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Research article

Similarity reduction and novel Jacobi wave solutions for the variable (4+1)-dimensional Fokas equation

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Abstract: In this study, the (4+1)-dimensional Fokas equation with variable coefficients, which describes water waves in deep and wider channels, was reduced to a sixth-order nonlinear ordinary differential equation using the direct similarity reduction method. Then, the Jacobi expansion method was used to obtain multiple novel types of traveling wave solutions, including solitons, periodic waves, and singular waves. The obtained Jacobi wave solutions were considered new and have never been obtained before. Last, the dynamic behavior of the periodic and soliton wave solutions was explained according to different variable coefficient values and visualized by 3D plots.

Keywords: (4+1)-dimensional variable-coefficient Fokas; similarity reduction; novel Jacobi elliptic wave solutions
Mathematics Subject Classification: 35-XX, 35C08

1. Introduction

The study of nonlinear evolution equations (NLEEs) in higher dimensions has gained significant interest due to their ability to model complex phenomena in many different fields of physics and engineering. One of these equations is the (4+1)-dimensional Fokas equation, which Fokas gave [1], because of Lax pairs; the higher higher-dimensional extension for the two nonlinear integrable equations, namely, the Kadomtsev-Petviashvili (KP) and Davey-Stewartson equations (DS). Therefore, the physical applications of both equations, as they model water waves in deep and wider channels, make the Fokas equation very important in that field.

$$4v_{xt} - 6v_{zw} - v_{xxxy} + v_{xyyy} + 12\left(v_x v_y + v v_{xy}\right) = 0, \tag{1.1}$$

where $v = v(x, y, z, w)$ represents the wave function, t is the time, and x, y, z, w are the space variables. Equation (1.1) has attracted researchers and been solved by different methodologies like the Lie group, modified tanh-coth method, extended Jacobi elliptic function method, the Exp-function method, generalized exponential rational function, etc. [2–4].

The existence of variable coefficients on higher dimensional NLEEs makes this equation significantly more complex to solve analytically compared to its constant-coefficient counterpart [5,6]. However, this complexity enables the modeling of more realistic and involved physical scenarios [7,8]. Among these, the Fokas equation, originally derived by Fokas to explain the evolution of water waves with surface tension, has been generalized and extended in numerous ways. One such extension is the (4+1)-dimensional variable-coefficient Fokas (vc-Fokas) equation, which introduces higher spatial dimensions and coefficients that depend on the time independent variable t.

$$\alpha(t)\, v_{xt} - \beta(t)\left(v_{xxxy} + v_{xyyy}\right) + \delta(t)\left(v_x v_y + v v_{xy}\right) + a(t)\, v_{zw} = 0, \tag{1.2}$$

where $\alpha(t) \neq 0, \beta(t) \neq 0, \delta(t) \neq 0$, and $a(t) \neq 0$ are arbitrary functions of t. The vc-Fokas equation, as an extension of the KP equation, can be used to describe the long water waves and small amplitude surface waves with weak nonlinearity, weak dispersion, and weak perturbation in fluid mechanics or two-dimensional matter-wave pulses in Bose-Einstein condensates with $\beta(t)$ as a dispersion coefficient, where $\delta(t)$ is the nonlinearity term and both $\alpha(t)$ and $a(t)$ are the perturbed effects [9, 10]. Equation (1.2) was solved in [9, 10] by the Hirota bilinear method and Bäcklund transformation, where one, two, and three soliton solutions were obtained and they used the test method to obtain lump and breather solutions.

According to the importance of the higher dimensional variable coefficients NLEEs and the few studies done on the (4+1)-dimensional variable coefficients Fokas equation, we study this equation using the direct similarity reduction method to reduce it from (4+1)-dimensional to one dimensional as a nonlinear ordinary differential equation and solve this equation using the Jacobi elliptic method to obtain novel periodic and solitary wave solutions. Finally, we discuss the dynamic behavior of the *sn* wave solution and its limit under the control of the variable coefficients.

2. Direct similarity method connected with HB

Lie group analysis is a powerful technique that is considered an important tool for finding transformations to reduce nonlinear or linear partial differential equations in a number of dimensions [11–13], but it was complicated, and many other related similarity methods appeared to be easier than the lie group such as the nonclassical symmetry method, conditional symmetries, non-Lie symmetry, direct similarity reduction, and others [5, 6]. In this paper, we use the direct similarity technique connected with the homogeneous balance (HB) method, which was modified to NEE with variable coefficients in [14–16]. This technique is simple, coincide, and direct, so we introduce the steps and apply it to the 4D vc-Fokas equation as follows:

Assume that a NLEE with variable coefficients takes the form:

$$NEE(t, x, y, z, w, v_t, v_x, \alpha(t), \beta(t), ...) = 0. \tag{2.1}$$

Equation (1.2) has a solution in the form:

$$v = \frac{\partial^n}{\partial x^n}\varkappa(\xi) + u_0, \tag{2.2}$$

where $\xi = \xi(t, x, y, z, w)$ is the similarity variable, $u_0 = u_0(t, x, y, z, w)$ is an arbitrary function, and n is an integer determined using the balance methodology. Substitute from Eq (2.2) into Eq (2.1), then collect the different terms of $\varkappa$ and its derivatives. After that, equate those terms with arbitrary functions say $\Xi_i(\xi), i = 1, 2, ...$ Thus, a system of partial differential equations in ξ is obtained. To solve it, we should make use of the following rules:

I) If $u_0 = u_0(x, y, z, w, t) + \frac{\partial^m}{\partial x^m}\Xi(\xi)$, therefore we may consider $\Xi = 0$;

II) If $\Xi(\xi) = \xi_0(t, x, y, z, w)$ takes the form, one could assume that $\Xi = \xi$.

Finally, Eq (2.1) is transformed into a nonlinear ordinary differential equation under some constrains between the variable coefficients.

3. Similarity reduction for the 4D vc-Fokas equation

As given in Section 2 and by using the assumption (2.1) in Eq (1.2), applying the homogeneous balance method, we get $n = 2$, and v takes the form:

$$v(x, y, z, w, t) = u_0(x, y, z, w, t) + \frac{d^2\varkappa(\xi)}{d\xi^2}(\frac{\partial \xi}{\partial x})^2 + \frac{d\varkappa(\xi)}{d\xi}\frac{\partial^2 \xi}{\partial x^2}. \tag{3.1}$$

Insert Eq (3.1) into the 4D vc-Fokas equation and collect all terms of $\varkappa$ and its derivatives as follows:

$$\begin{gathered}
\delta(t)\left(\xi_{xx}\xi_{xxxy} + \xi_{xxx}\xi_{xxy}\right)\chi'^2 + [\delta(t)\left(3\xi_x\xi_{xx}\xi_{xxy} + \xi_{xxx}\left(2\xi_x\xi_{xy} + \xi_{xx}\xi_y\right) + \xi_x^2\xi_{xxxy}\right.\\
\left.+\xi_{xx}\left(3\xi_x\xi_{xxy} + 3\xi_{xx}\xi_{xy} + \xi_y\xi_{xxx}\right)\right)\varkappa'' + \delta\left(\xi_x^3\xi_{xxy} + \xi_{xxx}\xi_x^2\xi_y + 3\xi_{xx}(\xi_x^2\xi_{xy} + \xi_x\xi_y\xi_{xx})\right)\chi'''\\
+\delta\xi_{xx}\xi_x^3\xi_y\varkappa'''' + \alpha(t)\xi_{xxxt} - \beta(t)\left(\xi_{xxxxxy} + \xi_{xxxyyy}\right) + \delta(u_{0_{xy}}\xi_{xx} + u_{0_y}\xi_{xxx} + u_{0_x}\xi_{xxy} + u_0\xi_{xxxy})\\
+a\xi_{xxwz}]\varkappa' + \delta(t)[3\xi_x\xi_{xx}\left(2\xi_x\xi_{xy} + \xi_{xx}\xi_y\right) + \xi_x^2\left(3\xi_x\xi_{xxy} + 3\xi_{xx}\xi_{xy} + \xi_{xxx}\xi_y\right)]\chi''^2\\
+[\delta\left(\xi_x^3\left(2\xi_x\xi_{xy} + \xi_{xx}\xi_y\right) + 3\xi_x^3\xi_{xx}\xi_y + \xi_x^2\left(3\xi_x^2\xi_{xy} + 3\xi_x\xi_{xx}\xi_y\right)\right)\chi''\chi''' + \delta\xi_x^5\xi_y\chi''''\\
+\alpha(3\xi_x\xi_{xxxt} + \xi_t\xi_{xxx} + 3\xi_{xt}\xi_{xx}) - \beta\left(3\xi_x\xi_{xxyyy} + 5\xi_x\xi_{xxxxy} + 3\xi_{xyyy}\xi_{xx} + 10\xi_{xx}\xi_{xxxy} + \xi_{yyy}\xi_{xxx}\right.\\
\left.+ 10\xi_{xxx}\xi_{xxy} + 3\xi_y\xi_{xxxyy} + \xi_y\xi_{xxxxx} + 3\xi_{yy}\xi_{xxxy} + 9\xi_{xy}\xi_{xxyy} + 5\xi_{xy}\xi_{xxxx} + 9\xi_{xyy}\xi_{xxy}\right)\\
+\delta\left(u_{0_x}\left(2\xi_x\xi_{xy} + \xi_{xx}\xi_y\right) + 3u_{0_y}\xi_x\xi_{xx} + u_0\left(3\xi_x\xi_{xxy} + 3\xi_{xx}\xi_{xy} + \xi_{xxx}\xi_y\right) + \xi_x^2u_{0_{xy}}\right) + a(2\xi_x\xi_{xwz}\\
+\xi_{xx}\xi_{wz} + \xi_z\xi_{xxw} + \xi_w\xi_{xxz} + 2\xi_{xz}\xi_{xw})]\chi'' + \delta\xi_x^5\xi_y\chi'''^2 + [3\alpha\left(\xi_x^2\xi_{xt} + \xi_x\xi_t\xi_{xx}\right) - \beta\left(3\xi_x^2\xi_{xyyy}\right.\\
+10\xi_x^2\xi_{xxxy} + 3\xi_x\xi_{xx}\xi_{yyy} + 30\xi_x\xi_{xx}\xi_{xxy} + 20\xi_x\xi_{xxx}\xi_{xy} + 9\xi_x\xi_y\xi_{xxyy} + 5\xi_x\xi_y\xi_{xxxx} + 9\xi_x\xi_{yy}\xi_{xxy}\\
18\xi_x\xi_{xy}\xi_{xyy} + 15\xi_{xx}^2\xi_{xy} + 10\xi_{xx}\xi_{xxx}\xi_y + 9\xi_{xx}\xi_y\xi_{xyy} + 9\xi_{xx}\xi_{yy}\xi_{xy} + 3\xi_{xxx}\xi_y\xi_{yy} + 3\xi_y^2\xi_{xxxy} + 18\xi_y\xi_{xy}\xi_{xxy}\\
\left.+6\xi_{xy}^2 + 10\xi_x^2\xi_{xxxy}\right) + \delta(t)\left(\xi_x^3u_{0_y} + u_{0_x}\xi_x^2\xi_y + 3u_0\left(\xi_x^2\xi_{xy} + \xi_x\xi_{xx}\xi_y\right)\right) + a(t)\left(\xi_x^2\xi_{wz} + 2\xi_x\xi_z\xi_{xw} + 2\xi_x\xi_w\xi_{xz}\right.\\
\left.+\xi_{xx}\xi_z\xi_w\right)]\chi''' + [\alpha\xi_x^3\xi_t - \beta\left(\xi_x^3\xi_{yyy} + 10\xi_x^3\xi_{xxy} + 30\xi_x^2\xi_{xx}\xi_{xy} + 10\xi_{xxx}\xi_x^2\xi_y + 9(\xi_x^2\xi_y\xi_{xyy} + \xi_x^2\xi_{yy}\xi_{xy}\right.\\
\left.+\xi_x\xi_{xx}\xi_y\xi_{yy} + \xi_x\xi_y^2\xi_{xxy} + 2\xi_x\xi_y\xi_{xy}^2 + \xi_{xx}\xi_y^2\xi_{xy} + \xi_{xxx}\xi_y^3) + 15\xi_x\xi_{xx}^2\xi_y + \xi_{xxx}\xi_y^3) + \delta u_0\xi_x^3\xi_y\\
+a\xi_x^2\xi_z\xi_w]\chi^{(4)} - \beta\left(5\left(\xi_x^4\xi_{xy} + 2\xi_x^3\xi_{xx}\xi_y\right) + 3\left(\xi_x^3\xi_y\xi_{yy} + 3\xi_x^2\xi_y^2\xi_{xy} + \xi_x\xi_{xx}\xi_y^3\right)\right)\chi^{(5)} - \beta\left(\xi_x^5\xi_y + \xi_x^3\xi_y^3\right)\chi^{(6)}
\end{gathered}$$

$$\alpha u_{0xt}-\beta\left(u_{0xxxy}+u_{0xyyy}\right)+\delta\left(u_{0x}u_{0y}+u_0u_{0xy}\right)+au_{0zw}=0. \tag{3.2}$$

Now, to transform the above Eq (3.2) to a NODE, assume that all coefficients of χ and its derivatives are arbitrary functions on the similarity variable ξ multiplied by the coefficient of the most dispersive term $\chi^{(6)}$, as follows:

$$\begin{gathered}
\beta\left(5\left(\xi_x^4\xi_{xy}+2\xi_x^3\xi_{xx}\xi_y\right)+3\left(\xi_x^3\xi_y\xi_{yy}+3\xi_x^2\xi_y^2\xi_{xy}+\xi_x\xi_{xx}\xi_y^3\right)\right)=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_1(\xi),\\
[\alpha\xi_x^3\xi_t-\beta\left(\xi_x^3\xi_{yyy}+10\xi_x^3\xi_{xxy}+30\xi_x^2\xi_{xx}\xi_{xy}+10\xi_{xxx}\xi_x^2\xi_y+9(\xi_x^2\xi_y\xi_{xyy}+\xi_x^2\xi_{yy}\xi_{xy}\right.\\
+\xi_x\xi_{xx}\xi_y\xi_{yy}+\xi_x\xi_y^2\xi_{xxy}+2\xi_x\xi_y\xi_{xy}^2+\xi_{xx}\xi_y^2\xi_{xy}+\xi_{xxx}\xi_y^3)+15\xi_x\xi_{xx}^2\xi_y+\xi_{xxx}\xi_y^3)+\delta u_0\xi_x^3\xi_y\\
+a\xi_x^2\xi_z\xi_w]=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_2(\xi),\\
[3\alpha\left(\xi_x^2\xi_{xt}+\xi_x\xi_t\xi_{xx}\right)-\beta\left(3\xi_x^2\xi_{xyyy}+10\xi_x^2\xi_{xxxy}+3\xi_x\xi_{xx}\xi_{yyy}+30\xi_x\xi_{xx}\xi_{xxy}+20\xi_x\xi_{xxx}\xi_{xy}+9\xi_x\xi_y\xi_{xxyy}\right.\\
+5\xi_x\xi_y\xi_{xxxx}+9\xi_x\xi_{yy}\xi_{xxy}+18\xi_x\xi_{xy}\xi_{xyy}+15\xi_{xx}^2\xi_{xy}+10\xi_{xx}\xi_{xxx}\xi_y+9\xi_{xx}\xi_y\xi_{xyy}+9\xi_{xx}\xi_{yy}\xi_{xy}+3\xi_{xxx}\xi_y\xi_{yy}\\
\left.+3\xi_y^2\xi_{xxxy}+18\xi_y\xi_{xy}\xi_{xxy}++6\xi_{xy}^2\right)+\delta(t)\left(\xi_x^3u_{0_y}+u_{0_x}\xi_x^2\xi_y+3u_0\left(\xi_x^2\xi_{xy}+\xi_x\xi_{xx}\xi_y\right)\right)\\
a(t)\left(\xi_x^2\xi_{wz}+2\xi_x\xi_z\xi_{xw}+2\xi_x\xi_w\xi_{xz}+\xi_{xx}\xi_z\xi_w\right)]=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_3(\xi),\\
\delta\xi_x^5\xi_y=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_5(\xi),\\
\delta\left(\xi_x^3\left(2\xi_x\xi_{xy}+\xi_{xx}\xi_y\right)+3\xi_x^3\xi_{xx}\xi_y+\xi_x^2\left(3\xi_x^2\xi_{xy}+3\xi_x\xi_{xx}\xi_y\right)\right)=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_6(\xi),\\
\alpha\left(3\xi_x\xi_{xxxt}+\xi_t\xi_{xxx}+3\xi_{xt}\xi_{xx}\right)-\beta\left(3\xi_x\xi_{xxyyy}+5\xi_x\xi_{xxxxy}+3\xi_{xyyy}\xi_{xx}+10\xi_{xx}\xi_{xxxy}+\xi_{yyy}\xi_{xxx}\right.\\
\left.+10\xi_{xxx}\xi_{xxy}+3\xi_y\xi_{xxxyy}+\xi_y\xi_{xxxxx}+3\xi_{yy}\xi_{xxxy}+9\xi_{xy}\xi_{xxyy}+5\xi_{xy}\xi_{xxxx}+9\xi_{xyy}\xi_{xxy}\right)\\
+\delta\left(u_{0_x}\left(2\xi_x\xi_{xy}+\xi_{xx}\xi_y\right)+3u_{0_y}\xi_x\xi_{xx}+u_0\left(3\xi_x\xi_{xxy}+3\xi_{xx}\xi_{xy}+\xi_{xxx}\xi_y\right)+\xi_x^2u_{0_{xy}}\right)+a\left(2\xi_x\xi_{xwz}\right.\\
\left.+\xi_{xx}\xi_{xz}+\xi_z\xi_{xxw}+\xi_w\xi_{xxz}+2\xi_{xz}\xi_{xw}\right)=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_7(\xi),\\
\delta(t)\left(\xi_{xx}\xi_{xxxy}+\xi_{xxx}\xi_{xxy}\right)=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_8(\xi),\\
\delta\xi_{xx}\xi_x^3\xi_y=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_9(\xi),\\
\delta\left(\xi_x^3\xi_{xxy}+\xi_{xxx}\xi_x^2\xi_y+3\xi_{xx}(\xi_x^2\xi_{xy}+\xi_x\xi_y\xi_{xx})\right)=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_{10}(\xi),\\
\delta(t)\left(3\xi_x\xi_{xx}\xi_{xxy}+\xi_{xxx}\left(2\xi_x\xi_{xy}+\xi_{xx}\xi_y\right)+\xi_x^2\xi_{xxxy}+\xi_{xx}\left(3\xi_x\xi_{xxy}+3\xi_{xx}\xi_{xy}+\xi_y\xi_{xxx}\right)\right)\\
=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_{11}(\xi),\\
\alpha(t)\,\xi_{xxxt}-\beta(t)\left(\xi_{xxxxxy}+\xi_{xxxyyy}\right)+\delta(u_{0_{xy}}\xi_{xx}+u_{0_y}\xi_{xxx}+u_{0_x}\xi_{xxy}+u_0\xi_{xxxy})+a\xi_{xxwz}\\
=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_{12}(\xi),\\
\delta(t)\left[3\xi_x\xi_{xx}\left(2\xi_x\xi_{xy}+\xi_{xx}\xi_y\right)+\xi_x^2\left(3\xi_x\xi_{xxy}+3\xi_{xx}\xi_{xy}+\xi_{xxx}\xi_y\right)\right]=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_{13}(\xi),\\
\alpha u_{0xt}-\beta\left(u_{0xxxy}+u_{0xyyy}\right)+\delta\left(u_{0x}u_{0y}+u_0u_{0xy}\right)+au_{0zw}=-\beta\left(\xi_x^5\xi_y+\xi_x^3\xi_y^3\right)\Xi_{14}(\xi),
\end{gathered} \tag{3.3}$$

where $\Xi_i(\xi), i=1,...,14$ are unknown functions that can be found from the direct method rules. Solving system (3.2) yields, $\Xi_2(\xi)=A$, $\Xi_5(\xi)=B$, $\Xi_1(\xi)=\Xi_3(\xi)=\Xi_4(\xi)=0$, $\Xi_i(\xi)=0$, $i=6,...,14, u_0(x,y,z,w,t)=u_0$,where A, B and u_0 are constants.

Theorem 3.1. *If $v(x,y,z,w,t) = u_0 + p^2\chi''(\xi)$ is a solution of Eq (1.2) with a similarity variable*

$$\xi = px + qy + rz + sw + \int \frac{\beta(t)\left(p^2+q^2\right)\left(Apq - Bu_0\frac{q}{p}\right) - rsa(t)}{p\alpha(t)}dt + \xi_0, \tag{3.4}$$

where neither A nor B are zero. The integrability conditions between the coefficients is given by

$$\delta(t) = B\left(1 + \left(\frac{q}{p}\right)^2\right)\beta(t), \tag{3.5}$$

then Eq (1.2) reduces to the following NODE:

$$\chi^{(6)} - A\chi^{(4)} - B\left(\chi^{(4)}\chi'' + \chi'''^2\right) = 0, \tag{3.6}$$

where p, q, r, s and ξ_0 are constants.

By integrating Eq (3.5) twice with respect to ξ

$$\chi^{(4)} - A\chi'' - \frac{B}{2}\chi''^2 = C_1\xi + C, \tag{3.7}$$

where C_1 and C are two integration constants. To find solutions for Eq (1.1), assume that

$$\chi'' = F(\xi) \text{ and } C_1 = 0. \tag{3.8}$$

Therefore, Eq (3.8) becomes

$$F'' - AF - \frac{B}{2}F^2 = C, \tag{3.9}$$

To solve Eq (3.9), we use the Jacobi expansion method [17, 18], assuming that this equation has a solution in the form

$$F(\xi) = \sum_{k=0}^{k=M} B_k\phi(\xi)^k, \tag{3.10}$$

where B_k are arbitrary constants and $\phi(\xi)$ can given by solving the Jacobi equation

$$\phi' = \sqrt{c_0 + c_2\phi^2(\xi) + c_4\phi^4(\xi)}. \tag{3.11}$$

Using the homogeneous balance between the most dispersive term F'' and the nonlinear term F^3, the positive integer $M = 2$, therefore ϕ takes the form:

$$F(\xi) = B_0 + B_1\phi(\xi) + B_2\phi^2(\xi), \tag{3.12}$$

Inserting Eq (3.12) into Eq (3.9) using Eq (3.11), we collect the same powers of $\phi(\xi)$ and $\phi'(\xi)$ and equate it to zero, so that the following algebraic system is yielded:

$$\begin{aligned} &6B_2c_4 - \frac{1}{2}BB_2^2 = 0, 2B_1c_4 - BB_1B_2 = 0, B_1c_2 - AB_1 - BB_0B_1 = 0,\\ &2B_2c_0 - AB_0 - \frac{1}{2}BB_0^2 - C = 0, 4B_2c_2 - 2BB_0B_2 - AB_2^2 - BB_2 = 0. \end{aligned} \tag{3.13}$$

By using the Maple program to solve system (3.13), the following solution is provided:

$$B_0 = \frac{4c_2 - A}{B}, B_1 = 0, B_2 = \frac{12c_4}{B}, \text{ under condition } C = \frac{A^2 + 48c_0c_4 - 16c_2^2}{2B}. \tag{3.14}$$

By back substitution from Eq (3.14) into Eq (3.12) and by using Eqs (3.2) and (3.6) in Eq (2.2), the following novel optical Jacobi wave solutions are obtained:

$$v_1 = p^2\left(-\frac{A + 4\left(1 + m^2\right)}{B} + \frac{12m^2}{A}\text{sn}^2\ (\xi, m)\right) + u_0, \tag{3.15}$$

$$v_2 = p^2\left(-\frac{A + 4\left(1 + m^2\right)}{B} + \frac{12m^2}{A}\text{cd}^2\ (\xi, m)\right) + u_0, \tag{3.16}$$

$$v_3 = -p^2\left(\frac{A + 4\left(1 - 2m^2\right)}{B} + \frac{12m^2}{B}\text{cn}^2\ (\xi, m)\right) + u_0, \tag{3.17}$$

$$v_4 = -p^2\left(\frac{4\left(m^2 - 2\right) + A}{B} - \frac{12}{B}\text{dn}^2\,(\xi, m)\right) + u_0, \tag{3.18}$$

$$v_5 = p^2\left(-\frac{A + 4\left(m^2 + 1\right)}{B} + \frac{12}{B}\text{ns}^2\,(\xi, m)\right) + u_0, \tag{3.19}$$

$$v_6 = p^2\left(-\frac{4(1 - 2m^2) + A}{B} + \frac{12\left(1 - m^2\right)}{B}\text{nc}^2\,(\xi, m)\right) + u_0, \tag{3.20}$$

$$v_7 = p^2\left(-\frac{4(m^2 - 2) + A}{B} + \frac{12\left(m^2 - 1\right)}{B}\text{nd}^2\,(\xi, m)\right) + u_0, \tag{3.21}$$

$$v_8 = p^2\left(-\frac{4(m^2 - 2) + A}{B} + \frac{12\left(1 - m^2\right)}{B}\text{sc}^2\,(\xi, m)\right) + u_0, \tag{3.22}$$

$$v_9 = p^2\left(-\frac{4(1 - 2m^2) + A}{B} - \frac{12m^2\left(1 - m^2\right)}{B}\text{sd}^2\,(\xi, m)\right) + u_0, \tag{3.23}$$

$$v_{10} = p^2\left(-\frac{4(m^2 - 2) + A}{B} + \frac{12}{B}\text{cs}^2\,(\xi, m)\right) + u_0, \tag{3.24}$$

$$v_{11} = p^2\left(-\frac{4(1 - 2m^2) + A}{B} + \frac{6}{A}\text{ds}^2\,(\xi, m)\right) + u_0, \tag{3.25}$$

where $C = \frac{A^2+16\left(m^2-m^4-1\right)}{2B}$,

$$v_{12} = p^2\left(-\frac{2(2m^2 - 1) + A}{B} + \frac{3}{B}(\text{ns}\,(\xi, m) \pm \text{cs}\,(\xi, m)^2\right) + u_0, \tag{3.26}$$

where $C = \frac{A^2+16m^2\left(1-m^2\right)-1}{2B}$,

$$v_{13} = p^2\left(\frac{A-2(m^2+1)}{B} + \frac{3(1-m^2)}{B}(\mathrm{nc}\,(\xi,m) \pm \mathrm{sc}\,(\xi,m)^2\right) + u_0, \tag{3.27}$$

where $C = \frac{A^2-m^2\left(14+m^2\right)-1}{2B}$,

When the module m finishes or becomes one, the Jacobi functions become hyperbolic and periodic functions as follows:

$$v_{14} = p^2\left(-\frac{A+8}{B} + \frac{12}{B}\ \tanh^2(\xi)\right) + u_0, \tag{3.28}$$

$$v_{15} = -p^2\left(\frac{A-4}{B} + \frac{12}{B}\ sech^2(\xi)\right) + u_0, \tag{3.29}$$

$$v_{16} = p^2\left(-\frac{A+8}{B} + \frac{12}{B}\ \coth^2(\xi)\right) + u_0, \tag{3.30}$$

$$v_{17} = p^2\left(-\frac{A-4}{B} + \frac{12}{A}\ csch^2(\xi)\right) + u_0, \tag{3.31}$$

$$v_{18} = p^2\left(-\frac{A+4}{B} + \frac{12}{B}\ \sec^2(\xi)\right) + u_0, \tag{3.32}$$

$$v_{19} = p^2\left(-\frac{A+4}{B} + \frac{12}{B}\ \csc^2(\xi)\right) + u_0, \tag{3.33}$$

$$v_{20} = p^2\left(\frac{8-A}{B} + \frac{12}{B}\ \tan^2(\xi)\right) + u_0, \tag{3.34}$$

$$v_{21} = p^2\left(\frac{8-A}{B} + \frac{12}{B}\ \cot^2(\xi)\right) + u_0, \tag{3.35}$$

where $C = \frac{B^2-16}{4A}$,

$$v_{22} = p^2\left(-\frac{(A+2)}{B} + \frac{3}{B}\ (\coth(\xi) \pm csch(\xi))^2\right) + u_0, \tag{3.36}$$

$$v_{23} = p^2\left(-\frac{A-2}{B} + \frac{3}{B}\ (\csc(\xi) \pm \cot(\xi))^2\right) + u_0, \tag{3.37}$$

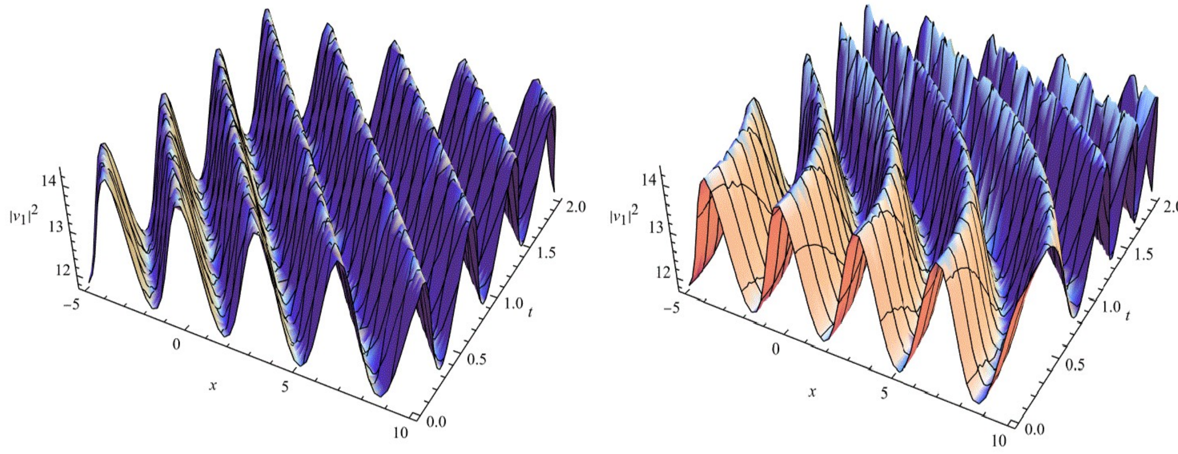
$$v_{24} = p^2\left(-\frac{A-2}{B} + \frac{3}{B}\ (\sec(\xi) \pm \tan(\xi))^2\right) + u_0, \tag{3.38}$$

where $C = \frac{B^2-1}{4A}$,

4. Physical applications and dynamic behavior

Solitary waves appear from a balance between nonlinear and dispersive effects in a medium. They are self-reinforcing wave packets that maintain their shape and speed over long distances. They are localized waves with a single peak and decay rapidly away from it [19, 20]. Soliton is a special type of solitary wave that keeps its shape even after interacting with other solitons [21]. In this section, we choose the periodic Jacobi SN function solution v_1 and its solitary wave limit as $m = 1$ given by v_{14} to plot the intensity of it, according to different values of both parameters and variable functions. We fix the parameters $A = p = q = r = s = 1, y = z = w = 0, u_0 = 2$.

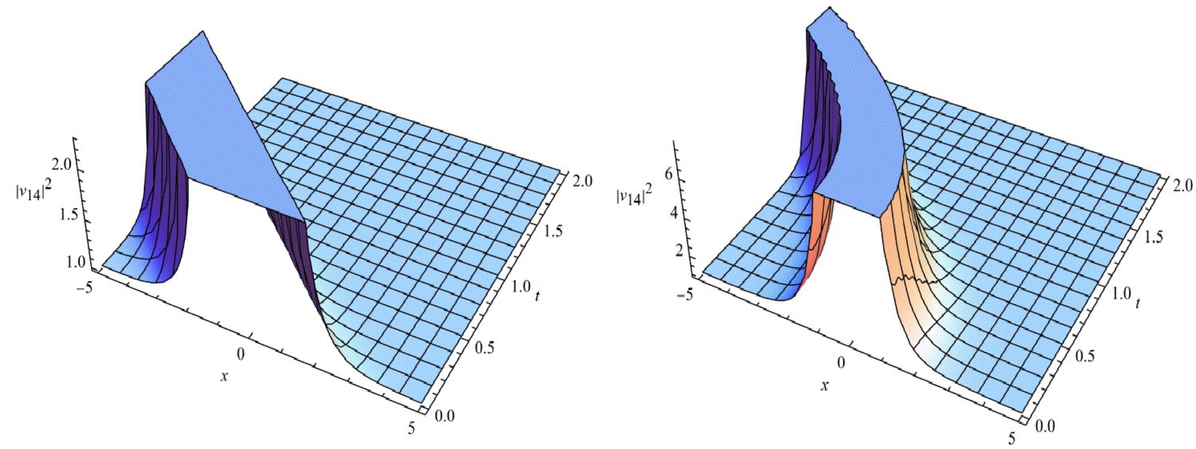
The intensity $|v_1|^2$ of the periodic wave solution v_1 when $B = -3, m = 0.3$ and the variable functions $\beta(t) = t, a(t) = 2t$ but $\alpha(t) = 2t$ in Figure 1(a) and $\alpha(t) = 2$ in Figure 1(b).



(a) (b)

Figure 1. (a) The intensity $|v_1|^2$ of the periodic wave solution, (b)The intensity of $|v_1|^2$ the periodic wave solution taking parabolic shape.

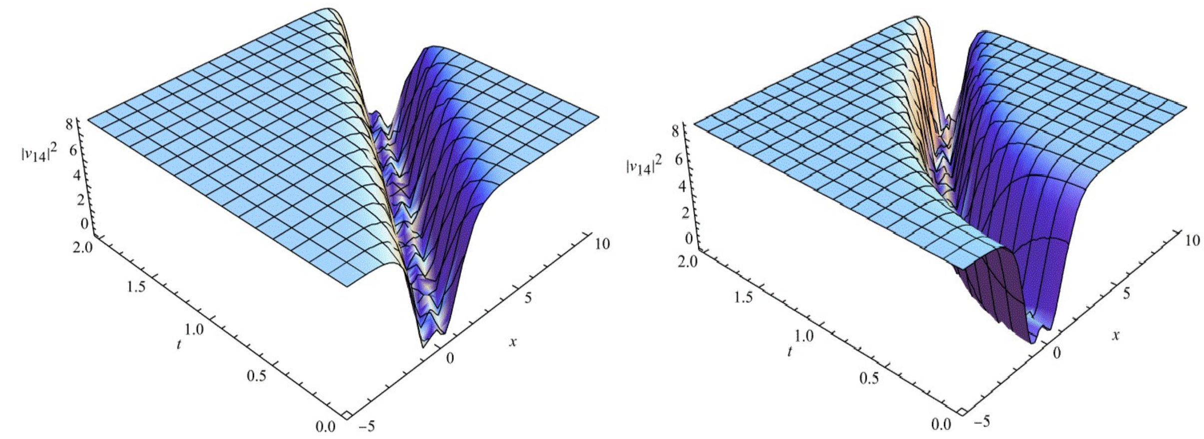
The intensity $|v_{14}|^2$ of the bright soliton solution v_{14} when $B = 3$ and the variable functions $\beta(t) = t, a(t) = 2t$ but $\alpha(t) = 2t$ in Figure 2(a) and $\alpha(t) = 2$ in Figure 2(b).



(a) (b)

Figure 2. (a) The intensity $|v_{14}|^2$ of the bright soliton solution, (b) The intensity $|v_{14}|^2$of the bright soliton with parabolic shape.

The intensity $|v_{14}|^2$of the dark soliton solution v_{14} when $B = -3$ and the variable functions $\beta(t) = t, a(t) = 2t$ but $\alpha(t) = 2t$ in Figure 3(a) and $\alpha(t) = 2$ in Figure 3(b).



(a) (b)

Figure 3. (a) The intensity of the dark soliton $|v_{14}|^2$, (b) The intensity $|v_{14}|^2$of the dark soliton with parabolic shape.

In all figures, we calculate and plot the intensity of the wave solution v_{14}. We can see that we have fixed $\beta(t) = t, a(t) = 2t$, $\alpha(t) = 2t$, and a periodic sn wave is given in Figure 1(a). A bright soliton is shown in Figure 2(a), where $B = 3$ but when $B = -3$, the soliton becomes a dark soliton; therefore, the sign of B affects the direction of the solitary wave. In figures (b), the variables are fixed as $\beta(t) = t, a(t) = 2t$,$\alpha(t) = 2$, so that the parabolic shape arises in all figures (b). Thus, It can be seen in the sn wave; bright and dark solitons and similar to figures (a), where the sign of B affects the direction of the soliton wave intensity.

5. Conclusions

The (4+1)-dimensional variable-coefficients Fokas equation, which describes the evolution of water waves with surface tension in higher dimensions, is investigated using the direct similarity reduction method. It is reduced to a sixth-order nonlinear ordinary differential equation under a constraint between the variable coefficients, utilizing a theorem that illustrated the similarity variables used to obtain that reduction. The previous studies on the 4D vc-Fokas equation are few [9, 10] and compared with the equation, the obtained periodic Jacobi wave solutions are novel and have not been obtained before. Finally, as an example, the intensity of the periodic *sn* wave solution v_1 and its limit v_{14} are plotted in 3D according to different choices of the variables $\alpha(t)$, $\beta(t)$, and $a(t)$ from the figures given. We conclude that the shape differs from figures (a) to figures (b) since the effect of the t^2 parabolic shape appears. Also, the sign of the B positive or negative parameter affects the soliton intensity direction from bright to dark soliton, respectively.

Author contributions

Rehab M. Elshiekh wrote and applied different methodologies and Mahmoud Gaballah made physical applications and figures.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no compact of interest.

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