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Research article

On the study of the hyperbolic $p(.)$-biharmonic equation with no flux boundary condition

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Abstract: In the present paper, we studied a hyperbolic $p(x)$-biharmonic problem with known no-flux values on the smooth part of the boundary in variable exponent Sobolev spaces. The global existence of a weak solution, using the Galerkin method, was proved. The blow-up of the solution with negative initial functional energy in finite time was discussed using the concavity method. The interaction between the variable exponent in the main equation and the boundary condition is crucial in determining the presence/absence of solutions.

Keywords: weak solution; p(x)-biharmonic equation; variable exponent; nonlinear equations; negative initial energy; energy and industry; global existence; blow-up of the solution
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1. Introduction

In 1943, A. N. Tikhonov [1] pointed out the practical importance of ill-posed problems and the possibility of their stable solutions. The concept of a regularizing algorithm and the related concept of a regularized family of approximate solutions, along with the introduction of a positive parameter a depending on the error of the initial data, were first noted by M. M. Lavrentiev [2, 3]. Biharmonic equations, which are fourth-order partial differential equations, play an important role in continuum mechanics when modeling the behavior of elastic plates under load, as well as in low-Reynolds-number hydrodynamics. Despite the linear nature of the biharmonic equation, the integration of some particular formulations of boundary value problems, for example, a plate clamped at the boundary with a uniform load or conductive-laminar free convection in a cavity, whose solution's existence and uniqueness were proven in [4], causes a number of difficulties. We consider a bounded open domain Ω of $\mathbb{R}^n$, with a regular boundary Γ and $I = [0, T]$, and T is a real number. The main objective is to show the global existence and finite time blow-up of the solution to the following hyperbolic $p(x)$-biharmonic problem subject to homogeneous Neumann boundary conditions.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \Delta(|\Delta u|^{p(x)-2}\Delta u) - \sigma\Delta u_t = \phi(x,u) + f(x,t), & \text{in } I \times \Omega, \\ u = c,\ \Delta u = 0, & \text{on } \Gamma, \\ \int_\Gamma \dfrac{\partial}{\partial \eta}(|\Delta u|^{p(x)-2}\Delta u) d\theta = 0, & \text{on } \Gamma, \\ u(0,x) = u_0, \dfrac{\partial u}{\partial t}(0,x) = u_1, & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $\sigma > 0$ and ϕ is a continuous function satisfying

$$|\phi(.,u)| \leq \beta |u|^{m(\cdot)-1} + \gamma, \tag{1.2}$$

for some $\gamma, \beta > 0$, $1 < m^- \leq m(x) \leq m^+ < \infty$ if $n \leq p(x)$ and $1 < m^- \leq m(x) \leq m^+ < \frac{np(x)}{n-p(x)}$ if $n > p(x)$, and

$$\Phi(x,u) = \int_0^u \phi(x,s) ds, \tag{1.3}$$

here, $p \in C(\overline{\Omega}, (1,\infty))$ such that

$$1 < p^- \leq p(x) \leq p^+ < \infty \tag{1.4}$$

with

$$\begin{cases} p^- = \inf\ p(x), x \in \overline{\Omega} \text{ a.e}, \\ p^+ = \sup\ p(x), x \in \overline{\Omega} \text{ a.e}, \end{cases}$$

and $m(x)$ satisfies the log-Hölder continuity condition

$$||m(u) - m(v)| \leq -\frac{B}{\log|a-b|},\ a, b \in \Omega \tag{1.5}$$

with

$$|a-b| < \delta,\ B > 0,\ 0 < \delta < 1.$$

The following problem was considered by M.-M. Boureanu et al. in [5]:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u), & \text{for } x \in \Omega, \\ u = c,\ \Delta u = 0 & \text{for } x \in \partial\Omega, \\ \int_{\partial\Omega} \dfrac{\partial}{\partial \nu}(|\Delta u|^{p(x)-2}\Delta u) dS = 0. \end{cases}$$

The authors studied a fourth-order variable-exponent $p(.)-$biharmonic problem subject to a no-flux boundary condition. They proved the existence and multiplicity of weak solutions by employing variational techniques, including the mountain-pass theorem. Authors in [6] proposed a nonlinear dissipative wave $p(.)-$bi-Laplace equation in variable-exponent Sobolev spaces,

$$u_{tt} + \Delta(\text{div}(|\Delta u|^{p(x)-2}\nabla u)) + \mu|u_t|^{m-2}u_t = b|u|^{r-2}u.$$

They derived sufficient conditions for the finite-time blow-up of solutions with negative initial energy using energy methods. N. T. Brahim et al. [7] considered the following high-order degenerate parabolic p-biharmonic equation with a memory term:

$$\frac{\partial\beta(u)(t,x)}{\partial t} + \Delta_p^2 u(t,x) = f(t,x) - K(t)(u)$$

with

$$(K(t)(u), v) = \int_\Omega \int_0^t a(t-s)k(s,x,\Delta u(s,x))ds\Delta v(t,x)dx.$$

The authors proved the existence, uniqueness, and qualitative properties of weak solutions using Rothe's method combined with a mixed finite element method. Through this framework, they established a priori estimates and verified the convergence of the numerical scheme. In [8], the authors addressed an evolution equation involving the nonlinear p-biharmonic operator

$$u_t + \Delta_p^2 u - \alpha\Delta u = f(x,t).$$

They focused on establishing the existence, uniqueness, and regularity of weak solutions by applying classical functional analysis. Then, the authors studied its fully discrete numerical formulation, proving the existence, uniqueness, stability, and convergence order of discrete solutions using Brouwer's fixed-point theorem and mixed finite element methods, and validated the theoretical findings with numerical experiments; see [9, 10].

Recent years have witnessed increasing interest in the study of high-order partial differential equations due to their many applications in various fields, such as elastic beam deformations [11], thin film theory [12, 13], mathematical modeling of non-Newtonian fluid motions, and image inpainting and restoration problems [14]. In this context, evolution p-biharmonic equations have drawn the focus of many mathematicians; for example, in [15], the authors proved the well-posedness of the solution and provided some numerical results for parabolic and degenerate parabolic p-biharmonic equations, respectively, with a constant exponent, using the M.F.E method combined with the backward-Euler method. Cömert et al. [16] has shown the global existence and exponential decay of solutions for higher-order parabolic equations with logarithmic nonlinearity. Recently, the authors in [17] investigated the blow-up of the solution for some parabolic $p(.)$-biharmonic equation with a variable exponent. Other interesting results can be found in [18–20].

In the present work, the critical criterion to identify finite time blow-up or global existence of a hyperbolic p-biharmonic equation with no-flux boundary conditions and a variable exponent is studied.

Note that the no-flux problems are related to impermeable surfaces for certain contaminants.

Our paper is organized as follows: In Section 2, we introduce some fundamental notations and materials required for our work. Then, we establish in Section 3 some necessary a priori estimates and discuss the existence of global weak solutions to problem (1.1) using the Galerkin method. Finally, we address the blow-up of the solution subject to negative initial functional energy in Section 4.

2. Preliminaries

Let $p \in C(\overline{\Omega}, (1,\infty))$ satisfy the growth condition

$$|p(\eta) - p(\xi)| \leq -\frac{c}{\ln|\eta-\xi|} \quad \forall |\eta-\xi| < \frac{1}{2}, \tag{2.1}$$

for some $c > 0$.

Definition 2.1. *We introduce the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ as follows*

$$L^{p(.)}(\Omega) = \Big\{\Phi \,:\, \Omega \to \mathbb{R}, \Phi \textit{ is measurable } : \int_\Omega |\Phi(x)|^{p(x)} dx < \infty\Big\}, \tag{2.2}$$

with the Luxembourg norm

$$\|\Phi\|_{L^{p(.)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{\Phi(x)}{\lambda}\Big|^{p(x)} dx \leq 1\Big\}. \tag{2.3}$$

Note that $(L^{p(x)}(\Omega), \|.\|_{L^{p(x)}(\Omega)})$ is a separable Banach space, see [21].

Definition 2.2. *Let $W_0^{2,p(.)}(\Omega)$ be the space defined by*

$$W_0^{2,p(.)}(\Omega) = \Big\{\Phi \in W^{2,p(.)}(\Omega)\,;\,;\, D^\alpha \Phi|_\Gamma = 0,\ \forall \alpha \in \mathbb{N}^n \textit{ such that } |\alpha| \leq 1\Big\}, \tag{2.4}$$

over the space $W_0^{2,p}(\Omega)$, $\|\Delta(\cdot)\|_{L^{p(.)}(\Omega)}$ and $\|\cdot\|_{W^{2,p(.)}(\Omega)}$ are equivalent.

Remark 2.1. *Let p and $q \in C(\overline{\Omega}, (1, \infty))$ such that*

$$\frac{1}{p(.)} + \frac{1}{q(.)} = 1.$$

1) *If $p(x) \in [q(x), \infty)$, then*
$$L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega). \tag{2.5}$$

2) *For $(u, v) \in L^{p(.)}(\Omega) \times L^{q(.)}(\Omega)$, then*
$$(u, v)_{L^2(\Omega)} \leq (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}. \tag{2.6}$$

3) *For all $u \in L^{p(x)}(\Omega)$, then*
$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\} \leq \int_\Omega |u|^{p(x)} dx \leq \max\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\}. \tag{2.7}$$

4) *Let $m(x) : \overline{\Omega} \to \mathbb{R}$ be a measurable function. Then, the embedding $W^{k,p(x)}(\Omega) \subset L^{m(x)}(\Omega)$ is continuous if*
$$p(x) \leq m(x) \leq \frac{np(x)}{n - kp(x)} a.e, \qquad \forall x \in \overline{\Omega}. \tag{2.8}$$

3. Existence results

Theorem 3.1. *Let $u_0 \in W^{2,p(.)}(\Omega) \cap W_0^{1,2}(\Omega), u_1 \in L^2(\Omega)$ be given functions and assume that*

$$\textit{either } m^+ \leq 2 \textit{ or } 2 < m^+ \leq p^-. \tag{3.1}$$

Then the problem (1.1) admits a weak solution $u : I \times \Omega \longrightarrow \mathbb{R}$, $T < \infty$, such that

1) $u \in L^{\infty}((0,T),V)$*,* $V = \{u \in W^{2,p(.)}(\Omega) \cap W_0^{1,p(.)}(\Omega), |u|^{\frac{m}{2}} \in L^2(\Omega)\}$*;*
2) $u' \in L^{\infty}((0,T),L^2(\Omega)) \cap L^{\infty}((0,T),V)$*;*
3) $u_{tt} + \Delta(|\Delta u|^{p(x)-2}\Delta u) - \sigma\Delta u_t = \phi(x,u) + f(x,t)$ *a.e. in* $L^2((0,T),V')$*.*

Proof. Let us demonstrate that the solution u exists in V. The weak solution of our problem will be $\tilde{u} = u + c$.

Let $\{e_i\}_{i=1}^{\infty}$ be the orthonormal basis of $W_0^{S,2}(\Omega)$ that satisfies

$$\begin{cases} -\Delta e_i = \lambda_i e_i, & \text{if } x \in \Omega, \\ -\Delta^2 e_i = -\lambda_i^2 e_i, & \text{if } x \in \Omega, \\ e_i = 0, & \text{if } x \in \Gamma. \end{cases} \tag{3.2}$$

Let $E_l = \langle e_1, ..., e_l \rangle$ and

$$\int_\Omega Muv dx = \int_\Omega u_{tt} v dx + \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + \sigma \int_\Omega \nabla u_t \nabla v dx - \int_\Omega \phi v dx - \int_\Omega f v dx, \ \forall v \in E_l. \tag{3.3}$$

For $l \in \mathbb{N}^*$, we consider the Galerkin approximate solution

$$u_l = \sum_{i=1}^{l} C_i(t) e_i(x), \tag{3.4}$$

such that

$$\begin{cases} \langle M u_l, e_i \rangle = 0, \ \ i = 1, ..., l, \\ u_l(0) = u_{0l}, \ u_{lt}(0) = u_{1l}, \end{cases} \tag{3.5}$$

where

$$u_{0l} = \sum_{i=1}^{l} (u_0, e_i) e_i, \quad u_{1l} = \sum_{i=1}^{l} (U_1, e_i) e_i, \quad C_i = (u(x,t), e_i(x)), \tag{3.6}$$

and

$$u_{0l} \longrightarrow u_0 \text{ in } W^{2,p(.)}(\Omega) \cap W^{1,2}(\Omega), \quad u_{1l} \longrightarrow u_1 \text{ in } L^2(\Omega). \tag{3.7}$$

From problem (1.1), we derive the system below consisting of l differential equations

$$\begin{cases} u_i''(t) = -(\Delta |u_l|^{p(x)-2}\Delta u_l, \Delta e_i) - \sigma(\nabla \partial_t u_l, \nabla e_i) + (\phi(u_l), e_i) + (f, e_i), \\ u_i(0) = (u_0, e_i), \ u_i'(0) = (U_1, e_i), \ i = 1...l, \end{cases} \tag{3.8}$$

where $(.,.)$ is the $\mathbb{L}^2$ inner product. Note that the solution u_l of (3.8) is ensured by the standard theory of ODE in $[0, t_l]$. A priori estimates that will be proved below demonstrate that the solution $u_l(t)$ can be extended to [0,T], $T > 0$. Multiplying each equation in the system (3.8) by $C_i'(t)$ and summing with respect to $i = 1, ..., l$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\int_\Omega \frac{1}{p(x)} |\Delta u_l(t)|^{p(x)} dx + \int_\Omega \frac{|\partial_t u_l(t)|^2}{2} dx - \int_\Omega \Phi(x, u_l(t)) dx \Big] \\ &+ \sigma \|\nabla \partial_t u_l(t)\|^2_{L^2(\Omega)} = \int_\Omega f \partial_t u_l(t) dx, \end{aligned} \tag{3.9}$$

which implies

$$J'(t) + \sigma\|\nabla\partial_t u_l\|^2_{L^2(\Omega)} = \int_\Omega f\partial_t u_l dx, \tag{3.10}$$

where

$$J(t) = \int_\Omega \frac{1}{p(x)}|\Delta u_l(t)|^{p(x)}dx + \int_\Omega \frac{|\partial_t u_l(t)|^2}{2}dx - \int_\Omega \Phi(x, u_l(t))dx, \tag{3.11}$$

is the energy functional.

To prove this theorem, we need some a priori estimates, which are given in this lemma.

Lemma 3.1. *There exists $C > 0$, such that*

$$\sup_{t\in[0,T]}\Big[\|u_l'\|^2_{L^2(\Omega)} + \int_\Omega |\Delta u_l(t)|^{p(x)}dx\Big] + \sigma\int_0^T \|\nabla u_l'\|^2_{L^2(\Omega)}ds \le C. \tag{3.12}$$

Proof. Substituting e_i in (3.5) by $u_l'(t)$, we get

$$(u_l'', u_l') + (|\Delta u_l|^{p(x)-2}\Delta u_l, \Delta u_l') + \sigma\|\nabla u_l'\|^2_{L^2(\Omega)} - (\phi, u_l') \le \|f\|_{L^2(\Omega)}\|u_l'\|_{L^2(\Omega)}. \tag{3.13}$$

Taking the integral on $[0, t]$, we get

$$\begin{aligned}&\frac{\|u_l'(s)\|^2_{L^2(\Omega)}}{2}\Big|_0^t + \int_\Omega \frac{1}{p(x)}|\Delta u_l(s)|^{p(x)}dx\Big|_0^t + \int_0^t \sigma\|\nabla u_l'(s)\|^2_{L^2(\Omega)}ds - \int_\Omega \Phi(x, u_l(s))dx\Big|_0^t\\ &\le \int_0^t \|f\|_{L^2(\Omega)}\|u_l'(s)\|_{L^2(\Omega)}ds.\end{aligned} \tag{3.14}$$

This implies

$$\begin{aligned}&\frac{1}{2}\|u_l'(t)\|^2_{L^2(\Omega)} + \int_\Omega \frac{|\Delta u_l(t)|^{p(x)}}{p(x)}dx + \int_0^t \sigma\|\nabla u_l'(s)\|^2_{L^2(\Omega)}ds + \int_\Omega \Phi(x, u_{0l})dx\\ &\le \int_0^t \|f\|_{L^2(\Omega)}\|u_l'(s)\|_{L^2(\Omega)}ds + \frac{1}{2}\|u_{1l}\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_{0l}|^{p(x)}dx + \int_\Omega \Phi(x, u_l(t))dx.\end{aligned} \tag{3.15}$$

Taking into account (1.2) and for $m^+ \le 2$, then we obtain

$$\begin{aligned}\int_\Omega |\Phi(x, u_l(t))|dx &= \int_\Omega \Big|\int_0^{u_l} \phi(x, s)ds\Big|dx\\ &\le \int_\Omega\int_0^{u_l} \Big(\beta|s|^{m(x)-1} + \gamma\Big)dsdx\\ &\le \beta\int_\Omega \frac{|u_l|^{m(x)}}{m(x)}dx + \gamma\|u_l\|_{L^1(\Omega)}\\ &\le C(\|u_l\|^2_{L^2(\Omega)} + 1)\\ &\le 2C\Big(t\int_0^t \|u_l'\|^2_{L^2(\Omega)}ds + \|u_{0l}\|^2_{L^2(\Omega)} + 1\Big)\\ &\le c\Big(t\int_0^t \Big[\frac{1}{2}\|u_l'\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_l(s)|^{p(x)}dx\Big]ds + 1\Big).\end{aligned}$$

Assume that

$$\Im(t) = \frac{1}{2}\|u_l'\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_l(t)|^{p(x)}dx. \tag{3.16}$$

Hence, the inequality (3.15) becomes

$$\begin{aligned} &\Im(t) + \int_0^t \sigma\|\nabla u_l'(s)\|^2_{L^2(\Omega)}ds + c\Big(t\int_0^t \Big[\frac{1}{2}\|u_{1l}\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_{0l}(s)|^{p(x)}dx\Big]ds + 1\Big) \\ \leq\ & \|f\|^2_{L^2(I,L^2(\Omega))}ds + c + (1+ct)\int_0^t \Im(s)ds + \frac{1}{2}\|u_{1l}\|^2_{L^2(\Omega)} + \int_\Omega \frac{|\Delta u_{0l}|^{p(x)}}{p(x)}dx. \end{aligned} \tag{3.17}$$

Now, for $2 < m^+ \leq p^-$, we have

$$\begin{aligned} \int_\Omega |\Phi(x,u_l(t))|dx &\leq \beta\int_\Omega \frac{1}{m(x)}|u_l|^{m(x)}dx + \gamma\|u_l\|_{L^1(\Omega)} \\ &\leq C\int_\Omega \frac{1}{m(x)}|\Delta u_l|^{m(x)}dx + C\|u_l\|_{L^{p(x)}(\Omega)} \\ &\leq C\int_\Omega \frac{|\Delta u_l|^{p^-}}{m^+}dx + C\|\Delta u_l\|_{L^p(\Omega)} \\ &\leq C\int_\Omega |\Delta u_l|^p dx + C\|u_l\|_{W^{2,p(x)}(\Omega)} \\ &\leq C\max\{\|\Delta u_l\|^{p^-}_{L^{p(x)}(\Omega)}, \|\Delta u_l\|^{p^+}_{L^{p(x)}(\Omega)}\} + C\|u_l\|_{W^{2,p(x)}(\Omega)} \\ &\leq C\max\{\|u_l\|^{p^-}_{W^{2,p(x)}(\Omega)}, \|u_l\|^{p^+}_{W^{2,p(x)}(\Omega)}, \|u_l\|_{W^{2,p(x)}(\Omega)}\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{2}\|u_l'(t)\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_l(t)|^{p(x)}dx + \int_0^t \sigma\|\nabla u_l'(s)\|^2_{L^2(\Omega)}ds \\ &+C\max\{\|u_{0l}\|^{p^-}_{W^{2,p(x)}(\Omega)}, \|u_{0l}\|^{p^+}_{W^{2,p(x)}(\Omega)}, \|u_{0l}\|_{W^{2,p(x)}(\Omega)}\} \\ \leq\ & \int_0^t \|f\|_{L^2(\Omega)}\|u_l'(s)\|_{L^2(\Omega)}ds + \frac{1}{2}\|u_{1l}\|^2_{L^2(\Omega)} + \int_\Omega \frac{1}{p(x)}|\Delta u_{0l}|^{p(x)}dx \\ &+C\max\{\|u_l\|^{p^-}_{W^{2,p(x)}(\Omega)}, \|u_l\|^{p^+}_{W^{2,p(x)}(\Omega)}, \|u_l\|_{W^{2,p(x)}(\Omega)}\}dx. \end{aligned} \tag{3.18}$$

Applying Gronwall's lemma in (3.17) and (3.18), we achieve the proof of the lemma. □

Return to the proof of Theorem 3.1. Using the Lions-Aubin lemma, the preceding a priori estimate enables us to deduce the existence of a subsequence of $\{u_l\}$ denoted $\{u_l\}$ and a function u where

$$\begin{aligned} &u_l \to u \text{ in } L^2(I, L^2(\Omega)), \\ &\partial_t u_l \to \partial_t u \text{ in } L^2(I, L^2(\Omega)), \\ &u_l \rightharpoonup u \text{ in } L^\infty(I, W_0^{2,p(x)}(\Omega)), \\ &\partial_t u_l \rightharpoonup \partial_t u \text{ in } L^\infty(I, L^2(\Omega)), \\ &\partial_t u_l \rightharpoonup \partial_t u \text{ in } L^2(I, H_0^1(\Omega)), \end{aligned}$$

$$|\Delta u_l|^{p(.)-2}\Delta u_l \rightharpoonup \zeta \text{ in } L^\infty(I,(W_0^{2,p(x)}(\Omega))'),$$
$$\phi(x,u_l) \rightharpoonup \phi(x,u) \text{ in } L^{m'}(I,L^{m'}(\Omega)) \text{ with } \frac{1}{m}+\frac{1}{m'}=1,$$
$$\partial_{tt}u_l \rightharpoonup u_{tt} \text{ in } L^2(I,(W_0^{2,p(x)}(\Omega))'). \tag{3.19}$$

Let

$$w=\sum_{i=1}^{l} k_i(t)e_i,$$

with $k_i(t)\in C^2[0,T]$. Taking Eq (3.5), multiplying it by $k_i(t)$ and summing over i, we get

$$(\partial_{tt}u_l,w)+(|\Delta u_l|^{p(x)-2}\Delta u_l,\Delta w)+\sigma(\nabla\partial_t u_l,\nabla w)-(\phi(u_l),w)-(f,w)=0. \tag{3.20}$$

Applying integration to (3.20) on $[0,T]$, we get

$$-\int_0^T(\partial_t u_l,\partial_t w)ds+(\partial_t u_l,w)\Big|_0^T+\int_0^T(|\Delta u_l|^{p(x)-2}\Delta u_l,\Delta w)ds$$
$$+\sigma\int_0^T(\nabla\partial_t u_l,\nabla w)ds-\int_0^T(\phi(u_l),w)ds-\int_0^T(f,w)ds=0. \tag{3.21}$$

Taking the limit $(m\to\infty)$ and using (3.19), we arrive at

$$-\int_0^T(\partial_t u,\partial_t w)ds+(\partial_t u,w)\Big|_0^T+\int_0^T(\zeta,\Delta w)ds+\sigma\int_0^T(\nabla\partial_t u,\nabla w)ds$$
$$-\int_0^T(\phi(u),w)ds-\int_0^T(f,w)ds=0 \text{ a.e.} \tag{3.22}$$

Let us prove that

$$\zeta=\frac{|\Delta u|^{p(x)}\Delta u}{|\Delta u|^2}.$$

We choose $w=u_l$ in (3.21) and $w=u$ in (3.22), we get

$$-\int_0^T(\partial_t u_l,\partial_t u_l)ds+(\partial_t u_l,u_l)\Big|_0^T+\int_0^T(|\Delta u_l|^{p(x)-2}\Delta u_l,\Delta u_l)ds$$
$$+\frac{\sigma}{2}(\nabla u_l,\nabla u_l)\Big|_0^T-\int_0^T(\phi(u_l),u_l)ds-\int_0^T(f,u_l)ds=0, \tag{3.23}$$

and

$$-\int_0^T(\partial_t u,\partial_t u)ds+(\partial_t u,u)\Big|_0^T+\int_0^T(\zeta,\Delta u)ds+\frac{\sigma}{2}(\nabla u,\nabla u)\Big|_0^T$$
$$-\int_0^T(\phi(u),u)ds-\int_0^T(f,u)ds=0 \text{ a.e.} \tag{3.24}$$

The monotonicity of $\triangle^2_{p(x)}:=\triangle(|\triangle u|^{p(x)-2}\triangle u)$ implies

$$\int_0^T\Big(\frac{|\Delta u_l|^{p(x)}\Delta u_l}{|\Delta u_l|^2}-\frac{|\Delta v|^{p(x)}\Delta v}{|\Delta v|^2},\Delta u_l-\Delta v\Big)ds\geq 0. \tag{3.25}$$

According to (3.19) and (3.23)–(3.25), we obtain

$$\int_0^T (\zeta - \frac{|\Delta v|^{p(x)}\Delta v}{|\Delta v|^2}, \Delta u - \Delta v)ds \geq 0, \qquad \forall v \in C_0^\infty(I \times \Omega). \tag{3.26}$$

Substituting v in (3.26) with $u - \kappa\varphi$ and $u + \kappa\varphi$, respectively, and passing to the limit ($\kappa \to 0$), we obtain

$$\int_0^T (\zeta - \frac{|\Delta v|^{p(x)}\Delta v}{|\Delta v|^2}, \Delta\varphi) \geq 0, \tag{3.27}$$

$$\int_0^T (\zeta - \frac{|\Delta v|^{p(x)}\Delta v}{|\Delta v|^2}, \Delta\varphi) \leq 0. \tag{3.28}$$

For some $\kappa > 0$ and $\varphi \in C_0^\infty(I \times \Omega)$, and by the density of $C_0^\infty(I \times \Omega)$, we conclude the proof. □

4. Blow-up of solutions

Here, we consider that

$$f = 0, \quad J(0) \leq 0, \quad (u_0, U_1) > 0, \quad 2 \leq p^- \leq p^+ < \kappa < m^-. \tag{4.1}$$

Assume that

$$2 \leq p(x) \leq \frac{2n}{n-2}, \; n > 2, \tag{4.2}$$

and

$$\phi(x, u) = \beta|u|^{m(x)-2}u, \; \beta > 0. \tag{4.3}$$

Let us define the energy function

$$J(t) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \frac{1}{2}\int_\Omega |u_t|^2 dx - \beta\int_\Omega \frac{1}{m(x)}|u|^{m(x)}dx. \tag{4.4}$$

It is easy to see that

$$J(t) + \sigma\int_0^T \int_\Omega |\nabla u_t|^2 dxds \leq J(0) \leq 0. \tag{4.5}$$

Theorem 4.1. *The weak solution u of (1.1), which satisfies (4.4) and (4.5), is blowing up on*

$$T^* = \frac{C}{(\rho - 1)(H'(0))^{\rho-1}},$$

where H is defined in (4.6).

Proof. To show the blow-up of the solution u, we need to define the following function

$$H(t) = ||u(t)||^2_{L^2(\Omega)} + \sigma\int_0^T \int_\Omega |\nabla u|^2 dxds, \tag{4.6}$$

thus

$$H'(t) = 2\int_\Omega u_t u dx + \sigma\int_\Omega |\nabla u|^2 dx, \tag{4.7}$$

and

$$
\begin{aligned}
H''(t) &= 2(u_{tt}, u) + 2\sigma(\nabla u_t, \nabla u) + 2\|u_t\|^2_{L^2(\Omega)} \\
&= 2\beta \int_\Omega |u|^{m(x)}dx - 2\int_\Omega |\Delta u|^{p(x)}dx + 2\int_\Omega |u_t|^2 dx.
\end{aligned} \tag{4.8}
$$

Consequently, by virtue of (4.1) and (4.5) and for $\kappa > 2$, $p^+ < \kappa < m^-$, we have

$$
\begin{aligned}
H''(t) &\geq H''(t) + 2\kappa\Big(J(t) + \sigma\int_0^T\int_\Omega |\nabla u_t|^2 dxds\Big) \\
&\geq 2\beta\int_\Omega |u|^{m(x)}dx - 2\int_\Omega |\Delta u|^{p(x)}dx + 2\int_\Omega |u_t|^2dx \\
&\quad +2\kappa\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \frac{1}{2}\int_\Omega |u_t|^2dx - 2\kappa\beta\int_\Omega \frac{|u|^{m(x)}}{m(x)}dx + 2\kappa\sigma\int_0^T\int_\Omega |\nabla u_t|^2dxds \\
&\geq 2\beta\int_\Omega \Big(1 - \frac{\kappa}{m(x)}\Big)|u|^{m(x)}dx + 2\int_\Omega\Big(\frac{\kappa}{p(x)} - 1\Big)|\Delta u|^{p(x)}dx \\
&\quad +(2+\kappa)\int_\Omega |u_t|^2dx + 2\kappa\sigma\int_0^T\int_\Omega |\nabla u_t|^2dxds > 0.
\end{aligned} \tag{4.9}
$$

Therefore, $H(t)$ is positive and strictly increasing for all $t > 0$. This implies that $H(t)$ diverges as $t \to T^*$. Inequality (4.9) leads us to

$$
\int_\Omega |u|^{m(x)}dx \leq CH''(t), \qquad \|u_t\|^2_{L^2(\Omega)} \leq CH''(t), \qquad \int_\Omega |\Delta u|^{p(x)}dx \leq CH''(t). \tag{4.10}
$$

On the other hand, we suppose that $T^* = \infty$. Taking into account (2.5), (2.7), and (4.10), we get

$$
\begin{aligned}
\int_\Omega |\nabla u|^2 dx &\leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \\
&\leq C\max\Big[\Big(\int_\Omega |u|^{p(\cdot)}dx\Big)^{\frac{1}{p^-}}, \Big(\int_\Omega |u|^{p(\cdot)}dx\Big)^{\frac{1}{p^+}}\Big] \\
&\leq C\max\Big[(H''(t))^{\frac{1}{p^-}}, (H''(t))^{\frac{1}{p^+}}\Big].
\end{aligned} \tag{4.11}
$$

Furthermore,

$$
\begin{aligned}
\int_\Omega |u|^2 dx &\leq C\|u\|_{L^{m(\cdot)}(\Omega)} \\
&\leq C\max\Big[\Big(\int_\Omega |u|^{m(\cdot)}dx\Big)^{\frac{1}{m^-}}, \Big(\int_\Omega |u|^{m(\cdot)}dx\Big)^{\frac{1}{m^+}}\Big] \\
&\leq C\max\Big[(H''(t))^{\frac{1}{m^-}}, (H''(t))^{\frac{1}{m^+}}\Big].
\end{aligned} \tag{4.12}
$$

Assume that $H' \geq 1$ and $H'' \geq 1$. Using Cauchy-Schwarz, (4.11), and (4.12), (4.7) becomes

$$
\begin{aligned}
H'(t) &\leq 2\int_\Omega |u_t|^2dx\int_\Omega |u|^2dx + \sigma\int_\Omega |\nabla u|^2dx \\
&\leq C\Big(H''(t)^{\frac{2+m^-}{2m^-}} + H''(t)^{\frac{2}{p^-}}\Big).
\end{aligned} \tag{4.13}
$$

From this, we obtain

$$C(H'(t))^{\theta} \le H''(t), \text{ with } \frac{1}{\theta} = \max\Big(\frac{2+m^-}{2m^-}, \frac{1}{p^-}\Big) \text{ if } p^- > 2, m^- > 2. \tag{4.14}$$

Simple calculations give us

$$H'(t) \ge \frac{H'(0)}{\Big(1 - \frac{t(\theta-1)}{C}(H'(0))^{\theta-1}\Big)^{\frac{1}{\theta-1}}} \to \infty \text{ as } t \to T^*, \tag{4.15}$$

with

$$T^* = \frac{C}{(\theta-1)(H'(0))^{\theta-1}} < \infty, \qquad \rho > 1, \tag{4.16}$$

which leads us to the contradiction.

Using inequality (4.13), and for $p \ge 2$ and $m \ge 2$, we obtain

$$\begin{aligned} H'(t) &\le 2\int_{\Omega} |u_t|^2 dx \int_{\Omega} |u|^2 dx + \sigma \int_{\Omega} |\nabla u|^2 dx \\ &\le 2\|u_t\|_{L^2(\Omega)} \|u\|_{L^{m(\cdot)}(\Omega)} + C\sigma \|\nabla u\|^2_{L^{p(.)}(\Omega)}. \end{aligned} \tag{4.17}$$

According to (4.5), we get

$$\|u_t\|^2_{L^2(\Omega)} dx \le C\rho_{m(x)}(u), \quad \rho_{p(x)}(\nabla u) \le C\rho_{m(x)}(u). \tag{4.18}$$

Then, the inequality (4.17) becomes

$$\begin{aligned} H'(t) &\le C\Big[\Big(\int_{\Omega} |u|^m dx\Big)^{\frac{1}{2}} + \Big(\int_{\Omega} |u|^m dx\Big)^{\frac{2}{p^-}}\Big] \\ &\le C\Big(\int_{\Omega} |u|^m dx\Big)^{\frac{2+p^-}{2p^-}}. \end{aligned} \tag{4.19}$$

Which implies that

$$\int_{\Omega} |u|^m dx \ge C\Bigg[\frac{H'(0)}{\Big(1 - \frac{t(\theta-1)}{C}(H'(0))^{\theta-1}\Big)^{\frac{1}{\theta-1}}}\Bigg]^{\frac{2p^-}{2+p^-}} \to \infty \text{ as } t \to T^*. \tag{4.20}$$

□

Theorem 4.2. *Let u be a weak solution of (1.1) with* $\alpha = 0$ *that satisfies (4.4) and (4.5). Then,* $\|u\|^2_{L^2(\Omega)}$ *is unbounded in a finite time interval* $(0, T^*)$*, with*

$$T^* = \frac{2\|u_0\|^2_{L^2(\Omega)}}{\kappa(u_0 - U_1) - \psi'(0)}, \ \kappa > 0,$$

where ψ *is defined in* (4.21).

Proof. Let ψ be a function defined by

$$\psi(t) = \int_{\Omega} |u(x,t)|^2 dx. \tag{4.21}$$

Thus

$$\psi'(t) = 2\int_{\Omega} u_t u dx. \tag{4.22}$$

Then

$$\begin{aligned} (\psi'(t))^2 &= 4\Big(\int_{\Omega} u_t u dx\Big)^2 \\ &\leq 4\Big(\int_{\Omega} |u_t|^2 dx\Big)\Big(\int_{\Omega} |u|^2 dx\Big) \\ &\leq 4\psi(t)\Big(\int_{\Omega} |u_t|^2 dx\Big), \end{aligned} \tag{4.23}$$

which gives us

$$\int_{\Omega} |u_t|^2 dx \geq \frac{(\psi'(t))^2}{4\psi(t)}. \tag{4.24}$$

Thanks to (4.5) and (4.24) for $\kappa > 2$, we have

$$\begin{aligned} \psi''(t) &= 2\int_{\Omega} |u_t|^2 dx + 2\beta\rho_{m(x)}(u) - 2\rho_{p(x)}(\Delta u) \\ &\geq \psi''(t) + 2\kappa J(t) \\ &\geq (\kappa+2)\int_{\Omega} |u_t|^2 dx \\ &\geq (\kappa+2)\frac{(\psi'(t))^2}{4\psi(t)}. \end{aligned} \tag{4.25}$$

This implies

$$\frac{\psi''(t)}{(\psi'(t))^2} \geq \frac{\kappa+2}{4\psi(t)} \geq 0. \tag{4.26}$$

Taking into account (4.1) and (4.26), we get

$$\psi'(t) > 0, \ t > 0. \tag{4.27}$$

Simple calculation applied to (4.26) leads to

$$\psi(t) \geq \frac{\|u_0\|^2_{L^2(\Omega)}}{\Big(1 - t\frac{\kappa(u_0-U_1)-\psi'(0)}{2\|u_0\|^2_{L^2(\Omega)}}\Big)^{\frac{4}{\kappa-2}}}, \ \kappa > 2. \tag{4.28}$$

□

5. Conclusions

We discussed a new type of boundary condition associated with the classical damped wave equation, featuring an unusual bi-harmonic Laplacian. Its importance is due to its wide applications in science, engineering, and real life. The novelty of our work is as follows:

(1) The first contribution is provided in Theorem 3.1, where we found that the solution exists globally in time under a condition on the exponent variable (3.1) ($m^+ \leq 2$ or $2 < m^+ \leq p^-$).
(2) The second novelty is in Theorems 4.1 and 4.2. Under conditions (4.1) and $2 \leq p^- \leq p^+ < \kappa < m^-$, the solutions blow up in two different finite times.
(3) The algebraic nonlinearity in sources is considered in a more general case with the function ϕ in (1.2).

We clearly found the impact of additional boundary conditions (no flux boundary condition) on the behavior of the solution.

Author contributions

Roumaissa Khalfallaoui and Abderrazak Chaoui: Writing–original draft preparation; Safa M. Mirgani: Writing–review and editing, visualization; Khaled Zennir: Visualization, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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