{2}p\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega(\upsilon,y)}{y^{p+1}}dy \leq 2\sqrt{2}\frac{\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}. \tag{3.17}$$

By a similar argument as above for Ω_y with $y=\varphi$ and $y=\chi$, we have

$$\frac{\sqrt{2}p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(x,\varphi)}{x^{p+1}}dx \leq 2\sqrt{2}\frac{\Omega(\tau,\varphi)+\Omega(\upsilon,\varphi)}{2}. \tag{3.18}$$

$$\frac{\sqrt{2}p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(x,\chi)}{x^{p+1}}dx \leq 2\sqrt{2}\frac{\Omega(\tau,\chi)+\Omega(\upsilon,\chi)}{2}. \tag{3.19}$$

Adding (3.16)–(3.19) gives

$$\begin{aligned}&p\left(\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(x,\varphi)}{x^{p+1}}dx+\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(x,\chi)}{x^{p+1}}dx\right.\\&\left.+\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega(\upsilon,y)}{y^{p+1}}dy+\frac{\varphi^p\chi^p}{\chi^p-\varphi^p}\int_\varphi^\chi \frac{\Omega(\tau,y)}{y^{p+1}}dy\right)\\&\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\upsilon,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\chi)}{2}\right).\end{aligned} \tag{3.20}$$

Combining (3.11), (3.14), and (3.20), we have the required inequality. □

Example 4. *To authenticate the result obtained in Theorem 4, we used Mathematica to get the following 2D graph in Figure 6 by choosing* $\Omega(x,y)=\frac{1}{x^py^p}$, $p=3$, *and* $\upsilon=2$, $\tau\in[1,1.2]$, $\varphi=1.5$, $\chi=2.5$, *of the Hermite-Hadamard inequality* (3.1).

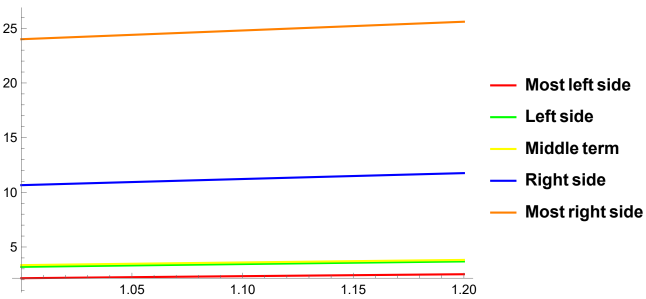

Figure 6. The graphs represent 2D view of the Hermite-Hadamard inequality (3.7).

Example 5. *To authenticate the result obtained in Theorem 4, we used Mathematica to get the following graph in Figure 7 by choosing* $\Omega(x, y) = \frac{1}{x^p y^p}$, $p = 3$, *and* $\upsilon = 2$, $\tau \in [1, 1.2]$, $\varphi = 1.5$, $\chi \in [2, 2.5]$, *of the Hermite-Hadamard inequality* (3.7).

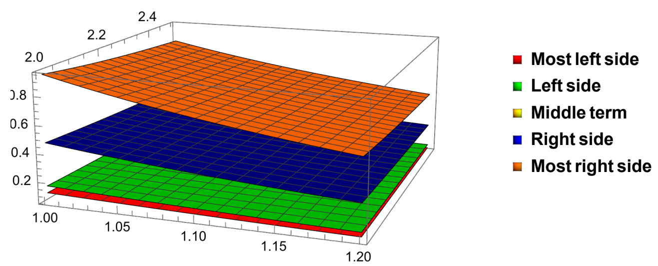

Figure 7. The graphs of functions represent the 3D view of the Hermite-Hadamard inequality (3.7).

Theorem 5. *Let* $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ *be an HT p-coordinated convex function and* $\tau, \upsilon, \varphi, \chi \in H_p$, $\tau < \upsilon$, $\varphi < \chi$. *If* $\Omega \in L(\Delta)$, *then the Fejér Hermite-Hadamard inequality*

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\upsilon^p + \tau^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\chi^p + \varphi^p}\right)^{\frac{1}{p}}\right)\int_\tau^\upsilon\int_\varphi^\chi \frac{\Phi(u, v)}{(uv)^{p+1}}dudv \\
&\leq 2\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(u, v)\Phi(u, v)}{(uv)^{p+1}}dudv \\
&\leq 4\left(\frac{\Omega(\tau, \varphi) + \Omega(\tau, \chi) + \Omega(\upsilon, \varphi) + \Omega(\upsilon, \chi)}{2}\right)\int_\tau^\upsilon\int_\varphi^\chi \frac{\Phi(u, v)}{(uv)^{p+1}}dudv, \qquad p \neq 0 \qquad (3.21)
\end{aligned}$$

holds, where $\Phi : [\tau, \upsilon] \times [\varphi, \chi] \to \mathbb{R}$ *is a non-negative, integrable function satisfying the harmonic p-symmetric property, i.e.,*

$$\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar}\right)^{\frac{1}{p}}, \left(\frac{\varphi^p\chi^p}{\ell}\right)^{\frac{1}{p}}\right) = \Phi\left(\left(\frac{\tau^p\upsilon^p}{\tau^p + \upsilon^p - \hbar}\right)^{\frac{1}{p}}, \left(\frac{\varphi^p\chi^p}{\varphi^p + \chi^p - \ell}\right)^{\frac{1}{p}}\right). \qquad (3.22)$$

Proof. Using a similar technique as in Theorem 3 and the definition of HT p-coordinated convexity of Ω, we get

$$\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p + \upsilon^p}\right)^{\frac{1}{p}}, \left(\frac{2\varphi^p\chi^p}{\varphi^p + \chi^p}\right)^{\frac{1}{p}}\right) \\
&\leq \frac{1}{2}\left(\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p + (1-\hbar)\upsilon^p}\right)^{\frac{1}{p}}, \left(\frac{\varphi^p\chi^p}{\ell\varphi^p + (1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\Bigg)\\
&\leq \frac{1}{2}\Big((\Omega(\upsilon,\varphi)+\Omega(\tau,\chi))\sin\frac{\pi}{2}(\hbar+\ell)\\
&+(\Omega(\upsilon,\chi)+\Omega(\tau,\varphi))\cos\frac{\pi}{2}(\hbar-\ell)\\
&+(\Omega(\upsilon,\chi)+\Omega(\tau,\varphi))\sin\frac{\pi}{2}(\hbar+\ell)\\
&+(\Omega(\upsilon,\varphi)+\Omega(\tau,\chi))\cos\frac{\pi}{2}(\hbar-\ell)\Big)\\
&=\frac{1}{2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)\\
&\times\left(\sin\frac{\pi}{2}(\hbar+\ell)+\cos\frac{\pi}{2}(\hbar-\ell)\right).
\end{aligned}
\tag{3.23}
$$

As Φ is non-negative, multiplying the double inequality (3.23) by $\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)$, we have

$$
\begin{aligned}
&\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)\\
&\leq\frac{1}{2}\Bigg(\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\\
&+\Omega\left(\left(\frac{\tau^p\upsilon^p}{(1-\hbar)\tau^p+\hbar\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)\Bigg)\\
&\times\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)
\end{aligned}
$$

$$
\leq\frac{1}{2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)
$$

$$\times\left(\sin\frac{\pi}{2}(\hbar+\ell)+\cos\frac{\pi}{2}(\hbar-\ell)\right)$$
$$\times\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right).$$

Integrating the above inequality over $[0,1]\times[0,1]$ and using condition (3.22) leads to

$$\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\int_0^1\int_0^1\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)d\hbar d\ell$$
$$\leq\int_0^1\int_0^1\left(\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right.$$
$$+\Omega\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{(1-\ell)\varphi^p+\ell\chi^p}\right)^{\frac{1}{p}}\right)$$
$$\times\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)d\hbar d\ell$$
$$\leq\frac{1}{2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)$$
$$\times\int_0^1\int_0^1\left(\sin\frac{\pi}{2}(\hbar+\ell)+\cos\frac{\pi}{2}(\hbar-\ell)\right)$$
$$\times\Phi\left(\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}\right)d\hbar d\ell. \tag{3.24}$$

Making the substitutions $e=\left(\frac{\tau^p\upsilon^p}{\hbar\upsilon^p+(1-\hbar)\tau^p}\right)^{\frac{1}{p}}$ and $f=\left(\frac{\varphi^p\chi^p}{\ell\chi^p+(1-\ell)\varphi^p}\right)^{\frac{1}{p}}$ in (3.24), we get

$$\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\int_\tau^\upsilon\int_\varphi^\chi\frac{\Phi(e,f)}{(ef)^{p+1}}dedf$$
$$\leq\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(e,f)\Phi(e,f)}{(ef)^{p+1}}dedf$$
$$\leq\frac{1}{2}\left(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)\right)$$
$$\times\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\left(\sin\frac{\pi}{2}\left(\frac{\upsilon^p-\tau^p\upsilon^pe^{-p}}{\upsilon^p-\tau^p}+\frac{\chi^p-\varphi^p\chi^pf^{-p}}{\chi^p-\varphi^p}\right)\right.$$
$$\left.+\cos\frac{\pi}{2}\left(\frac{\upsilon^p-\tau^p\upsilon^pe^{-p}}{\upsilon^p-\tau^p}-\frac{\chi^p-\varphi^p\chi^pf^{-p}}{\chi^p-\varphi^p}\right)\right)\frac{\Phi(e,f)}{(ef)^{p+1}}dedf.$$

Multiplying by $\frac{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{p^2\tau^p\upsilon^p\varphi^p\chi p}$ and taking into account the fact that $\sin x,\cos x\in[-1,1]$ for all $x\in\mathbb{R}$, leads to

$$\Omega\left(\left(\frac{2\tau^p\upsilon^p}{\tau^p+\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{2\varphi^p\chi^p}{\varphi^p+\chi^p}\right)^{\frac{1}{p}}\right)\int_\tau^\upsilon\int_\varphi^\chi\frac{\Phi(x,y)}{(xy)^{p+1}}dxdy$$

$$\leq 2\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Omega(x,y)\Phi(x,y)}{(xy)^{p+1}}dxdy$$
$$\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right)\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Phi(x,y)}{(xy)^{p+1}}dxdy.$$

Hence, the desired result is obtained. □

Corollary 3. *Choosing $p = 1$ in (3.21), we get the Féjer-Hermite-Hadamard inequality for HT-coordinated convexity:*

$$\Omega\left(\frac{2\tau\upsilon}{\upsilon+\tau},\frac{2\varphi\chi}{\chi+\varphi}\right)\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Phi(u,v)}{(uv)^{2}}dudv$$
$$\leq 2\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Omega(u,v)\Phi(u,v)}{(uv)^{2}}dudv$$
$$\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right)\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\frac{\Phi(u,v)}{(uv)^{2}}dudv.$$

Corollary 4. *Choosing $p = -1$ in (3.21), we get the Féjer-Hermite-Hadamard inequality for trigonometric coordinated convexity:*

$$\Omega\left(\frac{\upsilon+\tau}{2},\frac{\chi+\varphi}{2}\right)\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\Phi(u,v)dudv$$
$$\leq 2\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\Omega(u,v)\Phi(u,v)dudv$$
$$\leq 4\left(\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right)\int_{\tau}^{\upsilon}\int_{\varphi}^{\chi}\Phi(u,v)dudv.$$

Example 6. *To authenticate the result obtained in Theorem 5, we used Mathematica by choosing $\Omega(u,v)=\frac{1}{u^p v^p}$, $\Phi(u)=(\frac{1}{u^p}-\frac{1}{\upsilon^p})^{\frac{1}{p}}+(\frac{1}{\tau^p}-\frac{1}{u^p})^{\frac{1}{p}}+(\frac{1}{v^p}-\frac{1}{\chi^p})^{\frac{1}{p}}+(\frac{1}{\varphi^p}-\frac{1}{v^p})^{\frac{1}{p}}$, $p=1$, $\upsilon=4$, $\chi=9$, $\tau\in[2,3]$, and $\upsilon\in[7,8]$ for the 3D graph in Figure 8 of the Fejér Hermite-Hadamard inequality (3.21).*

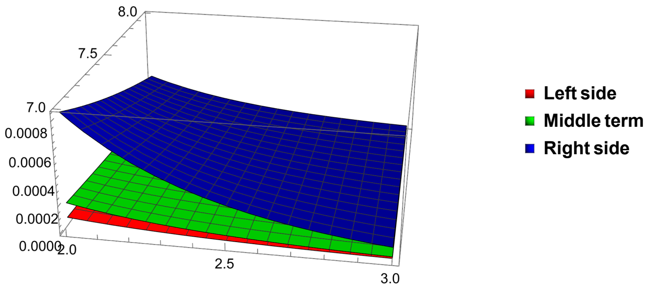

Figure 8. The graphs of functions represent the 3D view of the Fejér Hermite-Hadamard inequality (3.21).

Remark 1. *If we take $\Phi(x, y) = 1$, we reach the Hermite-Hadamard inequality given in Theorem 3.*

Lemma 2. *Let $\Omega : \Delta = H_p \times H_p \to \mathbb{R}$ be a partial differentiable function on Δ with $\tau, \upsilon, \varphi, \chi \in H_p$ and $\tau < \upsilon$, $\varphi < \chi$. If $\frac{\partial^2\Omega}{\partial\hbar\partial\ell} \in L(\Delta)$, then for $\hbar, \ell \in [0, 1]$, the relation*

$$\begin{aligned}
&\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2} \\
&-\left(\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon \frac{\Omega(u,\varphi)}{u^{p+1}}du\right. \\
&\left.+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi \frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi \frac{\Omega(\tau,v)}{v^{p+1}}dv\right) \\
&+\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi \frac{\Omega(u,v)}{(uv)^{p+1}}dudv \\
&=\frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2}\int_0^1\int_0^1 \frac{(1-2\hbar)(1-2\ell)}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}} \\
&\times\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p},\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}d\hbar d\ell
\end{aligned}$$

holds.

Proof. Substituting $u = \left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}}$ and $v = \left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}$ in to the integral

$$\begin{aligned}
I=&\int_0^1\int_0^1(1-2\hbar)(1-2\ell)\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{1+\frac{1}{p}}\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{1+\frac{1}{p}} \\
&\times\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p},\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}d\hbar d\ell,
\end{aligned}$$

leads to

$$\begin{aligned}
I=&\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)^2}\int_\tau^\upsilon\int_\varphi^\chi\left(\tau^p+\upsilon^p-2\tau^p\upsilon^pu^{-p}\right) \\
&\times\left(\varphi^p+\chi^p-2\varphi^p\chi^pv^{-p}\right)\frac{\partial^2\Omega(u,v)}{\partial u\partial v}dudv.
\end{aligned}\tag{3.25}$$

Consider

$$\begin{aligned}
J&=\int_\varphi^\chi\left(\varphi^p+\chi^p-2\varphi^p\chi^pv^{-p}\right)\frac{\partial^2\Omega(u,v)}{\partial u\partial v}dv \\
&=\left(\varphi^p+\chi^p\right)\int_\varphi^\chi\frac{\partial^2\Omega(u,v)}{\partial u\partial v}dv-2\varphi^p\chi^p\int_\varphi^\chi v^{-p}\frac{\partial^2\Omega(u,v)}{\partial u\partial v}dv.
\end{aligned}$$

Applying integration by parts to get

$$J=(\chi^p-\varphi^p)\left(\frac{\partial\Omega(u,\chi)}{\partial u}+\frac{\partial\Omega(u,\varphi)}{\partial u}\right)-2p\varphi^p\chi^p\int_\varphi^\chi\frac{\partial\Omega(u,v)}{\partial u}v^{-p-1}dv.\tag{3.26}$$

Putting (3.26) into (3.25), gives

$$I=\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)}\int_\tau^\upsilon(\tau^p+\upsilon^p-2\tau^p\upsilon^pu^{-p})\left(\frac{\partial\Omega(u,\chi)}{\partial u}+\frac{\partial\Omega(u,\varphi)}{\partial u}\right)du$$
$$-\frac{2p^3\tau^p\upsilon^p\varphi^{2p}\chi^{2p}}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)^2}\int_\tau^\upsilon(\tau^p+\upsilon^p-2\tau^p\upsilon^pu^{-p})\int_\varphi^\chi\frac{\partial\Omega(u,v)}{\partial u}v^{-p-1}dvdu.$$

Upon simplifying, we have

$$\begin{aligned}I=&\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p(\tau^p+\upsilon^p)}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)}\int_\tau^\upsilon\left(\frac{\partial\Omega(u,\chi)}{\partial u}+\frac{\partial\Omega(u,\varphi)}{\partial u}\right)du\\&-\frac{2p^2\tau^{2p}\upsilon^{2p}\varphi^p\chi^p}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)}\left(\int_\tau^\upsilon\frac{\partial\Omega(u,\chi)}{\partial u}u^{-p}du+\int_\tau^\upsilon\frac{\partial\Omega(u,\varphi)}{\partial u}u^{-p}du\right)\\&-\frac{2p^3\tau^p\upsilon^p\varphi^{2p}\chi^{2p}(\tau^p+\upsilon^p)}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)^2}\int_\tau^\upsilon\int_\varphi^\chi\frac{\partial\Omega(u,v)}{\partial u}v^{-p-1}dvdu\\&+\frac{4p^3\tau^{2p}\upsilon^{2p}\varphi^{2p}\chi^{2p}}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)^2}\int_\tau^\upsilon\int_\varphi^\chi\frac{\partial\Omega(u,v)}{\partial u}u^{-p}v^{-p-1}dvdu.\end{aligned}\tag{3.27}$$

Solving the integrals involved in (3.27) leads to

$$\begin{aligned}I=&\frac{p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}(\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi))\\&-\frac{2p^3\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\left(\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\varphi)}{u^{p+1}}du\right.\\&\left.+\frac{\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\tau,v)}{v^{p+1}}dv\right)\\&+\frac{4p^4\tau^{2p}\upsilon^{2p}\varphi^{2p}\chi^{2p}}{(\upsilon^p-\tau^p)^2(\chi^p-\varphi^p)^2}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(u,v)}{(uv)^{p+1}}dudv.\end{aligned}\tag{3.28}$$

Multiplying (3.28) by $\frac{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2\tau^p\upsilon^p\varphi^p\chi^p}$, the result follows. □

Now, by using Lemma 2, we develop the following theorem.

Theorem 6. *Let $\Omega:\Delta=H_p\times H_p\to\mathbb{R}$ be a partial differentiable function on Δ with $\tau,\upsilon,\varphi,\chi\in H_p$ and $\tau<\upsilon$, $\varphi<\chi$. If $\left|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\right|\in L(\Delta)$ is HT p-coordinated convex, then for $\hbar,\ell\in[0,1]$, the following result holds.*

$$\begin{aligned}&\left|\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}\right.\\&-\left(\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\varphi)}{u^{p+1}}du\right.\\&\left.+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\tau,v)}{v^{p+1}}dv\right)\\&\left.+\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(u,v)}{(uv)^{p+1}}dudv\right|\end{aligned}$$

$$\leq \frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2}$$
$$\times\left(F_1\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|+F_2\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|+F_3\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|+F_4\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|\right),$$

where

$$F_1=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{4})^{2n}\frac{1}{2(2n)!}\left(\left(\upsilon^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right)\right.\right.$$
$$\left.+\frac{\tau^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right)\right)$$
$$\times\left(\chi^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\varphi^p}{\chi^p})\right)\right.$$
$$\left.\left.+\frac{\varphi^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\chi^p}{\varphi^p})\right)\right)\right),$$

$$F_2=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{4})^{2n}\frac{1}{2(2n)!}\left(\left(\frac{\upsilon^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right)\right.\right.$$
$$\left.+\tau^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right)\right)$$
$$\times\left(\frac{\chi^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\varphi^p}{\chi^p})\right)\right.$$
$$\left.\left.+\varphi^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\chi^p}{\varphi^p})\right)\right)\right),$$

$$F_3=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{4})^{2n}\frac{1}{2(2n)!}\left(\left(\frac{\upsilon^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right)\right.\right.$$
$$\left.+\tau^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right)\right)$$
$$\times\left(\chi^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\varphi^p}{\chi^p})\right)\right.$$
$$\left.\left.+\frac{\varphi^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\chi^p}{\varphi^p})\right)\right)\right)$$

and

$$F_4=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{4})^{2n}\frac{1}{2(2n)!}\left(\left(\upsilon^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right)\right.\right.$$
$$\left.+\frac{\tau^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right)\right)$$
$$\times\left(\frac{\chi^p\pi}{4(2n+1)}\beta(2n+2,2)_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\varphi^p}{\chi^p})\right)\right.$$

$$+\varphi^p\beta(2n+1,2)_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\chi^p}{\varphi^p})\right)\Bigg)\Bigg)\Bigg).$$

Proof. Utilizing Lemma 2 with the property of the modulus for integrals and the HT p-coordinated convexity of $|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}|$, we have

$$\Bigg|\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}$$
$$-\left(\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\varphi)}{u^{p+1}}du\right.$$
$$\left.+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\tau,v)}{v^{p+1}}dv\right)$$
$$+\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(u,v)}{(uv)^{p+1}}dudv\Bigg|$$

$$\leq\frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2}$$
$$\times\int_0^1\int_0^1\left|\frac{(1-2\hbar)(1-2\ell)}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\right|$$
$$\times\left|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right|d\hbar d\ell$$
$$\leq\frac{1}{2}\int_0^1\int_0^1\left|\frac{(1-2\hbar)(1-2\ell)}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\right|$$
$$\times\left(\left(\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|-\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|\right)\cos\frac{\pi(\hbar+\ell)}{2}\right.$$
$$+\left(\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|+\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|\right)\sin\frac{\pi(\hbar+\ell)}{2}$$
$$+\left(\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|+\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|\right)\cos\frac{\pi(\hbar-\ell)}{2}$$
$$\left.+\left(\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|-\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|\right)\sin\frac{\pi(\hbar-\ell)}{2}\right)d\hbar d\ell,$$

and with the help of elementary geometry, we have

$$\leq\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|\int_0^1\int_0^1\frac{\left|(1-2\hbar)(1-2\ell)\right|}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}\cos\frac{\pi\ell}{2}d\hbar d\ell$$
$$+\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|\int_0^1\int_0^1\frac{\left|(1-2\hbar)(1-2\ell)\right|}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\sin\frac{\pi\hbar}{2}\sin\frac{\pi\ell}{2}d\hbar d\ell$$

$$+\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|\int_0^1\int_0^1\frac{\left|(1-2\hbar)(1-2\ell)\right|}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\sin\frac{\pi\hbar}{2}\cos\frac{\pi\ell}{2}d\hbar d\ell$$
$$+\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|\int_0^1\int_0^1\frac{\left|(1-2\hbar)(1-2\ell)\right|}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}\sin\frac{\pi\ell}{2}d\hbar d\ell. \quad (3.29)$$

By solving the integrals on the right-hand side of (3.29) leads to

$$\int_0^1\int_0^1\frac{\left|(1-2\hbar)(1-2\ell)\right|}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}\cos\frac{\pi\ell}{2}d\hbar d\ell$$
$$=\int_0^1\frac{\left|1-2\hbar\right|}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}d\hbar$$
$$\times\int_0^1\frac{\left|1-2\ell\right|}{(\ell\varphi^p+(1-\ell)\chi^p)^{1+\frac{1}{p}}}\cos\frac{\pi\ell}{2}d\ell. \quad (3.30)$$

Also,

$$\int_0^1\frac{\left|1-2\hbar\right|}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}d\hbar$$
$$=\int_0^{\frac{1}{2}}\frac{(1-2\hbar)}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}d\hbar$$
$$+\int_{\frac{1}{2}}^1\frac{(2\hbar-1)}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}d\hbar. \quad (3.31)$$

Here, substituting $X=2\hbar$ gives us the following integral:

$$\int_0^{\frac{1}{2}}\frac{(1-2\hbar)}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}}\cos\frac{\pi\hbar}{2}d\hbar$$
$$=\frac{1}{2}\int_0^1\frac{(1-X)}{\left(\frac{X}{2}\tau^p+(1-\frac{X}{2})\upsilon^p\right)^{1+\frac{1}{p}}}\cos\frac{\pi X}{4}dX$$
$$=\frac{1}{2}\int_0^1(1-X)\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi X}{4})^{2n}}{(2n)!}$$
$$\times\left(\frac{X}{2}\tau^p+(1-\frac{X}{2})\upsilon^p\right)^{-(1+\frac{1}{p})}dX$$
$$=\frac{\upsilon^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n}}{(2n)!}\int_0^1(1-X)X^{2n}$$
$$\times\left(1-\frac{1}{2}(1-\frac{\tau^p}{\upsilon^p}X)\right)^{-(1+\frac{1}{p})}dX$$
$$=\frac{\upsilon^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n}}{(2n)!}\beta(2n+1,2)$$

$$\times {}_2F_1\left(1+\frac{1}{p}, 2n+1; 2n+3; \frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right). \tag{3.32}$$

Now, substituting $X = 1 - \hbar$ into the following integral first and then substituting $X = \frac{y}{2}$ into the resulting integral, leads to

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \frac{(2\hbar-1)}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}} \cos\frac{\pi\hbar}{2} d\hbar \\
&= \int_0^{\frac{1}{2}} \frac{(1-2X)}{(X\upsilon^p+(1-X)\tau^p)^{1+\frac{1}{p}}} \sin\frac{\pi X}{2} dX \\
&= \frac{1}{2}\int_0^1 \frac{(1-y)}{\left(\frac{y}{2}\upsilon^p+(1-\frac{y}{2})\tau^p\right)^{1+\frac{1}{p}}} \sin\frac{\pi y}{4} dy \\
&= \frac{1}{2}\int_0^1 (1-y)\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi y}{4})^{2n+1}}{(2n+1)!} \\
&\times\left(\frac{y}{2}\upsilon^p+(1-\frac{y}{2})\tau^p\right)^{-(1+\frac{1}{p})} dy \\
&= \frac{\tau^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n+1}}{(2n+1)!}\int_0^1 (1-y)y^{2n+1} \\
&\times\left(1-\frac{1}{2}(1-\frac{\upsilon^p}{\tau^p}y)\right)^{-(1+\frac{1}{p})} dy \\
&= \frac{\tau^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n+1}}{(2n+1)!}\beta(2n+2,2) \\
&\times {}_2F_1\left(1+\frac{1}{p}, 2n+2; 2n+4; \frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right).
\end{aligned} \tag{3.33}$$

Hence, using (3.32) and (3.33) in (3.31), we have

$$\begin{aligned}
&\int_0^1 \frac{\left|1-2\hbar\right|}{(\hbar\tau^p+(1-\hbar)\upsilon^p)^{1+\frac{1}{p}}} \cos\frac{\pi\hbar}{2} d\hbar \\
&= \frac{\upsilon^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n}}{(2n)!}\beta(2n+1,2) \\
&\times {}_2F_1\left(1+\frac{1}{p}, 2n+1; 2n+3; \frac{1}{2}(1-\frac{\tau^p}{\upsilon^p})\right) \\
&+ \frac{\tau^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n+1}}{(2n+1)!}\beta(2n+2,2) \\
&\times {}_2F_1\left(1+\frac{1}{p}, 2n+2; 2n+4; \frac{1}{2}(1-\frac{\upsilon^p}{\tau^p})\right).
\end{aligned} \tag{3.34}$$

By symmetry,

$$\int_0^1 \frac{|1-2\ell|}{(\ell\varphi^p+(1-\ell)\chi^p)^{1+\frac{1}{p}}}\cos\frac{\pi\ell}{2}d\ell$$
$$=\frac{\chi^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n}}{(2n)!}\beta(2n+1,2)$$
$$\times {}_2F_1\left(1+\frac{1}{p},2n+1;2n+3;\frac{1}{2}(1-\frac{\varphi^p}{\chi^p})\right)$$
$$+\frac{\varphi^p}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{\pi}{4})^{2n+1}}{(2n+1)!}\beta(2n+2,2)$$
$$\times {}_2F_1\left(1+\frac{1}{p},2n+2;2n+4;\frac{1}{2}(1-\frac{\chi^p}{\varphi^p})\right). \tag{3.35}$$

Utilizing (3.34) and (3.35) in (3.30) gives F_1. By a similar approach as for F_1, one can find F_2–F_4. □

Example 7. *To authenticate the result obtained in Theorem 6, we have the following 3D graph in Figure 9 with the help of Mathematica corresponding to the function* $\Omega(u,v)=\frac{u^{1-p}v^{1-p}}{(1-p)^2}$ *with the choice of parameters* $p=-1$, $\chi=9$, $\upsilon=3$, $\tau\in[1,2]$, *and* $\varphi\in[7,8]$.

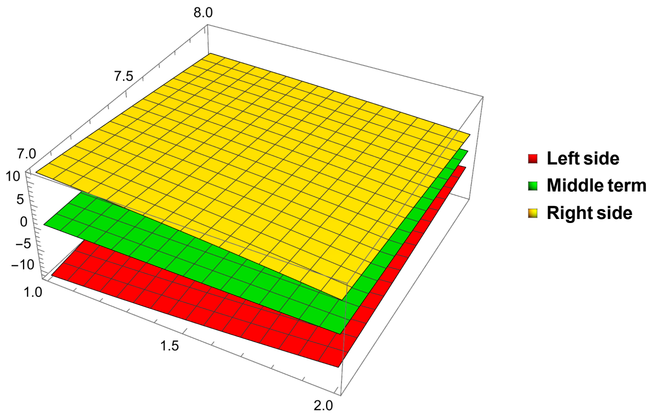

Figure 9. The graphs of functions present the 3D view of the inequality presented in Theorem 6.

Theorem 7. *Let* $\Omega:\Delta=H_p\times H_p\to\mathbb{R}$ *be a partial differentiable function on* Δ *with* $\tau,\upsilon,\varphi,\chi\in H_p$ *and* $\tau<\upsilon$, $\varphi<\chi$. *If* $\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\in L(\Delta)$ *and* $\left|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\right|^q$, $q>1$, $\frac{1}{q}+\frac{1}{r}=1$, *is an HT p-coordinated convex, then for* $\hbar,\ell\in[0,1]$, *the following result holds:*

$$\Bigg|\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}$$
$$-\left(\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\varphi)}{u^{p+1}}du\right.$$

$$+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\tau,v)}{v^{p+1}}dv\Bigg)$$
$$+\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(u,v)}{(uv)^{p+1}}dudv\Bigg|$$
$$\leq\frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{4(r+1)^{\frac{2}{r}}}$$
$$\times\left(F_5\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|^q+F_6\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|^q+F_7\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|^q+F_8\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|^q\right)^{\frac{1}{q}},$$

where

$$F_5=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{2})^{2n}\frac{1}{(2n)!}\beta(2n+1,1)\left(\upsilon^p\chi^p{}_2F_1\left(q(1+\frac{1}{p}),2n+1;2n+2;(1-\frac{\tau^p}{\upsilon^p})\right)\right.$$
$$\left.\times{}_2F_1\left(q(1+\frac{1}{p}),2n+1;2n+2;(1-\frac{\varphi^p}{\chi^p})\right)\right).$$

$$F_6=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{2})^{2n+1}\frac{1}{(2n+1)!}\beta(2n+2,1)\left(\upsilon^p\chi^p{}_2F_1\left(q(1+\frac{1}{p}),2n+2;2n+3;(1-\frac{\tau^p}{\upsilon^p})\right)\right.$$
$$\left.\times{}_2F_1\left(q(1+\frac{1}{p}),2n+1;2n+2;(1-\frac{\varphi^p}{\chi^p})\right)\right).$$
$$F_7=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{2})^{2n}\frac{1}{(2n)!}\left(\chi^p\beta(2n+1,1){}_2F_1\left(q(1+\frac{1}{p}),2n+1;2n+2;(1-\frac{\varphi^p}{\chi^p})\right)\right.$$
$$\left.\times\frac{\pi}{2(2n+1)}\upsilon^p\beta(2n+2,1){}_2F_1\left(q(1+\frac{1}{p}),2n+2;2n+3;(1-\frac{\tau^p}{\upsilon^p})\right)\right).$$
$$F_8=\sum_{n=0}^{\infty}(-1)^n(\frac{\pi}{2})^{2n}\frac{1}{(2n)!}\left(\upsilon^p\beta(2n+1,1){}_2F_1\left(q(1+\frac{1}{p}),2n+1;2n+2;(1-\frac{\tau^p}{\upsilon^p})\right)\right.$$
$$\left.\times\frac{\pi}{2(2n+1)}\chi^p\beta(2n+2,1){}_2F_1\left(q(1+\frac{1}{p}),2n+2;2n+3;(1-\frac{\varphi^p}{\chi^p})\right)\right).$$

Proof. Utilizing Lemma 2 with the property of the modulus for integrals, Hölder's inequality, and the HT p-coordinated convexity of $|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}|^q$, we have

$$\Bigg|\frac{\Omega(\tau,\varphi)+\Omega(\tau,\chi)+\Omega(\upsilon,\varphi)+\Omega(\upsilon,\chi)}{2}$$
$$-\Bigg(\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\chi)}{u^{p+1}}du+\frac{p\tau^p\upsilon^p}{\upsilon^p-\tau^p}\int_\tau^\upsilon\frac{\Omega(u,\varphi)}{u^{p+1}}du$$
$$+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\upsilon,v)}{v^{p+1}}dv+\frac{p\varphi^p\chi^p}{\varphi^p-\chi^p}\int_\varphi^\chi\frac{\Omega(\tau,v)}{v^{p+1}}dv\Bigg)$$
$$+\frac{2p^2\tau^p\upsilon^p\varphi^p\chi^p}{(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}\int_\tau^\upsilon\int_\varphi^\chi\frac{\Omega(u,v)}{(uv)^{p+1}}dudv\Bigg|$$

$$
\begin{aligned}
&\leq \frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2} \\
&\times \int_0^1\int_0^1 \left|\frac{(1-2\hbar)(1-2\ell)}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{1+\frac{1}{p}}}\right| \\
&\times \left|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right| d\hbar d\ell \\
&\leq \frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2}\left(\int_0^1\int_0^1 |(1-2\hbar)(1-2\ell)|^r d\hbar d\ell\right)^{\frac{1}{r}} \\
&\times \left(\int_0^1\int_0^1 \frac{\left|\frac{\partial^2\Omega}{\partial\hbar\partial\ell}\left(\left(\frac{\tau^p\upsilon^p}{\hbar\tau^p+(1-\hbar)\upsilon^p}\right)^{\frac{1}{p}},\left(\frac{\varphi^p\chi^p}{\ell\varphi^p+(1-\ell)\chi^p}\right)^{\frac{1}{p}}\right)\right|^q}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}} d\hbar d\ell\right)^{\frac{1}{q}} \\
&\leq \frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2(r+1)^{\frac{2}{r}}}\left(\int_0^1\int_0^1 \left|\frac{1}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}}\right|\right. \\
&\times\left(\left(\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|^q-\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|^q\right)\cos\frac{\pi(\hbar+\ell)}{2}\right. \\
&+\left(\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|^q+\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|^q\right)\sin\frac{\pi(\hbar+\ell)}{2} \\
&+\left(\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|^q+\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|^q\right)\cos\frac{\pi(\hbar-\ell)}{2} \\
&\left.\left.+\left(\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|^q-\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|^q\right)\sin\frac{\pi(\hbar-\ell)}{2}\right)d\hbar d\ell\right)^{\frac{1}{q}},
\end{aligned}
$$

and with the help of elementary geometry, we have

$$
\begin{aligned}
&\leq \frac{\tau\upsilon\varphi\chi(\upsilon^p-\tau^p)(\chi^p-\varphi^p)}{2p^2(r+1)^{\frac{2}{r}}} \\
&\times\left(\left|\frac{\partial^2\Omega(\tau,\varphi)}{\partial\hbar\partial\ell}\right|^q\int_0^1\int_0^1\frac{\cos\frac{\pi\hbar}{2}\cos\frac{\pi\ell}{2}}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}}d\hbar d\ell\right. \\
&+\left|\frac{\partial^2\Omega(\upsilon,\chi)}{\partial\hbar\partial\ell}\right|^q\int_0^1\int_0^1\frac{\sin\frac{\pi\hbar}{2}\sin\frac{\pi\ell}{2}}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}}d\hbar d\ell \\
&+\left|\frac{\partial^2\Omega(\upsilon,\varphi)}{\partial\hbar\partial\ell}\right|^q\int_0^1\int_0^1\frac{\sin\frac{\pi\hbar}{2}\cos\frac{\pi\ell}{2}}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}}d\hbar d\ell \\
&\left.+\left|\frac{\partial^2\Omega(\tau,\chi)}{\partial\hbar\partial\ell}\right|^q\int_0^1\int_0^1\frac{\cos\frac{\pi\hbar}{2}\sin\frac{\pi\ell}{2}}{((\hbar\tau^p+(1-\hbar)\upsilon^p)(\ell\varphi^p+(1-\ell)\chi^p))^{q(1+\frac{1}{p})}}d\hbar d\ell\right)^{\frac{1}{q}}. \qquad (3.36)
\end{aligned}
$$

Solving the integrals involved in (3.36) by using the same technique as in Theorem 6, we get the result. □

Example 8. *To authenticate the result obtained in Theorem 7, we have the following 3D graph in Figure 10 with the help of Mathematica corresponding to the choice of function* $\Omega(u, v) = \frac{u^{1-\frac{p}{q}} v^{1-\frac{p}{q}}}{(1-\frac{p}{q})^2}$ *and parameters* $p = 1$, $q = 2$, $\upsilon = 3$, $\chi = 9$, $\tau \in [1, 2]$ *and* $\varphi \in [7, 8]$.

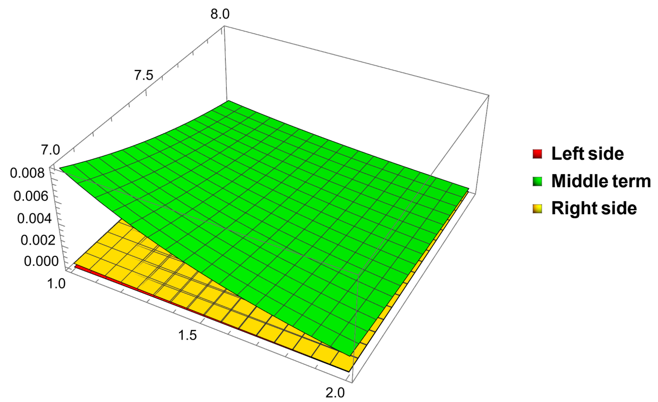

Figure 10. The graphs of functions show the 3D representation of the inequality presented in Theorem 7.

4. Conclusions

In the present study, we introduced two novel categories of convex functions called HT p-convex functions on a rectangular plan Δ and HT p-convex functions on its coordinates. We explored the relationship of the introduced convexities through examples, including 3D graphs, to validate the novel convexities. Moreover, we discussed two of its special cases which are also novel categories of convex functions in two dimensions. We also investigated the classical Hermite-Hadamard, Fejér-Hermite-Hadamard, and related inequalities by utilizing this convexity concept. The validity of our findings was confirmed through graphical representations. These findings have significant implications for convex analysis and various other fields within both pure and applied sciences. The potential applications of these results are wide ranging and include areas such as optimization, mathematical modeling and signal processing. By establishing connections between different classes of convex functions and presenting new integral inequalities, the article opens up avenues for further exploration and advancement in this fascinating domain. It is anticipated that these results will inspire future research efforts and contribute to the ongoing development of convex analysis and related disciplines.

Author contributions

S. Ali: Conceptualization, data curation, formal analysis, writing original draft; M. Samraiz: Investigation, methodology, writing original draft; S. Trabelsi: Conceptualization, resources, software, validation, writing-review & editing; H. Zaway: Conceptualization, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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