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Research article

Nonlinear exponential stability of traveling front solutions for a reaction-diffusion system with acidic nitrate-ferroin reaction

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Abstract: This paper is concerned with the nonlinear exponential stability of traveling front solutions for a reaction-diffusion system with acidic nitrate-ferroin reaction. For diffusion coefficient δ near 1, we first establish the perturbation relationship and precise spatial decay rates of traveling front solutions. Subsequently, in some exponentially weighted spaces, we demonstrate the nonlinear exponential stability of the traveling front solutions with all noncritical speeds $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$. This improves the stability results in [1], in which the authors obtained the stability of traveling front solutions for $c > \frac{1}{\sqrt{2\beta}}$.

Keywords: traveling front solutions; spectral analysis; exponentially weighted spaces; Evans function; exponential stability
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1. Introduction and statement of main results

In this paper, we investigate the stability of the following reaction-diffusion system

$$\begin{cases} u_t = \delta u_{xx} - \frac{2uv}{\beta+u}, \\ v_t = v_{xx} + \frac{uv}{\beta+u}. \end{cases} \tag{1.1}$$

Here β is a positive constant, u and v represent the concentrations of the ferroin and acidic nitrate, respectively, and δ denotes the ratio of the diffusion rates of the ferroin and acidic nitrate. To see more background and the research results of model (1.1), one can refer to [1–7] and references therein, among which [2,3,5] and [1] are concerned with the existence and stability of traveling front solutions of (1.1), respectively.

In [3], the author gave a necessary and sufficient condition of the existence of traveling front solutions for (1.1). A traveling wave solution of system (1.1) is a solution of the form $(u(x,t), v(x,t)) = (U_c(\xi,\delta), V_c(\xi,\delta))$ with $\xi = x - ct$ connecting the stable steady state $(0, \frac{1}{2})$ at $\xi = -\infty$ to the unstable steady state $(1,0)$ at $\xi = +\infty$ (see [1–3,5]). Hence, (U_c, V_c) satisfies

$$\begin{cases} \delta U'' + cU' - \frac{2UV}{\beta+U} = 0, \\ V'' + cV' + \frac{UV}{\beta+U} = 0, \\ (U,V)(-\infty) = (0, \frac{1}{2}), \\ (U,V)(+\infty) = (1,0), \end{cases} \tag{1.2}$$

with $' = \frac{d}{d\xi}$.

The following theorem is the main result obtained in [3].

Theorem A. [3] For $\delta > 0$, system (1.1) has a unique, up to translation, traveling wave solution with speed c iff $c \geq c_{\min} = \frac{2}{\sqrt{\beta+1}}$.

In the moving coordinates (ξ, t) $(\xi = x - ct)$, the initial value problem of system (1.1) can be written as

$$\begin{cases} u_t = \delta u_{\xi\xi} + cu_\xi - \frac{2uv}{\beta+u}, \\ v_t = v_{\xi\xi} + cv_\xi + \frac{uv}{\beta+u}, \\ u|_{t=0} = u_0(\xi), \\ v|_{t=0} = v_0(\xi). \end{cases} \tag{1.3}$$

It is easy to see that the traveling front solution $(U_c(\xi,\delta), V_c(\xi,\delta))$ is the steady state of (1.3). To investigate the stability of traveling front solutions of system (1.1), it is equivalent to investigating the stability of the steady states of system (1.3).

Let $f(\xi,t) = u(\xi,t) - U_c(\xi,\delta)$, $g(\xi,t) = v(\xi,t) - V_c(\xi,\delta)$. Then, (f,g) satisfies

$$\begin{pmatrix} f \\ g \end{pmatrix}_t = \mathcal{L}_\delta \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} -2R(f,g) \\ R(f,g) \end{pmatrix}, \tag{1.4}$$

where $\mathcal{L}_\delta$ is the linearized operator around $(U_c(\xi,\delta), V_c(\xi,\delta))$ and is defined by

$$\mathcal{L}_\delta = \begin{pmatrix} \delta \frac{\partial^2}{\partial \xi^2} + c\frac{\partial}{\partial \xi} - \frac{2\beta V_c}{(\beta+U_c)^2} & -\frac{2U_c}{\beta+U_c} \\ \frac{\beta V_c}{(\beta+U_c)^2} & \frac{\partial^2}{\partial \xi^2} + c\frac{\partial}{\partial \xi} + \frac{U_c}{\beta+U_c} \end{pmatrix}, \tag{1.5}$$

and

$$R(f,g) = \frac{-\beta V_c f^2}{(\beta+U_c+f)(\beta+U_c)^2} + \frac{-U_c f g}{(\beta+U_c+f)(\beta+U_c)} + \frac{fg}{\beta+U_c+f}. \tag{1.6}$$

In [1], the authors provided a stability analysis of traveling front solutions for system (1.4) with the wave speed $c > \frac{1}{\sqrt{2\beta}}$ and δ near 1 by using energy functionals. A direct computation shows that if $\beta > \frac{1}{7}$, then the wave speed c satisfies $c \geq \frac{2}{\sqrt{\beta+1}} > \frac{1}{\sqrt{2\beta}}$. To introduce the stability result of traveling front solutions in [1], we first give the working spaces used in [1]. Define the exponential weighted spaces as

$$Y_s = \{\omega \mid \omega \in H^1(\mathbb{R}),\ \omega(\xi)\cdot(1+e^{2s\xi})^{\frac{1}{2}} \in L^2(\mathbb{R}),\ \partial_\xi\omega\cdot(1+e^{2s\xi})^{\frac{1}{2}} \in L^2(\mathbb{R})\},$$

with the norm defined by

$$\|\omega\|_{Y_s} = \left(\|\omega(\xi)\cdot(1+e^{2s\xi})^{\frac{1}{2}}\|^2_{L^2(\mathbb{R})} + \|\partial_\xi\omega(\xi)\cdot(1+e^{2s\xi})^{\frac{1}{2}}\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.$$

Let $Y_s^2 = Y_s \times Y_s$ and the norm of Y_s^2 be

$$\|(\omega_1,\omega_2)\|_{Y_s^2} = (\|\omega_1\|^2_{Y_s} + \|\omega_2\|^2_{Y_s})^{\frac{1}{2}}.$$

Now, we give the main stability results obtained in [1].

Theorem B. [1] Let $s \in \left[\frac{c-\sqrt{c^2-\frac{4}{\beta+1}}}{2}, \frac{c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\right]$. Suppose the wave speed $c > \frac{1}{\sqrt{2\beta}}$. Then, there exist positive constants κ_0 ($\kappa_0 \ll 1$), ϵ, and $K^* \geq 1$ such that for any $\delta \in [1-\kappa_0, 1+\kappa_0]$, if $\|(f_0, g_0)\|_{Y_s^2} \leq \epsilon$, then we have $\|f_0\|_{L^\infty(\mathbb{R})} < \beta/2$, and system (1.4) admits a unique global classical solution $(f(\cdot,t), g(\cdot,t)) \in C^0([0,\infty), Y_s^2) \cap C^1((0,\infty), Y_s^2)$ with initial data (f_0, g_0). Moreover,

$$\|(f(\cdot,t), g(\cdot,t))\|_{Y_s^2} \leq K^*\|(f_0,g_0)\|_{Y_s^2}, \quad \forall t > 0 \tag{1.7}$$

and

$$\lim_{t\to\infty}\sup_{\xi\in\mathbb{R}}(|f(\xi,t)| + |g(\xi,t)|)(1+e^{s\xi}) = 0, \quad \forall\ t > 0. \tag{1.8}$$

In particular, if we further assume $s \in \left(\frac{c-\sqrt{c^2-\frac{4}{\beta+1}}}{2}, \frac{c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\right)$, then we have the results of exponential decay to the wave in time: There exists a positive constant M_1 such that

$$\|e^{s\xi}f(\xi,t)\|_{H^1(\mathbb{R})} + \|e^{s\xi}g(\xi,t)\|_{H^1(\mathbb{R})} \leq M_1 e^{p(s)t/2}\|(f_0,g_0)\|_{Y_s^2}, \quad \forall\ t > 0, \tag{1.9}$$

with $p(s) = s^2 - cs + \frac{1}{\beta+1} < 0$. Furthermore, for any given $L \in \mathbb{R}$, there exist positive constants M_2 and ω such that

$$\sup_{\xi\in[L,\infty]}(|f(\xi,t)| + |g(\xi,t)|) \leq M_2 e^{-\omega t}, \quad \forall\ t > 0. \tag{1.10}$$

Remark 1.1. *In Theorem B, suppose the wave speed* $c > \frac{1}{\sqrt{2\beta}}$. *For* $|\delta - 1|$ *small, when the weight function is chosen as* $(1+e^{2s\xi})^{\frac{1}{2}}$, $s \in \left[\frac{c-\sqrt{c^2-\frac{4}{\beta+1}}}{2}, \frac{c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\right]$, *(1.7) shows the Lyapunov stability of the traveling front solutions, and (1.8) shows that the solution* $(f(\xi,t), g(\xi,t))$ *to system (1.4) decays exponentially in space as* $\xi \to +\infty$ *when* $t \to +\infty$. *Furthermore, when the weight function is chosen as* $e^{s\xi}$, $s \in \left(\frac{c-\sqrt{c^2-\frac{4}{\beta+1}}}{2}, \frac{c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\right)$, *(1.9) and (1.10) show that the solution* $(f(\xi,t), g(\xi,t))$ *to system (1.4) decays exponentially in time.*

In this paper, by using spectral analysis and the Evans function method, when the weight function is chosen as $e^{s\xi}$, $s \in \left(\frac{c-\sqrt{c^2-\frac{4}{\beta+1}}}{2}, \frac{c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\right)$, *we can prove the nonlinear exponential stability of* $(f(\xi,t), g(\xi,t))$ *to system (1.4) with* $|\delta - 1|$ *small for all noncritical speeds* $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$.

For $\delta = 1$, it is easy to check that the traveling front solution (U_c, V_c) satisfies $U_c + 2V_c = 1$ from (1.2). Substituting $U_c = 1 - 2V_c$ into the second equation of (1.2), we have

$$\begin{cases} V'' + cV' + \frac{(1-2V)V}{\beta+1-2V} = 0, \\ V(-\infty) = \frac{1}{2}, \quad V(+\infty) = 0. \end{cases} \tag{1.11}$$

If the initial value $(u_0(\xi), v_0(\xi))$ satisfies $u_0(\xi) + 2v_0(\xi) = 1$, then the solution $(u(\xi, t), v(\xi, t))$ to system (1.3) with $\delta = 1$ also satisfies

$$u(\xi, t) + 2v(\xi, t) = 1. \tag{1.12}$$

In this case, the asymptotic behavior of the solution to (1.3) with $\delta = 1$ can be described by (1.12) and the asymptotic behavior of the solution to the scalar equation

$$v_t = v_{\xi\xi} + cv_\xi + \frac{(1-2v)v}{\beta + 1 - 2v}. \tag{1.13}$$

It is easy to check that (1.13) is a Fisher type equation. The stability of the traveling front solutions of Fisher type equations with critical or noncritical speeds has been extensively studied. It was first shown in [8] that the traveling front solutions with noncritical speeds are locally exponentially stable in some exponentially weighted spaces. In [9], the asymptotic stability of the critical Fisher type front is investigated, by using pointwise estimates. There is also much literature focusing on the asymptotic behavior of solutions to the Fisher type equation under more general initial conditions, see [10–14] and the references therein.

In this paper, we will also focus on the stability of the traveling front solutions with noncritical speeds to system (1.3) with δ near 1. First, we give the perturbation relationship of $(U_c(\xi, \delta), V_c(\xi, \delta))$ and $(U_c(\xi, 1), V_c(\xi, 1))$, and the precise spatial decay rates of $(U_c(\xi, \delta), V_c(\xi, \delta))$ for $c \geq c_{\min} = \frac{2}{\sqrt{\beta+1}}$ and δ near 1. Utilizing the spectral analysis combined with Evans function method, we can obtain the nonlinear exponential stability of traveling front solutions of (1.3) for all noncritical wave speeds $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ and δ near 1. This improves the stability results of traveling front solutions obtained in [1]. The ideas used in this paper are inspired by [15], in which the authors investigated the stability of traveling front solutions with algebraic spatial decay for some autocatalytic chemical reaction systems when the two diffusion rates are close. By using the spectral analysis and Evans function method, there are some recent works [16–19] on stability of waves for reaction-diffusion systems.

Before stating the main results of this paper, we introduce the working spaces used in this paper. Let $C_{unif}(\mathbb{R})$ denote the Banach space consisting of all bounded and uniformly continuous functions with bounded $L^\infty(\mathbb{R})$ norm, and denote

$$C^2_{unif}(\mathbb{R}) = \{u(\xi) \in C^2(\mathbb{R}) \mid u(\xi), u_\xi(\xi), u_{\xi\xi}(\xi) \in C_{unif}(\mathbb{R})\}.$$

Define the exponentially weighted space $C_{\alpha,unif}(\mathbb{R})$ and its norm as

$$C_{\alpha,unif}(\mathbb{R}) = \{u \mid u(\xi)w_\alpha(\xi) \in C_{unif}(\mathbb{R})\}, \quad w_\alpha(\xi) = e^{\alpha\xi}, \quad \alpha > 0,$$

$$\|u(\xi)\|_{C_{\alpha,unif}(\mathbb{R})} = \|u(\xi)w_\alpha(\xi)\|_{L^\infty(\mathbb{R})}.$$

Now we introduce the main results of this paper.

Theorem 1.2. *For each fixed wave speed* $c \geq c_{\min} = \frac{2}{\sqrt{\beta+1}}$*, let* $(U_c(\xi,\delta), V_c(\xi,\delta))$ *be the traveling front solutions of (1.1) stated in Theorem A. Then, there exists a family of wave fronts, still denoted by* $(U_c(\xi,\delta), V_c(\xi,\delta))$*,* $\xi = x - ct$*, satisfying* $V_c(0,\delta) = \frac{1}{4}$*, and* $(U_c(\xi,\delta), V_c(\xi,\delta))$ *are continuous in* δ *for* $\xi \in \mathbb{R}$ *and* δ *near* 1*; i.e.,*

$$\|U_c(\xi,\delta) - U_c(\xi,1)\|_{C_{unif}(\mathbb{R})} + \|V_c(\xi,\delta) - V_c(\xi,1)\|_{C_{unif}(\mathbb{R})} \to 0, \quad as \ \ \delta \to 1. \tag{1.14}$$

Furthermore, for small $|\delta - 1|$*,* $(U_c(\xi,\delta), V_c(\xi,\delta))$ *has the following asymptotic properties:*

(i) For $c \geq \frac{2}{\sqrt{\beta+1}}$*,*

$$\begin{pmatrix} U_c(\xi,\delta) \\ \frac{1}{2} - V_c(\xi,\delta) \end{pmatrix} = \begin{pmatrix} A_{1c} \\ A_{2c} \end{pmatrix} e^{\frac{-c+\sqrt{c^2+\frac{4\delta}{\beta}}}{2\delta}\xi} + o(e^{\frac{-c+\sqrt{c^2+\frac{4\delta}{\beta}}}{2\delta}\xi}), \tag{1.15}$$

as $\xi \to -\infty$*, where* A_{1c} *and* A_{2c} *are positive constants.*

(ii) For $c > \frac{2}{\sqrt{\beta+1}}$*,*

$$\begin{pmatrix} 1 - U_c(\xi,\delta) \\ V_c(\xi,\delta) \end{pmatrix} = \begin{pmatrix} \bar{A}_{1c} \\ \bar{A}_{2c} \end{pmatrix} e^{\frac{-c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\xi} + o(e^{\frac{-c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\xi}), \tag{1.16}$$

as $\xi \to +\infty$*, where* $\bar{A}_{1c}$ *and* $\bar{A}_{2c}$ *are positive constants.*

(iii) For $c = \frac{2}{\sqrt{\beta+1}}$*,*

$$\begin{pmatrix} 1 - U_c(\xi,\delta) \\ V_c(\xi,\delta) \end{pmatrix} = \begin{pmatrix} \tilde{A}_{1c} + \tilde{B}_{1c}\xi \\ \tilde{A}_{2c} + \tilde{B}_{2c}\xi \end{pmatrix} e^{-\frac{c}{2}\xi} + o(e^{-\frac{c}{2}\xi}), \tag{1.17}$$

as $\xi \to +\infty$*, where* $\tilde{A}_{ic} \in \mathbb{R}$ *and* $\tilde{B}_{ic}$ *are positive constants,* $i = 1, 2$*.*

Theorem 1.3. (Nonlinear exponential stability of the traveling front solutions) *Let* $(U_c(\xi,\delta), V_c(\xi,\delta))$ *be the traveling front solutions of (1.1) stated in Theorem A. For small* $|\delta - 1|$*, each fixed* $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$*, and* $\alpha > 0$ *satisfying (2.11), if for sufficiently small* $\varepsilon_1 > 0$*,*

$$\|u_0(\xi) - U_c(\xi,\delta)\|_{C_{\alpha,unif}(\mathbb{R})} + \|v_0(\xi) - V_c(\xi,\delta)\|_{C_{\alpha,unif}(\mathbb{R})} \leq \varepsilon_1,$$

then the solution $(u(\xi,t), v(\xi,t))$ *to system (1.3) with initial value* $(u_0(\xi), v_0(\xi))$ *exists globally and satisfies*

$$\|u(\xi,t) - U_c(\xi,\delta)\|_{C_{\alpha,unif}(\mathbb{R})} + \|v(\xi,t) - V_c(\xi,\delta)\|_{C_{\alpha,unif}(\mathbb{R})} \leq Ce^{-kt}, \quad \forall\ t \geq 0,$$

where $k > 0$ *is independent of* δ *and* $C > 0$ *is independent of* t *and* $(u_0(\xi), v_0(\xi))$*.*

Remark 1.4. *For* $\delta = 1$ *and the initial value* $(u_0(\xi), v_0(\xi))$ *satisfying* $u_0(\xi) + 2v_0(\xi) = 1$ *in (1.3), the authors in [9] proved the nonlinear asymptotic stability of the critical front* $(U_{c_{\min}}(\xi,1), V_{c_{\min}}(\xi,1))$ *with* $c_{\min} = \frac{2}{\sqrt{\beta+1}}$*, that is, perturbations of the critical front decay algebraically with rate* $t^{-\frac{3}{2}}$ *in a weighted* L^∞ *space.*

This paper is organized as follows: Section 2 is the location of essential spectrum and the uniform boundedness of unstable isolated eigenvalues of the linearized operator for $|\ \delta - 1\ |$ small. The proofs of Theorems 1.2 and 1.3 are given in Section 3. Finally, Section 4 is conclusions and future research.

2. Spectral analysis of the linearized operator for $|\,\delta-1\,|$ small

2.1. The location of essential spectrum of the linearized operator for $|\,\delta-1\,|$ small

Define $\mathbf{X}=C^2_{unif}(\mathbb{R})\times C^2_{unif}(\mathbb{R})$ and $\mathbf{Y}=C_{unif}(\mathbb{R})\times C_{unif}(\mathbb{R})$. Consider the linearized system of (1.3) around $(U_c(\xi,\delta),V_c(\xi,\delta))$, i.e.,

$$\begin{pmatrix} f \\ g \end{pmatrix}_t=\mathcal{L}_\delta\begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.1}$$

with $\mathcal{L}_\delta:\mathbf{X}\to\mathbf{Y}$ defined by (1.5).

The asymptotic operators of $\mathcal{L}_\delta$ at $\xi=\pm\infty$ are

$$\mathcal{L}^+_\delta=\begin{pmatrix} \delta\frac{\partial^2}{\partial\xi^2}+c\frac{\partial}{\partial\xi} & -\frac{2}{\beta+1} \\ 0 & \frac{\partial^2}{\partial\xi^2}+c\frac{\partial}{\partial\xi}+\frac{1}{\beta+1}\end{pmatrix},$$

and

$$\mathcal{L}^-_\delta=\begin{pmatrix} \delta\frac{\partial^2}{\partial\xi^2}+c\frac{\partial}{\partial\xi}-\frac{1}{\beta} & 0 \\ \frac{1}{2\beta} & \frac{\partial^2}{\partial\xi^2}+c\frac{\partial}{\partial\xi}\end{pmatrix}.$$

Let

$$A^+_\delta(\tau)=\begin{pmatrix} -\delta\tau^2+ci\tau & -\frac{2}{\beta+1} \\ 0 & -\tau^2+ci\tau+\frac{1}{\beta+1}\end{pmatrix},\quad \text{for}\ \ \tau\in\mathbb{R},$$

and

$$A^-_\delta(\tau)=\begin{pmatrix} -\delta\tau^2+ci\tau-\frac{1}{\beta} & 0 \\ \frac{1}{2\beta} & -\tau^2+ci\tau\end{pmatrix},\quad \text{for}\ \ \tau\in\mathbb{R}.$$

Define the curves $S^\pm_\delta$ by

$$\begin{aligned} S^+_\delta &= \{\lambda\in\mathbb{C}\mid \det(\lambda\mathrm{I}-A^+_\delta(\tau))=0,\ \ \text{for some}\ \ \tau\in\mathbb{R}\} \\ &= \{\lambda\in\mathbb{C}\,\big|\lambda=-\delta\tau^2+ci\tau \text{ or } \lambda=-\tau^2+ci\tau+\tfrac{1}{\beta+1},\ \ \text{for some}\ \ \tau\in\mathbb{R}\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} S^-_\delta &= \{\lambda\in\mathbb{C}\mid \det(\lambda\mathrm{I}-A^-_\delta(\tau))=0,\ \ \text{for some}\ \ \tau\in\mathbb{R}\} \\ &= \{\lambda\in\mathbb{C}\,\big|\lambda=-\delta\tau^2+ci\tau-\tfrac{1}{\beta} \text{ or } \lambda=-\tau^2+ci\tau,\ \ \text{for some}\ \ \tau\in\mathbb{R}\}. \end{aligned} \tag{2.3}$$

Denote $\sigma_n(\mathcal{L}_\delta)$ as the set of all the isolated eigenvalues of $\mathcal{L}_\delta$ with finite algebraic multiplicities, and define the *essential spectrum* of $\mathcal{L}_\delta$ as $\sigma_{\rm ess}(\mathcal{L}_\delta)=\sigma(\mathcal{L}_\delta)\setminus\sigma_n(\mathcal{L}_\delta)$, where $\sigma(\mathcal{L}_\delta)$ denotes the spectral set of $\mathcal{L}_\delta$. By applying classical spectral theory (see [20]), the boundary of the essential spectrum of $\mathcal{L}_\delta$ is characterized by the curves $S^\pm_\delta$.

From (2.2) and (2.3), it is easy to get

$$\sup\{\mathrm{Re}\,S^+_\delta\}=\frac{1}{\beta+1}>0\ \ \text{and}\ \ \sup\{\mathrm{Re}\,S^-_\delta\}=0,$$

thus we have

$$\sigma_{\rm ess}(\mathcal{L}_\delta)\cap\{\lambda\mid \mathrm{Re}\,\lambda>0\}\neq\emptyset,$$

which implies that there is no spectral stability of traveling front solutions in $\mathbf{Y} = C_{unif}(\mathbb{R}) \times C_{unif}(\mathbb{R})$. In the following, we shall investigate the spectral stability and the exponential stability of the traveling front solutions in some exponentially weighted spaces.

Choose the exponential weight function $w_\alpha(\xi) = e^{\alpha\xi}$ with $\alpha > 0$ to be determined later, and define

$$\mathbf{Y}_\alpha = C_{\alpha,unif}(\mathbb{R}) \times C_{\alpha,unif}(\mathbb{R}) \triangleq \{(\phi,\psi) \in \mathbf{Y} \mid (w_\alpha\phi, w_\alpha\psi) \in C_{unif}(\mathbb{R}) \times C_{unif}(\mathbb{R})\},$$

$\mathbf{X}_\alpha = C^2_{\alpha,unif}(\mathbb{R}) \times C^2_{\alpha,unif}(\mathbb{R}) \triangleq \{(\phi,\psi) \in \mathbf{X} \mid (w_\alpha\phi, w_\alpha\psi) \in C^2_{unif}(\mathbb{R}) \times C^2_{unif}(\mathbb{R})\}$.

Let $\mathcal{L}_{\delta,\alpha}: \ \mathbf{X}_\alpha \to \mathbf{Y}_\alpha$ be the operator of $\mathcal{L}_\delta$ restricted in $\mathbf{X}_\alpha$, i.e.,

$$\mathcal{L}_{\delta,\alpha}\begin{pmatrix} f(\xi) \\ g(\xi) \end{pmatrix} = \mathcal{L}_\delta \begin{pmatrix} f(\xi) \\ g(\xi) \end{pmatrix}, \quad \text{for } (f,g) \in \mathbf{X}_\alpha. \tag{2.4}$$

For each fixed $\alpha > 0$, we define the operator $\tilde{\mathcal{L}}_{\delta,\alpha}: \ \mathbf{X} \to \mathbf{Y}$ by

$$\tilde{\mathcal{L}}_{\delta,\alpha}\begin{pmatrix} f(\xi) \\ g(\xi) \end{pmatrix} = w_\alpha(\xi)\,\mathcal{L}_\delta \begin{pmatrix} w_\alpha^{-1}(\xi)f(\xi) \\ w_\alpha^{-1}(\xi)g(\xi) \end{pmatrix}, \quad \text{for } (f,g) \in \mathbf{X}.$$

Obviously, $\mathcal{L}_{\delta,\alpha}$ defined in (2.4) is equivalent to $\tilde{\mathcal{L}}_{\delta,\alpha}$ in the sense that $\sigma(\mathcal{L}_{\delta,\alpha}) = \sigma(\tilde{\mathcal{L}}_{\delta,\alpha})$ and $\sigma_{\text{ess}}(\mathcal{L}_{\delta,\alpha}) = \sigma_{\text{ess}}(\tilde{\mathcal{L}}_{\delta,\alpha})$.

Consider the asymptotic operator of $\tilde{\mathcal{L}}_{\delta,\alpha}$ at $\xi = \pm\infty$, which is denoted by $\tilde{\mathcal{L}}^{\pm}_{\delta,\alpha}$ as follows:

$$\tilde{\mathcal{L}}^{+}_{\delta,\alpha} = \begin{pmatrix} \delta(\frac{\partial}{\partial\xi} - \alpha)^2 + c(\frac{\partial}{\partial\xi} - \alpha) & -\frac{2}{\beta+1} \\ 0 & (\frac{\partial}{\partial\xi} - \alpha)^2 + c(\frac{\partial}{\partial\xi} - \alpha) + \frac{1}{\beta+1} \end{pmatrix},$$

and

$$\tilde{\mathcal{L}}^{-}_{\delta,\alpha} = \begin{pmatrix} \delta(\frac{\partial}{\partial\xi} - \alpha)^2 + c(\frac{\partial}{\partial\xi} - \alpha) - \frac{1}{\beta} & 0 \\ \frac{1}{2\beta} & (\frac{\partial}{\partial\xi} - \alpha)^2 + c(\frac{\partial}{\partial\xi} - \alpha) \end{pmatrix}.$$

Let

$$A^{+}_{\delta,\alpha}(\tau) = \begin{pmatrix} \delta(\mathrm{i}\tau - \alpha)^2 + c(\mathrm{i}\tau - \alpha) & -\frac{2}{\beta+1} \\ 0 & (\mathrm{i}\tau - \alpha)^2 + c(\mathrm{i}\tau - \alpha) + \frac{1}{\beta+1} \end{pmatrix}, \quad \text{for } \tau \in \mathbb{R},$$

and

$$A^{-}_{\delta,\alpha}(\tau) = \begin{pmatrix} \delta(\mathrm{i}\tau - \alpha)^2 + c(\mathrm{i}\tau - \alpha) - \frac{1}{\beta} & 0 \\ \frac{1}{2\beta} & (\mathrm{i}\tau - \alpha)^2 + c(\mathrm{i}\tau - \alpha) \end{pmatrix}, \quad \text{for } \tau \in \mathbb{R}.$$

Define the curves $S^{\pm}_{\delta,\alpha}$ by

$$\begin{aligned} S^{+}_{\delta,\alpha} &= \{\lambda \in \mathbb{C} \mid \det(\lambda \mathrm{I} - A^{+}_{\delta,\alpha}(\tau)) = 0, \ \text{for some } \tau \in \mathbb{R}\} \\ &= \{\lambda \in \mathbb{C} \,\big|\, \lambda = -\delta\tau^2 + c\tau\mathrm{i} - 2\alpha\delta\tau\mathrm{i} + \delta\alpha^2 - c\alpha, \ \text{or} \\ &\qquad \lambda = -\tau^2 + c\tau\mathrm{i} - 2\alpha\tau\mathrm{i} + \alpha^2 - c\alpha + \tfrac{1}{\beta+1}, \ \text{for some } \tau \in \mathbb{R}\}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} S^{-}_{\delta,\alpha} &= \{\lambda \in \mathbb{C} \mid \det(\lambda \mathrm{I} - A^{-}_{\delta,\alpha}(\tau)) = 0, \ \text{for some } \tau \in \mathbb{R}\} \\ &= \{\lambda \in \mathbb{C} \,\big|\, \lambda = -\delta\tau^2 + c\tau\mathrm{i} - 2\alpha\delta\tau\mathrm{i} + \delta\alpha^2 - c\alpha - \tfrac{1}{\beta}, \ \text{or} \\ &\qquad \lambda = -\tau^2 + c\tau\mathrm{i} - 2\alpha\tau\mathrm{i} + \alpha^2 - c\alpha, \ \text{for some } \tau \in \mathbb{R}\}. \end{aligned} \tag{2.6}$$

From (2.5), for $|\delta-1|$ small, if $\delta\alpha^2 - c\alpha < 0$ and $\alpha^2 - c\alpha + \frac{1}{\beta+1} < 0$, i.e., if α satisfies

$$0 < \alpha < \frac{c}{\delta} \quad \text{and} \quad \frac{c - \sqrt{c^2 - \frac{4}{\beta+1}}}{2} < \alpha < \frac{c + \sqrt{c^2 - \frac{4}{\beta+1}}}{2}, \tag{2.7}$$

then for each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ there exists $\bar{\delta}_\alpha > 0$ such that

$$S^+_{\delta,\alpha} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le -\bar{\delta}_\alpha < 0\}. \tag{2.8}$$

From (2.6), for $|\delta-1|$ small, if $\delta\alpha^2 - c\alpha - \frac{1}{\beta} < 0$ and $\alpha^2 - c\alpha < 0$, i.e., if α satisfies

$$0 < \alpha < c, \tag{2.9}$$

then for each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ there exists $\tilde{\delta}_\alpha > 0$ such that

$$S^-_{\delta,\alpha} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le -\tilde{\delta}_\alpha < 0\}. \tag{2.10}$$

By applying classical spectral theory (see [20]), the boundaries of $\sigma_{\text{ess}}(\mathcal{L}_{\delta,\alpha})$ and $\sigma_{\text{ess}}(\tilde{\mathcal{L}}_{\delta,\alpha})$ can be described by the curves $S^\pm_{\delta,\alpha}$, thus by (2.7)–(2.10), we have the following lemma.

Lemma 2.1. *For $|\delta-1|$ small, each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$, and $\alpha > 0$ satisfying the condition*

$$\frac{c - \sqrt{c^2 - \frac{4}{\beta+1}}}{2} < \alpha < \frac{c + \sqrt{c^2 - \frac{4}{\beta+1}}}{2}, \tag{2.11}$$

there exists $\delta_\alpha = \min\{\bar{\delta}_\alpha, \tilde{\delta}_\alpha\} > 0$ such that

$$\sup \operatorname{Re}\{\sigma_{\text{ess}}(\mathcal{L}_{\delta,\alpha})\} \le -\delta_\alpha < 0.$$

2.2. *The uniform boundedness of unstable isolated eigenvalues of the linearized operator for $|\delta-1|$ small*

Let $Y = (f, f_\xi, g, g_\xi)^{\mathrm{T}}$. For each $\lambda \in \mathbb{C}$, $\mathcal{L}_\delta \begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix}$ with $\mathcal{L}_\delta$ defined by (1.5) can be rewritten as the differential equations

$$Y' = B(\xi, \lambda, \delta)Y, \tag{2.12}$$

with the symbol $' = \frac{d}{d\xi}$ and

$$B(\xi,\lambda,\delta) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\delta}[\lambda + \frac{2\beta V_c}{(\beta+U_c)^2}] & -\frac{c}{\delta} & \frac{1}{\delta}\frac{2U_c}{\beta+U_c} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta V_c}{(\beta+U_c)^2} & 0 & \lambda - \frac{U_c}{\beta+U_c} & -c \end{pmatrix}.$$

Denote $B^{\pm}(\lambda,\delta) = \lim_{\xi\to\pm\infty} B(\xi,\lambda,\delta)$. Then,

$$B^{+}(\lambda,\delta)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\lambda}{\delta} & -\frac{c}{\delta} & \frac{1}{\delta}\frac{2}{\beta+1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda-\frac{1}{\beta+1} & -c \end{pmatrix},\quad B^{-}(\lambda,\delta)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\delta}(\lambda+\frac{1}{\beta}) & -\frac{c}{\delta} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{2\beta} & 0 & \lambda & -c \end{pmatrix}.$$

The characteristic equations of $B^{\pm}(\lambda,\delta)$ satisfy

$$(\sigma^2+\frac{c}{\delta}\sigma-\frac{\lambda}{\delta})(\sigma^2+c\sigma+\frac{1}{\beta+1}-\lambda)=0,\quad \text{at } \xi=+\infty,$$

$$[\sigma^2+\frac{c}{\delta}\sigma-\frac{1}{\delta}(\lambda+\frac{1}{\beta})](\sigma^2+c\sigma-\lambda)=0,\quad \text{at } \xi=-\infty.$$

Denote the eigenvalues of $B^{\pm}(\lambda,\delta)$ by $\sigma_i^{\pm}(\lambda,\delta)$, $i=1,2,3,4$, which are given as follows:

$$\begin{aligned} \sigma_2^{+}(\lambda,\delta) &= \frac{-c+\sqrt{c^2-\frac{4}{\beta+1}+4\lambda}}{2}, \sigma_3^{+}(\lambda,\delta)=\frac{-c-\sqrt{c^2-\frac{4}{\beta+1}+4\lambda}}{2}, \\ \sigma_1^{+}(\lambda,\delta) &= \frac{-c+\sqrt{c^2+4\lambda\delta}}{2\delta}, \qquad \sigma_4^{+}(\lambda,\delta)=\frac{-c-\sqrt{c^2+4\lambda\delta}}{2\delta}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \sigma_2^{-}(\lambda,\delta) &= \frac{-c+\sqrt{c^2+4\lambda}}{2}, \qquad \sigma_3^{-}(\lambda,\delta)=\frac{-c-\sqrt{c^2+4\lambda}}{2}, \\ \sigma_1^{-}(\lambda,\delta) &= \frac{-c+\sqrt{c^2+\frac{4\delta}{\beta}+4\delta\lambda}}{2\delta}, \quad \sigma_4^{-}(\lambda,\delta)=\frac{-c-\sqrt{c^2+\frac{4\delta}{\beta}+4\delta\lambda}}{2\delta}. \end{aligned} \tag{2.14}$$

For small $|\delta-1|$, each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$, and $\alpha > 0$ satisfying (2.11), there exists small $\beta_\alpha \in (0,\delta_\alpha)$ independent of δ such that the following spectral gap condition holds:

$$\begin{cases} \operatorname{Re}\sigma_4^{+}(\lambda,\delta) < \operatorname{Re}\sigma_3^{+}(\lambda,\delta) < -\alpha < \operatorname{Re}\sigma_2^{+}(\lambda,\delta) < \operatorname{Re}\sigma_1^{+}(\lambda,\delta), \\ \operatorname{Re}\sigma_4^{-}(\lambda,\delta) < \operatorname{Re}\sigma_3^{-}(\lambda,\delta) < -\alpha < \operatorname{Re}\sigma_2^{-}(\lambda,\delta) < \operatorname{Re}\sigma_1^{-}(\lambda,\delta), \end{cases} \quad \text{for } \operatorname{Re}\lambda \geq -\beta_\alpha. \tag{2.15}$$

Theorem 2.2. *For small $|\delta-1|$, each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$, and $\alpha > 0$ satisfying (2.11), there exists small $\beta_\alpha \in (0,\delta_\alpha)$ independent of δ such that $\mathcal{L}_{\delta,\alpha}$ has no eigenvalues in $\{\lambda\in\mathbb{C} \mid \operatorname{Re}\lambda \geq -\beta_\alpha$ and $\operatorname{Re}\lambda+|\operatorname{Im}\lambda|\geq M\}$, where $M=\frac{c^2}{4}+\frac{1}{\beta}+\frac{1}{2}\max\{\frac{1}{2\beta},\frac{2}{\beta}\}+1$ and M is independent of δ.*

Proof. Consider the eigenvalue problem (2.1), i.e.,

$$\begin{cases} \delta f_{\xi\xi}+cf_\xi-\frac{2\beta V_c}{(\beta+U_c)^2}f-\frac{2U_c}{\beta+U_c}g=\lambda f, \\ g_{\xi\xi}+cg_\xi+\frac{\beta V_c}{(\beta+U_c)^2}f+\frac{U_c}{\beta+U_c}g=\lambda g. \end{cases} \tag{2.16}$$

If λ is an eigenvalue of $\mathcal{L}_{\delta,\alpha}$, then the corresponding eigenfunction $(f(\xi), g(\xi)) \in C^2_{\alpha,unif}(\mathbb{R}) \times C^2_{\alpha,unif}(\mathbb{R})$. In fact, for small $|\delta - 1|$, by Lemma 2.1, (2.15), and (2.16), it is easy to check that $(f(\xi), g(\xi))$ tends to zero exponentially at both ends for $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \geq -\beta_\alpha\}$. Therefore, $(f(\xi), g(\xi)) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$.

Multiplying the first equality of (2.16) by $\bar{f}$ and the second equality of (2.16) by $\bar{g}$, then integrating over $\mathbb{R}$, we have

$$\begin{aligned} &\lambda \int_{\mathbb{R}} (|f|^2 + |g|^2) d\xi + \delta \int_{\mathbb{R}} |f_\xi|^2 d\xi + \int_{\mathbb{R}} |g_\xi|^2 d\xi + \int_{\mathbb{R}} \frac{2\beta V_c}{(\beta + U_c)^2} |f|^2 d\xi \\ = \ &c \int_{\mathbb{R}} (f_\xi \bar{f} + g_\xi \bar{g}) d\xi + \int_{\mathbb{R}} \frac{U_c}{\beta + U_c} |g|^2 d\xi + \int_{\mathbb{R}} \left(\frac{-2U_c}{\beta + U_c} g\bar{f} + \frac{\beta V_c}{(\beta + U_c)^2} f\bar{g} \right) d\xi. \end{aligned} \tag{2.17}$$

It follows from (2.17) that

$$\begin{aligned} &(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)(\|f\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}) + \delta \|f_\xi\|^2_{L^2(\mathbb{R})} + \|g_\xi\|^2_{L^2(\mathbb{R})} \\ \leq \ &c(\|f\|_{L^2(\mathbb{R})} \|f_\xi\|_{L^2(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})} \|g_\xi\|_{L^2(\mathbb{R})}) \\ &+ \frac{1}{\beta} \|g\|^2_{L^2(\mathbb{R})} + \frac{1}{2} \max\{\frac{1}{2\beta}, \frac{2}{\beta}\} (\|f\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}). \end{aligned} \tag{2.18}$$

By using the two inequalities

$$\|f\|_{L^2(\mathbb{R})} \|f_\xi\|_{L^2(\mathbb{R})} \leq \frac{\delta}{c} \|f_\xi\|^2_{L^2(\mathbb{R})} + \frac{c}{4\delta} \|f\|^2_{L^2(\mathbb{R})},$$

$$\|g\|_{L^2(\mathbb{R})} \|g_\xi\|_{L^2(\mathbb{R})} \leq \frac{1}{c} \|g_\xi\|^2_{L^2(\mathbb{R})} + \frac{c}{4} \|g\|^2_{L^2(\mathbb{R})},$$

it can be deduced from (2.18) that

$$\begin{aligned} &(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|)(\|f\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}) \\ \leq \ &\left(\frac{c^2}{4\delta} + \frac{1}{2} \max\{\frac{1}{2\beta}, \frac{2}{\beta}\} \right) \|f\|^2_{L^2(\mathbb{R})} + \left(\frac{c^2}{4} + \frac{1}{\beta} + \frac{1}{2} \max\{\frac{1}{2\beta}, \frac{2}{\beta}\} \right) \|g\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Since $|\delta - 1|$ is small enough, then

$$\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda| < M,$$

where $M = \frac{c^2}{4} + \frac{1}{\beta} + \frac{1}{2} \max\{\frac{1}{2\beta}, \frac{2}{\beta}\} + 1$ and M is independent of δ.

This completes the proof of Theorem 2.2. Theorem 2.2 implies that the unstable eigenvalues of the linearized operator $\mathcal{L}_{\delta,\alpha}$ are uniformly bounded.

3. The proofs of Theorems 1.2 and 1.3

3.1. The proof of Theorem 1.2

Proof of Theorem 1.2. By a similar argument as in the proof of Theorem 1.4 in [15], we can obtain that $(U_c(\xi, \delta), V_c(\xi, \delta))$ is the small perturbation of $(U_c(\xi, 1), V_c(\xi, 1))$ as $\delta \to 1$, i.e., (1.14) holds.

To give the spatial decay rates of the traveling front solutions $(U_c(\xi,\delta), V_c(\xi,\delta))$ of (1.2) with small $|\delta-1|$, we first show the spatial decay rates of $(U_c(\xi,1), V_c(\xi,1))$ of (1.2) with $\delta=1$. Note that when $\delta=1$ in (1.2), we have $U_c+2V_c=1$, and then $V_c(\xi,1)$ satisfies (1.11). By using Theorem 1.2.15 in [21], we can get the spatial decay rates of $V_c(\xi,1)$ as

$$\frac{1}{2}-V_c(\xi,1)\sim\exp\Big(\frac{-c+\sqrt{c^2+\frac{4}{\beta}}}{2}\xi\Big),\quad \text{as}\ \ \xi\to-\infty,\ \ \text{if } c\geq c_{\min}=\frac{2}{\sqrt{\beta+1}}, \tag{3.1}$$

$$V_c(\xi,1)\sim\exp\Big(\frac{-c+\sqrt{c^2-\frac{4}{\beta+1}}}{2}\xi\Big),\quad \text{as}\ \ \xi\to+\infty,\ \ \text{if } c> c_{\min}=\frac{2}{\sqrt{\beta+1}}, \tag{3.2}$$

$$V_c(\xi,1)\sim(A+B\xi)e^{-\frac{c}{2}\xi},\quad \text{as}\ \ \xi\to+\infty,\ \ \text{if } c= c_{\min}=\frac{2}{\sqrt{\beta+1}}, \tag{3.3}$$

with $A\in\mathbb{R}$ and $B>0$. $U_c(\xi,1)$ has the same spatial decay rates as $V_c(\xi,1)$.

Now, linearizing (1.2) at $(0,\frac{1}{2})$ and $(1,0)$, respectively, we can get the corresponding eigenvalues, which in fact are (2.13)–(2.14) with $\lambda=0$. By (3.1)–(3.3) and (1.14), we can obtain that (1.15)–(1.17) hold.

3.2. Definition and properties of Evans function

From Lemma 2.1 and Theorem 2.2, we see that for each fixed $c>c_{\min}=\frac{2}{\sqrt{\beta+1}}$ and $\alpha>0$ satisfying (2.11), there exists small $\varepsilon_0>0$ such that

$$\sup \mathrm{Re}\,\{\sigma_{\mathrm{ess}}(\mathcal{L}_{\delta,\alpha})\}\leq-\delta_\alpha<0,\quad \text{for } |\delta-1|\leq\varepsilon_0, \tag{3.4}$$

$$\sigma_n(\mathcal{L}_{\delta,\alpha})\subset Q,\quad \text{for } |\delta-1|\leq\varepsilon_0, \tag{3.5}$$

where

$$Q=\{\lambda\in\mathbb{C}\mid \mathrm{Re}\,\lambda>-\beta_\alpha\}\cap\{\lambda\in\mathbb{C}\mid \mathrm{Re}\,\lambda+|\mathrm{Im}\,\lambda|<M\},$$

$\beta_\alpha\in(0,\delta_\alpha)$ and β_α is independent of δ. Denote $\bar{Q}$ as the closure of Q. Now, we need to investigate the location of eigenvalues of $\mathcal{L}_{\delta,\alpha}$ in $\bar{Q}$. Here, the Evans function method is utilized. From the properties of the Evans function $D(\lambda,\tau)$ in Lemma 3.1, if we can get the location of zeros of the Evans function $D(\lambda,\tau)$ in $\bar{Q}$, then the location of the eigenvalues of $\mathcal{L}_{\delta,\alpha}$ in $\bar{Q}$ is known.

By applying standard asymptotic theory of ODEs to system (2.12), it follows from (2.15) that there exist two linearly independent solutions $Y_1^-(\xi,\lambda,\delta)$, $Y_2^-(\xi,\lambda,\delta)$ to system (2.12) which are analytic in λ for $\mathrm{Re}\lambda\geq-\beta_\alpha$ and satisfy

$$Y_i^-(\xi,\lambda,\delta)e^{-\sigma_i^-(\lambda,\delta)\xi}\to V_i^-(\lambda,\delta),\quad i=1,2,\quad \text{as}\ \ \xi\to-\infty,$$

and there also exist two linearly independent solutions $Y_3^+(\xi,\lambda,\delta)$, $Y_4^+(\xi,\lambda,\delta)$ to system (2.12), which are analytic in λ for $\mathrm{Re}\lambda\geq-\beta_\alpha$ and satisfy

$$Y_i^+(\xi,\lambda,\delta)e^{-\sigma_i^+(\lambda,\delta)\xi}\to V_i^+(\lambda,\delta),\quad i=3,4,\quad \text{as}\ \ \xi\to+\infty.$$

Here,

$$V_1^-(\lambda,\delta)=\Big(1,\sigma_1^-,\frac{1}{2\beta}\frac{1}{\lambda-c\sigma_1^- -(\sigma_1^-)^2},\frac{1}{2\beta}\frac{\sigma_1^-}{\lambda-c\sigma_1^- -(\sigma_1^-)^2}\Big),$$

$$V_3^+(\lambda,\delta)=\Big(\frac{2}{\delta(\beta+1)}\frac{1}{(\sigma_3^+)^2+\frac{c}{\delta}\sigma_3^+-\frac{\lambda}{\delta}},\frac{2}{\delta(\beta+1)}\frac{\sigma_3^+}{(\sigma_3^+)^2+\frac{c}{\delta}\sigma_3^+-\frac{\lambda}{\delta}},1,\sigma_3^+\Big),$$

and

$$V_2^-(\lambda,\delta)=(0,0,1,\sigma_2^-),\quad V_4^+(\lambda,\delta)=(1,\sigma_4^+,0,0).$$

Let $\mathbb{U}^-(\xi,\lambda,\delta)$ represent the 2-dimensional unstable manifold spanned by $Y_1^-(\xi,\lambda,\delta)$ and $Y_2^-(\xi,\lambda,\delta)$. Let $\mathbb{S}^+(\xi,\lambda,\delta)$ represent the 2-dimensional stable manifold spanned by $Y_3^+(\xi,\lambda,\delta)$ and $Y_4^+(\xi,\lambda,\delta)$. Obviously, $\lambda\in\bar{Q}$ is an eigenvalue of $\mathcal{L}_{\delta,\alpha}$ if and only if $\mathbb{S}^+(\xi,\lambda,\delta)\cap\mathbb{U}^-(\xi,\lambda,\delta)$ has a nonzero intersection. For $\mathrm{Re}\lambda\geq-\beta_\alpha$, we define the Evans function of $\mathcal{L}_{\delta,\alpha}$ by

$$D(\lambda,\delta)=e^{-\int_0^\xi \mathrm{Tr}B(s,\lambda,\delta)ds}\det\big(Y_1^-(\xi,\lambda,\delta),\ Y_2^-(\xi,\lambda,\delta),\ Y_3^+(\xi,\lambda,\delta),\ Y_4^+(\xi,\lambda,\delta)\big).$$

Applying the abstract results in [22, 23], we can get some properties of Evans function, which can be stated as follows.

Lemma 3.1. *For each $\lambda\in\bar{Q}$, $D(\lambda,\delta)$ has the following properties:*
(i) *$D(\lambda,\delta)$ is independent of ξ and analytic in λ;*
(ii) *λ is an eigenvalue of $\mathcal{L}_{\delta,\alpha}$ if and only if $D(\lambda,\delta)=0$;*
(iii) *the number of the zeros (counting the algebraic multiplicities) of $D(\lambda,\delta)$ in $\bar{Q}$ is equal to the number of eigenvalues of $\mathcal{L}_{\delta,\alpha}$ (counting the algebraic multiplicities) in $\bar{Q}$.*

3.3. Spectral stability for small $|\delta-1|$ and the proof of Theorem 1.3

Lemma 3.2. *For $\delta=1$, each fixed $c>c_{\min}=\frac{2}{\sqrt{\beta+1}}$, and $\alpha>0$ satisfying (2.11), there exists a small enough $\beta_\alpha\in(0,\delta_\alpha)$, such that there exists no eigenvalue of the linearized opertaor $\mathcal{L}_{1,\alpha}$ in $\mathrm{Re}\,\lambda\geq-\beta_\alpha$.*

Proof. For $\delta=1$, system (1.3) becomes

$$\begin{cases} u_t=u_{\xi\xi}+cu_\xi-\frac{2uv}{\beta+u},\\ v_t=v_{\xi\xi}+cv_\xi+\frac{uv}{\beta+u},\\ u|_{t=0}=u_0(\xi),\\ v|_{t=0}=v_0(\xi).\end{cases}$$

If $u_0(\xi)+2v_0(\xi)=1$, then $u+2v\equiv1$.

If $u_0(\xi)+2v_0(\xi)\neq1$, then $u+2v\not\equiv1$. Let $w=u+2v-1$. Then, (w,v) satisfies

$$\begin{cases} w_t=w_{\xi\xi}+cw_\xi,\\ v_t=v_{\xi\xi}+cv_\xi+\frac{(w-2v+1)v}{\beta+w-2v+1},\\ w|_{t=0}=u_0(\xi)+2v_0(\xi)-1,\\ v|_{t=0}=v_0(\xi).\end{cases}\tag{3.6}$$

Note that for $\delta = 1$, $U_c(\xi, 1) + 2V_c(\xi, 1) \equiv 1$, which yields $W_c(\xi, 1) = U_c(\xi, 1) + 2V_c(\xi, 1) - 1 = 0$. Therefore, the traveling front solution of (3.6) is $(0, V_c(\xi, 1))$. For $\mathrm{Re}\,\lambda \geq -\beta_\alpha$, linearizing (3.6) at $(0, V_c(\xi, 1))$, we can get the corresponding eigenvalue problem

$$\begin{cases} w_\lambda'' + cw_\lambda' = \lambda w_\lambda, \\ v_\lambda'' + cv_\lambda' + \frac{\beta V_c}{(\beta-2V_c+1)^2} w_\lambda + \frac{(-2V_c+1)(\beta-2V_c+1)-2\beta V_c}{(\beta-2V_c+1)^2} v_\lambda = \lambda v_\lambda, \end{cases} \tag{3.7}$$

with eigenfunction (w_λ, v_λ). From the first equation of (3.7), it is easy to see that for each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ and $\alpha > 0$ satisfying (2.11), if $(w_\lambda(\xi), v_\lambda(\xi)) \in C^2_{\alpha,unif}(\mathbb{R}) \times C^2_{\alpha,unif}(\mathbb{R})$ is a solution of (3.7) for $\mathrm{Re}\,\lambda \geq -\beta_\alpha$, then $w_\lambda(\xi) \equiv 0$. Thus, $v_\lambda(\xi)$ satisfies

$$v_\lambda'' + cv_\lambda' + \frac{(-2V_c + 1)(\beta - 2V_c + 1) - 2\beta V_c}{(\beta - 2V_c + 1)^2} v_\lambda = \lambda v_\lambda. \tag{3.8}$$

Theorem 10.3.12 in [21] shows that for each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ and $\alpha > 0$ satisfying (2.11), (3.8) has no eigenvalue in $\lambda \in \{\lambda \,|\, \mathrm{Re}\,\lambda \geq -\beta_\alpha\}$, with eigenfunction $(1 + e^{\alpha\xi})v_\lambda \in C_{unif}(\mathbb{R})$. In fact, by (2.15), it is easy to check that if $v_\lambda(\xi) \in C_{\alpha,unif}(\mathbb{R})$ is a solution of (3.8) with $\mathrm{Re}\,\lambda \geq -\beta_\alpha$, then $(1 + e^{\alpha\xi})v_\lambda \in C_{unif}(\mathbb{R})$. Therefore, (3.8) has no solution in $C_{\alpha,unif}(\mathbb{R})$ for $\mathrm{Re}\,\lambda \geq -\beta_\alpha$ and $\alpha > 0$ satisfying (2.11), which combines $w_\lambda(\xi) \equiv 0$ implies that there exists no eigenvalue of $\mathcal{L}_{1,\alpha}$ in $\mathrm{Re}\,\lambda \geq -\beta_\alpha$. This completes the proof of Lemma 3.2.

Lemma 3.3. (Spectral stability of the traveling front solutions $(U_c(\xi, \delta), V_c(\xi, \delta))$ *For small* $|\delta - 1|$, *each fixed* $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$, *and* $\alpha > 0$ *satisfying (2.11), there exists a small* $\beta_\alpha > 0$ *which is independent of* δ *such that*

$$\sup \mathrm{Re}\,\{\sigma(\mathcal{L}_{\delta,\alpha})\} < -\beta_\alpha < 0.$$

Proof. From (3.4)–(3.5) and property (ii) in Lemma 3.1, it is easy to see that

$$D(\lambda, \delta) \neq 0, \quad \text{for } \lambda \text{ on the boundary of } Q \text{ and } |\delta - 1| \leq \varepsilon_0.$$

By Lemmas 2.1 and 3.2, we get $D(\lambda, 1) \neq 0$ for $\lambda \in \bar{Q}$. Since $D(\lambda, \delta)$ is analytic in λ and continuous in δ for any $\lambda \in \bar{Q}$ and $|\delta - 1| \leq \varepsilon_0$, by applying the Rouchét theorem, it follows that $D(\lambda, \delta) \neq 0$ for $\lambda \in \bar{Q}$ and $|\delta - 1| \leq \varepsilon_0$. It can be deduced that $\mathcal{L}_{\delta,\alpha}$ has no eigenvalues in $\lambda \in \bar{Q}$ and $|\delta - 1| \leq \varepsilon_0$, by using the property (ii) in Lemma 3.1. Combining (3.4) and (3.5), $\mathcal{L}_{\delta,\alpha}$ has no eigenvalues for $\mathrm{Re}\,\lambda \geq -\beta_\alpha$ and small $|\delta - 1|$. Therefore, Lemma 3.3 holds. □

Proof of Theorem 1.3. Note that for small $|\delta - 1|$, each fixed $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$, and $\alpha > 0$ satisfying (2.11), the linearized operator $\mathcal{L}_{\delta,\alpha}$ generates an analytic semigroup on the weighted space $\mathbf{Y}_\alpha = C_{\alpha,unif}(\mathbb{R}) \times C_{\alpha,unif}(\mathbb{R})$, and the nonlinear inhomogeneous term $\begin{pmatrix} -2R(f,g) \\ R(f,g) \end{pmatrix}$ in (1.6) satisfies

$$\left\| \begin{pmatrix} -2R(f,g) \\ R(f,g) \end{pmatrix} \right\|_{\mathbf{Y}_\alpha} \leq \tilde{C} \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2_{\mathbf{Y}_\alpha}, \quad \tilde{C} > 0.$$

Then, by Lemma 3.3 and applying standard argument based on analytic semigroup theories, it is easy to prove that there exists a unique global solution $(f(\xi, t), g(\xi, t)) \in C([0, \infty), \mathbf{Y}_\alpha)$ to system (1.4) with initial value $(f_0(\xi), g_0(\xi))$ satisfying

$$\left\|(f(\xi, t), g(\xi, t))\right\|_{\mathbf{Y}_\alpha} \leq C\left\|(f_0, g_0)\right\|_{\mathbf{Y}_\alpha} e^{-kt}, \quad \forall\, t \geq 0,$$

if $\left\|(f_0, g_0)\right\|_{\mathbf{Y}_\alpha}$ is small enough. Here, $k > 0$ is independent of δ. This implies that Theorem 1.3 holds.

4. Conclusions and future research

For the reaction-diffusion system with acidic nitrate-ferroin reaction (1.1), by detailed spectral analysis and the Evans function method, we obtain the nonlinear exponential stability of the traveling front solutions with all noncritical speeds $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ and δ near 1 in some exponentially weighted spaces. Theorem A in [3] shows that for any $\delta > 0$, system (1.1) has a unique, up to translation, traveling wave solution with speed c iff $c \geq c_{\min} = \frac{2}{\sqrt{\beta+1}}$. However, for the case $c = c_{\min} = \frac{2}{\sqrt{\beta+1}}$ with $\delta \neq 1$ and for the case $c > c_{\min} = \frac{2}{\sqrt{\beta+1}}$ with δ not close to 1, there is no stability results of the traveling front solutions. We therefore leave these two problems as open questions for future studies.

Author contributions

Lina Wang: Conceptualization, methodology, formal analysis, writing–original draft, funding acquisition, project administration; Xiaofan Zhu: Conceptualization, methodology, formal analysis, validation, supervision; Yanxia Wu: Investigation, validation, supervision, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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