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Research article

Enhanced Kneser-type oscillation criteria for second-order functional quasilinear dynamic equations on time scales

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Abstract: This work presents new Kneser-type oscillation criteria for second-order quasilinear functional dynamic equations defined on arbitrary unbounded above time scales. Our approach employs the Riccati transformation technique in conjunction with the integral averaging method. The results show a significant improvement over recent Kneser-type oscillation criteria. We provided several illustrative examples to highlight the importance of our findings.

Keywords: oscillation; Kneser-type; quasilinear; differential equations; dynamic equations; time scales
Mathematics Subject Classification: 26E70, 34C10, 34K11, 34K42, 34N05

1. Introduction

In this work, we study the oscillation of the quasilinear functional dynamic equation

$$\left[\beta_1(s)\left|x^{\Delta}(s)\right|^{a-1}x^{\Delta}(s)\right]^{\Delta}+\beta_2(s)\left|x(\nu(s))\right|^{b-1}x(\nu(s))=0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, where $s\in[s_0,\infty)_{\mathbb{T}}$, $s_0\geq 0$, $s_0\in\mathbb{T}$; a and b are positive real numbers; $\beta_1,\ \beta_2\in \mathrm{C}_{\mathrm{rd}}\left([s_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$ such that

$$K(s):=\int_{s_0}^{s}\frac{\Delta r}{\beta_1^{1/a}(r)}\to\infty \qquad \text{as } s\to\infty;$$

and $\nu:\mathbb{T}\to\mathbb{T}$ satisfies $\lim_{s\to\infty}\nu(s)=\infty$. By a solution of Eq (1.1) we mean a nontrivial real-valued function $x\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x\in[s_0,\infty)_{\mathbb{T}}$ such that $\beta_1\left|x^{\Delta}\right|^{a-1}x^{\Delta}\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$ and x satisfy (1.1) on $[T_x,\infty)_{\mathbb{T}}$, where C_{rd} is the set of rd-continuous functions. A solution x of (1.1) is considered oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. We will exclude from consideration solutions that vanish in the vicinity of infinity.

Now, we provide some definitions on time scales. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ denoted by the symbol $\mathbb{T}$. It has the topology that it inherits from the real numbers with the standard topology.

Definition 1.1. *For $s\in\mathbb{T}$, the forward operator $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by*

$$\sigma(s)=\inf\{r\in\mathbb{R}:r>s\},$$

and the backward operator $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(s)=\sup\{r\in\mathbb{T}:r<s\}.$$

If $\sigma(s)>s$, we say that s is right-scattered, while if $\rho(s)<s$, we say that s is left-scattered. Points such that

$$\rho(s)<s<\sigma(s),\ \rho(s)<s=\sup\mathbb{T}\ \textit{or}\ \inf\mathbb{T}=s<\sigma(s)$$

are called isolated points. If a time scale consists of only isolated points, then it is an isolated (discrete) time scale. Also, if $s<\sup\mathbb{T}$ and $\sigma(s)=s$, then s is called right-dense, and if $s>\inf\mathbb{T}$ and $\rho(s)=s$, then s is called left-dense. Points that are either left-dense or right-dense are called dense.

The graininess operator $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(s)=\sigma(s)-s$ and if $f:\mathbb{T}\to\mathbb{R}$ is a function, then the function $f^{\sigma}:\mathbb{T}\to\mathbb{R}$ is defined by

$$f^{\sigma}(s)=f(\sigma(s))\text{ for all } s\in\mathbb{T}.$$

Finally, the set $\mathbb{T}^k$ is derived from $\mathbb{T}$, that is,

$$\mathbb{T}^k:=\begin{cases}\mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}), & \text{if}\quad \sup\mathbb{T}<\infty,\\[2ex] \mathbb{T}, & \text{if}\quad \sup\mathbb{T}=\infty.\end{cases}$$

Now, define the so-called delta (or Hilger) derivative of f at a point $s\in\mathbb{T}^k$.

Definition 1.2. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $s \in \mathbb{T}^k$. Then, we define $f^{\Delta}(s)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of s (i.e., $U = (s-\delta, s+\delta) \cap \mathbb{T}$ for some $\delta > 0$ such that*

$$|[f(\sigma(s)) - f(r)] - f^{\Delta}(s)[\sigma(s) - r]| \leq |\sigma(s) - r| \quad \textit{for all} \quad r \in U.$$

Theorem 1.1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $s \in \mathbb{T}^k$. Then, we have the following:*

(i) If f is continuous at s and s is right-scattered, then f is differential at s with

$$f^{\Delta}(s) = \frac{f(\sigma(s)) - f(s)}{\mu(s)}.$$

(ii) If s is right-dense, then f is differential at s if

$$\lim_{r \to s} \frac{f(s)) - f(r)}{s - r}$$

exists as a finite number.

We refer the reader to [1–3] for more details regarding the theory of time scales. In dynamical models, deviation and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [4–6]. We also refer the reader to [7–9] for the oscillation of differential equations, [10–12] for nonlinear differential equations for dynamic equations on time scales, [13–15] for nonlinear dynamic equations on time scales, and [16–18] for quasilinear dynamic equations [19–21]. Higher-order equations can be found in [22] and the references therein. In particular, we highlight several oscillation findings for differential equations, which are related to the oscillation results for (1.1) on time scales in the following. The theory of oscillation has fundamentally depended on Euler differential equations and their various generalizations. The second-order Euler equation

$$x''(s) + \frac{\alpha}{s^2} x(s) = 0, \ \alpha > 0 \tag{1.2}$$

is oscillatory if and only if

$$\alpha > \frac{1}{4}.$$

Kneser-type (see [23]) are considered some of the essential oscillation criteria for second-order differential equations, which utilize Sturmian comparison methods and the oscillatory behavior of (1.2) to demonstrate that the linear differential equation

$$x''(s) + \beta_2(s) x(s) = 0$$

is oscillatory if

$$\liminf_{s \to \infty} s^2 \beta_2(s) > \frac{1}{4}.$$

Numerous works that derive Kneser-type criteria for various kinds of differential equations have since been published in a similar manner. Some of these works are as follows, see [24–26]:

(I) The linear differential equation

$$[\beta_1(s)x'(s)]' + \beta_2(s)x(s) = 0$$

is oscillatory if

$$\liminf_{s\to\infty} \beta_1(s)K^2(s)\beta_2(s) > \frac{1}{4}.$$

(II) The half-linear differential equation

$$\left[|x'(s)|^{a-1}x'(s)\right]' + \beta_2(s)|x(s)|^{a-1}x(s) = 0$$

is oscillatory if

$$\liminf_{s\to\infty} s^{a+1}\beta_2(s) > \left(\frac{a}{a+1}\right)^{a+1}.$$

(III) The half-linear differential equation

$$\left[\beta_1(s)|x'(s)|^{a-1}x'(s)\right]' + \beta_2(s)|x(s)|^{a-1}x(s) = 0$$

is oscillatory if

$$\liminf_{s\to\infty} \beta_1^{\frac{1}{a}}(s)K^{a+1}(s)\beta_2(s) > \left(\frac{a}{a+1}\right)^{a+1}.$$

Recently, Hassan et al. [27] found some interesting Kneser-type oscillation criteria for the dynamic equation

$$\left[\beta_1(s)\left|x^{\Delta}(s)\right|^{a-1}x^{\Delta}(s)\right]^{\Delta} + \beta_2(s)|x(\nu(s))|^{a-1}x(\nu(s)) = 0 \tag{1.3}$$

as in the following theorem.

Theorem 1.2. *[27] If* $R := \liminf_{s\to\infty} \dfrac{K(s)}{K^{\sigma}(s)} > 0$ *and*

$$\liminf_{s\to\infty} \beta_1^{\frac{1}{a}}(s)K(s)K^{a}(\Gamma(s))\beta_2(s) > \frac{1}{R^{a(a+1)}}\left(\frac{a}{a+1}\right)^{a+1}, \tag{1.4}$$

where

$$\Gamma(s) := \begin{cases} \nu(s), & \nu(s) \le s, \\ s, & \nu(s) \ge s, \end{cases} \tag{1.5}$$

then every solution of Eq (1.3) *is oscillatory.*

Considering the above-mentioned theorem and influenced by the contributions of [23–25, 27], this paper aims to derive sharp Kneser-type oscillation conditions for the dynamic Eq (1.1) using the known Riccati transformation technique for the cases where $a \le b$, $a \ge b$, $\nu(s) \le s$, and $\nu(s) \ge s$.

2. Main results

At the beginning of this section, we will present a preliminary theorem and some lemmas that we will use later to demonstrate the main results.

Theorem 2.1. *Pötzsche chain rule, see [1, Theorem 1.90] Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then, $g \circ f : \mathbb{T} \to \mathbb{R}$ is delta differentiable and satisfies*

$$(g \circ f)^{\Delta}(s) = \left(\int_0^1 g'(f(s) + h\mu(s) f^{\Delta}(s)) dh \right) f^{\Delta}(s).$$

Lemma 2.1. *[28, Theorem 1] Let $x(s)$ be an eventually positive solution of* (1.1). *Then,*

$$x^{\Delta}(s) > 0,\ \left(\frac{x}{K}\right)^{\Delta}(s) < 0,\ x(s) \geq \left[\beta_1^{\frac{1}{a}} x^{\Delta} K\right](s),\ \text{and}\ \left[\beta_1 \left|x^{\Delta}\right|^{a-1} x^{\Delta}\right]^{\Delta}(s) < 0 \tag{2.1}$$

eventually.

Lemma 2.2. *Let $x(s)$ be an eventually positive solution of* (1.1). *Then,*

$$z^{\Delta}(s) \leq -\frac{x^b(\nu(s))}{x^a(s)} \beta_2(s) - a\beta_1^{-\frac{1}{a}}(s) \left(\frac{K(s)}{K^{\sigma}(s)}\right)^{1-c} z^{\frac{1}{a}}(s) z^{\sigma}(s) \tag{2.2}$$

eventually, where $c := \min\{1, a\}$ and

$$z(s) := \frac{\beta_1(s) \left(x^{\Delta}(s)\right)^a}{x^a(s)}. \tag{2.3}$$

Proof. Let $x(\nu(s)) > 0$ for $s \in [s_0, \infty)_{\mathbb{T}}$. From Lemma 2.1, there exists an $s_1 \in [s_0, \infty)_{\mathbb{T}}$ such that $\left(\frac{x}{K}\right)^{\Delta}(s) < 0$ on $[s_1, \infty)_{\mathbb{T}}$. It follows from (2.3) that

$$\begin{aligned} z^{\Delta}(s) &= \left(\frac{1}{x^a}\left[\beta_1 \left(x^{\Delta}\right)^a\right]\right)^{\Delta}(s) \\ &= \frac{1}{x^a(s)}\left[\beta_1 \left(x^{\Delta}\right)^a\right]^{\Delta}(s) - \frac{(x^a(s))^{\Delta}}{x^a(s) x^a(\sigma(s))}\left[\beta_1 \left(x^{\Delta}\right)^a\right]^{\sigma}(s) \\ &\overset{(1.1)}{=} -\frac{x^b(\nu(s))}{x^a(s)}\beta_2(s) - \frac{(x^a(s))^{\Delta}}{x^a(s)} z^{\sigma}(s). \end{aligned} \tag{2.4}$$

From (2.1) and the Pötzsche chain rule application, we get

$$\begin{aligned} \frac{(x^a(s))^{\Delta}}{x^a(s)} &= \frac{a}{x^a(s)}\left[\int_0^1 \left[(1-h)x(s) + hx^{\sigma}(s)\right]^{a-1} \mathrm{d}h\right] x^{\Delta}(s) \\ &\geq \begin{cases} a\left(\dfrac{x(s)}{x^{\sigma}(s)}\right)^{1-a} \dfrac{x^{\Delta}(s)}{x(s)}, & 0 < a \leq 1, \\ a\dfrac{x^{\Delta}(s)}{x(s)}, & a \geq 1, \end{cases} \end{aligned}$$

$$\geq \begin{cases} a\beta_1^{-\frac{1}{a}}(s)\left(\dfrac{K(s)}{K^{\sigma}(s)}\right)^{1-a} z^{\frac{1}{a}}(s), & 0 < a \leq 1, \\ \\ a\beta_1^{-\frac{1}{a}}(s)\, z^{\frac{1}{a}}(s), & a \geq 1, \end{cases}$$
$$= a\beta_1^{-\frac{1}{a}}(s)\left(\frac{K(s)}{K^{\sigma}(s)}\right)^{1-c} z^{\frac{1}{a}}(s).$$

Therefore, (2.4) becomes

$$z^{\Delta}(s) \leq -\frac{x^b(\nu(s))}{x^a(s)}\beta_2(s) - a\beta_1^{-\frac{1}{a}}(s)\left(\frac{K(s)}{K^{\sigma}(s)}\right)^{1-c} z^{\frac{1}{a}}(s)\, z^{\sigma}(s). \tag{2.5}$$

□

The first theorem is Kneser-type to the second-order quasilinear dynamic Eq (1.1) in the case of $a \geq b$.

Theorem 2.2. *Let $a \geq b$ and $R := \liminf_{s\to\infty} \dfrac{K(s)}{K^{\sigma}(s)} > 0$. If*

$$A_1 := \liminf_{s\to\infty} \beta_1^{\frac{1}{a}}(s)\, K^{1-a}(\Omega(s))\, K^{a\sigma}(s)\, K^b(\Gamma(s))\beta_2(s) > \frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1}, \tag{2.6}$$

where Γ is defined by (1.5) *and*

$$\Omega(s) := \begin{cases} s, & 0 < a \leq 1, \\ \\ \sigma(s), & a \geq 1, \end{cases} \tag{2.7}$$

then every solution of Eq (1.1) *is oscillatory.*

Proof. If not, let $x(s) > 0$ and $x(\nu(s)) > 0$ for $s \in [s_0, \infty)_{\mathbb{T}}$. From Lemma 2.1, there exists an $s_1 \in [s_0, \infty)_{\mathbb{T}}$ such that $\left(\dfrac{x}{K}\right)^{\Delta}(s) < 0$ for $s \in [s_1, \infty)_{\mathbb{T}}$. Therefore,

$$\frac{x^b(\nu(s))}{x^a(s)} = \left(\frac{x(\nu(s))}{x(s)}\right)^b x^{b-a}(s)$$
$$\geq \begin{cases} \left(\dfrac{K(\nu(s))}{K(s)}\right)^b \left(\dfrac{x(s_1)}{K(s_1)}\right)^{b-a} K^{b-a}(s), & \nu(s) \leq s, \\ \\ \left(\dfrac{x(s_1)}{K(s_1)}\right)^{b-a} K^{b-a}(s), & \nu(s) \geq s, \end{cases}$$
$$= \lambda_1 \frac{K^b(\Gamma(s))}{K^a(s)},$$

where $\lambda_1 := \left(\dfrac{x(s_1)}{K(s_1)}\right)^{b-a}$. Hence, (2.2) becomes

$$z^{\Delta}(s) \leq -\lambda_1 \frac{K^b(\Gamma(s))}{K^a(s)}\beta_2(s) - a\beta_1^{-\frac{1}{a}}(s)\left(\frac{K(s)}{K^{\sigma}(s)}\right)^{1-c} z^{\frac{1}{a}}(s)\, z^{\sigma}(s). \tag{2.8}$$

Integrating (2.8) from s to v and letting $v \to \infty$, we get

$$z(s) \geq \lambda_1 \int_s^\infty \frac{K^b(\Gamma(t))}{K^a(t)} \beta_2(t) \Delta t + a \int_s^\infty \beta_1^{-\frac{1}{a}}(t) \left(\frac{K(t)}{K^\sigma(t)} \right)^{1-c} z^{\frac{1}{a}}(t) z^\sigma(t) \Delta t.$$

Now,

$$0 \leq \psi := \liminf_{s\to\infty} K^a(s) z(s) \overset{(2.3)}{=} \liminf_{s\to\infty} K^a(s) \frac{\beta_1(s) \left(x^\Delta(s)\right)^a}{x^a(s)} \overset{(2.1)}{\leq} 1.$$

Then, from the definitions of R, ψ, and A_1, for any $\varepsilon \in (0,1)$, there exists an $s_2 \in [s_1, \infty)_\mathbb{T}$ such that, for $s \in [s_2, \infty)_\mathbb{T}$,

$$\frac{K(s)}{K^\sigma(s)} \geq \varepsilon R, \quad K^a(s) z(s) \geq \varepsilon \psi, \tag{2.9}$$

and

$$\beta_1^{\frac{1}{a}}(s) K^{1-a}(\Omega(s)) K^a(\sigma(s)) K^b(\Gamma(s)) \beta_2(s) \geq \varepsilon A_1. \tag{2.10}$$

Using (2.9) and (2.10) in (2.5), we have

$$\begin{aligned}
z(s) &\geq \lambda_1 \int_s^\infty \frac{K^b(\Gamma(t))}{K^a(t)} \beta_2(t) \Delta t \\
&\quad + a \int_s^\infty \left(\frac{K(t)}{K^\sigma(t)} \right)^{1-c} \frac{1}{\beta_1^{\frac{1}{a}}(t) K(t) K^{a\sigma}(t)} K(t) z^{\frac{1}{a}}(t) K^{a\sigma}(t) z^\sigma(t) \Delta t \\
&\geq \varepsilon \lambda_1 A_1 \int_s^\infty \frac{K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^{a\sigma}(t)} \Delta t \\
&\quad + a (\varepsilon\psi)^{1+\frac{1}{a}} \int_s^\infty \left(\frac{K(t)}{K^\sigma(t)} \right)^{1-c} \frac{1}{\beta_1^{\frac{1}{a}}(t) K(t) K^{a\sigma}(t)} \Delta t \\
&= \varepsilon \lambda_1 A_1 \int_s^\infty \frac{K^{a-1}(\Omega_1(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^{a\sigma}(t)} \Delta t \\
&\quad + a (\varepsilon\psi)^{1+\frac{1}{a}} \int_s^\infty \left(\frac{K(t)}{K^\sigma(t)} \right)^{|a-1|} \frac{K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^{a\sigma}(t)} \Delta t \\
&\geq \varepsilon \lambda_1 A_1 \int_s^\infty \frac{K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^{a\sigma}(t)} \Delta t \\
&\quad + (\varepsilon R)^{|a-1|} (\varepsilon\psi)^{1+\frac{1}{a}} \int_s^\infty \frac{a K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^{a\sigma}(t)} \Delta t.
\end{aligned}$$

The Pötzsche chain rule implies that

$$(K^a(t))^\Delta = a \int_0^1 [(1-h) K(t) + h K(\sigma(t))]^{a-1} \mathrm{d}h \, K^\Delta(t) \leq a \beta_1^{-\frac{1}{a}}(t) K^{a-1}(\Omega(t)). \tag{2.11}$$

Now, from (2.11) and the quotient rule for derivatives, see [1, Theorem 1.20] we have that

$$\left(\frac{-1}{K^a(t)} \right)^\Delta = \frac{(K^a(t))^\Delta}{K^a(t) K^a(\sigma(t))} \leq \frac{a K^a(\Omega(t))}{\beta_1^{\frac{1}{a}}(t) K^a(t) K^a(\sigma(t))}.$$

Therefore,

$$\begin{aligned} z(s) &\geq \frac{\varepsilon\lambda_1 A_1}{a}\int_s^\infty \left(\frac{-1}{K^a(t)}\right)^\Delta \Delta t + (\varepsilon R)^{|a-1|}(\varepsilon\psi)^{1+\frac{1}{a}}\int_s^\infty \left(\frac{-1}{K^a(t)}\right)^\Delta \Delta t \\ &= \frac{\varepsilon\lambda_1 A_1}{a}\frac{1}{K^a(s)} + (\varepsilon R)^{|a-1|}(\varepsilon\psi)^{1+\frac{1}{a}}\frac{1}{K^a(s)}, \end{aligned}$$

which implies

$$\varepsilon\lambda_1 A_1 \leq aK^a(s)\,z(s) - a(\varepsilon R)^{|a-1|}(\varepsilon\psi)^{1+\frac{1}{a}}.$$

Taking the lim inf of both sides as $s \to \infty$, we obtain

$$\varepsilon\lambda_1 A_1 \leq a\psi - a(\varepsilon R)^{|a-1|}(\varepsilon\psi)^{1+\frac{1}{a}}.$$

Since ε and λ_1 are arbitrary, we arrive at

$$A_1 \leq a\psi - aR^{|a-1|}\psi^{1+\frac{1}{a}}.$$

Let

$$Y = aR^{|a-1|}, \quad X = a, \quad \text{and} \quad U = \psi.$$

By the inequality

$$XU - YU^{1+\frac{1}{a}} \leq \frac{a^a}{(a+1)^{a+1}}\frac{X^{a+1}}{Y^a}, \quad X, Y > 0, \tag{2.12}$$

we have

$$A_1 \leq \frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1},$$

which gives the contradiction with (2.6). □

The next theorem is Kneser-type to the second-order quasilinear dynamic Eq (1.1) when $a \leq b$.

Theorem 2.3. *Let $a \leq b$ and $R := \liminf_{s\to\infty} \dfrac{K(s)}{K^\sigma(s)} > 0$. If*

$$A_2 := \liminf_{s\to\infty} \beta_1^{\frac{1}{a}}(s)\,K^{a-b}(s)\,K^{1-a}(\Omega(s))\,K^{a\sigma}(s)\,K^b(\Gamma(s))\beta_2(s) > \frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1}, \tag{2.13}$$

where Γ and Ω are defined by (1.5) *and* (2.7)*, respectively, then every solution of Eq* (1.1) *is oscillatory.*

Proof. If not, let $x(s) > 0$ and $x(\nu(s)) > 0$ for $s \in [s_0, \infty)_\mathbb{T}$. Then, Lemma 2.1 implies that there exists an $s_1 \in [s_0, \infty)_\mathbb{T}$ such that $x^\Delta(s) > 0$ and $\left(\dfrac{x}{K}\right)^\Delta(s) < 0$ for $s \in [s_1, \infty)_\mathbb{T}$. Thus,

$$\begin{aligned} \frac{x^b(\nu(s))}{x^a(s)} &= \left(\frac{x(\nu(s))}{x(s)}\right)^b x^{b-a}(s) \\ &\geq \begin{cases} \left(\dfrac{K(\nu(s))}{K(s)}\right)^b x^{b-a}(s_1), & \nu(s) \leq s, \\ x^{b-a}(s_1), & \nu(s) \geq s, \end{cases} \end{aligned}$$

$$= \lambda_2 \left(\frac{K(\Gamma(s))}{K(s)} \right)^b, \tag{2.14}$$

where $\lambda_2 := x^{b-a}(s_1)$. Substituting (2.14) into (2.2), we have

$$z^{\Delta}(s) \leq -\lambda_2 \left(\frac{K(\Gamma(s))}{K(s)} \right)^b \beta_2(s) - a\beta_1^{-\frac{1}{a}}(s) \left(\frac{K(s)}{K^{\sigma}(s)} \right)^{1-c} z^{\frac{1}{a}}(s)\, z^{\sigma}(s). \tag{2.15}$$

By integrating (2.15) from s to v and letting $v \to \infty$, we get

$$z(s) \geq \lambda_2 \int_s^{\infty} \left(\frac{K(\Gamma(t))}{K(t)} \right)^b \beta_2(t)\Delta t + a \int_s^{\infty} \frac{1}{\beta_1^{\frac{1}{a}}(t)} \left(\frac{K(t)}{K^{\sigma}(t)} \right)^{1-c} z^{\frac{1}{a}}(t)\, z(t)\, \Delta t.$$

Then, from the definitions of R, ψ, and A_2, for any $\varepsilon \in (0,1)$, there exists an $s_2 \in [s_1, \infty)_{\mathbb{T}}$ such that (2.9) holds and

$$\beta_1^{\frac{1}{a}}(s)\, K^{a-b}(s)\, K^{1-a}(\Omega(s))\, K^{a}(\sigma(s))\, K^{b}(\Gamma(s))\beta_2(s) \geq \varepsilon A_2 \tag{2.16}$$

for $s \in [s_2, \infty)_{\mathbb{T}}$. Using (2.9) and (2.16) in (2.5), we have

$$\begin{aligned}
z(s) \geq{}& \lambda_2 \int_s^{\infty} \left(\frac{K(\Gamma(t))}{K(t)} \right)^b \beta_2(t)\Delta t \\
&+ a \int_s^{\infty} \left(\frac{K(t)}{K^{\sigma}(t)} \right)^{1-c} \frac{1}{\beta_1^{\frac{1}{a}}(t)\, K(t)\, K^{a\sigma}(t)} K(t)\, z^{\frac{1}{a}}(t)\, K^{a\sigma}(t)\, z^{\sigma}(t)\, \Delta t \\
\geq{}& \varepsilon\lambda_2 A_2 \int_s^{\infty} \frac{K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t)\, K^{a}(t)\, K^{a\sigma}(t)} \Delta t \\
&+ a(\varepsilon\psi)^{1+\frac{1}{a}} \int_s^{\infty} \left(\frac{K(t)}{K^{\sigma}(t)} \right)^{1-c} \frac{1}{\beta_1^{\frac{1}{a}}(t)\, K(t)\, K^{a\sigma}(t)} \Delta t \\
={}& \varepsilon\lambda_2 A_2 \int_s^{\infty} \frac{K^{a-1}(\Omega_1(t))}{\beta_1^{\frac{1}{a}}(t)\, K^{a}(t)\, K^{a\sigma}(t)} \Delta t \\
&+ a(\varepsilon\psi)^{1+\frac{1}{a}} \int_s^{\infty} \left(\frac{K(t)}{K^{\sigma}(t)} \right)^{|a-1|} \frac{K^{a-1}(\Omega(t))}{\beta_1^{\frac{1}{a}}(t)\, K^{a}(t)\, K^{a\sigma}(t)} \Delta t.
\end{aligned}$$

The remainder of the proof follows as in the proof of Theorem 2.2 and hence is omitted. □

3. Examples

We clarify the strength of our results with the next examples.

Example 3.1. *The second-order Euler dynamic equations:*

(1) For $a \geq b$ and $a \geq 1$,

$$\left[\beta_1(s) \left| x^{\Delta}(s) \right|^{a-1} x^{\Delta}(s) \right]^{\Delta} + \frac{\alpha}{\beta_1^{\frac{1}{a}}(s) K^{\sigma}(s)\, K^{b}(s)} |x(s)|^{b-1} x(s) = 0.$$

(2) For $a \le b$ and $0 < a \le 1$,

$$\left[\beta_1(s)\left|x^{\Delta}(s)\right|^{a-1}x^{\Delta}(s)\right]^{\Delta}+\frac{\alpha}{\beta_1^{\frac{1}{a}}(s)K(s)\,K^{a\sigma}(s)}\left|x(\sigma(s))\right|^{b-1}x(\sigma(s))=0$$

are oscillatory if $R > 0$ and $\alpha > \frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1}$ by using Theorems 2.2 and 2.3, respectively. It is well known that this condition is ideal for the second-order Euler differential equations

$$\left[\beta_1(s)\left|x'(s)\right|^{a-1}x'(s)\right]'+\frac{\alpha}{\beta_1^{\frac{1}{a}}(s)K^{b+1}(s)}\left|x(s)\right|^{b-1}x(s)=0,$$

and

$$\left[\beta_1(s)\left|x'(s)\right|^{a-1}x'(s)\right]'+\frac{\alpha}{\beta_1^{\frac{1}{a}}(s)K^{a+1}(s)}\left|x(s)\right|^{b-1}x(s)=0.$$

We note that the second-order Euler differential equation

$$\left[\beta_1(s)\left|x'(s)\right|^{a-1}x'(s)\right]'+\frac{\alpha}{\beta_1^{\frac{1}{a}}(s)K^{a+1}(s)}\left|x(s)\right|^{a-1}x(s)=0 \tag{3.1}$$

has a nonoscillatory solution $x(s) = K^{\frac{a}{a+1}}(s)$ if $\alpha = \left(\frac{a}{a+1}\right)^{a+1}$. That is to say, the constant $\left(\frac{a}{a+1}\right)^{a+1}$ provides the lower bound of the oscillation for (3.1).

Example 3.2. *Consider the second-order delay dynamic equation*

$$\left[\sqrt[4]{s^3}(x^{\Delta}(s))^3\right]^{\Delta}+\frac{\alpha}{R^6\sqrt[4]{s\left(\sigma(s)\,\nu^2(s)\right)^3}}x^2(\nu(s))\operatorname{sgn}(x(\nu(s)))=0, \tag{3.2}$$

where $\alpha > 0$ is a constant, $R > 0$, and $\nu(s) \le s$ on $[s_0,\infty)_{\mathbb{T}}$. We have

$$\int_{s_0}^{\infty}\frac{\Delta r}{\beta_1^{1/a}(s)}=\int_{s_0}^{\infty}\frac{\Delta r}{\sqrt[4]{r}}=\infty$$

by [2, Example 5.60]. Also, by the Pötzsche chain rule, we obtain

$$K(s)=\int_{s_0}^{s}\frac{\Delta r}{\beta_1^{1/a}(r)}=\int_{s_0}^{s}\frac{\Delta r}{\sqrt[4]{r}}\ge\frac{4}{3}\int_{s_0}^{s}\left(\sqrt[4]{r^3}\right)^{\Delta}\Delta r=\frac{4}{3}\left(\sqrt[4]{s^3}-\sqrt[4]{s_0^3}\right),$$

and consequently,

$$\begin{aligned}&\liminf_{s\to\infty}\beta_1^{\frac{1}{a}}(s)\,K^{1-a}(\Omega(s))\,K^{a\sigma}(s)\,K^{b}(\Gamma(s))\beta_2(s)\\ =\;&\frac{\alpha}{R^6}\left(\frac{4}{3}\right)^3\liminf_{s\to\infty}\frac{\sqrt[4]{s}\left(\sqrt[4]{(\sigma(s))^3}-\sqrt[4]{s_0^3}\right)\left(\sqrt[4]{\nu^3(s)}-\sqrt[4]{s_0^3}\right)^2}{\sqrt[4]{s\left(\sigma(s)\,\nu^2(s)\right)^3}}\end{aligned}$$

$$= \frac{\alpha}{R^6}\left(\frac{4}{3}\right)^3 \liminf_{s\to\infty}\left(\left(1-\sqrt[4]{\left(\frac{s_0}{\sigma(s)}\right)^3}\right)\left(1-\sqrt[4]{\left(\frac{s_0}{\nu(s)}\right)^3}\right)^2\right)$$
$$= \frac{\alpha}{R^6}\left(\frac{4}{3}\right)^3.$$

Applying Theorem 2.2 leads to the oscillation of all solutions of Eq (3.2) *for* $\alpha > \left(\frac{3}{4}\right)^7$.

Example 3.3. *Consider the second-order advanced dynamic equation*

$$\left[\frac{1}{\sigma(s)}\frac{x^\Delta(s)}{\sqrt[4]{\left|x^\Delta(s)\right|^3}}\right]^\Delta + \alpha\frac{\sqrt[4]{(\sigma(s))^{11}}}{s^5}\sqrt[3]{x(\nu(s))} = 0, \tag{3.3}$$

where $\alpha > 0$ *is a constant and* $\nu(s) \geq s$ *on* $[s_0, \infty)_{\mathbb{T}}$. *By the Pötzsche chain rule, we obtain*

$$K(s) = \int_{s_0}^{s}\frac{\Delta r}{\beta_1^{1/a}(r)} = \int_{s_0}^{s}(\sigma(r))^4\,\Delta r \geq \frac{1}{5}\int_{s_0}^{s}\left(r^5\right)^\Delta\Delta r = \frac{1}{5}\left(s^5 - s_0^5\right),$$

and then

$$\liminf_{s\to\infty}\beta_1^{\frac{1}{a}}(s)\,K^{a-b}(s)\,K^{1-a}(\Omega(s))\,K^{a\sigma}(s)\,K^b(\Gamma(s))\beta_2(s)$$
$$\geq \alpha\sqrt[4]{\left(\frac{1}{5}\right)^5}\liminf_{s\to\infty}\left(1-\left(\frac{s_0}{s}\right)^5\right)\sqrt[4]{1-\left(\frac{s_0}{\sigma(s)}\right)^5}$$
$$= \alpha\sqrt[4]{\left(\frac{1}{5}\right)^5}.$$

Therefore, Theorem 2.3 yields that every solution of Eq (3.3) *is oscillatory if* $\alpha > \frac{1}{\sqrt[16]{R^3}}$.

4. Discussion

(1) In this paper, the findings presented are applicable across all time scales, including $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$, without any restrictive conditions.

(2) We present some sharp oscillation criteria of the Kneser-type for second-order quasilinear functional dynamic equations when $a \leq b$, $a \geq b$, $\nu(s) \leq s$, and $\nu(s) \geq s$. Our results represent an improvement over previously established Kneser-type criteria, as detailed below: if $a = b$, then criteria (2.6) and (2.13) reduce to

$$\liminf_{s\to\infty}\beta_1^{\frac{1}{a}}(s)\,K^{1-a}(\Omega(s))\,K^a(\sigma(s))\,K^a(\Gamma(s))\beta_2(s) > \frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1}.$$

By virtue of

$$\beta_1^{\frac{1}{a}}(s)\,K^{1-a}(\Omega(s))\,K^a(\sigma(s))\,K^a(\Gamma(s))\beta_2(s) \geq \beta_1^{\frac{1}{a}}(s)\,K(s)\,K^a(\Gamma(s))\beta_2(s),$$

and
$$\frac{1}{R^{a|a-1|}}\left(\frac{a}{a+1}\right)^{a+1} < \frac{1}{R^{a(a+1)}}\left(\frac{a}{a+1}\right)^{a+1} \quad \text{for } 0 < R < 1,$$
Theorems 2.2 and 2.3 improve Theorem 1.2 (criteria (2.6) and (2.13) improve (1.4)).

5. Conclusions

(1) In this paper, we investigate new Kneser-type oscillation criteria for second-order quasilinear functional dynamic equations defined on arbitrary unbounded above time scales. These results apply to all time scales and improve related results in the literature, as explained in the discussion section.

(2) Establishing Kneser-type oscillation criteria for a second-order dynamic Eq (1.1) would be interesting, assuming that
$$\int_{s_0}^{\infty} \frac{\Delta r}{\beta_1^{1/a}(r)} < \infty.$$

Author contributions

Taher S. Hassan: Supervision, writing-original draft, writing-review and editing, formal analysis, resources, investigation; Elvan Akin: Formal analysis, resources; Hasan Nihal Zaidi: Formal analysis, resources; Bassant M. El-Matary: Writing-review and editing, validation, formal analysis, resources, investigation; Ioan-Lucian Popa: Formal nalysis, resources; Mouataz Billah Mesmouli: Formal analysis, resources; Ismoil Odinaev: Formal analysis, resources; Akbar Ali: Formal analysis, resources. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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