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Research article

Parametric Marcinkiewicz integral on grand variable Herz-Morrey spaces

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Abstract: We establish the boundedness of the parametric Marcinkiewicz integral μ_Ω^ρ and its higher-order commutators $[\Lambda^m, \mu_\Omega^\rho]$ with BMO symbols on grand variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$. These results also apply to grand variable Herz spaces, including the case where $\alpha(\cdot)$ is constant.

Keywords: parametric Marcinkiewicz integral; higher-order commutator; grand Herz-Morrey space; variable exponent
Mathematics Subject Classification: 42B20, 42B35

1. Introduction

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n (n \geq 2)$ with the normalized Lebesgue measure $d\sigma = d\sigma(x')$, where $x' = \frac{x}{|x|}$ for any $x \neq 0$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.1}$$

Then, the parametric Marcinkiewicz integral μ_Ω^ρ is defined as

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $0 < \rho < n$. If $\rho = 1$, then μ_Ω^ρ is just the Marcinkiewicz integral μ_Ω introduced by Stein in [27]. Hörmander [11] proved that if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ with $0 < \alpha \leq 1$, then μ_Ω^ρ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Ding, Lu, and Yabuta [6] extended this result for $p = 2$ to $\Omega \in L\log^+ L(\mathbb{S}^{n-1})$, while Al-Salman [2] showed $L^p(\mathbb{R}^n)$-boundedness of μ_Ω^ρ for any $1 < p < \infty$ with $\Omega \in H^1(\mathbb{S}^{n-1})$.

The space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* := \sup_B \frac{1}{|B|} \int_B |f(y) - f_B| dy < \infty,$$

where and below $B = B(x,r) := \{z \in \mathbb{R}^n : |x - z| < r\}$, and $f_B := \frac{1}{|B|}\int_B f(t)dt$. Given a natural number m and $\Lambda \in \mathrm{BMO}(\mathbb{R}^n)$, the higher-order commutators $[\Lambda^m, \mu_\Omega^\rho]$ generated by μ_Ω^ρ and Λ are defined as follows:

$$[\Lambda^m, \mu_\Omega^\rho](f)(x) = \left(\int_0^\infty \left|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[\Lambda(x)-\Lambda(y)]^m f(y)dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

These commutators are of interest in various situations, as discussed in [18,22,31] and their references.

In recent years, there has been significant progress in the development of the theory of variable exponent function spaces. These spaces are of particular interest due to their applications in partial differential equations [7], electrorheological fluids [25], and image restoration [3]. The concept of variable Lebesgue spaces was first introduced by Orlicz [23], but it was Kováčik and Rákosník [17] who initiated the systematic study of these spaces and the corresponding variable Sobolev spaces. Subsequently, a variety of other function spaces and classical operators in harmonic analysis have been extensively investigated with variable exponents. See, for example, [10, 21, 33].

The Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ are crucial in harmonic analysis and partial differential equations. These spaces provide an effective substitute for the ordinary Hardy spaces in the study of singular integrals. They are also important in the regularity theory for elliptic and parabolic equations in divergence form, see [24]. Izuki [12, 13] introduced Herz spaces with variable exponent q and Almeida and Direhem [1] investigated Herz spaces with variable exponents $q(\cdot)$ and $\alpha(\cdot)$, and obtained boundedness results for a broad range of sublinear operators on these spaces. The most generalized Herz spaces $K_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ and $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p(\cdot)}(\mathbb{R}^n)$ were recently introduced in discrete and continuous settings by Izuki and Noi [14] and Samko [26], respectively, with all three main indices being variable exponents.

The grand variable Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$ introduced by Nafis et al. [20] represents a significant extension of the variable Herz space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(\mathbb{R}^n)$. In [20], the authors established the boundedness of the Marcinkiewicz integral μ_Ω on the space $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$. The grand variable Herz-Morrey space, introduced by Sultan et al. [28–30], extends the grand variable Herz space, and they showed the boundedness of higher-order commutators of the Marcinkiewicz integral operator and Hardy operators. Inspired by previous works [20, 29, 30], it is natural to consider the boundedness of the operators μ_Ω^ρ and $[\Lambda^m, \mu_\Omega^\rho]$ on grand variable Herz-Morrey spaces. It is noteworthy that, even in the case where $\alpha(\cdot)$ is constant, the results presented in Theorems 6-10 are still new.

In this paper, $\mathbb{Z}$ represents all integers. $\mathbf{1}_E$ denotes the characteristic function of a measurable set E. $p'(\cdot)$ is the conjugate exponent defined by $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. C represents a positive constant that may vary in different contexts. $\phi \lesssim \psi$ means $\phi \leqslant C\psi$, and $\phi \approx \psi$ means $\phi \lesssim \psi \lesssim \phi$.

2. Preliminaries

We start by briefly discussing the variable Lebesgue spaces; for more details see the recent monographs [4, 5, 15, 16].

Let $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function. We assume that

$$1 \leq q_- \leq q(x) \leq q_+ < \infty,$$

where $q_- := \operatorname{ess}\inf_{x\in\mathbb{R}^n} q(x)$ and $q_+ := \operatorname{ess}\sup_{x\in\mathbb{R}^n} q(x)$. The variable Lebesgue space $L^{q(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f defined on $\mathbb{R}^n$ such that

$$I_{q(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{q(x)}\mathrm{d}x < \infty.$$

This space is a Banach space endowed with the norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \inf\Big\{\mu > 0 : I_{q(\cdot)}\Big(\frac{f}{\mu}\Big) \le 1\Big\}.$$

If $q(x) \equiv q$ is constant, then $L^{q(\cdot)}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ is the standard Lebesgue space.

The set $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot)$ satisfying $q_- > 1$ and $q_+ < \infty$. If $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the generalized Hölder inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \lesssim \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)}, \tag{2.1}$$

holds for every $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, see [17].

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n}\int_{B(x,r)} |f(y)|dy.$$

The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

A function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous if there exists a constant $C_{\log} > 0$ such that

$$|\Phi(x) - \Phi(y)| \le \frac{C_{\log}}{-\log|x-y|}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \mathbb{R}^n. \tag{2.2}$$

If, for some $\Phi_\infty \in \mathbb{R}$, there holds

$$|\Phi(x) - \Phi_\infty| \le \frac{C_{\log}}{\log(e+|x|)}, \quad x \in \mathbb{R}^n, \tag{2.3}$$

then we say $\Phi(\cdot)$ is log-Hölder continuous at infinity.

The class $\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ consists of all exponents $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that satisfy the conditions (2.2) and (2.3), where $q_\infty := \lim_{|x|\to\infty} q(x)$. It is worth noting that when $q_- > 1$, $q(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ if and only if $q'(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$. Additionally, if $q(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then the boundedness of the Hardy-Littlewood maximal operator M on $L^{q(\cdot)}(\mathbb{R}^n)$ implies that $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, see [5, Theorem 4.3.8].

In the following, we denote $B_j = \{x \in \mathbb{R}^n : |x| \leqslant 2^j\}$, $R_j = B_j \backslash B_{j-1}$ and $\mathbf{1}_j = \mathbf{1}_{R_j}$, $j \in \mathbb{Z}$.

Definition 1. *Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < \gamma \le \infty$, and $\beta(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable Herz space $\dot{K}^{\alpha(\cdot),\gamma}_{\beta(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{\beta(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{\vec{0}_n\})$ such that*

$$\|f\|_{\dot{K}^{\alpha(\cdot),\gamma}_{\beta(\cdot)}(\mathbb{R}^n)} := \Big(\sum_{j=-\infty}^{\infty} \|2^{j\alpha(\cdot)} f\mathbf{1}_j\|^\gamma_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{1/\gamma} < \infty,$$

with the usual modifications when $\gamma = \infty$.

Obviously, if $\alpha(\cdot) \equiv \alpha$ is constant, then $\dot{K}^{\alpha(\cdot),\gamma}_{\beta(\cdot)}(\mathbb{R}^n)$ is equivalent to $\dot{K}^{\alpha,\gamma}_{\beta(\cdot)}(\mathbb{R}^n)$ as defined in [12, 13]. In the case where both α and β are constant functions, the Herz spaces $\dot{K}^{\alpha(\cdot),\gamma}_{\beta(\cdot)}(\mathbb{R}^n)$ are equal to $\dot{K}^{\alpha,\gamma}_{\beta}(\mathbb{R}^n)$ as defined in [19].

Definition 2. *Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < \gamma < \infty, \theta > 0, 0 \le \lambda < \infty$, and $\beta(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The grand variable Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$ consists of all $f \in L^{\beta(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{\vec{0}_n\})$ such that*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} := \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \|2^{j\alpha(\cdot)} f \mathbf{1}_j\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} < \infty.$$

If $\lambda = 0$, then grand variable Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$ reduces to grand variable Herz space $\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\beta(\cdot)}(\mathbb{R}^n)$. The result in [32, Proposition 2.4] shows that variable Herz space $\dot{K}^{\alpha(\cdot),\gamma}_{\beta(\cdot)}(\mathbb{R}^n)$ is a subset of grand variable Herz space $\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\beta(\cdot)}(\mathbb{R}^n)$.

3. Boundedness of the parametric Marcinkiewicz integral

In this subsection, we show that the parametric Marcinkiewicz integral μ^ρ_Ω is bounded on grand variable Herz-Morrey spaces. As a result, we also establish the boundedness of this operator on grand variable Herz spaces. We first present some lemmas necessary for proving the main theorems.

Lemma 3. *If $\beta(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and $\tau > 1$, then we have*

$$\frac{1}{\nu_0} R^{\frac{n}{\beta(0)}} \le \|\mathbf{1}_{B(0,\tau R)\backslash B(0,R)}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \le \nu_0 R^{\frac{n}{\beta(0)}}, \quad 0 < R \le 1,$$

and

$$\frac{1}{\nu_\infty} R^{\frac{n}{\beta_\infty}} \le \|\mathbf{1}_{B(0,\tau R)\backslash B(0,R)}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \le \nu_\infty R^{\frac{n}{\beta_\infty}}, \quad R \ge 1,$$

where $\nu_0, \nu_\infty \ge 1$ are dependent on τ but are independent of R.

Lemma 4. *If $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $p > q_+$, and $\frac{1}{q(x)} = \frac{1}{r(x)} + \frac{1}{p}$, then we have*

$$\|fg\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{r(\cdot)}(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}$$

for any $f \in L^{r(\cdot)}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$.

The Lemmas 3 and 4 are shown in [26] and [5], respectively. We refer to Izuki [12] for the following Lemma 5.

Lemma 5. *If $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have for all measurable subsets $A \subset B$,*

$$\frac{\|\mathbf{1}_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_A\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|A|}{|B|}, \quad \frac{\|\mathbf{1}_A\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \lesssim \Big(\frac{|A|}{|B|}\Big)^\kappa,$$

where κ is constant with $0 < \kappa < 1$.

Our main results can be summarized as follows.

Theorem 6. *Let $\rho > n/2$, $\theta > 0$, $0 < \lambda < \infty$, $1 < \gamma < \infty$, $\alpha(\cdot), \beta(\cdot) \in \mathcal{P}_\infty^{log}(\mathbb{R}^n)$, and $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > (\beta')_+$. If α satisfies*

(i) $\lambda - \frac{n}{\beta(0)} < \alpha(0) < n(\frac{1}{\beta'(0)} - \frac{1}{s})$,

(ii) $\lambda - \frac{n}{\beta_\infty} < \alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$,

then the parametric Marcinkiewicz integral μ_Ω^ρ is bounded on $M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that for any $f \in M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)$,

$$\|\mu_\Omega^\rho(f)\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \leq C\|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.$$

Remark 7. *For any f belonging to the grand Herz-Morrey spaces with variable exponent, we can demonstrate that these spaces are embedded into certain weighted Lebesgue spaces with Muckenhoupt weights. This conclusion is derived by employing the ideas from the grand Morrey spaces as outlined in (5.1) of Ho [8] and from the Herz-Morrey spaces with variable exponent discussed in (26) of Ho [9]. Consequently, the definition of $\mu_\Omega^\rho(f)$ is established based on the corresponding definition of the parametric Marcinkiewicz integral operators on the weighted Lebesgue spaces.*

If $\lambda = 0$, slight modifications to the proof of Theorem 6 yield the following result.

Theorem 8. *Let $\rho > n/2$, $\theta > 0$, $1 < \gamma < \infty$, $\alpha(\cdot), \beta(\cdot) \in \mathcal{P}_\infty^{log}(\mathbb{R}^n)$, and $\Omega \in L^s(\mathbb{S}^{n-1})$, $s > (\beta')_+$. If $-\frac{n}{\beta(0)} < \alpha(0) < n(\frac{1}{\beta'(0)} - \frac{1}{s})$ and $-\frac{n}{\beta_\infty} < \alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$, then there exists a constant $C > 0$ such that for any $f \in \dot{K}_{\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)$,*

$$\|\mu_\Omega^\rho(f)\|_{\dot{K}_{\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \leq C\|f\|_{\dot{K}_{\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.$$

Proof of Theorem 6. Let $f \in M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)$. We decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\mathbf{1}_l(x) = \sum_{l=-\infty}^{\infty} f_l(x).$$

By Minkowski's inequality, we have

$$\begin{aligned}
&\|\mu_\Omega^\rho(f)\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \\
&= \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \Big(\sum_{l=-\infty}^{j-1} \|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \Big(\sum_{l=j+1}^{\infty} \|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: E + F + G.
\end{aligned}$$

For F, we have

$$\begin{aligned}
F &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=-\infty}^{-1}\|2^{j\alpha(\cdot)}\mu_{\Omega}^{\rho}(f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\quad+\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=0}^{j_0}\|2^{j\alpha(\cdot)}\mu_{\Omega}^{\rho}(f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&=: F_1+F_2.
\end{aligned}$$

For F_1, by using the boundedness of μ_{Ω}^{ρ} on $L^{\beta(\cdot)}(\mathbb{R}^n)$ (see [31]) and $2^{j\alpha(x)}\approx 2^{j\alpha(0)}$ for $j<0$ and $x\in R_j$, it follows that

$$\begin{aligned}
F_1 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=-\infty}^{-1}2^{j\alpha(0)\gamma(1+\zeta)}\|\mu_{\Omega}^{\rho}(f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=-\infty}^{-1}2^{j\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=-\infty}^{-1}\|2^{j\alpha(\cdot)}f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

For F_2, when $j\geq 0$ and $x\in R_j$, we observe that $2^{j\alpha(x)}\approx 2^{j\alpha_\infty}$ and obtain

$$\begin{aligned}
F_2 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=0}^{j_0}2^{j\alpha_\infty\gamma(1+\zeta)}\|\mu_{\Omega}^{\rho}(f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=0}^{j_0}2^{j\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^{\theta}\sum_{j=0}^{j_0}\|2^{j\alpha(\cdot)}f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

Combining the estimates above, we get

$$F\lesssim \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.$$

To estimate E, we split the operator into

$$\begin{aligned}
|\mu_{\Omega}^{\rho}(f\mathbf{1}_l)(x)| &\leq \Big(\int_0^{|x|}\Big|\frac{1}{t^{\rho}}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\\
&\quad+\Big(\int_{|x|}^{\infty}\Big|\frac{1}{t^{\rho}}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}}
\end{aligned}$$

$$=: I_1 + I_2.$$

When $l \le j-1$, $x \in R_j$, and $y \in R_l$, noting that $|x-y| \sim |x| \sim 2^j$, we use the mean value theorem and obtain

$$\left| |x-y|^{-2\rho} - |x|^{-2\rho} \right| \lesssim \frac{|y||x-\xi y|^{2\rho-1}}{|x|^{2\rho}|x-y|^{2\rho}} \lesssim \frac{|y|}{|x-y|^{2\rho+1}}, \quad \text{for some } \xi \in (0,1).$$

Thus, by Minkowski's inequality for integrals, we get

$$\begin{aligned} I_1 &\lesssim \int_{R_l} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f(y)| \Big(\int_{|x-y|\le t, |x|\ge t} \frac{dt}{t^{2\rho+1}} \Big)^{\frac{1}{2}} dy \\ &\lesssim \int_{R_l} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f(y)| \left| |x-y|^{-2\rho} - |x|^{-2\rho} \right|^{\frac{1}{2}} dy \\ &\lesssim \int_{R_l} |\Omega(x-y)||f(y)| \frac{|y|^{\frac{1}{2}}}{|x-y|^{n+\frac{1}{2}}} dy \\ &\lesssim 2^{\frac{l-j}{2}} 2^{-jn} \int_{R_l} |\Omega(x-y)||f(y)| dy \\ &\lesssim 2^{\frac{l-j}{2}} 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Similarly, we get

$$I_2 \lesssim 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}.$$

Therefore, we have

$$|\mu_\Omega^\rho(f\mathbf{1}_l)(x)| \lesssim 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}. \tag{3.1}$$

Since $s > (\beta')_+$, it follows that there exists a variable exponent $r(\cdot) \in \mathcal{P}_\infty^{log}(\mathbb{R}^n)$ such that $\frac{1}{\beta'(x)} = \frac{1}{r(x)} + \frac{1}{s}$, then, using Lemma 4, Lemma 5, and the fact that $\|\mathbf{1}_{B_l}\|_{L^{r(\cdot)}(\mathbb{R}^n)} \approx 2^{-\frac{ln}{s}} \|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}$ (see [31]), we obtain

$$\begin{aligned} &\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{1}_l\|_{L^{r(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^s(\mathbb{R}^n)} \\ &\lesssim 2^{-\frac{ln}{s}} \|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \Big(\int_{|x|-2^l}^{|x|+2^l} \int_{\mathbb{S}^{n-1}} |\Omega(y')|^s d\sigma(y') \varrho^{n-1} d\varrho \Big)^{\frac{1}{s}} \\ &\lesssim 2^{\frac{(j-l)n}{s}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\ &\lesssim 2^{\frac{(j-l)n}{s}} \frac{|R_l|}{|B_l|} \|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\ &\lesssim 2^{\frac{(j-l)n}{s}} \|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.2}$$

By Minkowski's inequality, we get

$$\begin{aligned} E &\lesssim \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\zeta^\theta \sum_{j=-\infty}^{-1} \Big(\sum_{l=-\infty}^{j-1} \|2^{j\alpha(\cdot)} \mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \Big)^{\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\ &\quad + \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\zeta^\theta \sum_{j=0}^{j_0} \Big(\sum_{l=-\infty}^{j-1} \|2^{j\alpha(\cdot)} \mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \Big)^{\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\ &=: E_1 + E_2. \end{aligned}$$

For E_1, Lemma 3 yields

$$\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\frac{ln}{\beta'(0)}}2^{\frac{jn}{\beta(0)}} \lesssim 2^{jn}2^{\frac{(l-j)n}{\beta'(0)}}. \tag{3.3}$$

Using (3.1), (3.2), (3.3), and $2^{j\alpha(x)} \approx 2^{j\alpha(0)}$, we obtain

$$\begin{aligned}
E_1 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1} 2^{j\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{\frac{(j-l)n}{s}}2^{-jn}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{(j-l)[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

For simplicity, we set $\delta = \alpha(0) - n(\frac{1}{\beta'(0)} - \frac{1}{s})$. Since $2^{-\gamma(1+\zeta)} < 2^{-\gamma}$ and $\delta < 0$, it follows from Hölder's inequality that

$$\begin{aligned}
E_1 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{(j-l)\delta\gamma(1+\zeta)/2}\Big) \\
&\quad \times \Big(\sum_{l=-\infty}^{j-1} 2^{(j-l)\delta(\gamma(1+\zeta))'/2}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)} \sum_{j=l+1}^{-1} 2^{(j-l)\delta\gamma(1+\zeta)/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)} \sum_{j=l+1}^{-1} 2^{(j-l)\delta\gamma/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} \|2^{l\alpha(\cdot)}f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For E_2, Minkowski's inequality implies that

$$\begin{aligned}
E_2 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=-\infty}^{-1} \|2^{j\alpha(\cdot)}\mu^\rho_\Omega(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} \|2^{j\alpha(\cdot)}\mu^\rho_\Omega(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: E_{21} + E_{22}.
\end{aligned}$$

For E_{21}, by Lemma 3, we have

$$\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\frac{ln}{\beta'(0)}}2^{\frac{jn}{\beta_\infty}} \lesssim 2^{jn}2^{\frac{-jn}{\beta'_\infty}}2^{\frac{ln}{\beta'(0)}}. \tag{3.4}$$

Using (3.1), (3.2), (3.4), and $2^{j\alpha(x)} \approx 2^{j\alpha_\infty}$, we obtain

$$\begin{aligned}
E_{21} &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=-\infty}^{-1} 2^{j\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad \times 2^{\frac{(j-l)n}{s}}2^{-jn}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} 2^{-j[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma(1+\zeta)} \\
&\quad \times \Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{-l[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

Since $\alpha_\infty < n(\frac{1}{\beta'_\infty}-\frac{1}{s})$ and $\alpha(0) < n(\frac{1}{\beta'(0)}-\frac{1}{s})$, by Hölder's inequality, we get

$$\begin{aligned}
E_{21} &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} 2^{-j[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma} \\
&\quad \times \Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{-l[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{-l[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big) \\
&\quad \times \Big(\sum_{l=-\infty}^{-1} 2^{-l[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})](\gamma(1+\zeta))'}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-1} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-1} \|2^{l\alpha(\cdot)}f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For E_{22}, Lemma 3 gives

$$\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\frac{ln}{\beta'_\infty}}2^{\frac{jn}{\beta_\infty}} \lesssim 2^{jn}2^{\frac{(l-j)n}{\beta'_\infty}}. \tag{3.5}$$

Using (3.1), (3.2), (3.5), and $2^{j\alpha(x)} \approx 2^{j\alpha_\infty}$, we have

$$
\begin{aligned}
E_{22} &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{j\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad\times 2^{\frac{(j-l)n}{s}} 2^{-jn}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}
$$

Since $\alpha_\infty < n(\frac{1}{\beta'_\infty}-\frac{1}{s})$ and $2^{-\gamma(1+\zeta)} < 2^{-\gamma}$, we use Hölder's inequality and obtain

$$
\begin{aligned}
E_{22} &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma(1+\zeta)/2}\Big) \\
&\quad\times\Big(\sum_{l=0}^{j-1} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty](\gamma(1+\zeta))'/2}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad\times \sum_{j=l+1}^{j_0} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma(1+\zeta)/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)} \sum_{j=l+1}^{j_0} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} \|2^{l\alpha(\cdot)}f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Next, we estimate G. Observing that for $l \geq j+1$, if $x \in R_j$ and $y \in R_l$, it follows that $|x-y| \sim |y| \sim 2^l$. Similar to the proof of E, we have

$$
\begin{aligned}
|\mu^\rho_\Omega(f\mathbf{1}_l)(x)| &\lesssim \Big(\int_0^{|y|}\Big|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\quad+\Big(\int_{|y|}^{\infty}\Big|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\lesssim 2^{\frac{j-l}{2}}2^{-ln}\int_{R_l}|\Omega(x-y)||f(y)|dy + 2^{-ln}\int_{R_l}|\Omega(x-y)||f(y)|dy \\
&\lesssim 2^{-ln}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.6}
$$

As argued in (3.2), we have

$$\begin{aligned}
&\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^{-\frac{ln}{s}}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\Big(\int_0^{2^l+1}\int_{\mathbb{S}^{n-1}}|\Omega(y')|^s d\sigma(y')\varrho^{n-1}d\varrho\Big)^{\frac{1}{s}}\\
&\lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{3.7}$$

By Minkowski's inequality, we get

$$\begin{aligned}
G &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{-1}\Big(\sum_{l=j+1}^{\infty}\|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\quad+\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}\|2^{j\alpha(\cdot)}\mu_\Omega^\rho(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&=: G_1+G_2.
\end{aligned}$$

For G_2, by Lemma 3, we have

$$\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_k\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\lesssim 2^{\frac{ln}{\beta'_\infty}}2^{\frac{kn}{\beta_\infty}}\lesssim 2^{ln}2^{\frac{(k-l)n}{\beta_\infty}}.\tag{3.8}$$

Using (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned}
G_2 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}2^{j\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times 2^{-ln}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{(j-l)(\alpha_\infty+\frac{n}{\beta_\infty})}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}\tag{3.9}$$

On the other hand, we note that

$$\begin{aligned}
\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} &= 2^{-l\alpha_\infty}\Big(2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim 2^{-l\alpha_\infty}\Big(\sum_{j=-\infty}^{l}2^{j\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim 2^{-l\alpha_\infty}2^{l\lambda}\zeta^{-\frac{\theta}{\gamma(1+\zeta)}}2^{-l\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{l}2^{j\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim 2^{-l\alpha_\infty}2^{l\lambda}\zeta^{-\frac{\theta}{\gamma(1+\zeta)}}\|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}\tag{3.10}$$

Therefore, since $\alpha_\infty > -\frac{n}{\beta_\infty} + \lambda$, combining (3.9) and (3.10), we get

$$
\begin{aligned}
G_2 &\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=0}^{j_0} 2^{j\lambda\gamma(1+\zeta)}\Big(\sum_{l=j+1}^{\infty} 2^{(j-l)(\alpha_\infty+\frac{n}{\beta_\infty}-\lambda)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=0}^{j_0} 2^{j\lambda\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For G_1, Minkowski's inequality implies that

$$
\begin{aligned}
G_1 &\lesssim \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=j+1}^{-1} \|2^{j\alpha(\cdot)}\mu^\rho_\Omega(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty} \|2^{j\alpha(\cdot)}\mu^\rho_\Omega(f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: G_{11} + G_{12}.
\end{aligned}
$$

The estimation for G_{11} proceeds analogously to that of G_2, with the substitution of β_∞ by $\beta(0)$ and the utilization of the inequality $\alpha(0) > -\frac{n}{\beta(0)} + \lambda$.

For G_{12}, by Lemma 3, we get

$$
\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \lesssim 2^{\frac{ln}{\beta'_\infty}} 2^{\frac{jn}{\beta(0)}} \lesssim 2^{ln} 2^{\frac{jn}{\beta(0)}} 2^{\frac{-ln}{\beta_\infty}}. \tag{3.11}
$$

Using (3.6), (3.7), and (3.11), we have

$$
\begin{aligned}
G_{12} &\lesssim \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty} 2^{j\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times 2^{-ln}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty} 2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times 2^{j(\alpha(0)+\frac{n}{\beta(0)})} 2^{-l(\alpha_\infty+\frac{n}{\beta_\infty})}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}
$$

Since $\alpha(0) > -\frac{n}{\beta(0)} + \lambda$ and $\alpha_\infty > -\frac{n}{\beta_\infty} + \lambda$, using (3.10) and $2^{-\gamma(1+\zeta)} < 2^{-\gamma}$, we get

$$\begin{aligned}
G_{12} \lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=-\infty}^{-1} 2^{j(\alpha(0)+\frac{n}{\beta(0)})\gamma(1+\zeta)} \\
& \times \Big(\sum_{l=0}^{\infty} 2^{-l(\alpha_\infty+\frac{n}{\beta_\infty}-\lambda)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=-\infty}^{-1} 2^{j(\alpha(0)+\frac{n}{\beta(0)})\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=-\infty}^{-1} 2^{j[\alpha(0)+\frac{n}{\beta(0)}-\lambda]\gamma(1+\zeta)} 2^{j\lambda\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=-\infty}^{-1} 2^{j[\alpha(0)+\frac{n}{\beta(0)}-\lambda]\gamma} 2^{j\lambda\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=-\infty}^{j_0} 2^{j\lambda\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim & \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

This proves Theorem 6. □

4. Boundedness of higher-order commutators of the parametric Marcinkiewicz integral

In this subsection, we prove that the higher-order commutators $[\Lambda^m, \mu^\rho_\Omega]$ generated by μ^ρ_Ω and BMO functions are bounded on grand variable Herz-Morrey spaces. We also show that these operators are bounded on grand variable Herz spaces.

Theorem 9. *Let* $\Lambda \in \mathrm{BMO}(\mathbb{R}^n)$*,* $m \in \mathbb{N}$*,* $\rho > n/2$*,* $\theta > 0$*,* $0 < \lambda < \infty$*,* $1 < \gamma < \infty$*,* $\alpha(\cdot), \beta(\cdot) \in \mathcal{P}^{log}_\infty(\mathbb{R}^n)$*, and* $\Omega \in L^s(\mathbb{S}^{n-1})$*,* $s > (\beta')_+$*. If* α *satisfies*

(i) $\lambda - \frac{n}{\beta(0)} < \alpha(0) < n(\frac{1}{\beta'(0)} - \frac{1}{s})$*,*

(ii) $\lambda - \frac{n}{\beta_\infty} < \alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$*,*

then the higher-order commutator $[\Lambda^m, \mu^\rho_\Omega]$ *is bounded on* $M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$*. Moreover, there exists a constant* $C > 0$ *such that for any* $f \in M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$*,*

$$\|[\Lambda^m, \mu^\rho_\Omega](f)\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \le C\|\Lambda\|^m_* \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.$$

If $\lambda = 0$, we can obtain the following result using a proof similar to Theorem 9.

Theorem 10. *Let* $\Lambda \in \mathrm{BMO}(\mathbb{R}^n)$*,* $m \in \mathbb{N}$*,* $\rho > n/2$*,* $\theta > 0$*,* $1 < \gamma < \infty$*,* $\alpha(\cdot), \beta(\cdot) \in \mathcal{P}^{log}_\infty(\mathbb{R}^n)$*, and* $\Omega \in L^s(\mathbb{S}^{n-1})$*,* $s > (\beta')_+$*. If* $-\frac{n}{\beta(0)} < \alpha(0) < n(\frac{1}{\beta'(0)} - \frac{1}{s})$ *and* $-\frac{n}{\beta_\infty} < \alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$*, then there exists a constant* $C > 0$ *such that for any* $f \in \dot{K}^{\alpha(\cdot),\gamma),\theta}_{\beta(\cdot)}(\mathbb{R}^n)$*,*

$$\|[\Lambda^m, \mu^\rho_\Omega](f)\|_{\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\beta(\cdot)}(\mathbb{R}^n)} \le C\|\Lambda\|^m_* \|f\|_{\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\beta(\cdot)}(\mathbb{R}^n)}.$$

The following lemma is due to Izuki [13].

Lemma 11. *Let $\Lambda \in \mathrm{BMO}(\mathbb{R}^n)$, $j > l$, and $j, l, m \in \mathbb{N}$. If $\beta(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\mathbf{1}_B\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}} \|(\Lambda - \Lambda_B)^m \mathbf{1}_B\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \approx \|\Lambda\|_*^m,$$

and

$$\|(\Lambda - \Lambda_{B_l})^m \mathbf{1}_{B_j}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \lesssim (j-l)^m \|\Lambda\|_*^m \|\mathbf{1}_{B_j}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}.$$

Proof of Theorem 9. Let $f \in M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)$. We decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\mathbf{1}_l(x) = \sum_{l=-\infty}^{\infty} f_l(x).$$

It follows from Minkowski's inequality that

$$\begin{aligned}
&\|[\Lambda^m, \mu_\Omega^\rho](f)\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \Big(\sum_{l=-\infty}^{j-1} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_j)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{j_0} \Big(\sum_{l=j+1}^{\infty} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: U + V + W.
\end{aligned}$$

For U, we have

$$\begin{aligned}
|[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)(x)| &\le \Big(\int_0^{|x|} \Big|\frac{1}{t^\rho}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[\Lambda(x)-\Lambda(y)]^m f_l(y)dy\Big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{|x|}^{\infty} \Big|\frac{1}{t^\rho}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[\Lambda(x)-\Lambda(y)]^m f_l(y)dy\Big|^2 \frac{dt}{t}\Big)^{\frac{1}{2}} \\
&=: J_1 + J_2.
\end{aligned}$$

When $l \le j-1, x \in R_j$, and $y \in R_l$, as in the estimation of U_1, using Minkowski's inequality and the generalized Hölder's inequality, we obtain

$$J_1 \lesssim \int_{R_l} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |\Lambda(x)-\Lambda(y)|^m |f(y)| \Big(\int_{|x-y|\le t, |x|\ge t} \frac{dt}{t^{2\rho+1}}\Big)^{\frac{1}{2}} dy$$

$$
\begin{aligned}
&\lesssim \int_{R_l} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |\Lambda(x)-\Lambda(y)|^m |f(y)| \left| |x-y|^{-2\rho} - |x|^{-2\rho} \right|^{\frac{1}{2}} dy \\
&\lesssim \int_{R_l} |\Lambda(x)-\Lambda(y)|^m |\Omega(x-y)||f(y)| \frac{|y|^{\frac{1}{2}}}{|x-y|^{n+\frac{1}{2}}} dy \\
&\lesssim 2^{\frac{l-j}{2}} 2^{-jn} \int_{R_l} |\Lambda(x)-\Lambda(y)|^m |\Omega(x-y)||f(y)| dy \\
&\lesssim 2^{\frac{l-j}{2}} 2^{-jn} \Big(|\Lambda(x)-\Lambda_{B_l}|^m \int_{R_l} |\Omega(x-y)||f(y)| dy \\
&\quad + \int_{R_l} |\Lambda_{B_l}-\Lambda(y)|^m |\Omega(x-y)||f(y)| dy \Big) \\
&\lesssim 2^{\frac{l-j}{2}} 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \Big(|\Lambda(x)-\Lambda_{B_l}|^m \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|(\Lambda-\Lambda_{B_l})^m \Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \Big).
\end{aligned}
$$

Similarly, we have

$$
J_2 \lesssim 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \Big(|\Lambda(x)-\Lambda_{B_l}|^m \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} + \|(\Lambda-\Lambda_{B_l})^m \Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \Big). \tag{4.1}
$$

Therefore, we get

$$
\begin{aligned}
|[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)(x)| \lesssim 2^{-jn} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} &\Big(|\Lambda(x)-\Lambda_{B_l}|^m \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\
&+ \|(\Lambda-\Lambda_{B_l})^m \Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \Big).
\end{aligned} \tag{4.2}
$$

As argued in (3.2), by Lemmas 4 and 11, we have

$$
\begin{aligned}
&\|(\Lambda-\Lambda_{B_l})^m \Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|(\Lambda-\Lambda_{B_l})^m \mathbf{1}_l\|_{L^{r(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^s(\mathbb{R}^n)} \\
&\lesssim \|\Lambda\|_*^m \|\mathbf{1}_{B_l}\|_{L^{r(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^s(\mathbb{R}^n)} \\
&\lesssim \|\Lambda\|_*^m 2^{-\frac{ln}{s}} \|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \|\Omega(x-\cdot)\mathbf{1}_l\|_{L^s(\mathbb{R}^n)} \\
&\lesssim 2^{\frac{(j-l)n}{s}} \|\Lambda\|_*^m \|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{4.3}
$$

Thus, it follows from (3.2), (4.2), (4.3), and Lemmas 5 and 11 that

$$\begin{aligned}
&\|[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^{-jn}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big(\|(\Lambda-\Lambda_{B_l})^m\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|(\Lambda-\Lambda_{B_l})^m\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)\\
&\lesssim 2^{-jn}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big((j-l)^m\|\Lambda\|_*^m\|\mathbf{1}_{B_j}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{\frac{(j-l)n}{s}}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\\
&\quad+2^{\frac{(j-l)n}{s}}\|\Lambda\|_*^m\|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)\\
&\lesssim 2^{-jn}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}(j-l)^m\|\Lambda\|_*^m2^{\frac{(j-l)n}{s}}\Big(\frac{|B_j|}{|R_j|}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\\
&\quad+\frac{|B_l|}{|R_l|}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)\\
&\lesssim (j-l)^m\|\Lambda\|_*^m2^{\frac{(j-l)n}{s}}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{-jn}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{4.4}$$

By Minkowski's inequality, we get

$$\begin{aligned}
U&\lesssim\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\quad+\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=-\infty}^{j-1}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&=:U_1+U_2.
\end{aligned}$$

For U_1, by using (3.3), (4.4), and $2^{j\alpha(x)}\approx 2^{j\alpha(0)}$ for $j<0$ and $x\in R_j$, we derive

$$\begin{aligned}
U_1&\lesssim\|\Lambda\|_*^m\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1}2^{j\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times(j-l)^m2^{\frac{(j-l)n}{s}}2^{-jn}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}\\
&\lesssim\|\Lambda\|_*^m\sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{-1}\Big(\sum_{l=-\infty}^{j-1}2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times(j-l)^m2^{(j-l)[\alpha(0)-n(\frac{1}{\beta'(0)}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

Since $\delta = \alpha(0) - n(\frac{1}{\beta'(0)} - \frac{1}{s}) < 0$ and $2^{-\gamma(1+\zeta)} < 2^{-\gamma}$, we use Hölder's inequality and obtain

$$\begin{aligned}
U_1 &\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1} \Big(\sum_{l=-\infty}^{j-1} 2^{l\alpha(0)\gamma(1+\zeta)} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} 2^{(j-l)\delta\gamma(1+\zeta)/2}\Big) \\
&\quad \times \Big(\sum_{l=-\infty}^{j-1} (j-l)^{m(\gamma(1+\zeta))'} 2^{(j-l)\delta(\gamma(1+\zeta))'/2}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)\gamma(1+\zeta)} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} \sum_{j=l+1}^{-1} 2^{(j-l)\delta\gamma(1+\zeta)/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} 2^{l\alpha(0)\gamma(1+\zeta)} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} \sum_{j=l+1}^{-1} 2^{(j-l)\delta\gamma/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-2} \|2^{l\alpha(\cdot)} f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

For U_2, Minkowski's inequality implies that

$$\begin{aligned}
U_2 &\lesssim \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} \Big(\sum_{l=-\infty}^{-1} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} \Big(\sum_{l=0}^{j-1} \|2^{j\alpha(\cdot)}[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: U_{21} + U_{22}.
\end{aligned}$$

For U_{21}, using (3.4), (4.4), and $2^{j\alpha(x)} \approx 2^{j\alpha_\infty}$ for $j \geq 0$ and $x \in R_j$, we obtain

$$\begin{aligned}
U_{21} &\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} \Big(\sum_{l=-\infty}^{-1} 2^{j\alpha_\infty} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad \times (j-l)^m 2^{\frac{(j-l)n}{s}} 2^{-jn} \|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} 2^{j[\alpha_\infty - n(\frac{1}{\beta'_\infty} - \frac{1}{s})]\gamma(1+\zeta)} \\
&\quad \times \Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)} \|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} (j-l)^m 2^{l[n(\frac{1}{\beta'(0)} - \frac{1}{s}) - \alpha(0)]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

Since $\alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$ and $\alpha(0) < n(\frac{1}{\beta'(0)} - \frac{1}{s})$, by Hölder's inequality, we get

$$\begin{aligned}
U_{21} \lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0} 2^{j[\alpha_\infty - n(\frac{1}{\beta'_\infty}-\frac{1}{s})]\gamma} \\
& \times \Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}(j-l)^m 2^{l[n(\frac{1}{\beta'(0)}-\frac{1}{s})-\alpha(0)]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
& \times (j-l)^m 2^{l[n(\frac{1}{\beta'(0)}-\frac{1}{s})-\alpha(0)]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\Big(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big) \\
& \times \Big(\sum_{l=-\infty}^{-1} (j-l)^{m(\gamma(1+\zeta))'} 2^{l[n(\frac{1}{\beta'(0)}-\frac{1}{s})-\alpha(0)](\gamma(1+\zeta))'}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-1} 2^{l\alpha(0)\gamma(1+\zeta)}\|f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=-\infty}^{-1} \|2^{l\alpha(\cdot)}f\mathbf{1}_l\|^{\gamma(1+\zeta)}_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For U_{22}, using (3.5), (4.4), and $2^{j\alpha(x)} \approx 2^{j\alpha_\infty}$, we obtain

$$\begin{aligned}
U_{22} \lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{j\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
& \times (j-l)^m 2^{\frac{(j-l)n}{s}} 2^{-jn}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
& \times (j-l)^m 2^{(j-l)[\alpha_\infty - n(\frac{1}{\beta'_\infty}-\frac{1}{s})]}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

Since $\alpha_\infty < n(\frac{1}{\beta'_\infty} - \frac{1}{s})$, by Hölder's inequality, we have

$$
\begin{aligned}
U_{22} &\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=0}^{j_0}\Big(\sum_{l=0}^{j-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma(1+\zeta)/2}\Big) \\
&\quad \times \Big(\sum_{l=0}^{j-1}(j-l)^{m(\gamma(1+\zeta))'} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty](\gamma(1+\zeta))'/2}\Big)^{\gamma(1+\zeta)/(\gamma(1+\zeta))'}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} \sum_{j=l+1}^{j_0} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma(1+\zeta)/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} 2^{l\alpha_\infty\gamma(1+\zeta)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)} \sum_{j=l+1}^{j_0} 2^{(l-j)[n(\frac{1}{\beta'_\infty}-\frac{1}{s})-\alpha_\infty]\gamma/2}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{l=0}^{j_0-1} \|2^{l\alpha(\cdot)}f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}
$$

As in the proof of F, using the $L^{\beta(\cdot)}(\mathbb{R}^n)$ boundedness of $[\Lambda^m,\mu_\Omega^\rho]$ (see [31]), we can easily obtain that

$$
V \lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
$$

Now we estimate W. Observing that for $l \geq j+1$, if $x \in R_j$ and $y \in R_l$, it follows that $|x-y| \sim |y| \sim 2^l$. Similar to the proof of U, we have

$$
\begin{aligned}
|[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)(x)| &\lesssim \Big(\int_0^{|y|}\Big|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[\Lambda(x)-\Lambda(y)]^m f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{|y|}^{\infty}\Big|\frac{1}{t^\rho}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}[\Lambda(x)-\Lambda(y)]^m f_l(y)dy\Big|^2\frac{dt}{t}\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-ln}\int_{R_l}|\Lambda(x)-\Lambda(y)|^m|\Omega(x-y)||f(y)|dy \\
&\lesssim 2^{-ln}\Big(|\Lambda(x)-\Lambda_{B_j}|^m\int_{R_l}|\Omega(x-y)||f(y)|dy \\
&\quad + \int_{R_l}|\Lambda_{B_j}-\Lambda(y)|^m|\Omega(x-y)||f(y)|dy\Big) \\
&\lesssim 2^{-ln}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big(|\Lambda(x)-\Lambda_{B_j}|^m\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|(\Lambda-\Lambda_{B_j})^m\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\Big).
\end{aligned} \tag{4.5}
$$

As argued in (3.7), by Lemma 11, we have

$$
\|(\Lambda-\Lambda_{B_j})^m\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \lesssim (l-j)^m\|\Lambda\|_*^m\|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}. \tag{4.6}
$$

Thus, it follows from (4.5), (4.6), and Lemma 5 that

$$
\begin{aligned}
&\|[\Lambda^m, \mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\lesssim 2^{-ln}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big(\|\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|(\Lambda-\Lambda_{B_j})^m\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|(\Lambda-\Lambda_{B_j})^m\Omega(x-\cdot)\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big) \\
&\lesssim 2^{-ln}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big(\|\Lambda\|_*^m\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_{B_j}\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad + (l-j)^m\|\Lambda\|_*^m\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_{B_l}\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\Big) \\
&\lesssim 2^{-ln}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}(l-j)^m\|\Lambda\|_*^m\Big(\frac{|B_j|}{|R_j|}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \frac{|B_l|}{|R_l|}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big) \\
&\lesssim (l-j)^m\|\Lambda\|_*^m\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}2^{-ln}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{4.7}
$$

By Minkowski's inequality, we get

$$
\begin{aligned}
W &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=-\infty}^{-1}\Big(\sum_{l=j+1}^{\infty}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: W_1 + W_2.
\end{aligned}
$$

For W_2, using (3.8) and (4.7) we obtain

$$
\begin{aligned}
W_2 &\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}2^{j\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times (l-j)^m 2^{-ln}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta\sum_{j=0}^{j_0}\Big(\sum_{l=j+1}^{\infty}2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\qquad \times (l-j)^m 2^{(j-l)(\alpha_\infty+\frac{n}{\beta_\infty})}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}
\tag{4.8}
$$

Therefore, since $\alpha_\infty > -\frac{n}{\beta_\infty} + \lambda$, combining (3.10) and (4.8), we get

$$\begin{aligned}
W_2 &\lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=0}^{j_0} 2^{j\lambda\gamma(1+\zeta)} \\
&\quad \times \Big(\sum_{l=j+1}^{\infty}(l-j)^m 2^{(j-l)(\alpha_\infty+\frac{n}{\beta_\infty}-\lambda)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)} \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\sum_{j=0}^{j_0} 2^{j\lambda\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma),\theta}_{\lambda,\beta(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For W_1, using the Minkowski inequality we obtain

$$\begin{aligned}
W_1 &\lesssim \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=j+1}^{-1}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\quad + \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty}\|2^{j\alpha(\cdot)}[\Lambda^m,\mu_\Omega^\rho](f\mathbf{1}_l)\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&=: W_{11} + W_{12}.
\end{aligned}$$

The estimate for W_{11} is derived similarly to W_2, but with the substitution of β_∞ with $\beta(0)$ and using the condition $\alpha(0) > -\frac{n}{\beta(0)} + \lambda$.

For W_{12}, using (3.11) and (4.7), we have

$$\begin{aligned}
W_{12} &\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty} 2^{j\alpha(0)}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} \\
&\quad \times (l-j)^m 2^{-ln}\|\mathbf{1}_l\|_{L^{\beta'(\cdot)}(\mathbb{R}^n)}\|\mathbf{1}_j\|_{L^{\beta(\cdot)}(\mathbb{R}^n)}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}} \\
&\lesssim \|\Lambda\|_*^m \sup_{\zeta>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\zeta^\theta \sum_{j=-\infty}^{-1}\Big(\sum_{l=0}^{\infty} 2^{l\alpha_\infty}\|f\mathbf{1}_l\|_{L^{\beta(\cdot)}(\mathbb{R}^n)} 2^{j(\alpha(0)+\frac{n}{\beta(0)})} \\
&\quad \times (l-j)^m 2^{-l(\alpha_\infty+\frac{n}{\beta_\infty})}\Big)^{\gamma(1+\zeta)}\Big)^{\frac{1}{\gamma(1+\zeta)}}.
\end{aligned}$$

Since $\alpha(0) > -\frac{n}{\beta(0)} + \lambda$ and $\alpha_\infty > -\frac{n}{\beta_\infty} + \lambda$, using (3.10) and $2^{-\gamma(1+\zeta)} < 2^{-\gamma}$, we get

$$\begin{aligned}
W_{12} \lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\sum_{j=-\infty}^{-1} 2^{j(\alpha(0)+\frac{n}{\beta(0)})\gamma(1+\zeta)} \\
& \times \Big(\sum_{l=0}^{\infty} (l-j)^m 2^{-l(\alpha_\infty+\frac{n}{\beta_\infty}-\lambda)} \Big)^{\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\sum_{j=-\infty}^{-1} 2^{j(\alpha(0)+\frac{n}{\beta(0)})\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\sum_{j=-\infty}^{-1} 2^{j[\alpha(0)+\frac{n}{\beta(0)}-\lambda]\gamma(1+\zeta)} 2^{j\lambda\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\sum_{j=-\infty}^{-1} 2^{j[\alpha(0)+\frac{n}{\beta(0)}-\lambda]\gamma} 2^{j\lambda\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)} \sup_{\zeta>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda} \Big(\sum_{j=-\infty}^{j_0} 2^{j\lambda\gamma(1+\zeta)} \Big)^{\frac{1}{\gamma(1+\zeta)}} \\
\lesssim\ & \|\Lambda\|_*^m \|f\|_{M\dot{K}_{\lambda,\beta(\cdot)}^{\alpha(\cdot),\gamma),\theta}(\mathbb{R}^n)}.
\end{aligned}$$

This proves Theorem 9. □

5. Conclusions

In this paper, we prove the boundedness of the parametric Marcinkiewicz integral and its higher-order commutators with BMO symbols on grand variable Herz-Morrey spaces. These results also extend to grand variable Herz spaces, including when $\alpha(\cdot)$ is constant.

Author contributions

Liwei Wang: Conceptualization, Methodology, Writing–original draft, Funding acquisition, Project administration; Xiaoyan Li: Validation, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

It is declared that the authors do not have any competing interests.

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