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Research article

The Schur complement of SDD_1 matrices and its applications

Shiyun Wang*, **Yun Li and Wanfu Tian**

College of Science, Shenyang Aerospace University, Shenyang, Liaoning, 110136, China

* **Correspondence:** Email: wsy0902@163.com.

Abstract: The Schur complements of H-matrices have a wide range of practical applications. This paper explores the diagonally dominant degrees of the Schur complements of SDD_1 matrices and its applications. it concludes that the Schur complements can be also SDD_1 matrices under certain conditions. Moreover, the upper and lower bounds for the determinants of SDD_1 matrices are presented, and large-scale linear equations with the coefficient matrix being an SDD_1 matrix are solved by Schur-based methods. The numerical results are reported.

Keywords: SDD_1 matrix; Schur complement; diagonal-Schur complement; determinant; large scale linear equations; Schur-based method
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1. Introduction

Denote the set of complex matrices with order n by $C^{n\times n}$ and denote $\langle n\rangle = \{1, 2, \cdots, n\}$. Let α and β be given subsets of $\langle n\rangle$. The set $\bar{\alpha}$ is the complement of α, (i.e., $\bar{\alpha} = \langle n\rangle \setminus \alpha$). $A(\alpha,\beta)$ stands for the sub-matrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular, then the Schur complement of A with respect to $A(\alpha)$ is denoted by A/α, i.e.,

$$A/\alpha = A(\bar{\alpha}) - A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1}A(\alpha, \bar{\alpha}).$$

Schur complements of H-matrices have wide applications in numerical analysis, control theory, matrix theory, and statistics [1, 2]. Particularly, Schur complements can be used to reduce the order of large linear equations [2–4] and compute the determinant of a matrix [5,6]. Therefore, the closure properties of Schur complements have attracted a lot of attention.

Many results on the closure properties of Schur complements of the subclasses of H-matrices have been obtained. The result that the Schur complements of strictly diagonally dominant (SDD) matrices are also SDD matrices was determined by Carlson and Markham in 1979 [7]. The closure properties of Schur complements for doubly strictly doubly diagonally dominant (DSDD) matrices were obtained in 1997 [8, 9]. In 2004, it was proven that Schur complements of H-matrices are also H-matrices [1]. Subsequently, various results on Schur complements related to H-matrices appeared. For example, the closure properties of Schur complements for Σ-SDD matrices were obtained in [10, 11], the closure properties of Schur complements for Dashnic-Zumanovich (DZ) matrices were obtained in [11, 12], and the closure properties of Schur complements for γ-SDD were obtained in [3]. However, there are several subclasses of H-matrices whose Schur complements may not be in the same subclass. For instance, the Schur complements of Nekrasov matrices may not be Nekrasov matrices. It has been proven that Schur complements of Nekrasov matrices with respect to principal the submatrix are Nekrasov matrices by Liu et al. [5]. For the closure property of Schur complements for DZ type (DZT) matrices, Li et al. [2] pointed that the Schur complement of a DZT matrix is not necessarily a DZT matrix, and they reported several closure properties of Schur complements for DZT matrices [2, 13]. Some closure properties of Schur complements of Cvetković-Kostić-Varga type (CKV-type) matrices have been presented in [14]. We recommend [10] for surveys on Schur complements.

The definition of the diagonally dominant degree was first proposed in [15]. Using the diagonally dominant degree of SDD matrices, the location of eigenvalues for the Schur complements of SDD matrices and the bounds of the determinant were investigated [15]. In 2012 , the doubly diagonally dominant degrees for DSDD matrices and its application in the location of the eigenvalues were studied [4, 16]. The Nekrasov diagonally dominant degree of Nekrasov matrices was proposed in [5], and has been applied to estimate the bounds for the determinant of Nekrasov matrices. The γ-diagonally dominant degrees and their applications were obtained in [3,6]. Very recently, the dominant degree for Σ-SDD matrices and its applications were investigated in [17].

Let $A = (a_{ij})$ be a $n \times n$ complex matrix. We say that A is an SDD_1 matrix if

$$|a_{ii}| > \sum_{j \notin N^+(A), j \neq i} |a_{ij}| + \sum_{j \in N^+(A), j \neq i} \frac{r_j(A)}{|a_{jj}|} |a_{ij}|, \text{ for all } i \in \langle n \rangle,$$

where $r_i(A) = \sum_{j \neq i}^{n} |a_{ij}|$, and $N^+(A) = \{i \in \langle n \rangle : |a_{ii}| > r_i(A)\}$. The class of SDD_1 matrices was independently introduced by Huang in 1993 [18] and Peña in 2011 [19]. It is well-known that SDD_1 matrices belong to H-matrices and both DZT and DSDD matrices belong to SDD_1 matrices. Many scientific contributions on SDD_1 matrices have been obtained, such as bounds for the infinity norm of the inverse on SDD_1 matrices [20–22], and the estimation of the determinants for SDD_1 matrices [22, 23].

In [19], Peña found that the Schur complements of SDD_1 matrices may not be SDD_1 matrices, and proposed pivoting strategies of Gaussian eliminations to preserve the closure properties. This paper studies Shur complements of SDD_1 matrices by investigating the relationship between α and $N^+(A)$, and explores the SDD_1 diagonally dominant degree of the Schur complements for SDD_1 matrices. For the case $\alpha \subsetneq N^+(A)$, our results improves [19, Lemmas 3.1 and 3.2, and Theorem 3.4]. The results for the case $\alpha \supsetneq N^+(A)$ are not included in [19]. The proposed results are applied to estimate the determinants of SDD_1 matrices and solve liner equations with the coefficient matrices being an SDD_1 matrices.

The rest of this paper outlined as follows: In Section 2, we introduce some preliminaries and prove that the principal sub-matrix of an SDD_1 matrix is also an SDD_1 matrix; in Section 3, the SDD_1 diagonally dominant degrees of the Schur complements are proposed, and the conditions under which the Schur complement of an SDD_1 matrix is also an SDD_1 matrix are presented; and in Section 4, we estimate the upper and lower bounds of the determinant for an SDD_1 matrix, and we solve large scale linear equations with the coefficient matrices being SDD_1 matrices with Schur-based methods. The numerical results show that our bounds of the determinants for SDD_1 matrices may be better than the result in [23] in some cases, and the Schur-based Gauss-Seidel method performs well when the Schur complement is an SDD matrix.

2. The preliminaries

For $A = (a_{ij}) \in C^{n\times n}$, $|A| = (|a_{ij}|)$. $det(A)$ denotes the determinant of A. The comparison matrix of A is denoted by $\mu(A) = (u_{ij})_{n\times n}$, where $u_{ij} = |a_{ij}|$ for $i = j$ and $u_{ij} = -|a_{ij}|$ for $i \neq j$. Denote

$$r_i(A) = \sum_{j\neq i}^{n} |a_{ij}|, \tag{2.1}$$

$$N^+(A) = \{i \in \langle n\rangle : |a_{ii}| > r_i(A)\}, \tag{2.2}$$

$$N^-(A) = \{i \in \langle n\rangle : |a_{ii}| \leq r_i(A)\}, \tag{2.3}$$

$$p_i(A) = \sum_{j\in N^-(A), j\neq i} |a_{ij}| + \sum_{j\in N^+(A), j\neq i} \frac{r_j(A)}{|a_{jj}|}|a_{ij}|. \tag{2.4}$$

For any nonempty proper subset S of $\langle n\rangle$, and $i \in \langle n\rangle$, denote

$$r_i^S(A) = \sum_{j\neq i, j\in S}^{n} |a_{ij}|, \tag{2.5}$$

$$p_i^S(A) = \sum_{j\in N^-(A)\cap S, j\neq i} |a_{ij}| + \sum_{j\in N^+(A)\cap S, j\neq i} \frac{r_j(A)}{|a_{jj}|}|a_{ij}|. \tag{2.6}$$

It is easy to see that

$$r_i(A) = r_i^S(A) + r_i^{\bar{S}}(A), \quad p_i(A) = p_i^S(A) + p_i^{\bar{S}}(A).$$

Definition 2.1. Let $A \in C^{n\times n}$. The matrix A is called an M-matrix if it can be written in the form of $A = sI - B$, where I is the identity matrix, B is a nonnegative matrix, $s > \rho(B)$, and $\rho(B)$ is the spectral radius of B.

Definition 2.2. The matrix $A \in C^{n\times n}$ is called an H-matrix if $\mu(A)$ is an M-matrix.

Lemma 2.1. *[10, p. 5] Let $A \in C^{n\times n}$ and α be a nonempty proper subset of $\langle n\rangle$. If $A(\alpha)$ is nonsingular, then*

$$det(A) = det(A(\alpha))det(A/\alpha). \tag{2.7}$$

Definition 2.3. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a strictly diagonally dominant (SDD) matrix if for all $i \in \langle n\rangle$, then it holds that $|a_{ii}| > r_i(A)$.

Definition 2.4. [19] Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is an SDD_1 matrix if

$$|a_{ii}| > p_i(A), \text{ for all } i \in N^-(A).$$

Remark 2.1. A matrix $A \in C^{n\times n}$ is an SDD_1 matrix, if and only if

$$|a_{ii}| > p_i(A), \forall i \in \langle n\rangle.$$

Remark 2.2. For an SDD_1 matrix A, it trivially holds that $N^+(A) \neq \emptyset$. We always assume $N^+(A) \subsetneq \langle n\rangle$ since if $N^+(A) = \langle n\rangle$ (i.e., A is an SDD matrix), then the Schur and diagonal-Schur complements of SDD matrices are also SDD matrices [1, 7].

Lemma 2.2. *[20] Let $A \in C^{n\times n}$ be an SDD_1 matrix. Let $W = diag(w_1, w_2, \ldots, w_n)$, with $w_i = \frac{p_i(A)}{|a_{ii}|} + \varepsilon$ for $i \in N^+(A)$ and $w_i = 1$ for $i \in N^-(A)$, where*

$$0 < \varepsilon < \min_{i\in N^-(A)} \frac{|a_{ii}| - p_i(A)}{\sum\limits_{j\in N^+(A), j\neq i} |a_{ij}|}.$$

Then, AW is an SDD matrix.

Lemma 2.3. *[24, p. 131] If A is an H-matrix, then $[\mu(A)]^{-1} \geq |A^{-1}|$.*

Lemma 2.4. *[24, p. 117] If A is an M-matrix, then $det(A) > 0$.*

Lemma 2.5. *[25] Let $b > c \geq 0$, $r > 0$ and $a \geq rb$. Then,*

$$\frac{b-c}{a-rc} \leq \frac{b}{a}.$$

Lemma 2.6. *Let $A \in C^{n\times n}$ be an SDD_1 matrix. For any nonempty proper subset α of $\langle n\rangle$, we have $A(\alpha)$ is also an SDD_1 matrix. Particularly, $A(\alpha)$ is an SDD matrix if $\alpha \subseteq N^+(A)$ or $\alpha \subseteq N^-(A)$.*

Proof. Denote $A = (a_{ij})$. Let α and $\bar{\alpha}$ be defined as in (2.8) and (2.9), respectively. For any $t \in \langle k\rangle$, if $i_t \in N^+(A)$, then it trivially holds that $t \in N^+(A(\alpha))$. If $i_t \in N^-(A)$, it holds that

$$\begin{aligned}
|a_{i_t i_t}| &> p_{i_t}(A) \\
&\geq \sum_{i_u\in N^+(A)} \frac{r_{i_u}(A)}{|a_{i_u i_u}|}|a_{i_t i_u}| + \sum_{i_u\in N^-(A), u\neq t} |a_{i_t i_u}| \\
&\geq \sum_{i_u\in N^+(A)} \frac{r_u(A(\alpha))}{|a_{i_u i_u}|}|a_{i_t i_u}| + \sum_{i_u\in N^-(A), u\neq t} |a_{i_t i_u}| \\
&= \sum_{i_u\in N^+(A)} \frac{r_u(A(\alpha))}{|a_{i_u i_u}|}|a_{i_t i_u}| + \\
&\quad \Big(\sum_{i_u\in N^-(A), u\in N^-(A(\alpha)), u\neq t} |a_{i_t i_u}| + \sum_{i_u\in N^-(A), u\in N^+(A(\alpha)), u\neq t} |a_{i_t i_u}|\Big) \\
&\geq \sum_{i_u\in N^+(A)} \frac{r_u(A(\alpha))}{|a_{i_u i_u}|}|a_{i_t i_u}| + \sum_{i_u\in N^-(A), u\in N^-(A(\alpha)), u\neq t} |a_{i_t i_u}| + \\
&\quad \sum_{i_u\in N^-(A), u\in N^+(A(\alpha)), u\neq t} \frac{r_u(A(\alpha))}{|a_{i_u i_u}|}|a_{i_t i_u}| \\
&= p_t(A(\alpha)).
\end{aligned}$$

Hence, $A(\alpha)$ is an SDD_1 matrix.

If $\alpha \subseteq N^+(A)$, then

$$|a_{i_ti_t}| > r_{i_t}(A) \geq r_t(A(\alpha)), \forall t \in \langle k\rangle,$$

which implies that $A(\alpha)$ is an SDD matrix.

If $\alpha \subseteq N^-(A)$, then

$$|a_{i_ti_t}| > p_{i_t}(A) \geq \sum_{i_u \in N^-(A), u \neq t} |a_{i_ti_u}| \geq r_t(A(\alpha)), \forall t \in \langle k\rangle,$$

which implies that $A(\alpha)$ is an SDD matrix. □

For any given matrix $A = (a_{ij}) \in C^{n\times n}$, we always assume that α is a nonempty proper subset of $\langle n\rangle$, (i.e., $\emptyset \neq \alpha \subsetneq \langle n\rangle$). The elements in both of α and $\bar{\alpha} = \langle n\rangle \setminus \alpha$ are listed in increasing order, i.e.,

$$\alpha = \{i_1, i_2, \cdots, i_k\}, \quad i_1 < i_2 < \cdots < i_k, \tag{2.8}$$

$$\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}, \quad j_1 < j_2 < \cdots < j_l. \tag{2.9}$$

For any $j_u \in \bar{\alpha}$, denote

$$x_u = (a_{j_ui_1}, \ldots, a_{j_ui_k})^T, \tag{2.10}$$

$$y_u = (a_{i_1j_u}, \ldots, a_{i_kj_u})^T. \tag{2.11}$$

3. Schur complements of SDD_1 matrices

Given a matrix A, let $N^+(A)$, $N^-(A)$, α, $\bar{\alpha}$, x_u, and y_u be defined as in (2.2), (2.3), and (2.8)–(2.11), respectively. Denote $A/\alpha = (a'_{uv})_{l\times l}$. By the definition of the Schur complement, we have for any $u, v \in \langle l\rangle$,

$$a'_{uv} = a_{j_uj_v} - x_u^T[A(\alpha)]^{-1}y_v.$$

Denote

$$\sum_{v=1}^{l} |y_v| = (r_{i_1}^{\bar{\alpha}}(A), \ldots, r_{i_k}^{\bar{\alpha}}(A))^T, \tag{3.1}$$

$$p_{\alpha}^{\bar{\alpha}}(A) = (p_{i_1}^{\bar{\alpha}}(A), \ldots, p_{i_k}^{\bar{\alpha}}(A))^T. \tag{3.2}$$

By (2.6), we have the following:

$$p_{\alpha}^{\bar{\alpha}}(A) = \sum_{j_v \in N^+(A)} \frac{r_{j_v(A)}}{|a_{j_vj_v}|}|y_v| + \sum_{j_v \in N^-(A)} |y_v|.$$

Lemma 3.1. *Let $A \in C^{n\times n}$ and $\emptyset \neq \alpha \supseteq N^+(A)$. Let α, $\bar{\alpha}$, x_u, and y_u be defined as in (2.8)–(2.11), respectively. If $|a_{i_ti_t}| > p_{i_t}(A)$ for all $i_t \in N^-(A)$, then for any $u \in \langle l\rangle$, we have the following*

$$|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| \leq p_{j_u}^{\alpha}(A). \tag{3.3}$$

Proof. For any given $j_u \in \bar{\alpha} \subseteq N^-(A)$, we define a matrix as follows:

$$D = \begin{bmatrix} \mu(A(\alpha)) & -\sum\limits_{v=1}^{l} |y_v| \\ -|x_u^T| & p_{j_u}^{\alpha}(A) + \varepsilon \end{bmatrix} := (d_{tm}) \ (\varepsilon > 0). \tag{3.4}$$

Then, for any $t \in \langle k \rangle$, we have the following

$$t \in N^+(D) \Leftrightarrow i_t \in N^+(A), \quad t \in N^-(D) \Leftrightarrow i_t \in N^-(A).$$

We only need to consider the t-th row of D with $i_t \in N^-(A)$. It holds that

$$\begin{aligned} & |d_{tt}| = |a_{i_t i_t}| \\ > \ & p_{i_t}(A) = p_{i_t}^{\alpha}(A) + p_{i_t}^{\bar{\alpha}}(A) = p_{i_t}^{\alpha}(A) + r_{i_t}^{\bar{\alpha}}(A) \\ = \ & \sum_{i_m \in N^-(A), m \neq t} |a_{i_t i_m}| + \sum_{i_m \in N^+(A)} \frac{r_{i_m}(A)}{|a_{i_m i_m}|} |a_{i_t i_m}| + r_{i_t}^{\bar{\alpha}}(A) \\ = \ & \sum_{m \in N^-(D), m \neq t} |a_{i_t i_m}| + \sum_{m \in N^+(D)} \frac{r_m(D)}{|a_{i_m i_m}|} |a_{i_t i_m}| + |d_{t(k+1)}| \\ \geq \ & p_t(D). \end{aligned}$$

For the $(k+1)$-th row of D, it is clear that

$$\begin{aligned} & |d_{k+1 k+1}| = p_{j_u}^{\alpha}(A) + \varepsilon \\ > \ & p_{j_u}^{\alpha}(A) = \sum_{i_m \in N^-(A)} |a_{j_u i_m}| + \sum_{i_m \in N^+(A)} \frac{r_{i_m}(A)}{|a_{i_m i_m}|} |a_{j_u i_m}| \\ = \ & \sum_{m \in N^-(D)} |a_{j_u i_m}| + \sum_{m \in N^+(D)} \frac{r_m(D)}{|a_{i_m i_m}|} |a_{j_u i_m}| \\ = \ & p_{k+1}(D). \end{aligned}$$

Then, D is an SDD_1 matrix. It follows from Lemmas 2.4 and 2.1 that $det(D) = det(\mu(D)) > 0$, $det(\mu[A(\alpha)]) > 0$, and

$$det(D) = det(\mu[A(\alpha)])(p_{j_u}^{\alpha}(A) + \varepsilon - |x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|).$$

Then,

$$|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| < p_{j_u}^{\alpha}(A) + \varepsilon.$$

we take the limit by letting $\varepsilon \to 0$, which produces (3.3). □

Theorem 3.1. *Let $A \in C^{n \times n}$ and $\emptyset \neq \alpha \supseteq N^+(A)$. Let α, $\bar{\alpha}$, x_u, and y_u be defined as in (2.8)–(2.11), respectively. If $|a_{i_t i_t}| > p_{i_t}(A)$ for all $i_t \in N^-(A)$, then for any $u \in \langle l \rangle$, we have the following:*

$$|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_u j_u}| - p_{j_u}(A), \tag{3.5}$$

$$|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + p_{j_u}(A), \tag{3.6}$$

$$\frac{r_u(A/\alpha)}{|a'_{uu}|} \leq \frac{p_{j_u}(A)}{|a_{j_u j_u}|}. \tag{3.7}$$

Proof. For any $u \in \langle l \rangle$ (i.e., $j_u \in \bar{\alpha}$), we have the following:

$$\begin{aligned} & |a'_{uu}| - r_u(A/\alpha) \\ = \; & |a_{j_u j_u} - x_u^T [A(\alpha)]^{-1} y_u| - \sum_{v \neq u} |a_{j_u j_v} - x_u^T [A(\alpha)]^{-1} y_v| \\ \geq \; & |a_{j_u j_u}| - |x_u^T| [\mu(A(\alpha))]^{-1} |y_u| - \sum_{v \neq u} |a_{j_u j_v}| - \sum_{v \neq u} |x_u^T| [\mu(A(\alpha))]^{-1} |y_v| \\ = \; & |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) - |x_u^T| [\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|. \end{aligned} \tag{3.8}$$

Then, by (3.3), we obtain

$$|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) - p_{j_u}^{\alpha}(A) = |a_{j_u j_u}| - p_{j_u}(A),$$

which implies that (3.5) holds. Similarly, we can obtain (3.6). Meanwhile, it holds that

$$\begin{aligned} \frac{r_u(A/\alpha)}{|a'_{uu}|} & = \frac{\sum\limits_{v \neq u} |a_{j_u j_v} - x_u^T [A(\alpha)]^{-1} y_v|}{|a_{j_u j_u} - x_u^T [A(\alpha)]^{-1} y_u|} \\ & \leq \frac{\sum\limits_{v \neq u} |a_{j_u j_v}| + \sum\limits_{v \neq u} |x_u^T| [\mu(A(\alpha))]^{-1} |y_v|}{|a_{j_u j_u}| - |x_u^T| [\mu(A(\alpha))]^{-1} |y_u|} \\ & = \frac{\sum\limits_{v \neq u} |a_{j_u j_v}| + |x_u^T| [\mu(A(\alpha))]^{-1} \sum\limits_{v \neq u} |y_v|}{|a_{j_u j_u}| - |x_u^T| [\mu(A(\alpha))]^{-1} |y_u|} \\ & \leq \frac{\sum\limits_{v \neq u} |a_{j_u j_v}| + |x_u^T| [\mu(A(\alpha))]^{-1} \sum\limits_{v=1}^{l} |y_v|}{|a_{j_u j_u}|} \text{ (by Lemma 2.5)} \\ & = \frac{r_{j_u}^{\bar{\alpha}} + |x_u^T| [\mu(A(\alpha))]^{-1} \sum\limits_{v=1}^{l} |y_v|}{|a_{j_u j_u}|}. \end{aligned} \tag{3.9}$$

By (3.3), we have the following:

$$\frac{r_u(A/\alpha)}{|a'_{uu}|} \leq \frac{r_{j_u}^{\bar{\alpha}}(A) + p_{j_u}^{\alpha}(A)}{|a_{j_u j_u}|} = \frac{p_{j_u}(A)}{|a_{j_u j_u}|},$$

that is, (3.7) holds. □

Corollary 3.1. *Let $A \in C^{n \times n}$ be an SDD_1 matrix, and $\alpha \supseteq N^+(A)$. Then, A/α is an SDD matrix.*

Proof. By (3.5) we have the following $|a'_{uu}| - r_u(A/\alpha) > 0$ for any $u \in \langle l \rangle$, which implies that A/α is an SDD matrix. □

Remark 3.1. We can obtain Corollary 3.1 by using the scaling matrices. Let X be defined as in Lemma 2.2. Then $C = AX = (c_{ij})$ is an SDD matrix. Observe that

$$C/\alpha = (A/\alpha) X(\overline{\alpha}).$$

If $\alpha \supseteq N^+(A)$, then $X(\overline{\alpha}) = I$, which leads to $C/\alpha = A/\alpha$. We know that the Schur complement of an SDD matrix is also an SDD matrix. Hence, A/α is an SDD matrix.

Theorem 3.2. *Let $A \in C^{n\times n}$ and $\emptyset \neq \alpha \subseteq N^+(A)$. Let α, $\bar{\alpha}$, x_u, and y_u be defined as in (2.8)–(2.11), respectively. For any $j_u \in N^+(A)$, we have the following:*

$$|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_u j_u}| - r_{j_u}(A) + \omega_{j_u} \geq |a_{j_u j_u}| - r_{j_u}(A), \tag{3.10}$$

$$|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + r_{j_u}(A) - \omega_{j_u} \leq |a_{j_u j_u}| + r_{j_u}(A), \tag{3.11}$$

$$\frac{r_u(A/\alpha)}{|a'_{uu}|} \leq \frac{r_{j_u}(A) - \omega_{j_u}}{|a_{j_u j_u}|} \leq \frac{r_{j_u}(A)}{|a_{j_u j_u}|}, \tag{3.12}$$

where $\omega_{j_u} := r^{\alpha}_{j_u}(A) - p^{\alpha}_{j_u}(A)$.

Proof. For any given $j_u \in N^+(A)$, let D be defined as in (3.4). Since $\alpha \subseteq N^+(A)$, then $\langle k\rangle \subseteq N^+(D)$. For the $(k+1)$-th row of D, it holds that

$$|d_{k+1k+1}| = p^{\alpha}_{j_u}(A) + \varepsilon > p^{\alpha}_{j_u}(A) = \sum_{i_m\in N^+(A)} \frac{r_{i_m}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| = \sum_{m\in N^+(D)} \frac{r_m(D)}{|a_{i_m i_m}|}|a_{j_u i_m}| = p_{k+1}(D).$$

Then D is an SDD_1 matrix. By the similar deduction in Lemma 3.1, we obtain (3.3). Combining (3.8) and (3.9), we have the following:

$$\begin{aligned}
& |a'_{uu}| - r_u(A/\alpha) \\
\geq\ & |a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A) - |x_u^T|[\mu(A(\alpha))]^{-1}\sum_{v=1}^{l}|y_v| \\
\geq\ & |a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A) - p^{\alpha}_{j_u}(A) \text{ (by (3.3))} \\
=\ & |a_{j_u j_u}| - r_{j_u}(A) + \omega_{j_u},
\end{aligned}$$

which implies (3.10). Similarly, (3.11) holds. Meanwhile,

$$\frac{r_u(A/\alpha)}{|a'_{uu}|} \leq \frac{r^{\bar{\alpha}}_{j_u}(A) + p^{\alpha}_{j_u}(A)}{|a_{j_u j_u}|} = \frac{r_{j_u}(A) - \omega_{j_u}}{|a_{j_u j_u}|},$$

that is (3.12) hold. □

Theorem 3.3. *Let $A \in C^{n\times n}$ and $\alpha \subseteq N^+(A)$. Let α, $\bar{\alpha}$, x_u, and y_u be defined as in (2.8)–(2.11), respectively. For any $j_u \in N^-(A)$, we have the following:*

$$|a'_{uu}| - p_u(A/\alpha) \geq |a_{j_u j_u}| - p_{j_u}(A) + \delta_{j_u} \geq |a_{j_u j_u}| - p_{j_u}(A), \tag{3.13}$$

$$|a'_{uu}| + p_u(A/\alpha) \leq |a_{j_u j_u}| + p_{j_u}(A) - \delta_{j_u} \leq |a_{j_u j_u}| + p_{j_u}(A), \tag{3.14}$$

$$\frac{p_u(A/\alpha)}{|a'_{uu}|} \leq \frac{p_{j_u}(A) - \delta_{j_u}}{|a_{j_u j_u}|} \leq \frac{p_{j_u}(A)}{|a_{j_u j_u}|}, \tag{3.15}$$

where

$$\delta_{j_u} = p^{\alpha}_{j_u}(A) - \sum_{m=1}^{k} \frac{r^{\alpha}_{i_m}(A) + p^{\bar{\alpha}}_{i_m}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| = \sum_{m=1}^{k} \frac{r^{\bar{\alpha}}_{i_m}(A) - p^{\bar{\alpha}}_{i_m}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}|.$$

Proof. Since $\alpha \subseteq N^+(A)$, then $\bar{\alpha} \supseteq N^-(A)$. For any $j_u \in N^-(A)$, we define D as follows

$$D = \begin{bmatrix} \mu(A(\alpha)) & -p_{\alpha}^{\bar{\alpha}}(A) \\ -|x_u^T| & \sum\limits_{m=1}^{k} \frac{r_{i_m}^{\alpha}(A)+p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| + \varepsilon \end{bmatrix} := (d_{tm})\ (\varepsilon > 0),$$

where $p_{\alpha}^{\bar{\alpha}}(A)$ is defined as in (3.2) It is easy to see that $\langle k \rangle \subseteq N^+(D)$. For the $(k+1)$-th row of D, it holds that

$$\begin{aligned} |d_{k+1k+1}| &= \sum_{i_m \in N^+(A)} \frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| + \varepsilon \\ &> \sum_{i_m \in N^+(A)} \frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| \\ &= \sum_{m \in N^+(D)} \frac{r_m(D)}{|a_{i_m i_m}|}|a_{j_u i_m}| = p_{k+1}(D). \end{aligned}$$

Then D is an SDD_1 matrix. By the similar deduction of Lemma 3.1, we have the following:

$$|x_u^T|[\mu(A(\alpha))]^{-1} p_{\alpha}^{\bar{\alpha}}(A) \leq \sum_{m=1}^{k} \frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}|. \tag{3.16}$$

Noticing that $\alpha \subseteq N^+(A)$, we have the following:

$$\begin{aligned} p_{j_u}^{\alpha}(A) &= \sum_{i_m \in \alpha} \frac{r_{i_m}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| \\ &= \sum_{m=1}^{k} \frac{r_{i_m}^{\alpha}(A) + r_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}| \\ &\geq \sum_{m=1}^{k} \frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_m i_m}|}|a_{j_u i_m}|. \end{aligned}$$

Then we have the following:

$$\begin{aligned} & |a'_{uu}| - p_u(A/\alpha) \\ = \ & |a_{j_u j_u} - x_u^T[A(\alpha)]^{-1} y_u| - (\sum_{v \in N^+(A/\alpha), v \neq u} \frac{r_v(A/\alpha)}{|a'_{vv}|}|a'_{uv}| + \sum_{v \in N^-(A/\alpha), v \neq u} |a'_{uv}|) \\ \geq \ & |a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{v \in N^+(A/\alpha), v \neq u} \frac{r_v(A/\alpha)}{|a'_{vv}|}|a'_{uv}| - \sum_{v \in N^-(A/\alpha), v \neq u} |a'_{uv}| \\ = \ & |a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{v \in N^-(A/\alpha), v \neq u} |a'_{uv}| - \\ & (\sum_{j_v \in N^+(A)} \frac{r_v(A/\alpha)}{|a'_{vv}|}|a'_{uv}| + \sum_{j_v \in N^-(A), v \in N^+(A/\alpha), v \neq u} \frac{r_v(A/\alpha)}{|a'_{vv}|}|a'_{uv}|) \end{aligned}$$

$$
\begin{aligned}
&\geq\ |a_{j_uj_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{v\in N^-(A/\alpha),v\neq u}|a'_{uv}| - \sum_{j_v\in N^+(A)}\frac{r_{j_v}(A)}{|a_{j_vj_v}|}|a'_{uv}| - \\
&\quad \sum_{j_v\in N^-(A),v\in N^+(A/\alpha),v\neq u}|a'_{uv}| \text{ (by (3.12))} \\
&=\ |a_{j_uj_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{j_v\in N^+(A)}\frac{r_{j_v}(A)}{|a_{j_vj_v}|}|a'_{uv}| - \sum_{j_v\in N^-(A),v\neq u}|a'_{uv}| \\
&\geq\ |a_{j_uj_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{j_v\in N^+(A)}\frac{r_{j_v}(A)}{|a_{j_vj_v}|}(|a_{j_uj_v}| + |x_u^T|[\mu(A(\alpha))]^{-1}|y_v|) - \\
&\quad \sum_{j_v\in N^-(A),v\neq u}(|a_{j_uj_v}| + |x_u^T|[\mu(A(\alpha))]^{-1}|y_v|) \\
&=\ (|a_{j_uj_u}| - p_{j_u}^{\bar{\alpha}}) - |x_u^T|[\mu(A(\alpha))]^{-1}p_\alpha^{\bar{\alpha}} \\
&\geq\ (|a_{j_uj_u}| - p_{j_u}^{\bar{\alpha}}) - \sum_{m=1}^{k}\frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_mi_m}|}|a_{j_ui_m}| \text{ by (3.16)} \\
&=\ (|a_{j_uj_u}| - p_{j_u}(A)) + \delta_{j_u},
\end{aligned}
$$

where

$$
\delta_{j_u} = p_{j_u}^{\alpha}(A) - \sum_{m=1}^{k}\frac{r_{i_m}^{\alpha}(A) + p_{i_m}^{\bar{\alpha}}(A)}{|a_{i_mi_m}|}|a_{j_ui_m}| \geq 0.
$$

By similar to the deduction of Theorem 3.2, we can get (3.14) and (3.15). □

Corollary 3.2. *Let $A \in C^{n\times n}$ be an SDD_1 matrix and $\emptyset \neq \alpha \subseteq N^+(A)$. Then, A/α is an SDD_1 matrix.*

Proof. Since $\alpha \subseteq N^+(A)$, then $\bar{\alpha} \supseteq N^-(A)$. By (3.13), we have the following:

$$
|a'_{uu}| - p_u(A/\alpha) > 0,\ j_u \in N^-(A).
$$

By (3.10), we have

$$
|a'_{uu}| - p_u(A/\alpha) \geq |a'_{uu}| - r_u(A/\alpha) > 0,\ j_u \in N^+(A).
$$

Then A/α is an SDD_1 matrix. □

Remark 3.2. By Theorems 3.1 and 3.2, we obtain that the Schur complements of SDD matrices are also SDD matrices.

Remark 3.3. Corollary 3.2 can also be obtained from [19, Theorem 3.7]. In fact, our results (Theorems 3.2 and 3.3) improves [19, Lemmas 3.1, 3.2, and 3.4] when $\alpha \subseteq N^+(A)$. The condition $\alpha \supsetneq N^+(A)$ not necessarily included in [19, Theorem 3.7]. For example, we consider the following SDD_1 matrix:

$$A = \begin{bmatrix} 4 & -2 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 20 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 20 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ -1 & -1.5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1.5 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Taking $\alpha = \{1, 2, 3, 4, 5\} \supsetneq N^+(A)$, by [19, (3.2)], we know that

$$A^{(5)} = \begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -1.0000 & -1.2821 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 & -1.3782 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 & -1.0000 & -0.3782 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 & -1.0000 & -0.2821 & 0 \end{bmatrix},$$

where "*" represents non-zero entries in the matrix that are not important in studying the closure property of the Schur complement. For details, one can refer to [19]. It is easy to see that the 5*th* row is not strictly diagonally dominant. Therefore, it can't get the closure of A/α by [19, Theorem 3.7].

Example 3.1. *First, let us consider the following SDD_1 matrix*

$$M = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1.01 \end{bmatrix}. \tag{3.17}$$

Taking $\alpha = \{1\}$, then

$$M/\alpha = \begin{bmatrix} 2.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & -0.51 \end{bmatrix} = (m'_{ts}).$$

We have

$$|m'_{33}| = 0.51 < \frac{1}{2.5}0.5 + \frac{1}{1.5}0.5 = \frac{r_1(M/\alpha)}{|m'_{11}|}|m'_{31}| + \frac{r_2(M/\alpha)}{|m'_{22}|}|m'_{32}| = p_3(M/\alpha),$$

which implies that the Schur complements of M are not necessarily SDD_1 matrices. Let M be defined as in (3.17) and A be defined as follows:

$$A = \begin{bmatrix} M & O \\ O & M \end{bmatrix}. \tag{3.18}$$

Then, $N^-(A) = \{1, 5\}$, $N^+(A) = \{2, 3, 4, 6, 7, 8\}$, *and* A *is an* SDD_1 *matrix. There are* 3 *nonempty proper index subsets of* $\langle 8 \rangle$ *which satisfy* $\alpha \supseteq N^+(A)$, *and the* 3 *Schur complements are all SDD matrices. There are* 60 *nonempty proper index subsets of* $\langle 8 \rangle$ *which satisfy* $\alpha \subsetneq N^+(A)$, *and the* 60 *Schur complements are all* SDD_1 *matrices but not necessarily SDD, such as*

$$A/\{2\} = \begin{bmatrix} 1.6667 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1.01 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1.01 \end{bmatrix}.$$

One the hand, if $\alpha \nsupseteq N^+(A)$ *and* $\alpha \nsubseteq N^+(A)$, *then* A/α *may not be* SDD_1 *matrix. Taking* $\alpha = \{1, 5\}$, *we have the following:*

$$A/\{1,5\} = \begin{bmatrix} 2.5 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0.51 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & -0.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & 1.5 & -0.5 \\ 0 & 0 & 0 & -0.5 & -0.5 & 0.51 \end{bmatrix}.$$

Then, $A/\{1, 5\}$ *is not an* SDD_1 *matrix since* $|a'_{33}| < p_3(A/\alpha)$. *Taking* $\alpha = \{1, 5, 6\}$, *then*

$$A/\{1,5,6\} = \begin{bmatrix} 2.5 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & 1.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 0.51 & 0 & 0 \\ 0 & 0 & 0 & 1.4 & -0.6 \\ 0 & 0 & 0 & -0.6 & 0.41 \end{bmatrix}.$$

Then $A/\{1, 5, 6\}$ *is not an* SDD_1 *matrix since* $|a'_{33}| < p_3(A/\alpha)$. *Hence* A/α *is not necessarily an* SDD_1 *matrix under the condition* $\alpha \supseteq N^-(A)$.

Taking $\alpha = \{1\}$, *we have the following:*

$$A/\{1\} = \begin{bmatrix} 2.5 & -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & 1.5 & -0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0.51 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1.01 \end{bmatrix}.$$

A/α *is not an* SDD_1 *matrix* ($|a'_{33}| < p_3(A/\alpha)$), *which implies that* A/α *is not necessarily an* SDD_1 *matrix under the condition* $\alpha \subsetneq N^-(A)$.

Taking $\alpha = \{1, 6\}$*, then*

$$A/\{1,6\} = \begin{bmatrix} 2.5 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0.51 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.66671 & 1 & \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1.01 \end{bmatrix}.$$

We have A/α *is not an* SDD_1 *matrix* ($|a'_{33}| < p_3(A/\alpha)$)*, which implies that* A/α *is not necessarily an* SDD_1 *matrix under the conditions* $\emptyset \neq \alpha \cap N^+(A) \subsetneq N^+(A)$ *and* $\emptyset \neq \alpha \cap N^-(A) \subsetneq N^-(A)$*. Here we summarizes the results of Schur complements on* SDD_1 *matrices (Table 1).*

Table 1. The Schur complements of SDD_1 matrices for the matrix A.

α	Theorem	A/α
$\alpha \supseteq N^+(A)$	Theorem 3.1	SDD
$\alpha \subsetneq N^+(A)$	Theorem 3.2	SDD_1 (not necessarily SDD, such as $A/\{2\}$)
$\alpha \supseteq N^-(A)$	–	not necessarily SDD_1, such as $A/\{1,5\}$ and $A/\{1,5,6\}$
$\alpha \subsetneq N^-(A)$	–	not necessarily SDD_1, such as $A/\{1\}$
$\emptyset \neq \alpha \cap N^+(A) \subsetneq N^+(A)$, $\emptyset \neq \alpha \cap N^-(A) \subsetneq N^-(A)$	–	not necessarily SDD_1, such as $A/\{1,6\}$

4. The applications

In this section, we first estimate the bounds of the determinants of SDD_1 matrices by using Lemma 2.1 and our proposed results. Then, we solve large scale linear systems by Schur-based methods.

• **The determinants of SDD_1 matrices**

Here, we focus on the determinants of SDD_1 matrices. Let $A = (a_{ij})_{n\times n}$ be an SDD_1 matrix. In [23], Huang gave a lower bound and a upper bound of determinants for A.

Lemma 4.1. *[23, Theorem 4] Let* $A \in C^{n\times n}$ *be an* SDD_1 *matrix; then,*

$$\prod_{i\in\langle n\rangle} l_i \leq |det(A)| \leq \prod_{i\in\langle n\rangle} u_i,$$

where

$$l_i = |a_{ii}| - \frac{1}{x_i}\sum_{j>i} x_j|a_{ij}|,\ \ u_i = |a_{ii}| + \frac{1}{x_i}\sum_{j>i} x_j|a_{ij}|,\ \ a_{n,n+1} = 0,$$

$$x_i = \begin{cases} 1, & i \in N^-(A), \\ \min\limits_{j\in N^-(A)}\{\frac{|a_{jj}|-p_j(A)}{\sum\limits_{t\in N^+(A)}|a_{jt}|}\} + \frac{r_i(A)}{|a_{ii}|}, & i \in N^+(A). \end{cases}$$

Notice that the dominant degrees of the Schur complements can be used to estimate the determinants of matrices. For instance, the determinant bounds of SDD matrices were investigated by using diagonally dominant degrees of the Schur complements [15], the determinant bounds of γ-SDD matrices were investigatedby using γ-diagonally dominant degrees of the Schur complements [6], and the determinant bounds of Nekrasov matrices were investigated by using Nekrasov diagonally dominant degree of the Schur complements [5]. Motivated by the pioneering work, we estimate the SDD_1 matrices' determinants by using SDD_1 diagonally dominant degrees of the Schur complements. Now, we first introduce the following result related to the SDD matrices' determinants which are due to Ostrowski.

Lemma 4.2. *Let $A \in C^{n\times n}$ be an SDD matrix; then,*

$$\prod_{i\in\langle n\rangle}(|a_{ii}| - r_i(A)) \leq |det(A)| \leq \prod_{i\in\langle n\rangle}(|a_{ii}| + r_i(A)).$$

Lemma 4.3. *[23] Let $A \in C^{n\times n}$ be an SDD matrix; then,*

$$\prod_{i\in\langle n\rangle}(|a_{ii}| - \sum_{j>i}|a_{ij}|) \leq |det(A)| \leq \prod_{i\in\langle n\rangle}(|a_{ii}| + \sum_{j>i}|a_{ij}|).$$

Theorem 4.1. *Let $A \in C^{n\times n}$ be an SDD_1 matrix; then,*

$$|det(A)| \geq \prod_{i\in N^-(A)}(|a_{ii}| - p_i(A)) \prod_{i\in N^+(A)}(|a_{ii}| - \sum_{j>i, j\in N^+(A)}|a_{ij}|),$$
$$|det(A)| \leq \prod_{i\in N^-(A)}(|a_{ii}| + p_i(A)) \prod_{i\in N^+(A)}(|a_{ii}| + \sum_{j>i, j\in N^+(A)}|a_{ij}|).$$

Proof. By Corollary 3.1, $A(\alpha)$ and A/α are both SDD matrices when $\alpha = N^+(A)$. By Lemma 2.1, we have the following:

$$|det(A)| = det(A(N^+(A)))det(A/N^+(A)). \tag{4.1}$$

Then by Lemmas 4.2 and 4.3, we have the following:

$$\prod_{i\in N^+(A)}(|a_{ii}| - \sum_{j>i, j\in N^+(A)}|a_{ij}|) \leq det(A(N^+(A))) \leq \prod_{i\in N^+(A)}(|a_{ii}| + \sum_{j>i, j\in N^+(A)}|a_{ij}|), \tag{4.2}$$

and

$$\prod_{u\in\langle l\rangle}(|a'_{uu}| - r_u(A/\alpha)) \leq det(A/N^+(A)) \leq \prod_{u\in\langle l\rangle}(|a'_{uu}| + r_u(A/\alpha)). \tag{4.3}$$

By (3.5) and (3.6), we have the following:

$$|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_u j_u}| - p_{j_u}(A),\ |a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + p_{j_u}(A), \forall j_u \in N^-(A),$$

which, together with (4.1)–(4.3), we complete the proof. □

Theorem 4.2. *Let $A \in C^{n\times n}$ be an SDD_1 matrix. If $\sum_{j\in N^-(A)}|a_{ij}| = \sum_{j\in N^+(A), j<i}|a_{ij}| = 0$ for any $i \in N^-(A)$, and $\sum_{j>i}|a_{ij}| = 0$ for any $i \in N^+(A)$, then we have the following:*

$$\prod_{i\in\langle n\rangle} l_i \leq \prod_{i\in N^-(A)}(|a_{ii}| - p_i(A)) \prod_{i\in N^+(A)}(|a_{ii}| - \sum_{j>i, j\in N^+(A)}|a_{ij}|),$$

$$\prod_{i\in\langle n\rangle} u_i \geq \prod_{i\in N^-(A)} (|a_{ii}| + p_i(A)) \prod_{i\in N^+(A)} (|a_{ii}| + \sum_{j>i, j\in N^+(A)} |a_{ij}|),$$

where l_i and u_i are defined in Lemma 4.1.

Proof. For any $i \in N^+(A)$, it is clear that $x_i > \frac{r_i(A)}{|a_{ii}|}$. By $\sum_{j\in N^-(A)} |a_{ij}| = \sum_{j\in N^+(A), j<i} |a_{ij}| = 0$ for any $i \in N^-(A)$, we have the following:

$$l_i = |a_{ii}| - \sum_{j>i} x_j |a_{ij}| = |a_{ii}| - \sum_{j\in N^+(A), j>i} x_j |a_{ij}| \leq |a_{ii}| - \sum_{j\in N^+(A), j>i} \frac{r_j(A)}{|a_{jj}|} |a_{ij}|$$

and

$$|a_{ii}| - p_i(A) = |a_{ii}| - \sum_{j\in N^+(A)} \frac{r_j(A)}{|a_{jj}|} |a_{ij}| = |a_{ii}| - \sum_{j\in N^+(A), j>i} \frac{r_j(A)}{|a_{jj}|} |a_{ij}|,$$

that is,

$$l_i \leq |a_{ii}| - p_i(A), \forall i \in N^-(A).$$

Similarly, we have the following:

$$u_i \geq |a_{ii}| + p_i(A), \forall i \in N^-(A).$$

By $\sum_{j>i} |a_{ij}| = 0$, we have the following:

$$l_i = u_i = |a_{ii}|, \forall i \in N^+(A).$$

Then, it holds that

$$\begin{aligned} & \prod_{i\in\langle n\rangle} l_i \\ \leq & \prod_{i\in N^-(A)} (|a_{ii}| - p_i(A)) \prod_{i\in N^+(A)} |a_{ii}| \\ = & \prod_{i\in N^-(A)} (|a_{ii}| - p_i(A)) \prod_{i\in N^+(A)} (|a_{ii}| - \sum_{j>i, j\in N^+(A)} |a_{ij}|), \end{aligned}$$

and

$$\begin{aligned} & \prod_{i\in\langle n\rangle} u_i \\ \geq & \prod_{i\in N^-(A)} (|a_{ii}| + p_i(A)) \prod_{i\in N^+(A)} |a_{ii}| \\ = & \prod_{i\in N^-(A)} (|a_{ii}| + p_i(A)) \prod_{i\in N^+(A)} (|a_{ii}| + \sum_{j>i, j\in N^+(A)} |a_{ij}|). \end{aligned}$$

We complete the proof. □

Remark 4.1. Theorem 4.2 implies that the bounds in Theorem 4.1 are better than those in Lemma 4.1 under certain conditions. In fact, we may get better bounds under weaker conditions. Given an SDD_1 matrix A, let x_i, l_i, and u_i be defined in Lemma 4.1. If $\sum_{j\in N^-(A)} |a_{ij}|$ and $\sum_{j\in N^+(A), j<i} |a_{ij}|$ are very small for any $i \in N^-(A)$, and $\sum_{j>i, j\in N^-(A)} |a_{ij}|$ is very small for any $i \in N^+(A)$, then it is possible for Theorem 4.1 to get a larger lower bound and a smaller upper bound. Moreover, the result in Theorem 4.1 is concise and elegant.

Example 4.1. *Consider the following SDD_1 matrices:*

$$A_1 = \begin{bmatrix} 0.9 & -0.1 \\ -6 & 1.4 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.9 & 0.1 \\ -6 & 1.4 \end{bmatrix}, \ A_3 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1.01 \end{bmatrix}, \ A_5 = \begin{bmatrix} -2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 6 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 10 \end{bmatrix},$$

in which A_4 and A_5 satisfy the conditions in Theorem 4.2. The bounds of determinants for A_1–A_5 are estimated by Lemma 4.1 and Theorem 4.1. It can be seen from Table 2 that the results in Theorem 4.1 are better than the counterparts in Lemma 4.1 in some cases.

Table 2. The determinants of A_1–A_5.

	A_1	A_2	A_3	A_4	A_5
$\lvert det(A_i)\rvert$	0.66	1.86	19	4.07	9000
Lemma 4.1([23])	[0.66, 1.86]	[0.66, 1.86]	[8.6, 39.4]	[0,24.24]	[0,10800]
Theorem 4.1	[0.66, 1.86]	[0.66, 1.86]	[4.67, 54.67]	[1.07,23.17]	[1800,9000]

• **The large scale linear systems with the SDD_1 coefficient matrices**

Linear equations are widely used across numerous fields. For instance, the finite element method in fluid mechanics transforms the problem of solving partial differential equations into the problem of solving systems of linear equations.

We consider the following linear equation:

$$Ax = b, \tag{4.4}$$

where $A \in R^{n\times n}$, and $b \in R^n$. If there is a subset α of $\langle n\rangle$ such that $A(\alpha)$ being nonsingular, Then (4.4) is equivalent to the following linear system with a smaller size:

$$(A/\alpha)x(\bar{\alpha}) = b(\bar{\alpha}) - A(\bar{\alpha}, \alpha)A(\alpha)^{-1}b(\alpha), \tag{4.5}$$

$$A(\alpha)x(\alpha) = -A(\alpha, \bar{\alpha})x(\bar{\alpha}) + b(\alpha). \tag{4.6}$$

This method of reducing the order of large linear equations by Schur complements are called Schur-based method [2–4]. It has been known that Schur-based iteration methods may be more efficient than the classical iteration methods if A is a DZT, SDD, γ-SDD, DSDD or a Nekrasov matrix [2–4]. In this section, we adopt Schur-based iteration methods including Schur-based Jacobi (S-Jacobi) method and Schur-based Gauss-Seidel iteration (S-GS) method, to solve the large scale linear equations (4.4) with A being an SDD_1 matrix.

Here, we present a numerical example. Example 4.2 shows that when A/α and $A(\alpha)$ are both SDD matrices, the performance of S-GS method is significant. All experiments are carried out via MATLAB 2023b on a Windows 11 (64 bit) PC with the configuration: Intel(R) Core(TM) i5-8250U 1.60GHz CPU and 8 GB RAM. The cputime is the sum of time in computing A/α and solving (4.5) and (4.6). The GS, S-GS, and S-Jacobi methods terminate if the residual error satisfies $\|x^{k+1} - x^k\|_\infty < 10^{-6}$. The Jacobi terminates if the residual error satisfies $\|x^{k+1} - x^k\|_\infty < 10^{-6}$ or iteration=1000.

Example 4.2. *Now, we consider the linear system $Ax = b$. Let $b = (1, 1, \ldots, 1)^T \in R^{2n\times1}$,*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R^{2n\times2n},$$

where $n \geq 20$, and A_{ij} (i,j=1,2) is the sub-matrix with the dimension $n \times n$:

$$A_{11} = \begin{bmatrix} n+8 & 1 & \ldots & 1 \\ 1 & n+8 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & n+8 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 4 & \ldots & 4 \\ \vdots & \ddots & \vdots \\ 4 & \ldots & 4 \end{bmatrix}, A_{22} = \begin{bmatrix} (\frac{n}{2}+1)n & & & \\ & (\frac{n}{2}+2)n & & \\ & & \ddots & \\ & & & (\frac{n}{2}+n)n \end{bmatrix}.$$

It is clear that

$$N^-(A) = \langle n \rangle, \ N^+(A) = \langle 2n \rangle \setminus \langle n \rangle.$$

We can testify that A is an SDD_1 matrix. We applied four methods (Jacobi method, S-Jacobi method, GS method and S-GS method) to solve $Ax = b$ with $\alpha = N^+(A)$. In this case, both of $A(\alpha)$ and A/α are SDD matrices. In our implementation, when the GS and S-GS methods terminate, $\|Ax - b\|_2$ is very small (about $\|Ax - b\|_2 \leq 0.01$). When Jacobi method terminates, iteration=1000 ($n = 300$) which implies that the value of $\|Ax - b\|_2$ becomes very large for $n \geq 300$ with Jacobi method. When the S-Jacobi methods terminates, iteration=1002 ($n = 1000$), which implies that the value of $\|Ax - b\|_2$ becomes very large for $n \geq 1000$. See Table 3. The iteration changes of four methods as the order of the matrix increases are shown in Figure 1. The cputime changes of four methods as the order of the matrix increases are shown in Figure 2. We can see that the larger the scale of the equation $Ax = b$, the more efficient the S-GS method is.

Table 3. The value of $||Ax - b||_2$ in Example 4.2.

n	Jacobi method	S-Jacobi method	GS method	S-GS method
n=100	0.0040879	0.00083091	3.9543e-05	2.1112e-05
n=500	21.721	0.010876	0.00055659	0.00062348
n=1000	5423.9	0.099243	0.0016253	0.0020077
n=1500	45214	15.59	0.0030184	0.0038533
n=2000	1.4882e+05	222.49	0.0046727	0.0060642

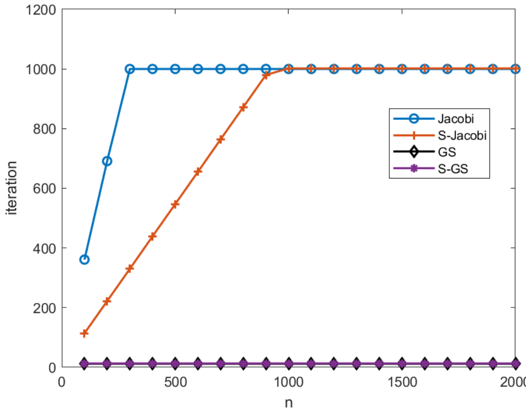

Figure 1. The iterations.

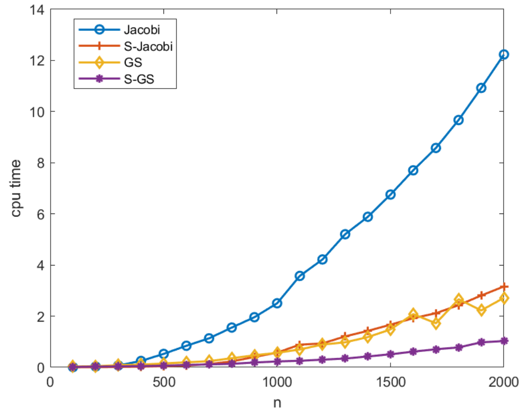

Figure 2. The cputime.

5. Conclusions

In this paper, we studied the Schur complement of SDD_1 matrices. The main results are summarized as follows (for any given SDD_1 matrix $A = (a_{ij}) \in C^{n\times n}$ and $\emptyset \neq \alpha \subseteq \langle n\rangle$):

• We proved that $A(\alpha)$ is also an SDD_1 matrix (Lemma 2.6). Particularly, $A(\alpha)$ is an SDD matrix if $\alpha \subseteq N^+(A)$ or $\alpha \subseteq N^-(A)$.

• We found that the Schur complement $A/\alpha = (a'_{ij})$ may not be an SDD_1 matrix, and proved that A/α was also an SDD_1 matrix if $\alpha \supseteq N^+(A)$ or $\alpha \subseteq N^+(A)$. Particularly, A/α was an SDD matrix if $\alpha \supseteq N^+(A)$. The relationship between

$$|a'_{uu}| - r_u(A/\alpha) \text{ and } |a_{j_u j_u}| - r_{j_u}(A), \quad \frac{r_u(A/\alpha)}{|a'_{uu}|} \text{ and } \frac{r_{j_u}(A)}{|a_{j_u j_u}|},$$

$$|a'_{uu}| - p_u(A/\alpha) \text{ and } |a_{j_u j_u}| - p_{j_u}(A), \quad \frac{p_u(A/\alpha)}{|a'_{uu}|} \text{ and } \frac{p_{j_u}(A)}{|a_{j_u j_u}|},$$

are given (Theorems 3.1–3.3).

• We obtained the upper and lower bounds for the determinants of SDD_1 matrices which were very elegant by applying the results on the Schur complements (Theorem 4.1).

• The proposed results that on the Schur complements were also be applied to solve large scale linear equations by the Schur-based methods. The S-GS method performed excellent when $A(\alpha)$ and A/α

are both SDD matrices. In the future research, we will study real applications from physics and engineering by using Schur complements.

Author contributions

Shiyun Wang: Conceptualization, methodology, writing-original draft; Yun Li: Writing review and editing; Wanfu Tian: Data duration, visualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article. All authors have read and approved the final version of the manuscript for publication.

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