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Research article

Perturbation analysis of eigenvalues for second Dirichlet-Neumann tridiagonal Toeplitz matrices

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Abstract: This study focused on tridiagonal Toeplitz matrices based on second Dirichlet-Neumann boundary conditions and conducted in-depth research on their eigenvalue sensitivity. After evaluating condition numbers from both individual and global perspectives using parameters, our framework further achieved spectral sensitivity assessment in the context of second Dirichlet-Neumann tridiagonal Toeplitz matrices. The analysis process examined the ε-pseudospectra. Finally, we explored an inverse eigenvalue problem embedded in a constrained optimization framework, where the trapezoidal second Dirichlet-Neumann tridiagonal Toeplitz matrix generated by this problem becomes the ultimate optimal computational solution.

Keywords: Toeplitz matrix; condition number; eigenvalue sensitivity; inverse eigenvalue problem
Mathematics Subject Classification: 15B05, 15A12, 15A18

1. Introduction

Perturbed tridiagonal Toeplitz matrices and their classical counterparts have extensive applications across multiple disciplines, including the modeling of resistor networks [1,2], the quantum anomalous Hall effect [3], atomic and molecular theory [4], numerical solutions of ordinary and partial differential equations [5–7], time series analysis [8], and Tikhonov regularization for discrete ill-posed problems [9, 10]. Consequently, a thorough investigation into the computational properties of tridiagonal Toeplitz matrices is of paramount importance. The structured distance, eigensystem, and condition number of the eigenvalues of PDNT Toeplitz matrices have been systematically studied in [11, 12]. For the numerical solution of different fractional diffusion equations under Dirichlet boundary conditions, the authors of [13, 14] proposed high-precision numerical methods.

There has been extensive research on bordered tridiagonal matrices and periodic tridiagonal matrices. A symbolic algorithm is proposed in [15] for inverting general bordered tridiagonal matrices, with an explicit formula for the determinant also presented therein. Sogabe [16] introduces a novel algorithm for solving periodic pentadiagonal linear systems and a corresponding method for determinant calculation. In [17], a fast algorithm for tridiagonal linear systems based on the quasi-Toeplitz structure is developed. Fonseca [18] addresses the eigenvalue problem of a specific class of symmetric tridiagonal matrices and provides a corresponding solution approach. Their investigations encompass the computation of determinants, the characterization of eigenvalues, and the solution of associated linear systems, contributing significantly to the field. Efficient algorithms tailored for solving tridiagonal quasi-Toeplitz linear systems were developed by Du et al. [19]. Perturbed tridiagonal Toeplitz matrices have been the subject of numerous studies concerning their determinants, inverses, and various norm equalities and inequalities [20–22]. Furthermore, two distinct types of regular matrix pairs have also been investigated in a series of works [23–25]. Studies [26–28] have elucidated distinctive spectral properties associated with ε-pseudospectral separations in banded Toeplitz matrices via systematic analysis. Biswa Datta's initial inquiries in [29, 30] encompassed a spectrum of topics, notably encompassing the class of inverse eigenvalue problems treated in this work.

The main contributions of this work are as follows: i) Closed-form expressions for the eigenvalues and eigenvectors of the matrix and the Frobenius norm of the Jacobian matrix are derived; ii) explicit expressions for the condition numbers of each eigenvalue are established; iii) the effect of structured perturbations is analyzed, and the associated rate-of-change formulas are developed; and iv) two types of inverse problems are investigated: One involves reconstructing the matrix elements from given extreme eigenvalues, and the other proposes a least-squares solution method for the inverse vector problem concerning trapezoidal second Dirichlet-Neumann tridiagonal Toeplitz matrices.

The structure of this paper is organized as follows: We begin by introducing the necessary definitions and preliminary knowledge. In Section 3, we undertake a detailed analysis of both the individual and global condition numbers for second Dirichlet-Neumann tridiagonal Toeplitz matrices. Section 4 is devoted to a numerical example illustrating the distribution of their eigenvalues. Finally, Section 5 addresses the inverse problems associated with two eigenvalues.

2. Preliminaries

Definition 1. *A square matrix $\mathbb{A}$ is called a second Dirichlet-Neumann tridiagonal Toeplitz matrix if it has the form:*

$$\mathbb{A} = \begin{bmatrix} \alpha_0 & \alpha_1 & & & & O \\ \alpha_2 & \alpha_0 & \alpha_1 & & & \\ & \alpha_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot \quad \cdot & \alpha_1 \\ O & & & & \alpha_2 & \alpha_0 + \sqrt{\alpha_1\alpha_2} \end{bmatrix} \in \mathbb{C}^{u \times u}, \tag{2.1}$$

denoted compactly as $\mathbb{A} = (u; \alpha_2, \alpha_0, \alpha_1, \sqrt{\alpha_1\alpha_2})$. *In the special case where* $\alpha_0 = 0$, *we write* $\mathbb{A}_0 = (u; \alpha_2, 0, \alpha_1, \sqrt{\alpha_1\alpha_2})$.

Theorem 1. *For the second Dirichlet-Neumann tridiagonal Toeplitz matrix* $\mathbb{A} = (u; \alpha_2, \alpha_0, \alpha_1, \sqrt{\alpha_1\alpha_2})$, *the* j*-th eigenvalue takes the form*

$$\lambda_j = \alpha_0 + 2\sqrt{\alpha_1\alpha_2}\cos\frac{(2j-1)\pi}{2u+1}, j = 1, \ldots, u. \tag{2.2}$$

Introducing the arguments

$$\psi_1 = \arg\alpha_2, \psi_2 = \arg\alpha_1, \psi_3 = \arg\alpha_0, \tag{2.3}$$

we obtain the equivalent expressions of eigenvalues by Eq (2.3)

$$\lambda_j(\mathbb{A}) = \alpha_0 + 2\sqrt{|\alpha_1\alpha_2|}e^{\frac{(\psi_1+\psi_2)i}{2}}\cos\frac{(2j-1)\pi}{2u+1}, j = 1, \ldots, u. \tag{2.4}$$

In the case where $\alpha_1\alpha_2 \neq 0$, *the right eigenvector corresponding to* $\lambda_j(\mathbb{A})$ *is given by*

$$\eta_k^{(j)} = \frac{2}{\sqrt{2u+1}}(\sqrt{\frac{\alpha_2}{\alpha_1}})^{k-1}\sin\frac{k(2j-1)\pi}{2u+1}, \quad k = 1, 2, \ldots, u, j = 1, 2, \ldots, u. \tag{2.5}$$

And the left eigenvector is

$$\theta_k^{(j)} = \frac{2}{\sqrt{2u+1}}(\sqrt{\frac{\overline{\alpha_1}}{\overline{\alpha_2}}})^{k-1}\sin\frac{k(2j-1)\pi}{2u+1}, k = 1, \ldots, u, j = 1, \ldots, u, \tag{2.6}$$

where $\bar{\cdot}$ *denotes the complex conjugate.*

Proof. Let

$$\Theta_u = \mathrm{diag}(1, \sqrt{\frac{\alpha_1}{\alpha_2}}, \cdots, (\sqrt{\frac{\alpha_1}{\alpha_2}})^{u-2}, (\sqrt{\frac{\alpha_1}{\alpha_2}})^{u-1}). \tag{2.7}$$

Whenever the magnitudes of α_1 and α_2 are equal ($|\alpha_1| = |\alpha_2|$), it is straightforward to demonstrate that Θ_u becomes unitary. This property implies $\Theta_u\Theta_u^H = I_u$ or equivalently $\Theta_u^{-1} = \Theta_u^H$, where I_u denotes the u-dimensional identity matrix, and Θ_u represents the Hermitian adjoin of Θ_u.

Let

$$\mathbb{S}_u^{VI} = \left(\frac{2}{\sqrt{2u+1}}\sin\frac{k(2j-1)\pi}{2u+1}\right)_{k,j=1}^{u}. \tag{2.8}$$

Orthogonality is a hallmark of the matrix $\mathbb{S}_u^{VI}$ [31], which is the sixth discrete sine transform matrix, as it inherently satisfies

$$(\mathbb{S}_u^{VI})^{-1} = (\mathbb{S}_u^{VI})^T = \mathbb{S}_u^{VII}, \tag{2.9}$$

where $\mathbb{S}_u^{VII}$ [31] represents the seventh discrete sine transform matrix.

The equation below is obtained by diagonalizing $\mathbb{A}$ and applying sequential matrix multiplications combined with advanced algebraic transformations.

$$(\mathbb{S}_u^{VI})^{-1}\Theta_u\mathbb{A}\Theta_u^{-1}(\mathbb{S}_u^{VI}) = \text{diag}(\lambda_1,\ldots,\lambda_u), \tag{2.10}$$

i.e.,

$$\mathbb{A} = \Theta_u^{-1}(\mathbb{S}_u^{VI})\text{diag}(\lambda_1,\ldots,\lambda_u)(\mathbb{S}_u^{VI})^{-1}\Theta_u, \tag{2.11}$$

where

$$\lambda_j = \alpha_0 + 2\sqrt{\alpha_1\alpha_2}\cos\frac{(2j-1)\pi}{2u+1}, \quad j=1,\ldots,u. \tag{2.12}$$

Equation (2.10) clearly demonstrates that the second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A} = (u;\alpha_2,\alpha_0,\alpha_1,\sqrt{\alpha_1\alpha_2})$ possessing these specified entries exhibits similarity to the diagonal matrix $\text{diag}(\lambda_1,\lambda_2,...,\lambda_u)$, where each λ_j corresponds to an eigenvalue of $\mathbb{A}$.

By performing a left-multiplying of Eq (2.10) with $\Theta_u^{-1}\mathbb{S}_u^{VI}$, we have

$$\mathbb{A}\Theta_u^{-1}\mathbb{S}_u^{VI} = \Theta_u^{-1}\mathbb{S}_u^{VI}\text{diag}(\lambda_1,\ldots,\lambda_u), \tag{2.13}$$

i.e.,

$$\mathbb{A}(\eta^{(1)},\eta^{(2)},\cdots,\eta^{(u)}) = (\eta^{(1)},\eta^{(2)},\cdots,\eta^{(u)})\text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_u), \tag{2.14}$$

where $\eta^{(j)} = (\eta_1^{(j)},\ldots,\eta_u^{(j)})^T$,

$$\eta_k^{(j)} = \frac{2}{\sqrt{2u+1}}(\sqrt{\frac{\alpha_2}{\alpha_1}})^{k-1}\sin\frac{k(2j-1)\pi}{2u+1}, \quad k=1,2,\ldots,u, \quad j=1,2,\ldots,u. \tag{2.15}$$

Equation (2.14) can be written as follows:

$$\mathbb{A}\eta^{(j)} = \lambda_j\eta^{(j)}, \quad j=1,2,\ldots,u. \tag{2.16}$$

From Eq (2.16), we get the right eigenvector $\eta^{(j)} = (\eta_1^{(j)},\ldots,\eta_u^{(j)})^\dagger$ corresponding to λ_j.

According to Eqs (2.3) and (2.12), we obtain

$$\lambda_j(\mathbb{A}) = \alpha_0 + 2\sqrt{|\alpha_1\alpha_2|}e^{\frac{(\psi_1+\psi_2)i}{2}}\cos\frac{(2j-1)\pi}{2u+1}, \quad j=1,\ldots,u. \tag{2.17}$$

When $\alpha_1\alpha_2 \neq 0$, Eq (2.15) is the right eigenvector $\eta^{(j)} = [\eta_1^{(j)},\ldots,\eta_u^{(j)}]^T$ associated with $\lambda_j(\mathbb{A})$.

According to Eq (2.10), we also get the corresponding left eigenvector $\theta^{(j)} = [\theta_1^{(j)},\ldots,\theta_u^{(j)}]^T$, where the components

$$\theta_k^{(j)} = \frac{2}{\sqrt{2u+1}}(\sqrt{\frac{\overline{\alpha_1}}{\overline{\alpha_2}}})^{k-1}\sin\frac{k(2j-1)\pi}{2u+1}, k=1,\ldots,u, \quad j=1,\ldots,u, \tag{2.18}$$

with the bar symbol are used to indicate complex conjugation.

□

Theorem 2. *Consider the normal second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A}^* = (u; \alpha_2^*, \alpha_0^*, \alpha_1^*, \sqrt{\alpha_1^* \alpha_2^*})$ that is closest to $\mathbb{A} = (u; \alpha_2, \alpha_0, \alpha_1, \sqrt{\alpha_1 \alpha_2})$. Its eigenvalues take the form*

$$\lambda_j(\mathbb{A}^*) = \alpha_0 + 2\phi^* e^{\frac{(\psi_1+\psi_2)i}{2}} \cos \frac{(2j-1)\pi}{2u+1}, \quad j = 1, \dots, u, \tag{2.19}$$

where ψ_1 and ψ_2 are as defined in Eq (2.3)*. The matrix parameters satisfy*

$$\alpha_0^* = \alpha_0, \quad \alpha_1^* = \phi^* e^{i\psi_2}, \quad \alpha_2^* = \phi^* e^{i\psi_1},$$

and $\phi^* = \frac{|\alpha_1|+|\alpha_2|}{2} - \frac{(\sqrt{|\alpha_1|} - \sqrt{|\alpha_2|})^2}{2(2u-1)}$.

Proof. To ensure that $\mathbb{A}^* = (u; \alpha_2^*, \alpha_0^*, \alpha_1^*, \sqrt{\alpha_1^* \alpha_2^*})$ is a normal second Dirichlet-Neumann tridiagonal Toeplitz matrix, the parameters must satisfy the following conditions:

$$\alpha_0^* = \alpha_0, \quad \alpha_1^* = \phi^* e^{i\psi_2}, \quad \alpha_2^* = \phi^* e^{i\psi_1},$$

where $\phi^* = |\alpha_1^*| = |\alpha_2^*|$.

To minimize $\|\mathbb{A}^* - \mathbb{A}\|_F$, the problem reduces to finding the optimal parameter ϕ^* that minimizes the objective function

$$g(\phi) = (\phi - \sqrt{|\alpha_1 \alpha_2|})^2 + (u-1)\left[(\phi - |\alpha_1|)^2 + (\phi - |\alpha_2|)^2\right].$$

Simplifying and solving the equation yields

$$\phi^* = \frac{|\alpha_1| + |\alpha_2|}{2} - \frac{(\sqrt{|\alpha_1|} - \sqrt{|\alpha_2|})^2}{2(2u-1)}.$$

Substitution yields the optimal normal approximation matrix $\mathbb{A}^*$. □

To facilitate subsequent proofs, we present the following equations.

Proposition 1. *The trigonometric identities hold true:*

$$\sum_{k=0}^{u} n^k \sin kx = \frac{n^{u+2} \sin ux - n^{u+1} \sin(u+1)x + n \sin x}{n^2 - 2n\cos x + 1}, \tag{2.20}$$

$$\sum_{k=0}^{u} n^k \cos kx = \frac{1 - n\cos x - n^{u+1}\cos(u+1)x + n^{u+2}\cos ux}{n^2 - 2n\cos x + 1}, \tag{2.21}$$

$$\sin \frac{2(u-1)(2j-1)\pi}{2u+1} = \sin \frac{3(2j-1)\pi}{2u+1}, \quad \sin \frac{2u(2j-1)\pi}{2u+1} = \sin \frac{(2j-1)\pi}{2u+1}, \tag{2.22}$$

$$\cos \frac{2(u-1)(2j-1)\pi}{2u+1} = -\cos \frac{3(2j-1)\pi}{2u+1}, \quad \cos \frac{2u(2j-1)\pi}{2u+1} = -\cos \frac{(2j-1)\pi}{2u+1}. \tag{2.23}$$

3. Eigenvalue sensitivity analysis of $\mathbb{A}_0$ and $\mathbb{A}$

Extensive research has been conducted on various aspects of eigenvalue analysis for general matrices, including conditional numbers and sensitivity characteristics, as evidenced by [32–34]. The bounds of structured eigenvalue condition numbers and the influence of algebraic structures on eigenvalue sensitivity are recognized as important topics in eigenvalue research. These two aspects have been investigated in [35]; different types of condition numbers have been compared in [36, 37], additionally, the eigenvalue sensitivity characteristics of Toeplitz matrices and Hankel matrices under patterned perturbations have been explored via numerical experiments.

The study analyzes eigenvalue sensitivity characteristics for both $\mathbb{A}_0$ and $\mathbb{A}$. First, we focus on the sensitivity of vectors composed for the eigenvalues of matrices $\mathbb{A}_0$,

$$\vec{\chi}(\mathbb{A}_0) = [\lambda_1(\mathbb{A}_0), \lambda_2(\mathbb{A}_0), \cdots, \lambda_u(\mathbb{A}_0)]$$

with respect to perturbations in α_1 and α_2. To facilitate this analysis, we define the function

$$q : F \subset \mathbb{C}^2 \to q(F) \subset \mathbb{C}^u, F = \left\{(\alpha_1, \alpha_2) \in \mathbf{C}^2 : \alpha_1\alpha_2 \neq 0\right\} : \vec{\chi}(\mathbb{A}_0) = q(\alpha_1, \alpha_2).$$

The Jacobian matrix of function q plays a key role in determining how perturbations in parameters α_1 and α_2 influence the sensitivity of the vector quantity $\vec{\chi}(\mathbb{A}_0)$. A comprehensive breakdown unfolds in the subsequent section.

Using Eq (2.12), we can obtain that the Jacobian matrix of q is represented by

$$T_q(\alpha_1, \alpha_2) = \begin{bmatrix} \sqrt{\frac{\alpha_1}{\alpha_2}} \cos\frac{\pi}{2u+1} & \sqrt{\frac{\alpha_2}{\alpha_1}} \cos\frac{\pi}{2u+1} \\ \sqrt{\frac{\alpha_1}{\alpha_2}} \cos\frac{3\pi}{2u+1} & \sqrt{\frac{\alpha_2}{\alpha_1}} \cos\frac{3\pi}{2u+1} \\ \vdots & \vdots \\ \sqrt{\frac{\alpha_1}{\alpha_2}} \cos\frac{(2u-1)\pi}{2u+1} & \sqrt{\frac{\alpha_2}{\alpha_1}} \cos\frac{(2u-1)\pi}{2u+1} \end{bmatrix} \in \mathbb{C}^{u\times 2}. \tag{3.1}$$

Applying Eq (3.1), we can derive that

$$\|T_q(\alpha_1, \alpha_2)\|_F = \frac{\sqrt{2u-1}}{2}\sqrt{|\frac{\alpha_2}{\alpha_1}| + |\frac{\alpha_1}{\alpha_2}|} = \frac{\sqrt{2u-1}}{2}\sqrt{\frac{|\alpha_1|^2 + |\alpha_2|^2}{|\alpha_1\alpha_2|}}. \tag{3.2}$$

If we consider the relative errors in the data α_1, α_2 and in $\lambda_j(\mathbb{A}_0)$, then the analogue of Eq (3.1) is the $u \times 2$ matrix $\Delta_q(\alpha_1, \alpha_2)$ given by

$$\Delta_q(\alpha_1, \alpha_2) = \begin{bmatrix} \frac{\alpha_2}{\vec{\chi}_1(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{1,1} & \frac{\alpha_1}{\vec{\chi}_1(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{1,2} \\ \frac{\alpha_2}{\vec{\chi}_2(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{2,1} & \frac{\alpha_1}{\vec{\chi}_2(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{2,2} \\ \vdots & \vdots \\ \frac{\alpha_2}{\vec{\chi}_u(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{u,1} & \frac{\alpha_1}{\vec{\chi}_u(\mathbb{A}_0)}(T_q(\alpha_1, \alpha_2))_{u,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Through calculation, we get

$$\Delta_q(\alpha_1, \alpha_2)^H \Delta_q(\alpha_1, \alpha_2) = \frac{u}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$\|\Delta_q(\alpha_1,\alpha_2)\|_2 = \|\Delta_q(\alpha_1,\alpha_2)\|_F = \sqrt{\frac{u}{2}}.$$

We now introduce a set of parameters that will greatly simplify the expressions in our forthcoming discussion:

$$n = \begin{cases} \left|\dfrac{\alpha_2}{\alpha_1}\right|, & |\alpha_2| < |\alpha_1|, \\ \left|\dfrac{\alpha_1}{\alpha_2}\right|, & |\alpha_1| < |\alpha_2|. \end{cases} \tag{3.3}$$

Remark 1. *It can be known from the above conclusion that the norm of Δ_q is independent of α_1 and α_2, whereas the norm of T_q is dependent on the ratio $|\frac{\alpha_2}{\alpha_1}|$. The norm of T_q achieves its minimum $\sqrt{\frac{2u-1}{2}}$ if and only if $|\alpha_1| = |\alpha_2|$, T_q is, if and only if $\mathbb{A}$ is normal. The norm of T_q tends to $+\infty$ when the ratio Eq* (3.3) *decreases to 0.*

3.1. Individual eigenvalue condition numbers

When $\alpha_1\alpha_2 \neq 0$, the condition numbers corresponding to each eigenvalue are discussed by using trigonometric identities and Eqs (2.15) and (2.18), specifically,

$$\sum_{k=1}^{u} \sin^2 \frac{k(2j-1)\pi}{2u+1} = \frac{2u+1}{4}, \quad j = 1,\ldots,u. \tag{3.4}$$

For $j = 1,\ldots,u$,

$$\|\eta^{(j)}\|_2^2 = \frac{4}{2u+1} \sum_{k=1}^{u} \left|\frac{\alpha_2}{\alpha_1}\right|^{k-1} \sin^2 \frac{k(2j-1)\pi}{2u+1}, \tag{3.5}$$

$$\|\theta^{(j)}\|_2^2 = \frac{4}{2u+1} \sum_{k=1}^{u} \left|\frac{\alpha_1}{\alpha_2}\right|^{k-1} \sin^2 \frac{k(2j-1)\pi}{2u+1}, \tag{3.6}$$

and

$$|(\theta^{(j)})^H \eta^{(j)}| = \frac{4}{2u+1} \sum_{k=1}^{u} \sin^2 \frac{k(2j-1)\pi}{2u+1} = 1. \tag{3.7}$$

The individual condition numbers can be derived by

$$\kappa(\lambda_j(\mathbb{A})) = \frac{\|\eta^{(j)}\|_2 \|\theta^{(j)}\|_2}{|(\theta^{(j)})^H \eta^{(j)}|} = \frac{4}{2u+1} \sqrt{\sum_{k=1}^{u} |\frac{\alpha_2}{\alpha_1}|^{k-1} \sin^2 \frac{k(2j-1)\pi}{2u+1} \sum_{k=1}^{u} |\frac{\alpha_1}{\alpha_2}|^{k-1} \sin^2 \frac{k(2j-1)\pi}{2u+1}}. \tag{3.8}$$

The equations for (3.4)–(3.8) can be found in [38].

Theorem 3. *If $|\alpha_1| = |\alpha_2|$, then $\kappa(\lambda_j(\mathbb{A})) = 1$.*
If $|\alpha_1| \neq |\alpha_2|$, then

$$\kappa(\lambda_j(\mathbb{A})) = \frac{2}{(2u+1)n^{(u-1)/2}} \sqrt{B_{u,n}(j) B_{u,\frac{1}{n}}(j)}, \quad j = 1,\ldots,u, \tag{3.9}$$

where $\kappa(\lambda_j(\mathbb{A}))$ represents the individual eigenvalue condition number of the matrix $\mathbb{A}$, n is defined by Eq (3.3),

$$B_{u,n}(j) = \frac{1-n^u}{1-n} - \frac{n^u \cos\frac{(2j-1)\pi}{2u+1}(1-n) - n + \cos\frac{2(2j-1)\pi}{2u+1}}{n^2 - 2n\cos\frac{2(2j-1)\pi}{2u+1} + 1},$$

and

$$B_{u,\frac{1}{n}}(j) = \frac{1-n^u}{1-n} - \frac{n^u(n\cos\frac{2(2j-1)\pi}{2u+1} - 1) + (n-1)\cos\frac{(2j-1)\pi}{2u+1}}{n^2 - 2n\cos\frac{2(2j-1)\pi}{2u+1} + 1}.$$

Proof. If $|\alpha_1| = |\alpha_2|$, $\mathbb{A}$ is a normal matrix. Moreover, as derived from Eqs (3.4)–(3.6), the squared magnitudes of both left and right eigenvectors are given by:

$$\|\eta^{(j)}\|_2^2 = \|\theta^{(j)}\|_2^2 = \frac{4}{2u+1}\sum_{k=1}^{u}\sin^2\frac{k(2j-1)\pi}{2u+1} = 1, \quad j = 1,\ldots,u. \tag{3.10}$$

According to Eqs (3.7) and (3.10), the relevant results can be obtained as follows:

$$\kappa(\lambda_j(\mathbb{A})) = \frac{\|\eta^{(j)}\|_2\|\theta^{(j)}\|_2}{|(\theta^{(j)})^H\eta^{(j)}|} = 1.$$

When $|\alpha_1| \neq |\alpha_2|$, for an eigenvalue λ_j, let $t = \frac{(2j-1)\pi}{2u+1}$, $j = 1,\ldots,u$, and it can be obtained that

$$\sum_{k=1}^{u}|\frac{\alpha_2}{\alpha_1}|^{k-1}\sin^2\frac{k(2j-1)\pi}{2u+1} = \frac{1}{2}(\sum_{k=1}^{u}n^{k-1} - \sum_{k=0}^{u-1}n^k\cos 2(k+1)t). \tag{3.11}$$

Subsequently, the expressions are simplified by applying trigonometric identities.

According to Eqs (2.20)–(2.23), the following formula can be derived

$$\begin{aligned}\sum_{k=0}^{u-1}n^k\cos 2(k+1)t &= \cos 2t\sum_{k=0}^{u-1}n^k\cos 2kt - \sin 2t\sum_{k=0}^{u-1}n^k\sin 2kt\\ &= \frac{\cos 2t - n + n^u\cos t - n^{u+1}\cos t}{n^2 - 2n\cos 2t + 1}.\end{aligned} \tag{3.12}$$

By Eqs (3.11) and (3.12), we have

$$\sum_{k=1}^{u}|\frac{\alpha_2}{\alpha_1}|^{k-1}\sin^2\frac{k(2j-1)\pi}{(2u+1)} = \frac{1}{2}\Big(\frac{1-n^u}{1-n} - \frac{\cos 2t - n + n^u\cos t(1-n)}{n^2-2n\cos 2t+1}\Big).$$

In the same way, it follows that

$$\sum_{k=1}^{u}|\frac{\alpha_1}{\alpha_2}|^{k-1}\sin^2\frac{k(2j-1)\pi}{(2u+1)} = \frac{1}{2n^{u-1}}\Big(\frac{1-n^u}{1-n} - \frac{n^u(n\cos 2t-1) + (n-1)\cos t}{n^2-2n\cos 2t+1}\Big).$$

Substituting the above results into Eq (3.8), then Eq (3.9) is obtained. □

Remark 2. *By simplifying Eq (3.9), we can obtain*

$$\kappa(\lambda_j(\mathbb{A})) = \frac{2}{(2u+1)n^{(u-1)/2}(1-n)(n^2-2n\cos 2t+1)}K, \tag{3.13}$$

where $K = \sqrt{CD}$, $t = \frac{(2j-1)\pi}{2u+1}$, $j = 1, \ldots, u$,

$$C = \left[(n+1)(1-\cos 2t) - n^u\left(n^2+1\right)(1+\cos t) + 2n^{u+1}(\cos 2t + \cos t)\right],$$

$$D = \left[-n^{u+1}(n+1)(1-\cos 2t) + \left(n^2+1\right)(1+\cos t) - 2n(\cos 2t + \cos t)\right].$$

The result

$$\frac{K}{n^2 - 2n\cos\frac{4j\pi}{2u+1} + 1} \geq 0 \tag{3.14}$$

is derived from Eq (3.13) in conjunction with numerical analysis.

3.2. The global eigenvalue condition number

For any diagonalizable matrix $\mathbb{A}$ where $|\alpha_1| = |\alpha_2|$, the bounds can be derived through Eqs (2.15) and (2.18). Additionally, when $|\alpha_1| \neq |\alpha_2|$, the global condition number is defined by Eq (3.13) in the following form:

$$\kappa_F(\lambda) = \sum_{j=1}^{u}\kappa(\lambda_j(\mathbb{A})) = \frac{2}{(2u+1)n^{u-1/2}(1-n)}\sum_{j=1}^{u}\frac{K}{n^2-2n\cos\frac{2(2j-1)\pi}{2u+1}+1}.$$

Using Eqs (3.13) and (3.14), we have $\kappa_F(\lambda) \geq 0$.

3.3. The ε-pseudospectrum

Following [28], the ε-pseudospectrum for a complex matrix $B \in \mathbb{C}^{u\times u}$ is defined as

$$\Gamma_\varepsilon(B) = \{\hat{d} : \| (\hat{d}I - B)^{-1} \|_2 \geq \varepsilon^{-1}\},$$

where $\varepsilon > 0$.

In Section 5, we will use the following alternative definition instead:

$$\Gamma_\varepsilon(B) = \{\hat{d} : \exists\vec{\zeta} \in \mathbb{C}^u,\ \| \vec{\zeta} \|_2 = 1,\ \text{such that}\ \| (\hat{d}I - B)\vec{\zeta} \|_2 \leq \varepsilon\}. \tag{3.15}$$

The vectors $\vec{\zeta}$ in the above definition are called ε-pseudoeigenvectors.

Next, we introduce the symbol, which is defined as

$$f(\hat{d}) = \alpha_1\hat{d} + \alpha_0 + \alpha_2\hat{d}^{-1}.$$

Then the ellipse

$$f(S) = \{f(\hat{d}) : \hat{d} \in \mathbb{C},\ |\hat{d}| = 1\} \tag{3.16}$$

is the boundary of the spectrum of $\mathbb{A}_\infty = (\infty; \alpha_2, \alpha_0, \alpha_1, \sqrt{\alpha_1\alpha_2})$.

The major axis of $f(S)$ is given by

$$S_a = \{\alpha_0 + he^{\frac{i(\psi_1+\psi_2)}{2}},\ h \in \mathbb{R},\ |\, h\, |\leq |\alpha_1| + |\alpha_2| - \frac{(\sqrt{|\alpha_1|} - \sqrt{|\alpha_2|})^2}{2u-1}\}, \tag{3.17}$$

and the interval between the foci of $f(S)$ is given by

$$S_b = \{\alpha_0 + he^{\frac{i(\psi_1+\psi_2)}{2}},\ h \in \mathbb{R},\ |\, h\, |\leq 2\sqrt{|\, \alpha_1\alpha_2\, |}\}. \tag{3.18}$$

For any finite $u \geq 1$, the spectrum of $\mathbb{A} = (u;\alpha_2,\alpha_0,\alpha_1, \sqrt{\alpha_1\alpha_2})$ is guaranteed to reside within the interval S_b, and there is no shorter interval with this property. Moreover, by Eq (2.19), the spectrum of the normal second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A}^*$ closest to $\mathbb{A}$ lives in the interval Eq (3.17).

3.4. Structured perturbations

Let $|\alpha_2| = \min\{|\alpha_1|, |\alpha_2|\}$ and consider the tridiagonal perturbation $E_p = (u; -p, 0, 0, -\sqrt{p\alpha_1})$ of the matrix $\mathbb{A} = (u;\alpha_2,\alpha_0,\alpha_1, \sqrt{\alpha_1\alpha_2})$. For $p = \gamma\alpha_2$ with $0 < \gamma < 1$, we obtain a family of diagonalizable matrices $\mathbb{A} + E_p$ with simple eigenvalues. As $\gamma \longrightarrow 1$, the matrices $\mathbb{A} + E_p$ converge to the defective matrix $\mathbb{A}^+ = (u; 0, \alpha_0, \alpha_1, 0)$. The latter matrix possesses a single distinct eigenvalue, α_0, and its geometric multiplicity is exactly one. Thus, the structured perturbation

$$E_{\alpha_2} = (u; -\alpha_2, 0, 0, -\sqrt{\alpha_1\alpha_2}), \quad \|\, E_{\alpha_2}\, \|_F = \sqrt{(u-1)\, |\alpha_2|^2 + |\alpha_1\alpha_2|},$$

moves all eigenvalues of matrix $\mathbb{A}$, and the eigenvalues are all α_0. For $0 < |\alpha_2| \leq |\alpha_1|$, the rate of change for the jth eigenvalue of $\mathbb{A}$ is given by

$$\frac{|\, \lambda_j(\mathbb{A} + E_{\alpha_2}) - \lambda_j(\mathbb{A})\, |}{\|\, E_{\alpha_2}\, \|_F} = \frac{2\sqrt{|\, \alpha_1\alpha_2\, |}\, |\cos\frac{(2j-1)\pi}{2u+1}\, |}{\sqrt{(u-1)\, |\alpha_2|^2 + |\alpha_1\alpha_2|}} = \frac{2\, |\cos\frac{(2j-1)\pi}{2u+1}\, |}{\sqrt{(u-1)n+1}}. \tag{3.19}$$

Similarly, let $E_{p,q} = (u; -p, 0, -q, -\sqrt{pq})$ with $p = \gamma\alpha_2$ and $q = \gamma\alpha_1$ for $0 < \gamma < 1$. We have

$$\lim_{\gamma\to 1}(\mathbb{A} + E_{p,q}) = \begin{pmatrix} \alpha_0 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ & & \alpha_0 & \\ 0 & & & \alpha_0 \end{pmatrix}_{u\times u}.$$

It is apparent that the limit matrix is normal. The structured perturbation

$$E_{\alpha_1,\alpha_2} = (u; -\alpha_2, 0, -\alpha_1, -\sqrt{\alpha_1\alpha_2}), \quad \|\, E_{\alpha_1,\alpha_2}\, \|_F = \sqrt{(u-1)\left(|\alpha_1|^2 + |\alpha_2|^2\right) + |\alpha_1\alpha_2|},$$

gives the limit matrix. When $j = 1, \ldots, u$, the rate of change of the eigenvalue under this perturbation is given by

$$\frac{|\, \lambda_j(\mathbb{A} + E_{\alpha_1,\alpha_2}) - \lambda_j(\mathbb{A})\, |}{\|\, E_{\alpha_1,\alpha_2}\, \|_F} = \frac{2\sqrt{|\alpha_1\alpha_2|}|\cos\frac{(2j-1)\pi}{2u+1}|}{\sqrt{(u-1)(|\alpha_1|^2 + |\alpha_2|^2) + |\alpha_1\alpha_2|}} = \frac{2\sqrt{2u-1}|\cos\frac{(2j-1)\pi}{2u+1}|}{\sqrt{4\,(u-1)\left\|T_f(\alpha_1,\alpha_2)\right\|_F^2 + 2u-1}}.$$

Consequently, the rate is inversely proportional to the norm of the Jacobian matrix defined in Eq (3.2). The rate attains its maximum value when $\mathbb{A}$ is normal. The structured perturbation induces increasingly significant sensitivity in the eigenvalues of $\mathbb{A}$ as they diverge farther from α_0.

4. Diagram illustrating the sensitivity of eigenvalues

Let

$$\mathbb{A}_{(n)} = (50; (4+3i)n, 16-3i, -5, \sqrt{-5n(4+3i)}) \tag{4.1}$$

across varying parameter values $0 < n < 1$, where n corresponds to the ratio defined in Eq (3.3). Notably, $\mathbb{A}_{(0)}$ is a defective matrix, whereas $\mathbb{A}_{(1)}$ is a normal matrix.

Figures 1–4 depict the eigenvalues of the second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A}_{(n)}$ and the closest normal second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A}^*_{(n)}$. These eigenvalues are computed using Eqs (2.12) and (2.19). The figures also display the image of the unit circle for the matrices $\mathbb{A}_{(n)}$; see Eq (3.16). The visualizations take the form of ellipses, each of which represents the boundary of the spectrum of the second Dirichlet-Neumann tridiagonal Toeplitz operator $\mathbb{A}_\infty = (\infty; (4+3i)n, 16-3i, -5, \sqrt{-5n(4+3i)})$. If the QR algorithm implemented by the MATLAB function eig is employed to find the eigenvalues of matrix $\mathbb{A}_{(0.1)}$ instead of using Eq (2.12), Figure 5 would exhibit a substantially altered appearance. Figures 1 and 5 do not visually reveal that $\mathbb{A}^T_{(0.1)}$ and $(\mathbb{A}^T_{(0.1)})^*$ share identical eigenvalues. As shown in Figure 5, the spectrum of $\mathbb{A}^T_{(0.1)}$ approximates closely to the boundary of the ε-pseudospectrum when ε is set to machine epsilon $2 \cdot 10^{-16}$. The horizontal and vertical axes correspond to the real and imaginary components of the eigenvalues.

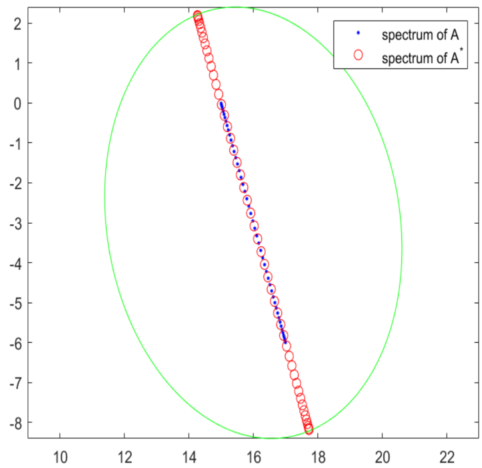

Figure 1. Spectra of matrix $\mathbb{A}_{(n)}$ and the closest normal matrix $\mathbb{A}^*_{(n)}$ when $n = 0.1$.

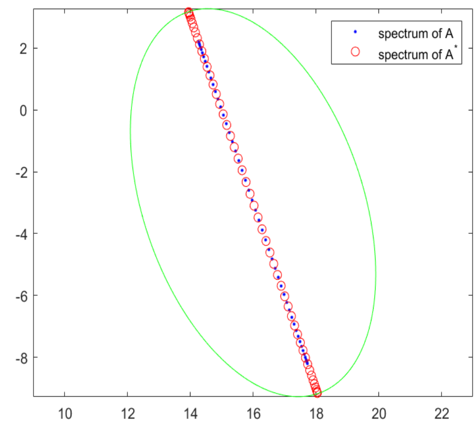

Figure 2. Spectra of matrix $\mathbb{A}_{(n)}$ and the closest normal matrix $\mathbb{A}^*_{(n)}$ when $n = 0.3$.

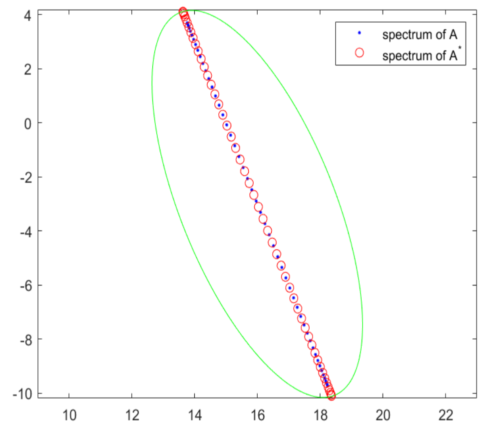

Figure 3. Spectra of matrix $\mathbb{A}_{(n)}$ and the closest normal matrix $\mathbb{A}^*_{(n)}$ when $n = 0.5$.

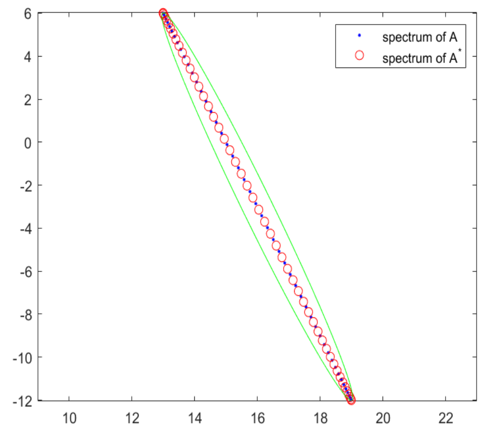

Figure 4. Spectra of matrix $\mathbb{A}_{(n)}$ and the closest normal matrix $\mathbb{A}^*_{(n)}$ when $n = 0.9$.

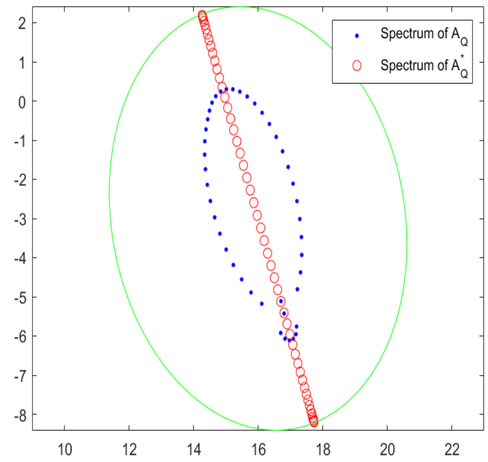

Figure 5. Spectra of $\mathbb{A}^T_{(0.1)}$(denoted $\mathbb{A}_Q$) and its closest normal matrix $(\mathbb{A}^T_{(0.1)})^*$ (denoted $\mathbb{A}^*_Q$).

5. Inverse problems for second Dirichlet-Neumann tridiagonal Toeplitz matrices

This section first explores the inverse eigenvalue problem for second Dirichlet-Neumann tridiagonal Toeplitz matrices and then investigates the inverse vector problem of this type of matrix. Among them, the inverse vector problem determines a perturbed trapezoidal second Dirichlet-Neumann tridiagonal Toeplitz matrix by minimizing the norm of the product of a given vector and the matrix vector.

Inverse problem 1. Given 2 complex numbers $\hat{x}$, $\hat{y}$, we discuss the conditions under which a matrix $\mathbb{A}$ of the form (2.1) can be constructed such that $\hat{x}$ and $\hat{y}$ are the minimum and maximum eigenvalues of $\mathbb{A}$, respectively.

Now we will solve this problem, given two distinct complex numbers $\hat{x}$ and $\hat{y}$ and a natural number u, and determine a second Dirichlet-Neumann tridiagonal Toeplitz matrix $\mathbb{A} = (u;\alpha_2,\alpha_0,\alpha_1, \sqrt{\alpha_1\alpha_2})$ configured as $\hat{x}$ and $\hat{y}$ being its extremal eigenvalues. While no unique solution exists for the problem itself, all eigenvalues of $\mathbb{A}$ are uniquely determined by the provided data. From the equations

$$\lambda_1 = \hat{x} = \alpha_0 + 2\sqrt{\alpha_1\alpha_2}\cos\frac{\pi}{2u+1}, \quad \lambda_u = \hat{y} = \alpha_0 + 2\sqrt{\alpha_1\alpha_2}\cos\frac{(2u-1)\pi}{2u+1},$$

we can deduce that the diagonal entry α_0 and the product of the sub-diagonal and super-diagonal entries, $\alpha_1\alpha_2$, are uniquely determined as follows:

$$\sqrt{\alpha_1\alpha_2} = \frac{\hat{x}-\hat{y}}{2(\cos\frac{\pi}{2u+1} - \cos\frac{(2u-1)\pi}{2u+1})}, \quad \alpha_0 = \frac{\hat{y}\cos\frac{\pi}{2u+1} - \hat{x}\cos\frac{(2u-1)\pi}{2u+1}}{\cos\frac{\pi}{2u+1} - \cos\frac{(2u-1)\pi}{2u+1}}. \tag{5.1}$$

The absolute value $|\alpha_1\alpha_2|$ and the sum of arguments $\arg(\alpha_1) + \arg(\alpha_2)$ are uniquely specified by the provided data. The angle of the sub-diagonal or super-diagonal entry can be specified arbitrarily. Additionally, the ratio $0 < n \leq 1$, governed by Eq (3.3), is also selectable. As n approaches zero, the eigenvalues exhibit heightened ill-conditioning. Choosing $n = 1$, i.e., $|\alpha_1| = |\alpha_2|$, results in a normal matrix. By allowing arbitrary selection of the sub-diagonal or super-diagonal angle of entry, the normal matrix loses uniqueness. Uniqueness can be achieved, for example, by also specifying $\arg(\alpha_1)$ or $\arg(\alpha_2)$.

Example 1. *Given two numbers $\hat{x}$ and $\hat{y}$, find a 5th-order second Dirichlet-Neumann tridiagonal Toeplitz matrix with these two numbers as the maximum and minimum eigenvalues,*

$$\hat{x} = 1 + 2\sqrt{2}\cos\frac{\pi}{11}, \quad \hat{y} = 1 + 2\sqrt{2}\cos\frac{9\pi}{11}. \tag{5.2}$$

Based on Eqs (5.1) and (5.2), it can be concluded that

$$\sqrt{\alpha_1\alpha_2} = \frac{\hat{x} - \hat{y}}{2(\cos\frac{\pi}{11} - \cos\frac{9\pi}{11})} = \frac{(1 + 2\sqrt{2}\cos\frac{\pi}{11}) - (1 + 2\sqrt{2}\cos\frac{9\pi}{11})}{2(\cos\frac{\pi}{11} - \cos\frac{9\pi}{11})} = \sqrt{2},$$

$$\alpha_0 = \frac{\hat{y}\cos\frac{\pi}{11} - \hat{x}\cos\frac{9\pi}{11}}{\cos\frac{\pi}{11} - \cos\frac{9\pi}{11}} = \frac{(1 + 2\sqrt{2}\cos\frac{9\pi}{11})\cos\frac{\pi}{11} - (1 + 2\sqrt{2}\cos\frac{\pi}{11})\cos\frac{9\pi}{11}}{\cos\frac{\pi}{11} - \cos\frac{9\pi}{11}} = 1.$$

Let $\arg(\alpha_1) = -\frac{\pi}{4}$ *and* $\arg(\alpha_2) = \frac{\pi}{4}$, *we have* $\alpha_1 = 1 - i$ *and* $\alpha_2 = 1 + i$.

The matrix is expressed in the form of

$$\mathbb{A} = \begin{bmatrix} 1 & 1-i & & & O \\ 1+i & 1 & 1-i & & \\ & 1+i & 1 & 1-i & \\ & & 1+i & 1 & 1-i \\ O & & & 1+i & 1+\sqrt{2} \end{bmatrix}. \tag{5.3}$$

Inverse problem 2. For any vector $\mathbf{x} \in \mathbb{C}^u$, there exists a trapezoidal second Dirichlet-Neumann tridiagonal Toeplitz matrix

$$\mathbb{A} = \begin{bmatrix} \alpha_2 & 1 & \alpha_1 & & & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & \alpha_2 & 1 & \alpha_1 \\ 0 & & & & \alpha_2 & 1+\sqrt{\alpha_1\alpha_2} & \alpha_1 \end{bmatrix} \in \mathbb{C}^{(u-2)\times u},$$

which solves the optimization problem:

$$\min_{\alpha_2,\alpha_1} \|\mathbb{A}\mathbf{x}\|_2 . \tag{5.4}$$

To address this problem, let $\mathbf{x} = [\varsigma_1, \varsigma_2, ..., \varsigma_u]^T$. Then the minimization problem (5.4) can be represented as

$$\min_{\alpha_1,\alpha_2} \left\| \begin{bmatrix} \varsigma_1 & \varsigma_3 \\ \varsigma_2 & \varsigma_4 \\ \vdots & \vdots \\ \vdots & \vdots \\ \varsigma_{u-2} & \varsigma_u \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix} + \begin{bmatrix} \varsigma_2 \\ \varsigma_3 \\ \vdots \\ \varsigma_{u-2} \\ (1+\sqrt{\alpha_1\alpha_2})\varsigma_{u-1} \end{bmatrix} \right\|_2 . \tag{5.5}$$

This least-squares problem admits a unique solution provided that the matrix columns are linearly independent. The columns are linearly dependent when the entries of **x** follow

$$\varsigma_{k+2} = \alpha\varsigma_k, \quad k = 1, \ldots, u-2,$$

for some $\alpha \in \mathbb{C}$.

Let $\hat{\mathbb{A}} \in \mathbb{C}^{u\times u}$ represent the second Dirichlet-Neumann tridiagonal Toeplitz matrix constructed by augmenting $\mathbb{A}$ with suitable rows at its upper and lower boundaries. Equation (3.15) implies that the ε-pseudoeigenvectors of $\hat{\mathbb{A}}$ corresponding to $\hat{d} = 0$ are members of the set

$$\{\vec{\zeta} : \|\mathbb{A}\vec{\zeta}\|_2 \leq \varepsilon, \|\vec{\zeta}\|_2 = 1\}.$$

6. Conclusions

In summary, this study systematically investigates the eigenvalues and sensitivity of tridiagonal Toeplitz matrices under the second Dirichlet–Neumann boundary conditions. First, explicit analytical expressions for the eigenvalues and eigenvectors are derived rigorously. Based on these results, accurate analytical expressions for the condition numbers of each eigenvalue are obtained, thereby completing a theoretical quantitative analysis of eigenvalue perturbations. Second, the stability of eigenvalues is analyzed using a combination of parameter studies and ε-pseudospectrum theory. In addition, we preliminarily discuss two types of inverse problems associated with the eigenvalue problem.

Author contributions

Zhaolin Jiang: Conceptualization, methodology, validation; Hongxiao Chu: Conceptualization, writing-original draf; Qiaoyun Miao: Project administration, writing-review and editing; Ziwu Jiang: Investigation, supervision, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors declare no conflicts of interest.

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