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Research article

Continuity and analyticity for a generalized two-component short pulse system

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Abstract: In this paper, we first established the local well-posedness of the generalized short pulse system in the critical Besov space $B^{\frac{3}{2}}_{2,1}(\mathbb{T})$, improving upon the local well-posedness result obtained in [S. Yu, X. Yin, *J. Math. Anal. Appl.*, **475** (2019), 1427–1447]. We then proved that the solution map was Hölder continuous in $B^{\mu}_{2,r}(\mathbb{T})$. Finally, by a generalized Ovsyannikov theorem combined with fundamental properties of Sobolev-Gevrey spaces, we established the Gevrey regularity and analyticity of solutions and further obtained a lower bound of the lifespan and the continuity of the data-to-solution map.

Keywords: local well-posedness; Hölder continuous; Gevrey regularity and analyticity
Mathematics Subject Classification: 35B65, 35B10, 35M31, 35G25

1. Introduction

In the present paper, we consider the Cauchy problem for the following generalized short pulse equations with high-order nonlinearities:

$$\begin{cases} u_{tx} = a(uv^{p-1}\partial_x u)_x + u, & x \in \mathbb{T}, t \in \mathbb{R}, \\ v_{tx} = b(vu^{q-1}\partial_x v)_x + v, & x \in \mathbb{T}, t \in \mathbb{R}, \\ u(x,0) = u_0, & x \in \mathbb{T}, t = 0, \\ v(x,0) = v_0, & x \in \mathbb{T}, t = 0, \end{cases} \tag{1.1}$$

where $p, q \in \mathbb{Z}^+$ and a, b are two constant parameters.

The special case where $a = b = \frac{1}{2}$, $p = q = 2$, and $u = v$ transforms system (1.1) into the classical short-pulse equation:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \tag{1.2}$$

This equation derived as a nonlinear approximation from Maxwell's equations, characterizes the propagation of ultrashort optical pulses in isotropic fibers [1]. Here, the real-valued function $u(t, x)$ characterizes the electric field's magnitude. As a prominent example of integrable loop-soliton systems, it has attracted considerable attention in the research community over recent decades. Equation (1.2) has a Lax pair [2] and bi-Hamiltonian structure [3]. In [4], their analysis revealed that the Eq (1.2) admits a Wadati-Konno-Ichikawa-type Lax pair and can be connected to the sine-Gordon equation via a series of transformations. This finding positions Eq (1.2) as an integrable model for ultrashort pulses, serving as an alternative to the nonlinear Schrödinger (NLS) equation. Furthermore, researchers identified an appropriate hodograph transformation that reduces Eq (1.2) to the renowned sine-Gordon (SG) equation, enabling the derivation of multi-loop solitary wave solutions [5]. Consequently, diverse solution type in Eq (1.2) have been established, including: periodic and solitary wave solutions [6]; two-loop soliton solutions [7]; and bilinear forms, multi-loop solutions, multi-breather solutions, and periodic solutions [8, 9]. Notably, the loop soliton solutions of Eq (1.2) can also be obtained through a Darboux transformation approach [10].

Considering the influence of polarization and anisotropy, researchers have developed different versions of the short pulse equation for two-component systems. In a recent study, the two-component short pulse system was proposed by Matsuno [11] as follows:

$$\begin{cases} u_{xt} = u + \frac{1}{2}(uvu_x)_x, \\ v_{xt} = v + \frac{1}{2}(uvv_x)_x, \end{cases} \tag{1.3}$$

where Eq (1.1) to be discussed in this paper is precisely a high-order generalization of this two-component short pulse system (1.3). It was shown that Eq (1.3) could be generated from the negative-order Wadati-Konno-Ichikawa hierarchy in [12, 13]. Besides, Eq (1.3) is integrable, with corresponding Lax pairs given by $\Psi_x = P\Psi$, $\Psi_t = Q\Psi$, where the given matrices P and Q are:

$$P = \lambda\begin{pmatrix} 1 & u_x \\ v_x & -1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 0 & v \\ -u & 0 \end{pmatrix} + \frac{1}{4\lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} uv & uvu_x \\ uvv_x & -uv \end{pmatrix}.$$

For the Cauchy problem, numerous works, such as [14–17], have examined the properties of solutions to nonlinear Camassa-Holm-type equations or chemotaxis models in Besov spaces. Their core strategy integrates Littlewood-Paley decomposition techniques with transport equation theory. Zhaqilao et al. [18] initially established the existence and uniqueness of a solution for Eq (1.3) with an estimate of the analytic lifespan, and then deduced the continuity of the data-to-solution map in the space of an analytic function. Later, in [19], the authors studied the local well-posedness of the Eq (1.1) in the Besov space $B^s_{p,r} \times B^s_{p,r}$ with $s > \{\frac{3}{2}, 1 + \frac{1}{p}\}$. Due to $B^s_{p,r} \hookrightarrow B^{1+\frac{1}{p}}_{p,1}$ for $s > 1 + \frac{1}{p}$, our first aim is to establish the local well-posedness of Eq (1.1) in the Besov space $B^{\frac{3}{2}}_{2,1}(\mathbb{T})$ to improve the local well-posedness result in [19]. The specific theorem is as follows:

Theorem 1.1. *Suppose that* $(u_0, v_0) \in B^{\frac{3}{2}}_{2,1}(\mathbb{T})$*, and then there exists a time* $T > 0$ *such that Eq* (1.1) *has a unique strong solution* (u, v) *belonging to* $E^{\frac{3}{2}}_{2,1} := C([0, T]; B^{\frac{3}{2}}_{2,1}) \cap C^1([0, T]; B^{\frac{1}{2}}_{2,1}) \times C([0, T]; B^{\frac{3}{2}}_{2,1}) \cap C^1([0, T]; B^{\frac{1}{2}}_{2,1})$*. Besides, the data-to-solution map* $(u_0, v_0) \to (u, v)$ *is continuous from* $B^{\frac{3}{2}}_{2,1}$ *into* $E^{\frac{3}{2}}_{2,1}$.

Remark 1.1. *According to the Littlewood-Paley theory [20], we have the continuous embedding* $B^{\frac{3}{2}}_{2,1} \hookrightarrow B^{\frac{3}{2}}_{2,2} \approx H^s$*, where the critical exponent* $s = \frac{3}{2}$ *is the minimal regularity index for which the Besov space* $B^s_{2,1}$ *is embedded into* Lip.

In [19], it has been proven that the data-to-solution mapping is continuous but not uniformly continuous. The following results will provide information about the stability of the data-to-solution map, that is, the data-to-solution map for Eq (1.1) to be Hölder continuous in $B^{\mu}_{2,r} \times B^{\mu}_{2,r}$.

Theorem 1.2. *Suppose that* $(u_0, v_0) \in B^s_{2,r} \times B^s_{2,r}$ *and* $r > 1$, $s > 3/2$ *or* $r = 1$, $s \geq 3/2$*, if* $\mu \in \mathbb{R}$ *such that* $s - 1 < \mu < s$*. Then the data-to-solution map is Hölder continuous from* $B^{\mu}_{2,r} \times B^{\mu}_{2,r}$ *to* $C([0, T]; B^{\mu}_{2,r}) \times C([0, T]; B^{\mu}_{2,r})$.

Next, based on the generalized Ovsyannikov theorem, we deduce the local analyticity and Gevrey regularity of the solutions to Eq (1.1) on the circle and the whole space, and we see the continuity of the data-to-solution map.

Theorem 1.3. *Let* $\sigma \geq 1$, $s > \frac{3}{2}$*. Assume that* $u_0 \in G^1_{\sigma,s}$*. Then for any* $0 < \delta < 1$*, there exists a* $T_0 > 0$ *such that Eq* (1.1) *has a unique solution u which is holomorphic in* $|t| < \frac{T_0(1-\delta)^{\frac{\sigma}{2}}}{\sigma-1}$ *with values in* $G^{\delta}_{\sigma,s}(\mathbb{R})$*. Moreover,*

$$T_0 = \frac{1}{2^{2\sigma+4+m}\left((a+b)e^{-\sigma}\sigma^{\sigma} + \sqrt{2}\right)(1 + \|z_0\|_{G^1_{\sigma,s}})^m},$$

where the positive constant C' *depends on* s, α, β, γ, λ, Γ.

Theorem 1.4. *Let* $\sigma \geq 1$ *and* $s > \frac{3}{2}$*. For initial data* $(u_0, v_0) \in G^1_{\sigma,s} \times G^1_{\sigma,s}$*, the data-to-solution map* $(u_0, v_0) \mapsto (u, v)$ *of Eq* (1.1) *is continuous as a map from* $G^1_{\sigma,s} \times G^1_{\sigma,s}$ *into the solution space.*

This paper is structured as follows: Section 2 establishes foundational preliminaries. Sections 3 and 4 investigate the local well-posedness of system (1.1) in the critical Besov space, and Hölder continuity of the data-to-solution map. The local analyticity and Gevrey regularity are examined in Section 5.

2. Preliminaries

In this section, we introduce the lemmas and definitions that will be employed in the subsequent proofs. First, we briefly review some fundamental Besov space properties that will be essential for the proof of the local well-posedness and Hölder continuous for the generalized short pulse equations with high-order nonlinearities.

To establish our work in Besov space, we present the Littlewood-Paley decomposition for the definition of Besov spaces.

Lemma 2.1. *(Littlewood-Paley decomposition) (see Proposition 2.10 in [20]) Assume that the ball* $\mathcal{B} \doteq \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ *and the ring* $\mathcal{R} \doteq \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$*. Then there exist two radial functions* $\chi \in C^{\infty}_c(\mathcal{B})$ *and* $\varphi \in C^{\infty}_c(\mathcal{R})$*, valued in the interval* $[0, 1]$*, such that*

$$\forall \xi \in \mathbb{R}^n,\ \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,$$

$$|q-q'|\geq 2 \Rightarrow \ Supp\ \varphi(2^{-q}\cdot)\cap Supp\ \varphi(2^{-q'}\cdot)=\emptyset,$$
$$q\geq 1 \Rightarrow \ Supp\ \chi(\cdot)\cap \ Supp\ \varphi(2^{-q}\cdot)=\emptyset.$$

Next, let $h\doteq\mathcal{F}^{-1}\varphi$ and $\tilde{h}\doteq\mathcal{F}^{-1}\chi$. Then for all $f\in\mathcal{S}'(\mathbb{R}^n)$, the nonhomogeneous dyadic blocks Δ_q and low-frequency cut-off operator S_q can be defined as follows:

$$\Delta_q f=0\ for\ q\leq -2,$$
$$\Delta_{-1}f=\chi(D)f=\int_{\mathbb{R}^n}\tilde{h}(y)u(x-y)dy,$$
$$\Delta_q f\doteq\varphi(2^{-q}D)f=2^{qn}\int_{\mathbb{R}^n}h(2^q y)f(x-y)dy\ for\ q\geq 0,$$
$$S_q f\doteq\sum_{p=-1}^{q-1}\Delta_p u=\chi(2^{-q}D)f=\int_{\mathbb{R}^n}\tilde{h}(2^q y)f(x-y)dy. \tag{2.1}$$

Moreover, it is easily shown that $\varphi(\xi)=1$ if $\frac{4}{3}\leq|\xi|\leq\frac{3}{2}$.

Next, we introduce the definition of Besov spaces as follows:

Definition 2.1. *(Besov spaces) (see Definition 2.68 in [20]) Let $s\in\mathbb{R}, 1\leq p,r\leq\infty$. The nonhomogenous Besov space $B^s_{p,r}(\mathbb{R}^n)$ can be characterized by*

$$B^s_{p,r}(\mathbb{R}^n)\doteq\{f\in\mathcal{S}'(\mathbb{R}^n);\ \ \|f\|_{B^s_{p,r}}<\infty\},$$

where

$$\|f\|_{B^s_{p,r}}\doteq\begin{cases}\left(\sum\limits_{q\geq -1}2^{qsr}\|\Delta_q f\|^r_{L^p}\right)^{\frac{1}{r}}, & for\ r<\infty,\\ \sup\limits_{q\geq -1}2^{qs}\|\Delta_q f\|_{L^p}, & for\ r=\infty.\end{cases}$$

In order to study the local well-posedness of the Eq (1.1) in Besov spaces, we need to the following transport theory.

Lemma 2.2. *(See Theorem 3.14 in [20]) Let $1\leq p,r\leq\infty$, and $s\geq-\min(\frac{n}{p},1-\frac{n}{p})$. Assume that $f_0\in B^s_{p,r}(\mathbb{R}^n)$ and $g\in L^1([0,T];B^s_{p,r}(\mathbb{R}^n))$. Let $f\in L^\infty([0,T];B^s_{p,r}(\mathbb{R}^n))$ be the solution to the transport equations*

$$\begin{cases}\partial_t f+v\cdot\nabla f=g,\\ f|_{t=0}=f_0\end{cases} \tag{2.2}$$

with $\nabla v\in L^1([0,T];B^{s-1}_{p,r}(\mathbb{R}^n))$ for $s>1+\frac{n}{p}$ or $\nabla v\in L^1([0,T];B^{\frac{n}{p}}_{p,r}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n))$ otherwise. Then,

(1) If $r=1$ or $s\neq 1+\frac{n}{p}$, then there exists $C>0$ depending only on s, p, and r such that

$$\|f\|_{B^s_{p,r}}\leq e^{C\tilde{V}_p(t)}\|f_0\|_{B^s_{p,r}}+\int_0^t e^{C\tilde{V}_p(t)-C\tilde{V}_p(s)}\|g(s)\|_{B^s_{p,r}}ds,$$

or

$$\|f\|_{B^s_{p,r}}\leq\|f_0\|_{B^s_{p,r}}+\int_0^t\|g(s)\|_{B^s_{p,r}}ds+C\int_0^t\tilde{V}'_p(s)\|f(s)\|_{B^s_{p,r}}ds,$$

with

$$\tilde{V}_p(t) := \begin{cases} \int_0^t \|\nabla v(\tau)\|_{B^{\frac{n}{p}}_{p,\infty}\cap L^\infty} d\tau, & \text{if } s < 1 + \dfrac{n}{p}; \\ \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}} d\tau, & \text{if } s > 1 + \dfrac{n}{p} \text{ or } s = 1 + \dfrac{n}{p}, r = 1. \end{cases}$$

(2) If $s > 0$, then there exists a constant $C = C(n, p, r, s)$ such that

$$\begin{aligned} \|f(t)\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|g(\tau)\|_{B^s_{p,r}} d\tau \\ + C \int_0^t (\|f(\tau)\|_{B^s_{p,r}} \|\nabla v(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{B^s_{p,r}} \|f(\tau)\|_{L^\infty}) d\tau. \end{aligned} \tag{2.3}$$

(3) If $r < \infty$, then $f \in C([0, T]; B^s_{p,r}(\mathbb{R}^n))$; and if $r = \infty$, then $f \in C([0, T]; B^{s'}_{p,1}(\mathbb{R}^n))$ for all $s' < s$.

(4) If $v = f$ and $s > 0$, the inequality in (1) holds true with $\tilde{V}'_p(t) := \|\nabla v(t)\|_{L^\infty}$.

Next, we introduce some useful properties of the Besov spaces to prove Theorems 1.1-1.2 as follows:

Lemma 2.3. *(See Proposition 1.3.5 in [21]) Assume that $s \in \mathbb{R}, 1 \le p, r, p_i, r_i \le \infty$ $(i = 1, 2)$. We get:*

(1) $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.

(2) If $r < \infty$, then $\lim_{q\to\infty} \|S_q u - u\|_{B^s_{p,r}} = 0$. The space C^∞_c is dense in $B^s_{p,r}$ if and only if $p, r < \infty$.

(3) If $p_1 \le p_2$, $r_1 \le r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. If $s > \frac{n}{p}$ or $s = \frac{n}{p}$, $r = 1$, we have $B^s_{p,r} \hookrightarrow L^\infty$.

(4) For $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Furthermore, $B^s_{p,r}$ is an algebra, provided that $s > \frac{n}{p}$ or $s \ge \frac{n}{p}$ and $r = 1$.

(5) Fatou lemma: If $(u_n)_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and $u_n \to u$ in $\mathcal{S}'$, then $u \in B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

(6) Let $m \in \mathbb{R}$ and f be an S^m-multiplier (i.e., $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and satisfies that $\forall \alpha \in \mathbb{N}^n$, there exists a constant C_α, s.t. $|\partial^\alpha f(\xi)| \le C_\alpha (1 + |\xi|^{m-|\alpha|})$, $\forall \xi \in \mathbb{R}^d$). Then the operator $f(D) = \mathcal{F}^{-1}(f\mathcal{F})$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.

Lemma 2.4. *(See Corollary 2.86 in [20]) Let $1 \le p, r \le +\infty$, and the following estimates hold:*

(i) For $s > 0$,

$$\|fg\|_{B^s_{p,r}} \le C(\|f\|_{B^s_{p,r}} \|g\|_{L^\infty} + \|g\|_{B^s_{p,r}} \|f\|_{L^\infty});$$

(ii) For all $s_1 \le \frac{n}{p} < s_2$ ($s_2 \ge \frac{n}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, we have

$$\|fg\|_{B^{s_1}_{p,r}} \le C\|f\|_{B^{s_1}_{p,r}} \|g\|_{B^{s_2}_{p,r}};$$

(iii) If $s > \frac{n}{p}$ or $s = \frac{n}{p}, r = 1$, we get

$$\|fg\|_{B^s_{p,r}} \le C\|f\|_{B^s_{p,r}} \|g\|_{B^s_{p,r}}.$$

Now we describe two available interpolation inequalities.

Lemma 2.5. *(See Theorem 2.80 and Corollary 2.86 in [20]) (1) Complex interpolation: If* $s_1 < s_2$*,* $\theta \in (0,1)$*, and* $1 \leq p, r \leq \infty$*, then we get*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq C\|u\|^{\theta}_{B^{s_1}_{p,r}}\|u\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall u \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r},$$

and

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_1 - s_2}\Big(\frac{1}{\theta} + \frac{1}{1-\theta}\Big)\|u\|^{\theta}_{B^{s_1}_{p,r}}\|u\|^{1-\theta}_{B^{s_2}_{p,r}}.$$

(2) If $s \in \mathbb{R}$*,* $\epsilon > 0$*, and* $1 \leq p \leq \infty$*, there exists a constant* $C > 0$ *such that*

$$\|u\|_{B^{s}_{p,1}} \leq C\frac{\epsilon+1}{\epsilon}\|u\|_{B^{s}_{p,\infty}}\left(1 + \log\frac{\|u\|_{B^{s+\epsilon}_{p,\infty}}}{\|u\|_{B^{s}_{p,\infty}}}\right).$$

Lemma 2.6. *(See Lemma 2.100 in [20]) Let* $\sigma > 0$*,* $1 \leq r \leq \infty$*, and* $1 \leq p \leq p_1 \leq \infty$*. Let v be a vector field over* $\mathbb{R}$*. Then the following estimates hold:*

$$\|(2^{j\sigma}\|[v\partial_x, \Delta_j]f\|_{L^p})_{j\in\mathbb{N}}\|_{l^r} \leq C(\|v_x\|_{L^\infty}\|f\|_{B^{\sigma}_{p,r}} + \|f_x\|_{L^{p_2}}\|v\|_{B^{\sigma-1}_{p_1,r}}),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$*. In addition, if* $\sigma < 1$*, we have*

$$\|(2^{j\sigma}\|[v\partial_x, \Delta_j]f\|_{L^p})_{j\in\mathbb{N}}\|_{l^r} \leq C\|v_x\|_{L^\infty}\|f\|_{B^{\sigma}_{p,r}}.$$

We also need to present the Osgood lemma, which is a generalization of the Gronwall lemma.

Lemma 2.7. *(See Lemma 3.4 in [20]) Let* ρ *be a measurable function from* $[t_0, T]$ *to* $[0, c]$*,* γ *be a locally integrable function from* $[t_0, T]$ *to* $\mathbb{R}^+$*, and* μ *be an increasing continuous function from* $[0, c]$ *to* $\mathbb{R}^+$*. Suppose that*

$$\rho(t) \leq a + \int_{t_0}^{t}\gamma(s)\mu(\rho(s))ds, \quad \text{for some } a \geq 0.$$

(1) If $a > 0$*, then we have*

$$-\mathcal{M}(\rho(t)) + \mathcal{M}(a) \leq \int_{t_0}^{t}\gamma(s)ds, \quad \mathcal{M}(x) \triangleq \int_{x}^{c}\frac{1}{\mu(r)}dr.$$

(2) If $a = 0$ *and* μ *satisfy the condition* $\int_0^c \frac{dr}{\mu(r)}dr = +\infty$*, then* $\rho \equiv 0$*.*

Remark 2.1. *If* $\mu(r) = r(1 - \ln r)$*,* $r \in [0, 1]$*, we have* $\mathcal{M} = \ln(1 - \ln x)$*, and* $\rho(t) \leq ec^{e^{-\int_{t_0}^{t}\gamma(t')dt'}}$ *with* $c > 0$*.*

Finally, we introduce the Sobolev-Gevrey spaces and some basic properties to study the analytic solution.

Definition 2.2. *(See (1.10) in [22]) A function* $u \in G^{\delta}_{\sigma,s}(\mathbb{R}^d)$ *if and only if*

$$\|u\|_{G^{\delta}_{\sigma,s}(\mathbb{R}^d)} = (\int_{R^d}(1 + |\xi|^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}}|\hat{u}|^2 d\xi) < \infty, \ \ u \in C^{\infty}(\mathbb{R}^d), \tag{2.4}$$

where $\kappa, s > 0$ *and s is a real number. Furthermore, when* $0 < \delta' < \delta$*,* $0 < \sigma' < \sigma$*, and* $s' < s$*, we can find* $G^{\delta}_{\sigma,s}(\mathbb{R}^d) \hookrightarrow G^{\delta'}_{\sigma,s}(\mathbb{R}^d)$*,* $G^{\delta}_{\sigma',s}(\mathbb{R}^d) \hookrightarrow G^{\delta}_{\sigma,s}(\mathbb{R}^d)$*,* $G^{\delta}_{\sigma,s}(\mathbb{R}^d) \hookrightarrow G^{\delta}_{\sigma,s'}(\mathbb{R}^d)$*.*

Remark 2.2. *For periodic domains, the Sobolev-Gevrey norm can be defined as:*

$$\|u\|_{G^{\delta}_{\sigma,s}(\mathbb{T})} = \Big(\sum_{k\in\mathbb{Z}}(1+|k|^2)^s e^{2\delta|k|^{\frac{1}{\sigma}}}|\hat{u}(k)|^2\Big)^{\frac{1}{2}} = \|e^{\delta(-\Delta)^{\frac{1}{2\sigma}}}u\|_{H^s(\mathbb{T})}.$$

For our Gevrey regularity analysis of (1.1), the following generalized Ovsyannikov theorem is fundamental.

Lemma 2.8. *(See Theorem 3.1 in [23] and Theorem 3.1 in [24]) Let $\{X_\delta\}_{0<\delta<1}$ be a scale of decreasing Banach spaces, namely, we have $X_\delta \subset X'_\delta$ with $\|\cdot\|_{\delta'} < \|\cdot\|_\delta$ for any $\delta < \delta'$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t,u(t)), \\ u|_{t=0} = u_0. \end{cases} \tag{2.5}$$

Let $T, R > 0$ and $\sigma \geq 1$. For given $u_0 \in X_1$, assume that F satisfies the following conditions:
(i) For $0 < \delta' < \delta < 1$, the function $t \longmapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| < T$ with values in X_s and

$$\sup_{|t|<T}\|u(t)\|_\delta < R,$$

and then $t \longmapsto F(t,u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{\delta'}$.
(ii) For any $0 < \delta' < \delta < 1$ and any $u, v \in \overline{B(u_0,R)} \subset X_\delta$, there exists a positive constant L depending on u_0 and R such that

$$\sup_{|t|\leq T}\|F(t,u) - F(t,v)\|_{\delta'} < \frac{L}{(\delta-\delta')^\sigma}\|u-v\|_\delta.$$

(iii) There exists an $M > 0$ depending on u_0 and R such that for any $0 < \delta < 1$,

$$\sup_{|t|\leq T}\|F(t,0)\|_\delta \leq \frac{M}{(1-\delta)^\sigma}.$$

Then the Cauchy problem (2.5) has a $T_0 \in (0,T)$ and a unique solution $u(t)$, which is holomorphic in $|t| < D_\sigma\frac{(1-\delta)^\sigma T_0}{2^{\sigma+1}}$ with values in X_δ for every $\delta \in (0,1)$.

Remark 2.3. *In particular, $T_0 = \min\left\{\frac{1}{2^{2\sigma+4}L}, \frac{2^{\sigma+1}R}{2^{\sigma+1}2^{2\sigma+3}LR+MD_\sigma}\right\}$, which gives a lower bound of the lifespan, where $D_\sigma = \frac{1}{2^{\sigma-2+\frac{2}{\sigma+1}}}$.*

Remark 2.4. *When $\sigma = 1$, Lemma 2.8 becomes equivalent to the classical abstract Cauchy-Kovalevsky theorem.*

Lemma 2.9. *(See Proposition 2.5 in [23]) For $s > \frac{1}{2}$, $\sigma \geq 1$, and $\delta > 0$, there is an algrbra for $G^\delta_{\sigma,s}$ and there is a constant C_s such that*

$$\|uu'\|_{G^\delta_{\sigma,s}} \leq C_s\|u\|_{G^\delta_{\sigma,s}}\|u'\|_{G^\delta_{\sigma,s}}. \tag{2.6}$$

Proof. The proof of this theorem can be found in Proposition 2.5 of [23]. For the reader's convenience, the detailed process is provided below.

Since $\widehat{fg} = \hat{f} * \hat{g}$, it follows that

$$
\begin{aligned}
\|uu'\|^2_{G^\delta_{\sigma,s}} &= \int (1+|\xi|^2)^s e^{2\delta|\xi|^{\frac{1}{\sigma}}} |\hat{u} * \hat{u'}|^2 d\xi \\
&= \int (1+|\xi|^2)^s \left| \int e^{\delta|\xi|^{\frac{1}{\sigma}}} \hat{u}(\eta)\hat{u'}(\xi-\eta)\, d\eta \right|^2 d\xi \\
&\le \int (1+|\xi|^2)^s \left| \int e^{\delta|\xi-\eta|^{\frac{1}{\sigma}}} e^{\delta|\eta|^{\frac{1}{\sigma}}} \hat{u}(\eta)\hat{u'}(\xi-\eta) d\eta \right|^2 d\xi \quad \text{(Here we use the fact that } \sigma \ge 1\text{).} \\
&= \int (1+|\xi|^2)^s \left| F\left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\right) * F\left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u'\right) \right|^2 d\xi \\
&= \left\| \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\right) \cdot \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u'\right) \right\|^2_{H^s} \\
&\le C_s \left\| e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u \right\|^2_{H^s} \left\| e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u' \right\|^2_{H^s} \quad \text{(Here we use the fact that } s > \frac{1}{2}\text{).} \\
&= C_s \|u\|^2_{G^\delta_{\sigma,s}} \|u'\|^2_{G^\delta_{\sigma,s}}.
\end{aligned}
\tag{2.7}
$$

□

Lemma 2.10. *(See Proposition 2.6 in [23]) Suppose that $s > \frac{1}{2}$, $\sigma \ge 1$, and $\delta > 0$, and there exists a constant C'_s such that $\|uu'\|_{G^\delta_{\sigma,s-1}} \le C'_s \|u\|_{G^\delta_{\sigma,s-1}} \|u'\|_{G^\delta_{\sigma,s}}$.*

Proof. While the proof of this theorem is available in Proposition 2.6 of [23], it is reproduced here for the convenience of the reader.

By a similar argument as in Lemma 2.9, we obtain

$$
\|uu'\|^2_{G^\delta_{\sigma,s-1}} \le \left\| \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\right) \cdot \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u'\right) \right\|^2_{H^{s-1}}. \tag{2.8}
$$

Using the fact that $\|ab\|_{H^{s-1}} \le C_s \|a\|_{H^{s-1}} \|b\|_{H^s}$ if $s > \frac{1}{2}$, we get

$$
\begin{aligned}
\left\| \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\right) \cdot \left(e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u'\right) \right\|^2_{H^{s-1}} &\le C'_s \left\| e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u \right\|^2_{H^{s-1}} \left\| e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u' \right\|^2_{H^s} \\
&= C'_s \|u\|^2_{G^\delta_{\sigma,s-1}} \|u'\|^2_{G^\delta_{\sigma,s}}.
\end{aligned}
\tag{2.9}
$$

□

Lemma 2.11. *(See Proposition 2.4 in [23]) When $\sigma > 0$, $s > 0$, and $0 < \delta' < \delta$, we can have*

$$
\|u_x\|_{G^{\delta'}_{\sigma,s}} \le \frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma} \|u\|_{G^\delta_{\sigma,s}}, \tag{2.10}
$$

where s is a real number and $u \in G^\delta_{\sigma,s}$.

The proof of Theorem 1.4 employs a fixed-point argument in a suitable Banach space. We now define a new such space.

Definition 2.3. *(See Definition 3.4 in [23]) Let $\sigma \geq 1$. For any $a > 0$, we denote by E_a the function space consisting of X_δ-valued holomorphic and continuous functions $u(t)$, defined for all $0 < \delta < 1$ and $|t| < \frac{a(1-\delta)^\sigma D_\sigma}{2^{\sigma-1}}$, where $D_\sigma = 2^{\sigma-2+\frac{1}{2^{\sigma+1}}}$. The norm in this space is given by:*

$$\|u\|_{E_T} := \sup_{|t|<\frac{a(1-\delta)^\sigma}{2^{\sigma-1}}} \left(\|u(t)\|_{G^{\sigma,s}_\delta}(1-\delta)^\sigma \sqrt{1 - \frac{|t|}{a(1-\delta)^\sigma}} \right) < +\infty.$$

The proof of Theorem 1.4 will rely on the following crucial lemma.

Lemma 2.12. *(See Lemma 3.7 in [23] and Lemma 3.7 in [24]) Let $\sigma \geq 1$. For any $a > 0$, $u \in E_a$, $0 < \delta < 1$, and $0 \leq t < \frac{a(1-\delta)^\sigma}{2^{\sigma+1}} D_\sigma$ with $D_\sigma = \frac{1}{2^{\sigma-2+\frac{2}{\sigma+1}}}$, the following estimate holds:*

$$\int_0^t \frac{\|u(\tau)\|_{\delta(\tau)}}{(\delta(\tau)-\delta)^\sigma} d\tau \leq \frac{a 2^{2\delta+3} \|u\|_{E_a}}{(1-\delta)^\sigma} \sqrt{\frac{a(1-\delta)^\sigma}{a(1-\delta)^\sigma - t}},$$

where the intermediate parameter $\delta(\tau)$ is given by

$$\delta(\tau) = \frac{1}{2}(1+\delta) + \left(\frac{1}{2}\right)^{2+\frac{1}{\sigma}} \left(\left[(1-\delta)^\sigma - \frac{t}{a} \right]^{\frac{1}{\sigma}} - \left[(1-\delta)^\sigma + (2^{\sigma+1}-1)\frac{t}{a} \right]^{\frac{1}{\sigma}} \right) \in (0,1).$$

The above constitutes all the properties required for this article. Lemmas 2.1–2.7 concern the properties in Besov spaces used in proving Theorems 1.1-1.2, while the latter part, Lemmas 2.8–2.12, contains the properties necessary for the proofs of Theorems 1.3-1.4.

3. Local well-posedness

The purpose of this section is to establish the existence, uniqueness, and continuity of strong local solutions to Eq (1.1). To utilize the transport theory, we first need to rewrite Eq (1.1) in the following equivalent form:

$$\begin{cases} u_t = auv^{p-1}\partial_x u + \partial_x^{-1} u, & x \in \mathbb{T}, t \in \mathbb{R}, \\ v_t = bvu^{q-1}\partial_x v + \partial_x^{-1} v, & x \in \mathbb{T}, t \in \mathbb{R}, \\ u(x,0) = u_0, & x \in \mathbb{T}, t = 0, \\ v(x,0) = v_0, & x \in \mathbb{T}, t = 0, \end{cases} \tag{3.1}$$

where the inverse derivative operator ∂_x^{-1} is a mean-zero, 2π-periodic pseudo differential operator. The exact definition is as follows:

$$\begin{aligned} \partial_x^{-1} f(x) \doteq & \int_0^x f(y)dy - \frac{x}{2\pi}\int_0^{2\pi} f(y)dy \\ & - \frac{1}{2\pi}\int_0^{2\pi} \Big[\int_0^y f(x)dx - \frac{y}{2\pi}\int_0^{2\pi} f(x)dx \Big] dy. \end{aligned} \tag{3.2}$$

Moreover, in [25], we can find $\|\partial_x^{-1} f\|_{B^s_{2,r}} \leq \|f\|_{B^{s-1}_{2,r}}$.

Proof of Theorem 1.1. First step: existence of a local solution.

We let $u_0 = v_0 \triangleq 0$ and develop a sequence of smooth functions $(u_n, v_n)_{n\in\mathbb{N}}$, which serve as smooth approximations to the solutions of the following linear transport system:

$$T_n \begin{cases} (\partial_t - au_n v_n^{p-1}\partial_x)u_{n+1} = \partial_x^{-1}u_n, \\ (\partial_t - bv_n u_n^{q-1}\partial_x)v_{n+1} = \partial_x^{-1}v_n, \\ u_{n+1}(0) = \Phi_{n+1}u(0), v_{n+1}(0) = \Phi_{n+1}v(0). \end{cases} \tag{3.3}$$

Based on $\|\partial_x^{-1}u_n\|_{B^{\frac{3}{2}}_{2,1}} \le \|u_n\|_{B^{\frac{1}{2}}_{2,1}}$, we use Lemma 2.2 for system (3.3) to get

$$\begin{aligned}\|u_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} \le{}& e^{C\int_0^t \|\partial_x(u_n v_n^{p-1})(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau} \|\Phi_{n+1}u(0)\|_{B^{\frac{3}{2}}_{2,1}} \\ &+ c\int_0^t e^{C\int_\tau^t \|\partial_x(u_n v_n^{p-1})(\tau')\|_{B^{\frac{1}{2}}_{2,1}} d\tau'} \|u_n\|_{B^{\frac{3}{2}}_{2,1}} d\tau,\end{aligned}$$

and

$$\begin{aligned}\|v_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} \le{}& e^{C\int_0^t \|\partial_x(v_n u_n^{q-1})(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau} \|\Phi_{n+1}v(0)\|_{B^{\frac{3}{2}}_{2,1}} \\ &+ c\int_0^t e^{C\int_\tau^t \|\partial_x(v_n u_n^{p-1})(\tau')\|_{B^{\frac{1}{2}}_{2,1}} d\tau'} \|v_n\|_{B^{\frac{3}{2}}_{2,1}} d\tau.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|u_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} + \|v_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} \le{}& e^{C\int_0^t (\|u_n\|_{B^{\frac{3}{2}}_{2,1}} \|v_n\|^{p-1}_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}} \|u_n\|^{q-1}_{B^{\frac{3}{2}}_{2,1}})d\tau} (\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}) \\ &+C_s\int_0^t e^{C\int_\tau^t (\|u_n\|_{B^{\frac{3}{2}}_{2,1}} \|v_n\|^{p-1}_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}} \|u_n\|^{q-1}_{B^{\frac{3}{2}}_{2,1}})d\tau'} (\|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})d\tau.\end{aligned}$$

Let $m = \max\{p, q\}$, and we deduce

$$\|u_n\|_{B^{\frac{3}{2}}_{2,1}} \|v_n\|^{p-1}_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}} \|u_n\|^{q-1}_{B^{\frac{3}{2}}_{2,1}} \le (1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^m,$$

and

$$\|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}} \le (1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^{m+1}.$$

So we get

$$\begin{aligned}1 + \|u_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} + \|v_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} \le{}& e^{C\int_0^t (1+\|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau)} (1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}) \\ &+ C_s\int_0^t e^{C\int_\tau^t (1+\|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau'} (1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^{m+1} d\tau.\end{aligned} \tag{3.4}$$

We can choose a $T > 0$ such that $0 < T < \frac{3}{8Cm(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}$, and assume that

$$1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}} \leq \frac{C(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})}{[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m]^{\frac{1}{m}}}.$$

Therefore, we have

$$\int_\tau^t (1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau \leq -\frac{1}{2Cm} \ln \frac{1 - 2mCt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}{1 - 2mC\tau(1\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}. \tag{3.5}$$

We also obtain

$$\int_0^t (1 + \|u_n\|_{B^{\frac{3}{2}}_{2,1}} + \|v_n\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau \leq -\frac{1}{2Cm} \ln\{1 - 2mCt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\}. \tag{3.6}$$

So we calculate

$$\begin{aligned}
1 + \|u_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} + \|v_{n+1}\|_{B^{\frac{3}{2}}_{2,1}} &\leq \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m]^{\frac{1}{2m}}} \\
&+ C\int_0^t \frac{(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^{m+1}}{[1 - 2Cm\tau(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m]^{1+\frac{1}{m}}} \times \left[\frac{1 - 2mCt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}{1 - 2Cm\tau(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}\right]^{\frac{-1}{2m}} d\tau \\
&\leq \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}} + \frac{C}{\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}} \\
&\times \int_0^t \frac{(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^{m+1}}{\left[1 - 2Cm\tau(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{2m+1}{2m}}} d\tau \\
&\leq \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}} + \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{-2m\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}} \\
&\times \int_0^t \frac{d(1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m)}{\left[1 - 2Cm\tau(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{2m+1}{2m}}} d\tau \\
&\leq \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}} + \frac{1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}}}{\left[1 - 2Cmt(1 + \|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\right]^{\frac{1}{2m}}}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\frac{1}{\Big[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\Big]^{\frac{1}{2m}}}-1\Big)\\
&\le (1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})\Big[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\Big]^{\frac{-1}{2m}}\\
&\quad+(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})\Big[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\Big]^{\frac{-1}{m}}\\
&\quad-(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})\Big[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\Big]^{\frac{-1}{2m}}\\
&\le (1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})\Big[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m\Big]^{\frac{-1}{m}}. \qquad (3.7)
\end{aligned}
$$

We have thus established that the sequence $(u_n, v_n)_{n\ge1}$ is uniformly bounded in the space $C([0,T]; B^{\frac{3}{2}}_{2,1}(\mathbb{T}))$. Given this uniform bound for (u_n, v_n) and the Banach algebra property of $B^{\frac{3}{2}}_{2,1}(\mathbb{T})$, we can infer that the terms $au_nv_n^{p-1}\partial_x u^{n+1}$, $bv_nu_n^{p-1}\partial_x v^{n+1}$, $\partial_x^{-1}u$, and $\partial_x^{-1}v$ belong to $C([0,T]; B^{\frac{3}{2}}_{2,1}(\mathbb{T}))$. Together with the linear equation T_n, this implies that $(\partial_t u_{n+1}, \partial_t v_{n+1}) \in C([0,T]; B^{\frac{3}{2}}_{2,1}(\mathbb{T})))$. Consequently, we conclude that $(u_n, v_n) \in E^{\frac{3}{2}}_{2,1}(\mathbb{T}))$ for all $n \in \mathbb{N}^+$.

Second step: convergence of the approximate solutions.

We claim that the approximation solutions $(u_n, v_n)_{n\ge1}$ is a Cauchy sequence in larger Banach spaces $C([0,T]; B^{\frac{1}{2}}_{2,\infty})(\mathbb{T})$. For this purpose, given any $(m,n) \in N^+ \times N^+$, we get from equation (T_n) that

$$
\begin{cases}
\partial_t(u_{n+k+1}-u_{n+1}) - a_{n+k}u_{n+k}v^{p-1}_{n+k}\partial_x(u_{n+k+1}-u_{n+1})\\
\qquad -(a_{n+k}v^{p-1}_{n+k}-a_nu_nv^{p-1}_n)\partial_x u_{n+1} = \partial_x^{-1}(u_{n+k}-u_n),\\
\partial_t(v_{n+k+1}-u_{n+1}) - b_{n+1}v_{n+k}u^{q-1}_{m+n}\partial_x(v_{n+k+1}-u_{n+1})\\
\qquad -(a_{n+k}u^{q-1}_{n+k}-a_nv_nu^{p-1}_n)\partial_x v_{n+1} = \partial_x^{-1}(v_{n+k}-v_n),\\
(u_{n+k+1}-u_{n+1})(0,x) = \Phi_{n+k+1}u_0 - \Phi_{n+1}u_0,\\
(v_{n+k+1}-v_{n+1})(0,x) = \Phi_{n+k+1}v_0 - \Phi_{n+1}v_0.
\end{cases} \qquad (3.8)
$$

We have already obtained that the smooth approximation u_n of (3.3) is uniformly bounded in $E^{\frac{3}{2}}_{2,1}$. The uniform bound (3.7) and $B^{\frac{1}{2}}_{2,1} \hookrightarrow B^{\frac{1}{2}}_{2,\infty}$ yield that for $0 \le t \le T < \frac{3}{8Cm(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^{\frac{1}{m}}}$,

$$
\begin{aligned}
\|(u_{n+k+1}-u_{n+1})(t)\|_{B^{\frac{3}{2}}_{2,\infty}} &\le \|u_{n+k+1}(t)\|_{B^{\frac{3}{2}}_{2,1}} + \|u_{n+1}(t)\|_{B^{\frac{3}{2}}_{2,1}}\\
&\le \frac{2C(1+\|u_0\|_{B^{\frac{3}{2}}_{2,\infty}}+\|v_0\|_{B^{\frac{3}{2}}_{2,\infty}})}{[1-2Cmt(1+\|u_0\|_{B^{\frac{3}{2}}_{2,\infty}}+\|v_0\|_{B^{\frac{3}{2}}_{2,\infty}})^m]^{\frac{1}{m}}} \qquad (3.9)\\
&\le 2C(1+\|u_0\|_{B^{\frac{3}{2}}_{2,\infty}}+\|v_0\|_{B^{\frac{3}{2}}_{2,\infty}}) := M.
\end{aligned}
$$

Similarly, we also derive $\|(v^{n+m+1}-v^{n+1})(t)\|_{B^{\frac{3}{2}}_{2,\infty}} \le M$. In particular, $\|u_n(t)\|_{B^{\frac{3}{2}}_{2,\infty}} \le 2C(1+\|u_0\|_{B^{\frac{3}{2}}_{2,\infty}}+\|v_0\|_{B^{\frac{3}{2}}_{2,\infty}})$. Next, we use Lemma 2.2 for Eq (3.8) to get

$$\begin{aligned}\|(u_{n+k+1}-u_{n+1})(t)\|_{B^{\frac{1}{2}}_{2,\infty}} &\le e^{C\int_0^t \|\partial_x(a_n u_{n+k} v^{p-1}_{n+k})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau} \|\Phi_{n+k+1}u_0-\Phi_{n+1}u_0\|_{B^{\frac{1}{2}}_{2,\infty}} \\ &+ C\int_0^t e^{c\int_\tau^t \|\partial_x(a_n u_{n+k} v^{p-1}_{n+k})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} d\tau'} \|\partial_x^{-1}(u_{n+k}-u_n)\|_{B^{\frac{1}{2}}_{2,\infty}}.\end{aligned} \tag{3.10}$$

Next, we estimate the terms on the right-hand side of the above inequality (3.10). According to the uniform bound for approximations, we deduce for any $0\le\tau\le t\le T$,

$$\begin{aligned}\int_\tau^t \|\partial_x(a_n u_{n+k} v^{p-1}_{n+k})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} dr &\le C\int_\tau^t \|u_{n+k}v^{p-1}_{n+k}\|_{B^{\frac{3}{2}}_{2,\infty}\cap Lip} dr \\ &\le C\int_\tau^t \|u_{n+k}\|_{B^{\frac{3}{2}}_{2,1}} \|v_{n+k}\|^{p-1}_{B^{\frac{3}{2}}_{2,1}} dr \\ &\le C\int_\tau^t (1+\|u_{n+k}\|^m_{B^{\frac{3}{2}}_{2,1}}+\|v_{n+k}\|^m_{B^{\frac{3}{2}}_{2,1}}) dr \\ &\le CT+\int_0^T \frac{C^{m+1}(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m}{1-2Cmr(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m} dr \\ &\le \frac{3C}{8mC^m(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}}} + 4C^{m+1}(1+\|u_0\|_{B^{\frac{3}{2}}_{2,1}}+\|v_0\|_{B^{\frac{3}{2}}_{2,1}})^m \\ &< \infty.\end{aligned} \tag{3.11}$$

Besides, referring to the definition of the cut-off low frequency operator,

$$\begin{aligned}\|\Phi_{n+k+1}u_0-\Phi_{n+1}u_0\|_{B^{\frac{3}{2}}_{2,\infty}} &\le C\|\sum_{n+1\le k\le n+k}\Delta_k u_0\|_{B^{\frac{3}{2}}_{2,1}} \\ &\le C\sum_{j\ge -1} 2^{\frac{1}{2}j}\|\Delta_j(\sum_{n+1\le k\le n+k}\Delta_k u_0)\|_{L^2} \\ &\le C\sum_{n\le j\le k+n+1} 2^{-j}2^{j(1+\frac{3}{2})}(\|\Delta_j\Phi_{n+k+1}u_0\|_{L^2}+\|\Delta_j\Phi_{n+1}u_0\|_{L^2}) \\ &\le C2^{-n}\sum_{n\le j\le k+n+1} 2^{j(1+\frac{3}{2})}\|\Delta_j u_0\|_{L^2} \\ &= C2^{-n}\|u_0\|_{B^{\frac{3}{2}}_{2,1}},\end{aligned} \tag{3.12}$$

where we have used $\Delta_j\Phi_i=\Phi_i\Delta_j$ and if $|j-i|\ge 2$, $\Delta_j\Delta_i=0$. Next, we estimate the remaining terms,

$$\|\partial_x^{-1}(u_{n+k}-u_n)\|_{B^{\frac{1}{2}}_{2,\infty}} \le \|u_{n+k}-u_n\|_{B^{\frac{1}{2}}_{2,1}}. \tag{3.13}$$

Hence, we obtain

$$\begin{aligned}\|u_{n+k+1}-u_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} &\le C2^{-n}\|u_0\|_{B^{\frac{1}{2}}_{2,1}} + C\int_0^t \|u^{n+k}-u^n\|_{B^{\frac{1}{2}}_{2,1}}\, d\tau \\ &\le CM2^{-n} + C\int_0^t \|u^{n+k}-u^n\|_{B^{\frac{1}{2}}_{2,1}}.\end{aligned} \tag{3.14}$$

We aim to use the logarithmic interpolation inequality on $u_{n+m} - u_n$ to derive

$$\|u_{n+k}-u_n\|_{B^{\frac{1}{2}}_{2,1}} \le C\|u_{n+k}-u_n\|_{B^{\frac{1}{2}}_{2,\infty}} \ln(e + \frac{\|u_{n+k}-u_n\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|u_{n+k}-u_n\|_{B^{\frac{1}{2}}_{2,\infty}}}). \tag{3.15}$$

Similarly, we also derive

$$\|v_{n+k}-v_n\|_{B^{\frac{1}{2}}_{2,1}} \le C\|v_{n+k}-v_n\|_{B^{\frac{1}{2}}_{2,\infty}} \ln(e + \frac{\|v_{n+k}-v_n\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|v_{n+k}-v_n\|_{B^{\frac{1}{2}}_{2,\infty}}}). \tag{3.16}$$

For convenience, we introduce the following notation:

$$\begin{aligned}&D_{n,k} = \|(u^{n+k}-u^n)(t)\|_{B^{\frac{1}{2}}_{2,\infty}} + \|(v^{n+k}-v^n)(t)\|_{B^{\frac{1}{2}}_{2,\infty}}, \\ &D_n(t) = \sup_{k\in N^+} D_{n,k}(t), D(t) = \limsup_{n\to\infty} D_n(t),\end{aligned} \tag{3.17}$$

and this leads to the conclusion from (3.14) that

$$D_{n+1}(t) \le CM2^{-n} + C\int_0^t D_n(\tau)\ln(e + \frac{M}{D_n(\tau)})d\tau. \tag{3.18}$$

According to the definite of $D(t)$, $\forall\varepsilon > 0$, $\exists N = N(\varepsilon) > 0$ such that $D_n(t) < D(t) + \varepsilon$ for $n > N$. Hence,

$$D_{n+1}(t) \le CM2^{-n} + C\int_0^t (D(\tau)+\varepsilon)\ln(e + \frac{M}{D(\tau)+\varepsilon})d\tau.$$

First, take the supremum over n, and then let n approach 0. We can deduce from the previous inequality that

$$D(t) \le C\int_0^t D(\tau)\ln(e + \frac{M}{D(\tau)})d\tau,$$

where if $x \in (0, M]$, $\mu_x = x\ln(e + \frac{M}{x})$, when $x = 0$, $\mu(x) = 0$. Note that $\int_0^M \frac{1}{x\ln(e+\frac{M}{x})}dx = \int_1^{+\infty} \frac{dy}{y\ln(e+y)} = +\infty$. Based on the Osgood lemma, $D(t) = 0$, $t \in (0, T]$, so $\limsup_{n\to\infty} D_n(t) \le \limsup_{n\to\infty} D_n(t) = 0$ This shows $\lim_{n\to\infty} D_n(t) = 0$, so $(u^n, v^n)_{n\ge1}$ is the Cauchy sequence in $C([0,T]; B^{\frac{1}{2}}_{2,\infty})$. Next, we prove that the approximate solution $\{u^n\}_{n\ge1}$ converges strongly to the space $C([0,T]; B^{\frac{1}{2}}_{2,1})$. If $0 < \varepsilon < 1$, $\theta \in (0, 1)$ such that $\frac{1}{2} + \varepsilon = \frac{\theta}{2} + \frac{3}{2}(1-\theta)$. Because $B^{\frac{1}{2}+\varepsilon}_{2,1} \hookrightarrow B^{\frac{1}{2}}_{2,1}$ for $\varepsilon > 0$, we can apply Lemma 2.5 to get

$$\begin{aligned}\|u_{n+k+1}-u_{n+1}\|_{B^{\frac{1}{2}}_{2,1}} &\le \|u_{n+k+1}-u_{n+1}\|_{B^{\frac{1}{2}+\varepsilon}_{2,1}}\\ &\le C(\frac{1}{\theta}+\frac{1}{1-\theta})\|u_{n+k+1}-u_{n+1}\|^{\theta}_{B^{\frac{1}{2}}_{2,\infty}}\|u_{n+k+1}-u_{n+1}\|^{1-\theta}_{B^{\frac{3}{2}}_{2,\infty}}\\ &\le C(\frac{1}{\theta}+\frac{1}{1-\theta})M^{1-\theta}\|u_{n+k+1}-u_{n+1}\|^{\theta}_{B^{\frac{1}{2}}_{2,1}}\\ &\to 0, n,k\to\infty,\end{aligned} \tag{3.19}$$

and

$$\|v_{n+k+1}-v_{n+1}\|_{B^{\frac{1}{2}}_{2,1}}\to 0, n,k\to\infty, \tag{3.20}$$

where the last limitation is based on the fact that $(u_n(t),v_n(t))_{n\ge1}$ forms a Cauchy sequence in $B^{\frac{1}{2}}_{2,\infty}$, as demonstrated previously. As a consequence, we have shown that $(u_n(t),v_n(t))_{n\ge1}$ is a Cauchy sequence in $C([0,T];B^{\frac{1}{2}}_{2,1})$, which implies that there exists a function u such that (u_n,v_n) converges strongly to (u,v) in $C([0,T];B^{\frac{1}{2}}_{2,1})$ as n approaches infinity. Taking the limit as $n\to\infty$ in (3.3), we confirm that the element is a valid solution to the Eq (1.1). Furthermore, due to the algebraic property of $B^{\frac{1}{2}}_{2,1}$, it is not hard to confirm that $auv^{p-1}\partial_x u$, $\partial_x^{-1}u$, $bvu^{q-1}\partial_x v$, $\partial_x^{-1}v$ together with Eq (1.1) itself imply that $\partial_t u$, $\partial_t v$ belongs to $C([0,T];B^{\frac{1}{2}}_{2,1})$. Therefore, we have demonstrated that the solution (u,v) belongs to $E^{\frac{3}{2}}_{2,1}(T)$. This completes the proof of the existence part of Theorem 1.1.

Third step: uniqueness and stability. Suppose that (u,v) and $(\tilde u,\tilde v)$ are solutions for Eq (1.1) with the same initial data (u_0,v_0). Let $w=u-\tilde u$, $z=v-\tilde v$, and the solution (w,z) satisfies the following equations:

$$\begin{cases}\partial_t w = auv^{p-1}\partial_x w+(auv^{p-1}-a\tilde u\tilde v^{p-1})\partial_x\tilde u+\partial_x^{-1}w,\\ \partial_t z = bvu^{q-1}\partial_x z+(bvu^{q-1}-b\tilde v\tilde u^{q-1})\partial_x\tilde v+\partial_x^{-1}z,\\ w(x,0)=u_0-\tilde u_0,\\ z(x,0)=v_0-\tilde v_0.\end{cases} \tag{3.21}$$

By utilizing the a priori estimate for the linear transport equation in Besov spaces to the above system, we obtain

$$\begin{cases}\|w(t)\|_{B^{\frac{1}{2}}_{2,\infty}}\le e^{c\eta(t)}(\|w_0\|_{B^{\frac{1}{2}}_{2,\infty}}+\int_0^t e^{-c\eta(\tau)}\|E(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}}d\tau),\\ \|z(t)\|_{B^{\frac{1}{2}}_{2,\infty}}\le e^{cI(t)}(\|z_0\|_{B^{\frac{1}{2}}_{2,\infty}}+\int_0^t e^{-cI(\tau)}\|F(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}}d\tau),\end{cases} \tag{3.22}$$

where $\eta(t)=\int_0^t\|\partial_x(auv^{p-1})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}d\tau$, $I(t)=\int_0^t\|\partial_x(bvu^{q-1})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}d\tau$, $E(t,x)=(auv^{p-1}-a\tilde u\tilde v^{p-1})\partial_x\tilde u+\partial_x^{-1}w$, and $F(t,x)=(bvu^{q-1}-b\tilde v\tilde u^{q-1})\partial_x\tilde v+\partial_x^{-1}z$. Based on $B^{\frac{3}{2}}_{2,1}\subset Lip$ and $B^{\frac{3}{2}}_{2,1}\subset B^{\frac{3}{2}}_{2,\infty}$, we have

$$\|\partial_x(auv^{p-1})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}+\|\partial_x(bvu^{q-1})\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\le C(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m. \tag{3.23}$$

Next, the estimates for the terms $E(t,x)$ and $F(t,x)$ are obtained by using the space $B^{\frac{1}{2}}_{2,1}\cap L^{\infty}$ as a Banach algebra;

$$\begin{aligned}\|E(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}} &= \|[au(v^{p-1}-\tilde{v}^{p-1})+a(u-\tilde{u})\tilde{v}^{p-1}]\partial_x\tilde{u}\|_{B^{\frac{1}{2}}_{2,\infty}}+\|\partial_x^{-1}w\|_{B^{\frac{1}{2}}_{2,\infty}}\\ &= \|[au(v^{p-1}-\tilde{v}^{p-1})+aw\tilde{v}^{p-1}]\partial_x\tilde{u}\|_{B^{\frac{1}{2}}_{2,\infty}}+\|w\|_{B^{\frac{1}{2}}_{2,1}}\\ &\le c\|u(v^{p-1}-v^{p-1})\partial_x\tilde{u}\|_{B^{\frac{1}{2}}_{2,1}}+\|w\tilde{v}^{p-1}\partial_x\tilde{u}\|_{B^{\frac{1}{2}}_{2,1}}+\|w\|_{B^{\frac{1}{2}}_{2,1}}\\ &\lesssim \|u\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}}\|z\|_{B^{\frac{1}{2}}_{2,1}}\sum_{i=1}^{p-2}\|v\|^i_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|^{p-2-i}_{B^{\frac{1}{2}}_{2,1}}+\|w\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|^{p-1}_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}}+\|w\|_{B^{\frac{1}{2}}_{2,1}},\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}\|F(t,x)\|_{B^{\frac{1}{2}}_{2,\infty}} \lesssim{}& \|v\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}}\|w\|_{B^{\frac{1}{2}}_{2,1}}\sum_{i=1}^{p-2}\|u\|^i_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|^{p-2-i}_{B^{\frac{1}{2}}_{2,1}}\\ &+\|z\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|^{p-1}_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}}+\|z\|_{B^{\frac{1}{2}}_{2,1}}.\end{aligned}\tag{3.25}$$

We let $\Psi=(w,z)$ and $\|\Psi(t)\|=\|w(t)\|+\|z(t)\|$, so we get

$$\begin{aligned}\|\Psi(t)\|_{B^{\frac{1}{2}}_{2,\infty}} \le{}& e^{c\int_0^t(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau}\Big(\|\Psi_0\|_{B^{\frac{1}{2}}_{2,\infty}}+c\int_0^t e^{-c\int_0^\tau(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau}\\ &\Gamma(\tau)\times\|\Psi(\tau)\|_{B^{\frac{1}{2}}_{2,1}}d\tau\Big),\end{aligned}\tag{3.26}$$

where

$$\begin{aligned}\Gamma(\tau) ={}& \|u\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}}\sum_{i=1}^{p-2}\|v\|^i_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|^{p-2-i}_{B^{\frac{1}{2}}_{2,1}}+\|\tilde{v}\|^{p-1}_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}}+2\\ &+\|v\|_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}}\sum_{i=1}^{q-2}\|u\|^i_{B^{\frac{1}{2}}_{2,1}}\|\tilde{u}\|^{q-2-i}_{B^{\frac{1}{2}}_{2,1}}+\|\tilde{u}\|^{q-1}_{B^{\frac{1}{2}}_{2,1}}\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}}\\ \lesssim{}& 4+\|u\|^4_{B^{\frac{3}{2}}_{2,1}}+\|v\|^4_{B^{\frac{3}{2}}_{2,1}}+\|\tilde{u}\|^4_{B^{\frac{3}{2}}_{2,1}}+\|\tilde{v}\|^4_{B^{\frac{3}{2}}_{2,1}}+(1+\|v\|_{B^{\frac{3}{2}}_{2,1}}+\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}})^{2m}\\ &+(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}})^{2m}+(1+\|\tilde{u}\|_{B^{\frac{3}{2}}_{2,1}}+\|\tilde{v}\|_{B^{\frac{3}{2}}_{2,1}})^m\triangleq\mathcal{A}(\tau).\end{aligned}\tag{3.27}$$

Suppose that $\sup\limits_{t\in[0,T^*]}\mathcal{H}(\tau)=\sup\limits_{t\in[0,T^*]}\exp\big(-c\int_0^t(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau\big)\|\Psi(t)\|_{B^{\frac{1}{2}}_{2,\infty}}$, and then from (3.26), we arrive at

$$\mathcal{H}(t)\le\mathcal{H}(0)+c\int_0^t e^{-c\int_0^\tau(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau}(\mathcal{A}(\tau)\times\|\Psi(\tau)\|_{B^{\frac{1}{2}}_{2,1}}d\tau).\tag{3.28}$$

We use interpolation inequalities of Lemma 2.5 to obtain

$$\|\Psi(\tau)\|_{B^{\frac{1}{2}}_{2,1}}\le\|\Psi(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}}\ln\Big(e+\frac{\|\Psi(\tau)\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|\Psi(\tau)\|_{B^{\frac{1}{2}}_{2,\infty}}}\Big).\tag{3.29}$$

Since $\|\Psi(\tau)\|_{B^{\frac{3}{2}}_{2,\infty}} \leq \mathcal{A}(\tau)$, we deduce

$$\begin{aligned} e^{-c\int_0^t(\|u\|_{B^{\frac{1}{2}}_{2,1}}\|v\|^{p-1}_{B^{\frac{1}{2}}_{2,1}})d\tau}\|\Psi\|_{B^{\frac{1}{2}}_{2,1}} &\leq e^{-c\int_0^t(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau}\Big(\|\Psi\|_{B^{\frac{1}{2}}_{2,\infty}}\times\ln\Big(e+\frac{\|\Psi(t)\|_{B^{\frac{3}{2}}_{2,\infty}}}{\|\Psi(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}\Big)\Big) \\ &\leq e^{-c\int_0^\tau(1+\|u\|_{B^{\frac{3}{2}}_{2,1}}+\|v\|_{B^{\frac{3}{2}}_{2,1}})^m d\tau}\Big(\|\Psi\|_{B^{\frac{1}{2}}_{2,\infty}}\times\ln\Big(e+\frac{\mathcal{A}(\tau)}{\|\Psi(t)\|_{B^{\frac{1}{2}}_{2,\infty}}}\Big) \\ &\leq \mathcal{H}(\tau)\ln(e+\mathcal{A}(\tau))(1-\ln\mathcal{H}(\tau)), \end{aligned} \tag{3.30}$$

where the final estimate relies on the inequality $\ln(e+\frac{a}{x}) \leq \ln(e+a)(1-\ln x)$, which holds for all $x \in (0,1]$ and any $a > 0$. Combining this with (3.28) , we derive

$$\mathcal{H}(\tau) \leq \mathcal{H}(0) + c\int_0^t \mathcal{H}(\tau)(1-\ln\mathcal{H}(\tau))\mathcal{A}(\tau)\ln(e+\mathcal{A}(\tau))d\tau. \tag{3.31}$$

The following properties hold:

(1) The function $\mu(x) = x\ln(1-x)$ is positive and increasing on $(0,1]$.

(2) The solution $(u,v) \in C([0,T];B^{\frac{3}{2}}_{2,1})$ ensures that the function $\gamma(t) := \mathcal{A}t\ln(e+\mathcal{A}t)$ is continuous (hence locally integrable) on $[0,T]$.

(3) The integral evaluates to $\int_x^1 \frac{dr}{\mu(r)} = \ln(1-\ln x)$.

Under $\sup_{t\in[0,T^*]} \mathcal{H}(t) \leq 1$, Osgood's lemma (Lemma 2.7) gives

$$\begin{aligned} C\int_0^t \mathcal{A}(\tau)\ln(e+\mathcal{A}(\tau))d\tau &\leq \ln(1-\ln\mathcal{H}(0)) - \ln(1-\ln\mathcal{H}(t)) \\ &= \ln\left(\frac{\ln\mathcal{H}(0)-1}{\ln\mathcal{H}(t)-1}\right) = \ln\left(\frac{\ln(\mathcal{H}(0)/e)}{\ln(\mathcal{H}(t)/e)}\right). \end{aligned} \tag{3.32}$$

To analyze the stability of the data-to-solution mapping, we will utilize the following fundamental lemma for the linear transport equation within the framework of Besov spaces.

Lemma 3.1. *([26]) Assume that $d \in \mathbb{N}^+$, $1 \leq p \leq \infty$, and $\{u_n\}_{n\geq 1}$ is a sequence of functions belonging to $C([0,T];B^{1+\frac{d}{p}}_{p,1})$. Let u_n be a solution to*

$$\begin{cases} \partial_t u_n + a_n\cdot\nabla u_n = h, \\ u_n(0,x) = u_0(x), \end{cases} \tag{3.33}$$

with $u_0 \in B^{\frac{d}{p}}_{p,1}$, $h \in L^1(0,T;B^{\frac{d}{p}}_{p,1})$, and $\sup_{n\in\mathbb{N}_+}\|a_n\|_{B^{1+\frac{d}{p}}_{p,1}} \leq \gamma(t)$, for some $\gamma \in L^1(0,T)$. If in addition a_n tends to a_∞ in $L^1(0,T;B^{\frac{d}{p}}_{p,1})$, then u_n tends to u_∞ in $C([0,T];B^{\frac{d}{p}}_{p,1})$.

Fourth step: the countinuous dependence. Assume that we are provided with (u_n,v_n) and (u_∞,v_∞), two solutions of Eq (1.1), with initial data (u_n^0,v_n^0) and (u_0^∞,v_0^∞) such that (u_0^n,v_0^n) converges to (u_0^∞,v_0^∞) in $B^{\frac{3}{2}}_{2,1}$. The above steps ensure that (u_n,v_n) and (u_∞,v_∞) are uniformly bounded in $L^\infty([0,T];B^{\frac{3}{2}}_{p,1})$, and $\|u^n-u^\infty\|_{B^0_{p,\infty}} + \|v^n-v^\infty\|_{B^0_{p,\infty}} \leq C(\|u_0^n-u_0^\infty\|_{B^{\frac{3}{2}}_{2,1}} + \|v_0^n-v_0^\infty\|_{B^{\frac{3}{2}}_{2,1}})$.

By applying the interpolation inequality, we deduce that (u^n, v^n) converges to (u^∞, v^∞) in $C([0,T]; B^{\frac{3}{2}-\delta}_{2,1})$ for $\delta > 0$. Selecting $\delta = 1$, we obtain an improved convergence result: $(u^n, v^n) \to (u^\infty, v^\infty)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$.

Next, to prove $(u^n, v^n) \to (u^\infty, v^\infty)$ in $C([0,T]; B^{\frac{3}{2}}_{2,1})$, it is sufficient to demonstrate the convergence of $(u^n_x, v^n_x) \to (u^\infty_x, v^\infty_x)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. For convenience, let $z^n = u^n_x$, $w^n = v^n_x$, $z^n = \rho^n + \theta^n$, $w^n = g^n + h^n$, and $(\rho^n, \theta^n, g^n, h^n)$ satisfies

$$\begin{cases} \partial_t \rho^n - a u^n (v^n)^{p-1} \rho^n_x = F^\infty, \\ \partial_t g^n - b v^n (u^n)^{q-1} g^n_x = f^\infty, \\ \rho^n(0,x) = u^\infty_{0x}, \\ g^n(0,x) = v^\infty_{0x}. \end{cases} \tag{3.34}$$

In addition,

$$\begin{cases} \partial_t \theta^n - a u^n (v^n)^{p-1} \theta^n_x = F^n - F^\infty, \\ \partial_t h^n - b v^n (u^n)^{q-1} h^n_x = f^n - f^\infty, \\ \theta^n(0,x) = u^n_{0x} - u^\infty_{0x}, \\ g^n(0,x) = v^n_{0x} - v^\infty_{0x}, \end{cases} \tag{3.35}$$

where $F^n = a u^n v^n z^n + u^n$, $f^n = b v^n u^n w^n + v^n$. We define $A^n = -a u^n (v^n)^{p-1}$ and $B^n = -b v^n (u^n)^{q-1}$ to deduce

$$\begin{aligned} \|A^n\|_{B^{\frac{1}{2}}_{2,1}} &\le C\|(u^n - u^\infty)(v^n)^{p-1}\|_{B^{\frac{1}{2}}_{2,1}} + \|u^\infty (v^n - v^\infty) \sum_{i=0}^{p=2} (v^n)^{p-2-i} (v^\infty)^2\|_{B^{\frac{1}{2}}_{2,1}} \\ &\le C(\|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}}), \end{aligned} \tag{3.36}$$

and

$$\|B^n\|_{B^{\frac{1}{2}}_{2,1}} \le C(\|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}}). \tag{3.37}$$

Hence, in light of the condition $(u^n, v^n) \to (u^\infty, v^\infty)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, we find that $(A^n, B^n) \to (A^\infty, B^\infty)$ in $L^1([0,T]; B^{\frac{1}{2}}_{2,1})$ as $n \to \infty$. By using Lemma 3.1, we gain $(\rho^n, g^n) \to (\rho^\infty, g^\infty)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, and find that $(A^n, B^n) \to (A^\infty, B^\infty)$ in $L^1([0,T]; B^{\frac{1}{2}}_{2,1})$.

In view of Lemma 2.2, we arrive at

$$\begin{aligned} \|\theta^n\|_{B^{\frac{1}{2}}_{2,1}} &\le e^{c\int_0^t \|\partial_x (u^n (v^n)^{p-1})\|_{B^{\frac{1}{2}}_{2,\infty}} d\tau} \|u^n_{0x} - u^\infty_{0x}\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad + c\int_0^t e^{\int_\tau^t \|\partial_x (u^n (v^n)^{p-1})\|_{B^{\frac{1}{2}}_{2,\infty}} d\eta} \times \|F^n - F^\infty\|_{B^{\frac{1}{2}}_{2,1}} d\tau, \end{aligned} \tag{3.38}$$

where we have

$$\begin{aligned} F^n - F^\infty &= a u^n v^n z^n + u^n - (a u^\infty v^\infty z^\infty + u^\infty) \\ &= a u^n v^n (z^n - z^\infty) + a u^n (v^n - v^\infty) z^\infty + a(u^n - u^\infty) v^\infty z^n + u^n - u^\infty. \end{aligned} \tag{3.39}$$

Next, we need to deduce

$$\begin{aligned}\|F^n - F^\infty\|_{B^{\frac{1}{2}}_{2,1}} &\leq (\|u^n\|_{B^{\frac{1}{2}}_{2,1}}\|v^n\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n\|_{B^{\frac{1}{2}}_{2,1}}\|z^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^\infty\|_{B^{\frac{1}{2}}_{2,1}}\|z^n\|_{B^{\frac{1}{2}}_{2,1}} + 1) \\ &\quad \times (\|z^n - z^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}}) \\ &\leq (1 + \|u^n\|^2_{B^{\frac{1}{2}}_{2,1}} + \|v^n\|^2_{B^{\frac{1}{2}}_{2,1}} + \|u^\infty\|^2_{B^{\frac{3}{2}}_{2,1}} + \|v^\infty\|^2_{B^{\frac{1}{2}}_{2,1}} + \|u^n\|^2_{B^{\frac{3}{2}}_{2,1}}) \\ &\quad \times (\|z^n - z^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}}) \\ &\leq C(\|u^n_x - u^\infty_x\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}}).\end{aligned} \tag{3.40}$$

Likewise, we have $\|f^n - f^\infty\|_{B^{\frac{1}{2}}_{2,1}} \leq C(\|v^n_x - v^\infty_x\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}})$. Therefore, we deploy

$$\begin{aligned}\|\theta^n\|_{B^{\frac{1}{2}}_{2,1}} &\leq C(\|u^n_{0x} - u^\infty_{0x}\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|u^n_x - u^\infty_x\|_{B^{\frac{1}{2}}_{2,1}} + \|v^n_x - v^\infty_x\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}} d\tau) \\ &\leq C(\|u^n_{0x} - u^\infty_{0x}\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|\theta^n\|_{B^{\frac{1}{2}}_{2,1}} + \|\rho^n - \rho^\infty\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}} d\tau),\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\|h^n\|_{B^{\frac{1}{2}}_{2,1}} &\leq C(\|v^n_{0x} - v^\infty_{0x}\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|h^n\|_{B^{\frac{1}{2}}_{2,1}} + \|g^n - g^\infty\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad + \|v^n - v^\infty\|_{B^{\frac{1}{2}}_{2,1}} + \|u^n - u^\infty\|_{B^{\frac{1}{2}}_{2,1}} d\tau).\end{aligned} \tag{3.42}$$

Applying the facts of $(u^n, v^n) \to (u^\infty, v^\infty)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, $(u^n_{0x}, v^n_{0x}) \to (u^\infty_{0x}, v^\infty_{0x})$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, and $(\rho^n, g^n) \to (\rho^\infty, g^\infty)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$, we derive that $(\theta^n, h^n) \to 0$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Based on Lemma 2.2, we get $(\theta^\infty, h^\infty) = 0$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. Hence,

$$\begin{aligned}&\|z^n - z^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} + \|w^n - w^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \\ &\quad \leq \|\rho^n - \rho^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} + \|\theta^n - \theta^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \\ &\qquad + \|g^n - g^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} + \|h^n - h^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \\ &\quad \leq \|\rho^n - \rho^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} + \|g^n - g^\infty\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} \\ &\quad + \|\theta^n\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})} + \|h^n\|_{L^\infty([0,T];B^{\frac{1}{2}}_{2,1})}.\end{aligned} \tag{3.43}$$

Therefore, we conclude that $(u^n_x, v^n_x) \to (u^\infty_x, v^\infty_x)$ in $C([0,T]; B^{\frac{1}{2}}_{2,1})$. In conclusion, the proof of the Theorem 1.1 has been completed. □

4. The Hölder continuity

In this section, we mainly focus on Hölder continuity. First, we need to deduce that the data-to-solution map is Lipschitz continuous in the Besov space $B^{s-1}_{2,r}$.

Proof of Theorem 1.2. Let $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ be the solutions of Eq (1.1) with the initial data $z_1(0) = (u_1(0), v_1(0)) \in B(0,R), z_2(0) = (u_2(0), v_2(0)) \in B(0,R)$, and $\tilde{u} = u_1 - u_2$, $\tilde{v} = v_1 - v_2$. Then $(\tilde{u}, \tilde{v})$ satisfies the equations

$$\begin{cases} \tilde{u}_t = au_1 v_1^{p-1}\partial_x \tilde{u} + (au_1 v_1^{p-1} - au_2 v_2^{p-1})\partial_x u_2 + \partial_x^{-1}\tilde{u}, \\ \tilde{v}_t = bv_1 u_1^{q-1}\partial_x \tilde{v} + (bv_1 u_1^{q-1} - bv_1 v_1^{q-1})\partial_x v_2 + \partial_x^{-1}\tilde{v}. \end{cases} \tag{4.1}$$

Based on Lemma 2.2, the first equation of (4.1) yields

$$\|\tilde{u}\|_{B^{s-1}_{2,r}} \le \|\tilde{u}(0)\|_{B^{s-1}_{2,r}} e^{c\int_0^t I_1(\tau)d\tau} + CR\int_0^t e^{c\int_\tau^t I(\tau')d\tau'}\|\tilde{u}(\tau)\|_{B^{s-1}_{2,r}}d\tau, \tag{4.2}$$

where $I_1(t) = \|\partial_x(au_1 v_1^{p-1})\|_{B^{s-1}_{2,r}}$ and R is a constant which depends only on $\|z_1(0)\|_{B^s_{2,r}}$, $\|z_2(0)\|_{B^s_{2,r}}$, and p. Multiplying the inequality by $e^{-c\int_0^t I_1(\tau)d\tau}$ and differentiating yields the differential inequality

$$\frac{d}{dt}(\|\tilde{u}\|_{B^{s-1}_{2,r}} e^{-c\int_0^t I_1(\tau)}d\tau) \le CRe^{-c\int_0^t I(t)dt}\|\tilde{u}(t)\|_{B^{s-1}_{2,r}}. \tag{4.3}$$

Next, we apply Gronwall's lemma, and we get

$$\|\tilde{u}\|_{B^{s-1}_{2,r}} \le \|\tilde{u}(0)\|_{B^{s-1}_{2,r}} e^{Rt+c\int_0^t I_1(\tau)d\tau} \le \|\tilde{u}(0)\|_{B^{s-1}_{2,r}} e^{R't}. \tag{4.4}$$

Similarly, utilizing the above steps in the second equation yields the estimate:

$$\|\tilde{v}\|_{B^{s-1}_{2,r}} \le \|\tilde{v}(0)\|_{B^{s-1}_{2,r}} e^{Rt+c\int_0^t I_1(\tau)d\tau} \le \|\tilde{v}(0)\|_{B^{s-1}_{2,r}} e^{R''t}, \tag{4.5}$$

where R', R'' depend on $p, q, \|u_0\|_{B^s_{2,r}}, \|v_0\|_{B^s_{2,r}}$, and the lifespan of the solution.

Consider two solutions $u, v \in C([0,T]; B^s_{2,r})$ with initial data $u_0, v_0 \in B^s_{2,r}$ such that $\|u_0\|_{B^s_{2,r}}, \|v_0\|_{B^s_{2,r}} \le R$. We study the difference $u - v$ in the $B^\sigma_{2,r}$ norm. Using the interpolation lemma, we derive

$$\|z_1 - z_2\|_{B^\mu_{2,r}} \le \|z_1 - z_2\|^{s-\mu}_{B^{s-1}_{2,r}} \|z_1 - z_2\|^{\mu+1-s}_{B^s_{2,r}}, \tag{4.6}$$

where we get $\|z_1 - z_2\|^{\mu+1-s}_{B^s_{2,r}} \le (2R)^{\mu+1-s}$. Then we obtain

$$\|z_1 - z_2\|_{B^\mu_{2,r}} \le (2R)^{\mu+1-s}\|z_1 - z_2\|^{s-\mu}_{B^{s-1}_{2,r}}, \tag{4.7}$$

and

$$\|z_1 - z_2\|_{B^\mu_{2,r}} \le (2R)^{\mu+1-s}\|z_1(0) - z_2(0)\|^{s-\mu}_{B^{s-1}_{2,r}}. \tag{4.8}$$

By exploiting the continuous embedding properties of Besov spaces, we obtain

$$\|z_1 - z_2\|_{B^\mu_{2,r}} \le (2R)^{\mu+1-s}\|z_1(0) - z_2(0)\|^{s-\mu}_{B^\mu_{2,r}}. \tag{4.9}$$

Thus, the proof of the theorem is finished. □

5. Local Gevrey regularity and analyticity

5.1. Analytic solutions in $G^{\delta}_{\sigma,s}$

In this section, we investigate the local Gevrey regularity and analyticity on the circle and the whole space. Below we mainly provide the proof in the whole space, but a similar argument also yields the periodic case. To this end, we first introduce a key lemma as follows:

Lemma 5.1. *If* $0 < \delta \leq 1$, $s \geq 0$, *and* $u \in G^{\delta}_{\sigma,s}$, *there holds the following estimate:*

$$\|\partial_x^{-1} u\|^2_{G^{\delta}_{\sigma,s}} \leq 2\|u\|^2_{G^{\delta}_{\sigma,s}}.$$

Besides, it is also true for the periodic case.

Proof. According to the definition of Sobolev-Gevrey spaces, we deduce

$$\begin{aligned}
\|\partial_x^{-1} u\|^2_{G^{\delta}_{\sigma,s}} &= \int_{\mathbb{R}} (1+|\xi|^2)^s e^{\delta|\xi|^{\frac{1}{\sigma}}} |\widehat{\partial_x^{-1} u}|^2 d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)^s \frac{e^{\delta|\xi|^{\frac{1}{\sigma}}}\hat{u}}{i\xi} d\xi \\
&= \int_{\mathbb{R}} (1+|\xi|^2)^s |\partial_x^{-1} e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u|^2 d\xi \\
&= \|\partial_x^{-1} e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\|^2_{H^s}.
\end{aligned} \tag{5.1}$$

We let $f = e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u$, due to $H^s = L^2 \cap \dot{H}^s$ with $s \geq 0$, and we find

$$\begin{aligned}
\|\partial_x^{-1} u\|^2_{G^{\delta}_{\sigma,s}} &= \|\partial_x^{-1} f\|^2_{L^2} + \|\partial_x^{-1} f\|^2_{\dot{H}^s} \\
&= \|f\|^2_{\dot{H}^{-1}} + \|f\|^2_{\dot{H}^{s-1}} \\
&\leq 2\|f\|^2_{H^s} = 2\|e^{\delta(-\Delta)^{\frac{1}{2\sigma}}} u\|^2_{H^s} \\
&= 2\int_{\mathbb{R}} (1+|\xi|^2)^s e^{2\delta|\xi|} |\hat{u}|^2 d\xi = 2\|u\|^2_{G^{\delta}_{\sigma,s}}.
\end{aligned} \tag{5.2}$$

□

Next, the main proof of Theorem 1.3 is as follows:

Proof of Theorem 1.3. System (1.1) takes the following form:

$$\begin{cases}
\frac{d}{dt} u = E(t, u(t)), \\
\frac{d}{dt} v = F(t, v(t)), \\
u(0,x) = u_0, v(0,x) = v_0,
\end{cases} \tag{5.3}$$

where $E(t, u(t)) = auv^{p-1}u_x + \partial_x^{-1} u$ and $F(t, u(t)) = bvu^{q-1}v_x + \partial_x^{-1} v$.

Fix $\sigma \geq 1$ and $s > \frac{3}{2}$, and $\{G^{\delta}_{\sigma,s}\}_{0<\delta<1}$ is the decreasing Banach space. For any $0 < \delta' < \delta$, we arrive at

$$\begin{aligned}
\|E(t,u(t))\|_{G^{\delta'}_{\sigma,s}} &\leq \|auv^{p-1}u_x\|_{G^{\delta'}_{\sigma,s}} + \|\partial_x^{-1}u\|_{G^{\delta'}_{\sigma,s}} \\
&\leq a\|uv^{p-1}u_x\|_{G^{\delta'}_{\sigma,s}} + \|\partial_x^{-1}u\|_{G^{\delta'}_{\sigma,s}} \\
&\leq a\|u\|_{G^{\delta'}_{\sigma,s}}\|v\|^{p-1}_{G^{\delta'}_{\sigma,s}}\|u_x\|_{G^{\delta'}_{\sigma,s}} + \sqrt{2}\|u\|_{G^{\delta'}_{\sigma,s}} \\
&\leq a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u\|^2_{G^{\delta}_{\sigma,s}}\|v\|^{p-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u\|_{G^{\delta}_{\sigma,s}}.
\end{aligned} \tag{5.4}$$

Similarly, we also get $\|F(t,v(t))\|_{G^{\delta'}_{\sigma,s}} \leq b\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|v\|^2_{G^{\delta}_{\sigma,s}}\|u\|^{p-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|v\|_{G^{\delta}_{\sigma,s}}$. We use similar steps to deduce

$$\|E(u_0)\|_{G^{\delta}_{\sigma,s}} \leq a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u_0\|^2_{G^{\delta}_{\sigma,s}}\|v_0\|^{p-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_0\|_{G^{\delta}_{\sigma,s}},$$

$$\|F(u_0)\|_{G^{\delta}_{\sigma,s}} \leq b\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|v_0\|^2_{G^{\delta}_{\sigma,s}}\|u_0\|^{q-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|v_0\|_{G^{\delta}_{\sigma,s}}.$$

Then we satisfy Lemma 2.8 (iii), where

$$M_1 = a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u_0\|^2_{G^{\delta}_{\sigma,s}}\|v_0\|^{p-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_0\|_{G^{\delta}_{\sigma,s}},$$

and

$$M_2 = b\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|v_0\|^2_{G^{\delta}_{\sigma,s}}\|u_0\|^{q-1}_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|v_0\|_{G^{\delta}_{\sigma,s}}.$$

Below we want to prove that condition (ii) of Lemma 2.8 is satisfied. Let $z_0 = (u_0, v_0)$, $z_1, z_2 \in G^{\delta}_{\sigma,s}$, $\|z - z_0\|_{G^{\delta}_{\sigma,s}} < R$, and $\|z_2 - z_0\|_{G^{\delta}_{\sigma,s}} < R$, and we arrive at

$$\begin{aligned}
&\|E(z_1) - E(z_2)\|_{G^{\delta'}_{\sigma,s}} \leq \|au_1v_1^{p-1}u_x - au_2v_2^{p-1}u_2\|_{G^{\delta'}_{\sigma,s}} + \|\partial_x^{-1}u_1 - \partial_x^{-1}u_2\|_{G^{\delta'}_{\sigma,s}} \\
&\leq \|au_1v_1^{p-1}(\partial_x u_1 - \partial_x u_2)\|_{G^{\delta'}_{\sigma,s}} + \|(au_1v_1^{p-1} - au_2v_2^{p-2})\partial_x u_2\|_{G^{\delta'}_{\sigma,s}} \\
&\leq a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u_1\|_{G^{\delta}_{\sigma,s}}\|v_1\|^{p-1}_{G^{\delta}_{\sigma,s}}\|u_1 - u_2\|_{G^{\delta}_{\sigma,s}} + a\|u_1v_1^{p-1} - u_2v_2^{p-1}\|_{G^{\delta}_{\sigma,s}}\|u_2\|_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_1 - u_1\|_{G^{\delta}_{\sigma,s}} \\
&\leq a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}(\|z_0\|_{G^{\delta}_{\sigma,s}} + R)^p\|u_1 - u_2\|_{G^{\delta}_{\sigma,s}} + a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u_1v_1^{p-1} - u_2v_2^{p-1}\|_{G^{\delta}_{\sigma,s}}\|u_2\|_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_1 - u_2\|_{G^{\delta}_{\sigma,s}},
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
\|u_1v_1^{p-1} - u_2v_2^{p-1}\|_{G^{\delta}_{\sigma,s}} &= \|u_1(v_1^{p-1} - v_2^{p-1} + (u_1 - u_2)v_2^{p-1})\|_{G^{\delta}_{\sigma,s}} \\
&= \|u_1(v_1 - v_2)\sum_{k=0}^{p-2} v_1^{p-2-k}v_2^k + (u_1 - u_2)v_2^{p-1})\|_{G^{\delta}_{\sigma,s}} \\
&\leq \|u_1\|_{G^{\delta}_{\sigma,s}}\|v_1 - v_2\|_{G^{\delta}_{\sigma,s}}\sum_{k=0}^{p-2}\|v_1\|^{p-2-k}_{G^{\delta}_{\sigma,s}}\|v_2\|^k_{G^{\delta}_{\sigma,s}} + \|u_1 - u_2\|_{G^{\delta}_{\sigma,s}}\|v_2\|_{G^{\delta}_{\sigma,s}} \\
&\leq \|z_1 - z_2\|_{G^{\delta}_{\sigma,s}}(R + \|z_0\|_{G^{\delta}_{\sigma,s}})^p.
\end{aligned} \tag{5.6}$$

Hence, we conclude that

$$\|E(z_1)-E(z_2)\|_{G^{\delta'}_{\sigma,s}} \le a\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}(R+\|z_0\|_{G^{\delta}_{\sigma,s}})^p\|z_1-z_2\|_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_1-u_2\|_{G^{\delta}_{\sigma,s}}, \tag{5.7}$$

and

$$\|F(z_1)-F(z_2)\|_{G^{\delta'}_{\sigma,s}} \le b\frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}(R+\|z_0\|_{G^{\delta}_{\sigma,s}})^q\|z_1-z_2\|_{G^{\delta}_{\sigma,s}} + \sqrt{2}\|u_1-u_2\|_{G^{\delta}_{\sigma,s}}. \tag{5.8}$$

So we combine (5.7) and (5.8) to obtain

$$\begin{aligned}&\|E(z_1)-E(z_2)\|_{G^{\delta'}_{\sigma,s}} + \|F(z_1)-F(z_2)\|_{G^{\delta'}_{\sigma,s}}\\ &\qquad \le \left(\frac{(a+b)e^{-\sigma}\sigma^{\sigma}+\sqrt{2}}{(\delta'-\delta)^{\sigma}}(R+\|z_0\|_{G^{\delta}_{\sigma,s}})^m\right)\|z_1-z_2\|_{G^{\delta}_{\sigma,s}},\end{aligned} \tag{5.9}$$

where $m=\max\{p,q\}$. Therefore, we deduce that $L=\left(\frac{(a+b)e^{-\sigma}\sigma^{\sigma}+\sqrt{2}}{(\delta'-\delta)^{\sigma}}(R+\|z_0\|_{G^{\delta}_{\sigma,s}})^m\right)$. Since

$$T_0=\min\{\frac{1}{2^{2\sigma+4}L},\frac{2^{\sigma+1}R}{2^{\sigma+1}2^{2\sigma+3}LR+MD_{\sigma}}\}$$

and $D_{\sigma}=\frac{1}{2^{\sigma}-2+\frac{1}{2^{\sigma+1}}}$, we choose $R=\|z_0\|_{G^{\delta}_{\sigma,s}}$, and then $L=2^m[(a+b)e^{-\sigma}\sigma^{\sigma}+\sqrt{2}]\|z_0\|^m_{G^{\delta}_{\sigma,s}}$. So we have finished Theorem 1.3. □

5.2. Continuity of the data-to-solution map in $G^1_{\sigma,s}$

In this part, from Definition 2.3, we study the continuity of the data-to-solution map from $G^1_{\sigma,s}\times G^1_{\sigma,s}$ into the solution space.

Proof of Theorem 1.4. Define that

$$\begin{aligned}T^{\infty}&=\frac{1}{2^{2\sigma+4+m}\left((a+b)e^{-\sigma}\sigma^{\sigma}+\sqrt{2}\right)\|z_0^{\infty}\|^m_{G^1_{\sigma,s}}},\\ T^{n}&=\frac{1}{2^{2\sigma+4+m}\left((a+b)e^{-\sigma}\sigma^{\sigma}+\sqrt{2}\right)\|z_0^{n}\|^m_{G^1_{\sigma,s}}}.\end{aligned} \tag{5.10}$$

Due to $\|z_0^n-z_0^{\infty}\|_{G^1_{\sigma,s}}\to 0$ as $n\to\infty$, this implies that there exists N if $n\ge N$, and we have $\|z_0^n\|_{G^1_{\sigma,s}}\le \|z_0^{\infty}\|_{G^1_{\sigma,s}}+1$. Let $T=\frac{1}{2^{2\sigma+4+m}(a+b)e^{-\sigma}\sigma^{\sigma}(\|z_0^{\infty}\|_{G^1_{\sigma,s}}+1)^m}$ such that $T\le\min\{T^n,T^{\infty}\}$ with $n\ge N$. Based on Theorem 1.3, we claim T^n,T^{∞} are the existence times of the solutions z^n,z^{∞} corresponding to the initial data z_0^n,z_0^{∞}. Then we see that for $n\ge N$,

$$\begin{aligned}z^{\infty}(t,x)&=z_0^{\infty}+\int_0^t H(\tau,z^{\infty}(t,\tau))d\tau, 0\le t<\frac{T(1-\delta)^{\sigma}}{2^{\sigma}-1},\\ z^{n}(t,x)&=z_0^{n}+\int_0^t H(\tau,z^{n}(t,\tau))d\tau, 0\le t<\frac{T(1-\delta)^{\sigma}}{2^{\sigma}-1},\end{aligned} \tag{5.11}$$

where $H = E + F$ in Theorem 1.3. Then if $0 \le t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1}$ and $0 \le \delta \le 1$, we get

$$\|z^n(t) - z^\infty(t)\|_\delta \le \|z_0^n - z_0^\infty\|_\delta + \int_0^t \|H(z^n(\tau)) - H(z^\infty)(\tau)\|_\delta d\tau. \tag{5.12}$$

Define $\delta(t) = \frac{1}{2}(1+\delta) + (\frac{1}{2})^{2+\frac{1}{\sigma}}((1-\delta)^\sigma - \frac{t}{T}^{\frac{1}{\sigma}} - [(1-\delta)^\sigma + (2^{\sigma+1}-1)\frac{t}{T}]^{\frac{1}{\sigma}})$, and setting $a = T$ in Lemma 2.12, we have $\delta < \delta(\tau) < 1$. Together with (5.12), this implies

$$\|H(z^n)(t) - H(z^\infty)(\tau)\|_\delta \le \frac{L\|z^n(\tau) - z^\infty(\tau)\|_\delta}{(\delta(\tau) - \delta)^\sigma}, \tag{5.13}$$

where $L = 2^m[(a+b)e^{-\sigma}\sigma^\sigma + \sqrt{2}]\|z_0\|^m_{G^\delta_{\sigma,s}}$, and we deduce

$$\|z^n(t) - z^\infty(t)\|_\delta \le \|z_0^n - z_0^\infty\|_\delta + LT2^{2\delta+3}\frac{\|z^n(\tau) - z^\infty(t)\|_{E_T}}{(1-\delta)^\sigma}\sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}}.$$

Since $LT2^{2\sigma+3} < \frac{1}{2}$, we get

$$\|z^n(t) - z^\infty(t)\|_\delta \le \|z_0^n - z_0^\infty\|_\delta + \frac{\|z^n(\tau) - z^\infty(t)\|_{E_T}}{2(1-\delta)^\sigma}\sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}},$$

which implies

$$\|z^n(t) - z^\infty(t)\|_{E_T}(1-\delta)^\sigma\sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}} \le \|z_0^n - z_0^\infty\|_\delta(1-\delta)^\sigma\sqrt{\frac{T(1-\delta)^\sigma}{T(1-\delta)^\sigma - t}} + \frac{1}{2}\|z^n(\tau) - z^\infty(t)\|_{E_T}.$$

Through taking the supremum over $0 < \delta < 1$, $0 < t < \frac{T(1-\delta)^\sigma}{2^\sigma - 1}$, we obtain

$$\|z^n - z^\infty\|_{E_T} \le \|z_0^n - z_0^\infty\|_1 + \frac{1}{2}\|z^n(t) - z^\infty(t)\|_{E_T},$$

which is

$$\|z^n - z^\infty\|_{E_T} \le 2\|z^n(t) - z^\infty(t)\|_{E_T}.$$

Therefore, we have completed the proof of Theorem 1.4. □

6. Conclusions

Our work primarily investigates the Cauchy problem for the generalized short pulse system on the periodic domain. Utilizing transport theory, we establish the existence and uniqueness of solutions and prove the continuity of the solution map. Furthermore, we demonstrate that this solution map is Hölder continuous. Finally, we also establish the Gevrey regularity and analyticity of the solutions.

Author contributions

Shanshan Zheng: Methodology, writing–original draft; Li Yang: review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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