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Research article

Constructing a metallic semi-Riemannian manifold from an almost paracontact metric manifold

Mehmet Solgun and Yasemin Karababa*

Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Turkey

* **Correspondence:** Email: yaseminn2604@gmail.com.

Abstract: In this paper, we introduce a new method to obtain a metallic semi-Riemannian structure $(\Psi, \widetilde{g})$ from the given almost paracontact metric structure (φ, ξ, η, g) on a smooth manifold M^{2n+1}, and give the relations between these structures via Levi-Civita connections of g and $\widetilde{g}$. After, we discuss these relations for para-Kenmotsu and para-Sasakian structures and state the properties of obtained metallic semi-Riemannian structure. Furthermore, we show that metallic structures obtained from para-Kenmotsu and para-Sasakian structures are integrable and non-integrable, respectively. Also, we study the curvature properties of the obtained metallic structure and give explicit examples for the results of the study.

Keywords: metallic semi-Riemannian structure; almost paracontact metric structure; Riemannian curvature tensor
Mathematics Subject Classification: 53C15, 53C25

1. Introduction

The golden mean is recognized as a remarkable mathematical constant due to its intrinsic aesthetic and structural properties, appearing in fields as diverse as geometry, art, architecture, and nature. In 1999, De Spinadel introduced a comprehensive generalization of the golden mean, known as the metallic means family [1, 2]. This family encompasses an infinite set of positive quadratic irrational numbers, each defined as the positive root of the quadratic equation $x^2 - px - q = 0$, where $p, q \in \mathbb{N}^+$. The positive solution of this equation is given by $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$, and is called the (p, q)-metallic number and $\sigma_{p,q}$, $\sigma^*_{p,q} = (p - \sigma_{p,q})$ eigenvalues. Each such number is analogously associated with a metal, leading to denominations, such as the golden mean, silver mean, bronze mean, copper mean, and nickel mean, among others.

Crasmareanu and Hretcanu [3], also conducted an exhaustive study on metallic structures on Riemannian manifolds. A metallic structure on a differentiable manifold M is characterized by a

(1, 1)-tensor field Ψ satisfying the algebraic condition:

$$\Psi^2 = p\Psi + qI,$$

where I denotes the identity transformation and $p, q \in \mathbb{N}^+$. When such a tensor field Ψ is compatible with a Riemannian metric g, in the sense that $g(\Psi X, Y) = g(X, \Psi Y)$, the pair (Ψ, g) defines a metallic Riemannian structure, and the manifold M is said to be a metallic Riemannian manifold.

In recent years, metallic structures have attracted significant attention due to their rich geometric properties and their ability to model various curvature conditions. In [4], Gezer and Karaman analyzed integrability conditions and curvature characteristics of metallic Riemannian structures. Blaga and Nannicini also introduced metallic semi-Riemannian structures and investigated certain classes of these structures along with their associated curvature tensors [5, 6].

Almost paracontact structures were first studied by Kaneyuki [7], and after the work of Zamkovoy in [8], many authors have made contributions to the subject. In the literature, there are many studies on almost paracontact manifolds from different perspectives in various dimensions. In [9–11], relations of almost paracontact strucutures with different geometric structures are studied. Also, for recent studies, one can see [12–14]. In [15], Zamkovoy and Nakova classified almost paracontact metric structures into the 2^{12} classes by considering the covariant derivative of the fundamental $2-$form Φ of the structure with respect to the Levi-Civita connection. This classification has inspired numerous studies for specific classes, such as para-Kenmotsu structures that examined in [16, 17].

Metallic structures, similar to almost product and almost complex structures, naturally occur in complementary pairs. Specifically, if Ψ is a metallic structure, then $pI - \Psi$ also constitutes a metallic structure. In this context, the metallic structure $pI - \Psi$ also possesses the properties of having σ and $\sigma^* = p - \sigma$ as eigenvalues, acting as an isomorphism transformation on the tangent space of the underlying manifold, and being invertible. In [18], Ghereci also constructed a metallic Riemannian structure on a Riemannian manifold by using the elements of almost contact metric structure, namely a reeb vector field ξ and its dual 1-form η, and studied the conditions of parallelism and integrability between these structures. Some studies have been conducted on para-Sasakian manifolds and para-Sasaki-like para-Norden manifold based on the relation between metallic structure and paracontact structure on a Riemannian manifolds [19, 20].

The aim of this paper is to investigate the relations between metallic semi-Riemannian structures and almost paracontact structures on (2n+1) dimensional semi-Riemannian manifolds. Since semi-Riemannian manifolds are of fundamental importance in mathematical physics and relativity, the construction in this work may inspire further research on the role of metallic semi-Riemannian structures in such areas.The remainder of this paper is organized as follows: In Section 2, some primary notions for almost paracontact metric manifolds is given. Section 3 is devoted to the definition of metallic semi-Riemannian manifolds, along with an examination the curvature properties of locally metallic semi-Riemannian manifolds [5, 6]. In Section 4, we constructed a metallic semi-Riemannian structure from an almost paracontact metric structure on a semi-Riemannian manifold and stated the relations between their respective Levi-Civita connections via the Koszul formula. Subsequently, we gave the curvature properties and integrability conditions that arise in metallic structure depending on certain special classes of the almost paracontact metric structure, such as when the Reeb vector field ξ is parallel, or in the cases of para-Sasakian and para-Kenmotsu structures. In the final section, we will provide certain examples.

2. Preliminaries

A (2n+1) dimensional smooth manifold M has an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η satisfying the followings:

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \text{rank}\varphi = 2n. \tag{2.1}$$

If a manifold M with (φ, ξ, η)-structure admits a semi-Riemannian metric g, such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.2}$$

Then, M is called an almost paracontact metric manifold. Any compatible metric g with a given almost paracontact structure is necessarily of signature $(n + 1, n)$. From the definition, it follows that η is the g-dual of ξ: $\eta(X) = g(X, \xi)$, ξ is a vector field: $g(\xi, \xi) = 1$, and φ is a g-skew-symmetric operator: $g(\varphi X, Y) = -g(X, \varphi Y)$.

An almost paracontact metric manifold is called integrable if its Nijenhuis tensor field of φ, N_φ vanishes, where $N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$, for $X, Y \in \mathfrak{X}(M)$.

The study presented in [12] investigates specific classes of almost paracontact metric manifolds characterized by the condition that the Reeb vector field ξ is parallel ($\nabla\xi = 0$) with respect to the Levi-Civita connection.

It is important to note that in [21], Zamkovoy studied a para-Sasakian manifold and determined its curvature conditions, as well as the ξ-sectional and φ-paraholomorphic sectional curvatures. Later, in [22], he studied the curvature properties of para-Kenmotsu manifolds and established the conditions η-Einstein manifolds.

An almost paracontact metric structure (φ, ξ, η, g) is called para-Kenmotsu if the Levi-Civita connection ∇ of g satisfies

$$(\nabla_X\varphi)Y = \eta(Y)\varphi X + g(X, \varphi Y)\xi, \tag{2.3}$$

for any $X, Y \in \mathfrak{X}(M)$. This aspect has also been discussed in [22].

On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, the following relations hold [22]:

$$\nabla_X\xi = -X + \eta(X)\xi, \quad (\nabla_X\eta)Y = -g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$(L_\xi g)(X, Y) = -2g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$\eta(\nabla_X\xi) = 0, \quad \nabla_\xi\xi = 0, \quad \nabla_\xi\eta = 0, \quad L_\xi\varphi = 0, \quad L_\xi\eta = 0, \quad L_\xi(\eta \otimes \eta) = 0. \tag{2.6}$$

Where L, the Lie differentiation of g and ∇, is the Levi-Civita connection associated to g.

Also, the following equalities hold for para-Kenmotsu manifolds [22]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad K(X, \xi) = -1, \tag{2.7}$$

$$R(X, Y)\varphi Z = \varphi(R(X, Y)Z) + g(Y, Z)\varphi X - g(X, Z)\varphi Y + g(X, \varphi Z)Y - g(Y, \varphi Z)X, \tag{2.8}$$

$$R(\varphi X,\varphi Y)Z = -R(X,Y)Z - g(Y,Z)X + g(X,Z)Y + g(Y,\varphi Z)\varphi - g(X,\varphi Z)\varphi Y, \tag{2.9}$$

$$R(\varphi X,\varphi Y)\varphi Z = -\varphi(R(X,Y)Z) + \eta(Z)\{\eta(X)\varphi Y - \eta(Y)\varphi X\}, \tag{2.10}$$

$$R(\varphi X,Y)Z + R(X,\varphi Y)Z = g(X,Z)\varphi Y - g(Y,Z)\varphi X - g(X,\varphi Z)Y + g(Y,\varphi Z)X, \tag{2.11}$$

where R is the Riemannian curvature tensor field, K is the sectional curvature.

A paracontact metric manifold (M,φ,ξ,η,g) is said to be a para-Sasakian manifold, if $N_{\varphi}(X,Y) = 2d\eta(X,Y)\xi$ [15].

Let (M,φ,ξ,η,g) para-Sasakian manifold. The following relations are provided for (M,φ,ξ,η,g) [21]:

$$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X, \quad \nabla_X\xi = -\varphi X, \quad (\nabla_X\eta)Y = d\eta(X,Y) = g(X,\varphi Y), \tag{2.12}$$

$$R(X,Y)\xi = (\eta(X)Y - \eta(Y)X), \quad R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.13}$$

$$\eta(R(X,Y)Z) = g(X,X)\eta(Y) - g(Y,Z)\eta(X), \quad \eta(R(X,Y)\xi) = 0, \tag{2.14}$$

$$R(X,Y)\varphi Z = \varphi(R(X,Y)Z) + g(Y,Z)\varphi X - g(X,Z)\varphi Y + g(X,\varphi Z)Y - g(Y,\varphi Z)X, \tag{2.15}$$

$$R(\varphi X,\varphi Y)Z = -R(X,Y)Z - g(Y,Z)X + g(X,Z)Y + g(Y,\varphi Z)\varphi - g(X,\varphi Z)\varphi Y. \tag{2.16}$$

3. Metallic semi-Riemannian manifolds

In [5], Blaga and Nannicini generalized the notion of a metallic Riemannian manifold to that of a metallic semi-Riemannian manifold and presented a mild classificaton of the Ψ-metallic structure with respect to the Levi-Civita covariant derivative, and constructed a metallic natural connection in a manner analogous to the approach in the work of Ganchev and Mihova [23].

Definition 3.1. *Let $(M,\widetilde{g})$ be a semi-Riemannian manifold and let Ψ be a $\widetilde{g}$-symmetric ($\widetilde{g}(\Psi X,Y) = \widetilde{g}(X,\Psi Y)$) (1,1) tensor field on M such that*

$$\Psi^2 = p\Psi + qI, \tag{3.1}$$

for some p,q real numbers. Then, $(M,\Psi,\widetilde{g})$ is called a metallic semi-Riemannian manifold, where I is the identity transformation [5].

Definition 3.2. *Let Ψ be a metallic structure on M and Levi-Civita connection $\widetilde{\nabla}$ corresponding to the metric $\widetilde{g}$. The defining relations of the classes of this structure are given as follows:*
(i) If $\widetilde{\nabla}\Psi = 0$, then $(M,\Psi,\widetilde{g})$ is called a locally metallic semi-Riemannian manifold.
(ii) If $(\widetilde{\nabla}_X\Psi)Y + (\widetilde{\nabla}_Y\Psi)X = 0$, then $(M,\Psi,\widetilde{g})$ is called a nearly locally metallic semi-Riemannian manifold for all vector fields $X,Y \in \mathfrak{X}(M)$.
(iii) The structure defined by $\Psi := \mu I$, where $\mu = \dfrac{p \pm \sqrt{p^2+4q}}{2}$ with $\sqrt{p^2+4q} \geq 0$, is called a trivial metallic structure [5].

It is known that there exist locally metallic semi-Riemannian structures on any semi-Riemannian manifold.

A metallic structure Ψ is called integrable if its Nijenhuis tensor field $\widetilde{N}_\Psi$ vanishes, where $\widetilde{N}_\Psi(X,Y) = \Psi^2[X,Y] + [\Psi X, \Psi Y] - \Psi[\Psi X, Y] - \Psi[X, \Psi Y]$, for $X, Y \in \mathfrak{X}(M)$.

It is known that if $(M, \widetilde{g}, \Psi)$ is a locally metallic semi-Riemannian manifold, then Ψ is integrable [5].

It has been shown that nearly locally metallic semi-Riemannian manifold $(M, \widetilde{g}, \Psi)$ such that $\Psi^2 = p\Psi + qI$ with $\sqrt{p^2+4q} \geq 0$ is locally metallic semi-Riemannian manifold if and only if Ψ is integrable (see Proposition 2.1 [5])

Let $(M, \Psi, \widetilde{g})$ be locally metallic semi-Riemannian manifold. Then, the following holds:

$$\widetilde{R}(X,Y)\Psi Z = \Psi(\widetilde{R}(X,Y)Z),$$

$$\widetilde{R}(\Psi X, Y, Z, W) = \widetilde{R}(X, \Psi Y, Z, W),$$

$$\widetilde{R}(X, Y, \Psi Z, W) = \widetilde{R}(X, Y, Z, \Psi W),$$

$$\widetilde{R}(\Psi X, \Psi Y, Z, W) = p\widetilde{R}(X, \Psi Y, Z, W) + q\widetilde{R}(X, Y, Z, W),$$

$$\widetilde{R}(X, Y, \Psi Z, \Psi W) = p\widetilde{R}(X, Y, Z, \Psi W) + q\widetilde{R}(X, Y, Z, W),$$

$$\widetilde{R}(X, \Psi Y, \Psi Z, W) = \widetilde{R}(\Psi X, Y, \Psi Z, W) = \widetilde{R}(\Psi X, Y, Z, \Psi W) = \widetilde{R}(X, \Psi Y, Z, \Psi W),$$

$$\widetilde{K}^\Psi(X) = 0, \quad \widetilde{K}^\Psi(X,Y) = 0,$$

where $\widetilde{R}$ is the Riemannian curvature tensor field, $\widetilde{K}$ is the sectional curvature [6].

4. Induced metallic semi-Riemannian structures by almost paracontact metric structures

In this section, we detail how to obtain a metallic semi-Riemannian structure $(\Psi, \tilde{g})$ from an almost paracontact metric structure (φ, ξ, η, g) on a manifold M and use the notation "∼" on the elements of the obtained metallic semi-Riemannian structure to ensure consistency with the literature.

Let M be a manifold equipped with an almost paracontact structure (φ, ξ, η). In what follows, we construct a metallic structure Ψ on M that is induced by (φ, ξ, η).

Theorem 1. *Let (M, φ, ξ, η) be a (2n+1)-dimensional almost paracontact manifold and Ψ be a tensor field of (1,1)-type on M, defined as,*

$$\begin{aligned} \Psi : \mathfrak{X}(M) &\rightarrow \mathfrak{X}(M) \\ X &\mapsto \Psi X := \sigma X + (p - 2\sigma)\eta(X)\xi. \end{aligned} \tag{4.1}$$

Then, Ψ is a metallic structure on M (with $\Psi^2 = p\Psi + qI$, where $\sigma = \dfrac{p + \sqrt{p^2+4q}}{2}$ and $p, q \in \mathbb{R}$). The complement of Ψ is $\widetilde{\Psi}X = \sigma^* X + (p - 2\sigma^*)\eta(X)\xi$, and ξ is the unique eigenvector of Ψ and $\widetilde{\Psi}$ associated with σ^* and σ, respectively.

Proof. It can be seen that the tensor field Ψ given with (4.1) satisfies the Eq (3.1). Also $\Psi(\xi) = \sigma\xi + (p - 2\sigma)\xi = \sigma^*\xi$, $\widetilde{\Psi}\xi = \sigma\xi$. □

Now on, Ψ is called $\varphi-$related metallic structure or related structure in short.

Proposition 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold and Ψ be the related metallic structure. Define the metric $\widetilde{g}$ as:*

$$\widetilde{g}(X, Y) = g(\Psi X, \Psi Y) + qg(X, Y). \tag{4.2}$$

Then, the metric $\widetilde{g}$ is compatible with Ψ.

Proof. By the definition of Ψ, the Eq (4.2) becomes as follows:

$$\widetilde{g}(X, Y) = (p\sigma + 2q)g(X, Y) + p(p - 2\sigma)\eta(X)\eta(Y). \tag{4.3}$$

And so, by straight calculation, it can be seen that $\widetilde{g}(\Psi X, Y) = \widetilde{g}(X, \Psi Y)$. □

Example 2. *We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$ and the vector fields $E = \frac{\partial}{\partial x}$, $\varphi E = \frac{\partial}{\partial y}$, $\xi = (x + 2y)\frac{\partial}{\partial x} + (2x + y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where x, y, z are the standard coordinates in $\mathbb{R}^3$. Let the semi-Riemannian metric g and the (1,1)-tensor field φ given by*

$$g = \begin{pmatrix} 1 & 0 & -(x+2y) \\ 0 & -1 & (2x+y) \\ -(x+2y) & (2x+y) & 1-(2x+y)^2+(x+2y)^2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & 1 & -(2x+y) \\ 1 & 0 & -(x+2y) \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta = dz.$$

with respect to basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Then, (φ, ξ, η, g) is an almost paracontact metric structure on M.

Using Eq (4.1), we get:

$$\Psi = \begin{pmatrix} \sigma & 0 & (p-2\sigma)(x+2y) \\ 0 & \sigma & (p-2\sigma)(2x+y) \\ 0 & 0 & p-\sigma \end{pmatrix}.$$

On the other hand, by Eq (4.2), the metric $\tilde{g}$ is as follows:

$$\widetilde{g} = \begin{pmatrix} \sigma & 0 & -(p\sigma+2q)(x+2y) \\ 0 & -(p\sigma+2q) & (p\sigma+2q)(2x+y) \\ -(p\sigma+2q)(x+2y) & (p\sigma+2q)(2x+y) & (p\sigma+2q)(1-(2x+y)^2+(x+2y)^2)+p(p-2\sigma) \end{pmatrix}.$$

And so, $(\Psi, \widetilde{g})$ is the related metallic structure on M.

From Eq 4.1, the Nijenhuis tensor of the $(\Psi, \widetilde{g})$ related metallic structure is given by

$$\widetilde{N}_\Psi(X, Y) = (p - 2\sigma)^2\{\eta([X, Y])\xi - \eta([\eta(X)\xi, Y])\xi - \eta([X, \eta(Y)\xi])\xi + [\eta(X)\xi, \eta(Y)\xi]\}. \tag{4.4}$$

Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold and $(\Psi, \widetilde{g})$ be the related metallic structure and let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections of g and $\widetilde{g}$, respectively. By Koszul formula, we get the following relation for general case:

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &= 2\{Ag(\nabla_X Y, Z) + Bg(\nabla_X Y, \xi)\eta(Z)\} + B\{\eta(Z)[g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)] \\ &+ \eta(Y)[g(\nabla_X \xi, Z) - g(\nabla_Z \xi, X)] + \eta(X)[g(\nabla_Y \xi, Z) - g(\nabla_Z \xi, Y)]\}, \end{aligned} \tag{4.5}$$

where $A = p\sigma + 2q$, $B = p(p - 2\sigma)$ and $A \neq 0, B \neq 0$.

Based on Eq (4.5), we proceed to study the properties of the metallic structure induced by the almost paracontact structure, specifically when it belongs to special classes such as ξ is parallel, para-Kenmotsu or para-Sasakian.

ξ parallel case:

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold, $(\Psi, \tilde{g})$ be the related structure and ξ be parallel. Then, $\widetilde{\nabla}_X Y = \nabla_X Y$.*

Proof. Consider Eq (4.5) with $\nabla\xi = 0$.
Then, we have

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = Ag(\nabla_X Y, Z) + Bg(\nabla_X Y, \xi)\eta(Z).$$

By the definition of $\widetilde{g}$ and non degeneracy of g, it can be seen that $\nabla = \widetilde{\nabla}$. □

Corollary 3. *Let $(M, \varphi, \xi, \eta, g)$ be an almost paracontact metric manifold and ξ be parallel. Then, the related metallic structure $(\Psi, \widetilde{g})$ is locally metallic structure.*

Proof. Since ξ is parallel, we have $\nabla\eta = 0$. Thus,

$$\begin{aligned} (\widetilde{\nabla}_X \Psi)(Y) &= \widetilde{\nabla}_X \Psi Y - \Psi(\widetilde{\nabla}_X Y) \\ &= \widetilde{\nabla}_X(\sigma Y + (p - 2\sigma)\eta(Y)\xi) - (\sigma\widetilde{\nabla}_X Y + (p - 2\sigma)\eta(\widetilde{\nabla}_X Y)\xi) \\ &= (p - 2\sigma)(\nabla_X \eta)(Y)\xi \\ &= 0. \end{aligned}$$

Hence, $(\Psi, \widetilde{g})$ is locally metallic. □

Para-Kenmotsu case:

Let $(M, \varphi, \xi, \eta, g)$ be para-Kenmotsu manifold and $(\Psi, \widetilde{g})$ be the related metallic structure. By applying (2.4) in (4.5), we obtain the following:

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = Ag(\nabla_X Y, Z) + Bg(\nabla_X Y, \xi)\eta(Z) + Bg(\varphi X, \varphi Y)\eta(Z). \tag{4.6}$$

Also, for $Z = \xi$, we get:

$$\eta(\widetilde{\nabla}_X Y) = \eta(\nabla_X Y) + \frac{p}{\sigma^*} g(\varphi X, \varphi Y). \tag{4.7}$$

Proposition 4.3. *Let $(M, \varphi, \xi, \eta, g)$ be para-Kenmotsu manifold and $(\Psi, \widetilde{g})$ be the related metallic structure. Then,*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{p}{\sigma^*} g(\varphi X, \varphi Y)\xi. \tag{4.8}$$

Proof. It can be seen directly, by applying Eq (4.7) into Eq (4.6) and non-degeneracy of g. □

On the other hand, since $\eta(X) = \frac{1}{A+B}\widetilde{g}(X,\xi)$, from the definition of $\widetilde{g}$, we get the following:

$$\widetilde{\nabla}_X \eta(Y) = \eta(\nabla_X Y) + (A+B)g(\varphi X, \varphi Y). \tag{4.9}$$

From Eqs (4.7)–(4.9), the covariant derivative of the metallic structure Ψ becomes

$$(\widetilde{\nabla}_X \Psi)Y = [\sigma^*(p^2+4q) - \frac{B}{\sigma^*}]g(\varphi X, \varphi Y)\xi + (p-2\sigma)\eta(Y)\nabla_X \xi,$$

for the para-Kenmotsu case.

Corollary 4. *If (M,φ,ξ,η,g) is a para-Kenmotsu manifold, then the related metallic structure $(\Psi,\widetilde{g})$ is integrable (i.e., $\widetilde{N}_\Psi = 0$).*

Proof. By Eqs (4.8) and (4.9), after a long calculation, it can be seen that $\widetilde{N}_\Psi = 0$, and so Ψ is integrable. □

Proposition 4.4. *Let (M,φ,ξ,η,g) be para-Kenmotsu manifold and $(\Psi,\widetilde{g})$ be the related metallic structure. Then, the following holds:*

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi, \quad \widetilde{R}(X,\xi)Y = R(X,\xi)Y + \frac{p}{\sigma^*}g(\varphi X,\varphi Y)\xi, \quad \widetilde{R}(X,Y)\Psi\xi = \sigma^* R(X,Y)\xi, \tag{4.10}$$

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + \frac{p}{\sigma^*}\{g(\varphi X,\varphi Z)Y - g(\varphi Y,\varphi Z)X\}, \tag{4.11}$$

$$\begin{aligned} \widetilde{R}(X,Y)\Psi Z &= \sigma R(X,Y)Z + (p-2\sigma)(1-\sigma^*(p-2\sigma))\eta(R(X,Y)Z)\xi \\ &+ (p-2\sigma)\eta(Z)R(X,Y)\xi + \frac{p\sigma}{\sigma^*}\{g(\varphi X,\varphi Z)Y - g(\varphi Y,\varphi Z)X\}, \end{aligned} \tag{4.12}$$

where $R, \widetilde{R}$ denote the Riemannian curvature of g and $\widetilde{g}$, respectively.

Proof. From Eq (4.8) we get,

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X(\nabla_Y Z + \frac{p}{\sigma^*}g(\varphi Y,\varphi Z)\xi) - \widetilde{\nabla}(\nabla_X Z + \frac{p}{\sigma^*}g(\varphi X,\varphi Z)\xi) - (\nabla_{[X,Y]}Z + \frac{p}{\sigma^*}g(\varphi[X,Y],\varphi Z)\xi)$$

and

$$\widetilde{\nabla}_X(g(\varphi Y,\varphi Z)\xi) = g(\nabla_X\varphi Y,\varphi Z)\xi + g(\varphi Y,\nabla_X\varphi Z)\xi + g(\varphi Y,\varphi Z)\nabla_X\xi. \tag{4.13}$$

By the Eqs (4.8) and (4.13), we obtain Eq (4.11).

For the proof of Eq (4.12), consider the Eqs (4.1), (4.8) and (4.9) that yield the following result:

$$\begin{aligned} \widetilde{R}(X,Y)\Psi Z &= \sigma\{\widetilde{\nabla}_X(\nabla_Y Z + \frac{p}{\sigma^*}g(\varphi Y,\varphi Z)\xi) - \widetilde{\nabla}_Y(\nabla_X Z + \frac{p}{\sigma^*}g(\varphi X,\varphi Z)\xi) \\ &- (\nabla_{[X,Y]} + \frac{p}{\sigma^*}g(\varphi([X,Y]),\varphi Z)\xi)\} \\ &+ (p-2\sigma)\{\widetilde{\nabla}_X(\eta(\nabla_Y Z)\xi + \sigma^*(p-2\sigma)g(\varphi Y,\varphi Z)\xi + \eta(Z)\nabla_Y\xi) \\ &- \widetilde{\nabla}_Y(\eta(\nabla_X Z)\xi + \sigma^*(p-2\sigma)g(\varphi X,\varphi Z)\xi + \eta(Z)\nabla_X\xi) \\ &- (\eta(\nabla_{[X,Y]}Z)\xi + \sigma^*(p-2\sigma)g(\varphi([X,Y]),\varphi Z)\xi + \eta(Z)\nabla_{[X,Y]}\xi)\}. \end{aligned}$$

And so, Eq (4.12) is obtained by direct calculations.

By setting $Z = \xi$ in Eqs (4.11) and (4.12), Eq (4.10) is readily holds. □

Para-Sasakian case:

Suppose that $(M, \varphi, \xi, \eta, g)$ is para-Sasakian manifold, and $(\Psi, \widetilde{g})$ is the related metallic structure. By applying Eq (2.12) into (4.5), we obtain the following:

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = Ag(\nabla_X Y, Z) + Bg(\nabla_X Y, \xi)\eta(Z) + B[\eta(Y)g(X, \varphi Z) + \eta(X)g(Y, \varphi Z)]. \tag{4.14}$$

In Eq (4.14), setting $Z = \xi$ yields $\eta(\widetilde{\nabla}_X Y) = \eta(\nabla_X Y)$, and by direct calculation, we can state the following proposition:

Proposition 4.5. *Let $(M, \varphi, \xi, \eta, g)$ be para-Sasakian manifold and $(\Psi, \widetilde{g})$ be the related metallic structure. Then,*

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{B}{A}[\eta(X)\varphi Y + \eta(Y)\varphi X]. \tag{4.15}$$

By considering Eq (4.15), the following holds:

$$\widetilde{\nabla}_X(\eta(Y)\xi) = \eta(\nabla_X Y)\xi + g(X, \varphi Y)\xi + \frac{A+B}{A}\eta(Y)\nabla_X \xi. \tag{4.16}$$

So, if an almost paracontact metric structure is para-Sasakian, the Levi-Civita connection derivative of the related metallic structure $(\Psi, \widetilde{g})$ is given as follows:

$$(\widetilde{\nabla}_X \Psi)Y = (p - 2\sigma)\{g(X, \varphi Y)\xi + \frac{A+B}{A}\eta(Y)\nabla_X \xi\}. \tag{4.17}$$

Corollary 5. *If $(M, \varphi, \xi, \eta, g)$ is a para-Sasakian manifold, then the related metallic structure $(\Psi, \widetilde{g})$ is not integrable (i.e. $\widetilde{N}_\Psi \neq 0$).*

Proof. By Eqs (4.1) and (4.15), after a long calculation, it can be seen that $\widetilde{N}_\Psi = 2(p - 2\sigma)^2 g(\varphi X, Y)\xi$, and so Ψ is not integrable. □

Proposition 4.6. *Let $(M, \varphi, \xi, \eta, g)$ be a para-Sasakian manifold and $(\Psi, \widetilde{g})$ be the related metallic structure, then the curvature properties of Ψ are as follows*

$$\widetilde{R}(X, Y)\xi = (\frac{\sigma^*}{\sigma})^2 R(X, Y)\xi, \quad \widetilde{R}(X, Y)\Psi\xi = \sigma^*(\frac{\sigma^*}{\sigma})^2 R(X, Y)\xi,$$

$$\begin{aligned} \widetilde{R}(X, Y)Z &= R(X, Y)Z + \frac{B}{A}\eta(R(X, Y)Z)\xi + [\frac{2AB + B^2}{A^2}]\eta(Z)R(X, Y)\xi \\ &+ \frac{B}{A}[2g(\varphi X, Y)\varphi Z - g(\varphi Y, Z)\varphi X - g(\varphi Z, X)\varphi Y], \end{aligned}$$

$$\begin{aligned} \widetilde{R}(X, Y)\Psi Z &= \sigma R(X, Y)Z + \frac{\sigma^* B}{A}\eta(R(X, Y)Z)\xi + [\frac{((\sigma^*)^3 - \sigma^3)}{\sigma^2}]\eta(Z)R(X, Y)\xi \\ &+ \frac{\sigma B}{A}\{2g(\varphi X, Y)\varphi Z - g(X, \varphi Z)\varphi Y + g(Y, \varphi Z)\varphi X\} \end{aligned}$$

Proof. It can be seen directly in a similar manner to the calculations in the proof of Proposition (4.4) □

Example 6. *Consider the 3-dimensional manifold* $M = \{(x, y, z) \in \mathbb{R} | z \neq 0\}$*, and the linearly independent vector fields:*

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z},$$

where (x, y, z) *are the standart coordinates of* $\mathbb{R}^3$*. Let g be the semi-Riemannian metric defined by*

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0, \quad g(e_1, e_1) = -1, \quad g(e_2, e_2) = g(e_3, e_3) = 1,$$

and η *be the 1-form defined by* $\eta(X) = g(X, e_3), \quad$ *for any* $\quad X \in \mathfrak{X}(M)$.
Define the $(1, 1)$ *tensor field* φ *by*

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Then, we have $\eta(e_3) = g(e_3, e_3) = 1, \quad \varphi^2 X = X - \eta(X)e_3$*. Moreover, the metric g satisfies*

$$g(\varphi X, Y) + g(X, \varphi Y) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$ *and thus,* $(\varphi, \eta, \xi = e_3, g)$ *is an almost paracontact metric structure on* M^3.

One can verify that $[\varphi, \varphi](e_i, e_j) + 2d\eta(e_i, e_j) = 0$*, for* $1 \leq i < j \leq 3$*, which implies that the structure is normal.*

Let ∇ *be the Levi-Civita connection with respect to g. Then, we compute the Lie brackets*

$$[e_1, e_3] = -e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_1] = [e_2, e_2] = [e_3, e_3] = 0.$$

The Koszul formula for the Levi-Civita connection is

$$2g(\nabla_X Y, Z) = X[g(Y, Z)] + Y[g(Z, X)] - Z[g(X, Y)] - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using this, we compute

$$(\nabla_{e_i} e_j) = \begin{pmatrix} \nabla_{e_1} e_1 & \nabla_{e_1} e_2 & \nabla_{e_1} e_3 \\ \nabla_{e_2} e_1 & \nabla_{e_2} e_2 & \nabla_{e_2} e_3 \\ \nabla_{e_3} e_1 & \nabla_{e_3} e_2 & \nabla_{e_3} e_3 \end{pmatrix} = \begin{pmatrix} -e_3 & 0 & -e_1 \\ 0 & e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, $(M, \varphi, \eta, \xi, g)$ *is a para-Kenmotsu manifold. From Eq (4.8), we have*

$$\widetilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \frac{p}{\sigma^*} g(\varphi e_i, \varphi e_j) e_3$$

and therefore,

$$(\widetilde{\nabla}_{e_i} e_j) = \begin{pmatrix} \widetilde{\nabla}_{e_1} e_1 & \widetilde{\nabla}_{e_1} e_2 & \widetilde{\nabla}_{e_1} e_3 \\ \widetilde{\nabla}_{e_2} e_1 & \widetilde{\nabla}_{e_2} e_2 & \widetilde{\nabla}_{e_2} e_3 \\ \widetilde{\nabla}_{e_3} e_1 & \widetilde{\nabla}_{e_3} e_2 & \widetilde{\nabla}_{e_3} e_3 \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{\sigma^*} e_3 & 0 & -e_1 \\ 0 & -\frac{\sigma}{\sigma^*} e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}$$

Accordingly, the Riemannian curvature tensor of the related metallic structure derived from the para-Kenmotsu structure is given as follows.

$$\begin{pmatrix} \widetilde{R}(e_1,e_2)e_1 & \widetilde{R}(e_1,e_2)e_2 & \widetilde{R}(e_1,e_2)e_3 \\ \widetilde{R}(e_2,e_3)e_1 & \widetilde{R}(e_2,e_3)e_2 & \widetilde{R}(e_2,e_3)e_3 \\ \widetilde{R}(e_3,e_1)e_1 & \widetilde{R}(e_3,e_1)e_2 & \widetilde{R}(e_3,e_1)e_3 \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{\sigma^*}e_2 & \frac{\sigma}{\sigma^*}e_1 & 0 \\ 0 & -\frac{\sigma}{\sigma^*}e_3 & -e_2 \\ -\frac{\sigma}{\sigma^*}e_3 & 0 & e_1 \end{pmatrix}$$

An example for the parasasakian case can also be considered as follows.

Example 7. *Consider the manifold $M = \{(x, y, z) \in \mathbb{R}^3$ with $z \neq 0\}$. The linearly independent vector fields are given by*

$$u_1 = e^z\frac{\partial}{\partial y}, \quad u_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad u_3 = \frac{\partial}{\partial z},$$

where x, y and z are the standard coordinates. Let g be the semi-Riemannian metric defined by

$$g(u_1,u_1) = -g(u_2,u_2) = g(u_3,u_3) = 1, \quad g(u_1,u_2) = g(u_1,u_3) = g(u_2,u_3) = 0.$$

Let η and ϕ be defined by: $\eta(Z) = g(Z, u_3)$, for all $Z \in \mathfrak{X}(M)$

$$\phi(u_1) = u_2, \quad \phi(u_2) = u_1, \quad \phi(u_3) = 0.$$

Using the linearity of ϕ and g, we find

$$\eta(u_3) = 1, \quad \phi^2 Z = Z - \eta(Z)u_3, \quad g(\phi Z, \phi V) = -g(Z,V) + \eta(Z)\eta(V),$$

for any vector fields $Z, V \in \mathfrak{X}(M)$.

Thus, for $u_3 = \xi$, the structure (η, ξ, ϕ, g) defines a paracontact structure on M.

We compute the Lie brackets

$$[u_1,u_2] = 0, \quad [u_1,u_3] = -u_1, \quad [u_2,u_3] = -u_2.$$

Using Koszul's formula, the Levi-Civita connection ∇ is determined as

$$\begin{pmatrix} \nabla_{u_1}u_1 & \nabla_{u_1}u_2 & \nabla_{u_1}u_3 \\ \nabla_{u_2}u_1 & \nabla_{u_2}u_2 & \nabla_{u_2}u_3 \\ \nabla_{u_3}u_1 & \nabla_{u_3}u_2 & \nabla_{u_3}u_3 \end{pmatrix} = \begin{pmatrix} u_3 & 0 & -u_1 \\ 0 & -u_3 & -u_2 \\ 0 & 0 & 0 \end{pmatrix}$$

From Eq (4.15), we have

$$\widetilde{\nabla}_{u_i}u_j = \nabla_{u_i}u_j - \frac{B}{A}\{\eta(u_i)\phi(u_j) + \eta(u_j)\phi(u_i)\}$$

and hence,

$$(\widetilde{\nabla}_{u_i}u_j) = \begin{pmatrix} \widetilde{\nabla}_{u_1}u_1 & \widetilde{\nabla}_{u_1}u_2 & \widetilde{\nabla}_{u_1}u_3 \\ \widetilde{\nabla}_{u_2}u_1 & \widetilde{\nabla}_{u_2}u_2 & \widetilde{\nabla}_{u_2}u_3 \\ \widetilde{\nabla}_{u_3}u_1 & \widetilde{\nabla}_{u_3}u_2 & \widetilde{\nabla}_{u_3}u_3 \end{pmatrix} = \begin{pmatrix} u_3 & 0 & -(u_1 + \frac{B}{A}u_2) \\ 0 & -u_3 & -(u_2 + \frac{B}{A}u_1) \\ -\frac{B}{A}u_2 & -\frac{B}{A}u_1 & 0 \end{pmatrix}$$

Thus, the Riemann curvature tensor corresponding to the metallic structure obtained from the para-Sasakian structure is expressed as follows:

$$\begin{pmatrix} \widetilde{R}(u_1,u_2)u_1 & \widetilde{R}(u_1,u_2)u_2 & \widetilde{R}(u_1,u_2)u_3 \\ \widetilde{R}(u_2,u_3)u_1 & \widetilde{R}(u_2,u_3)u_2 & \widetilde{R}(u_2,u_3)u_3 \\ \widetilde{R}(u_3,u_1)u_1 & \widetilde{R}(u_3,u_1)u_2 & \widetilde{R}(u_3,u_1)u_3 \end{pmatrix} = \begin{pmatrix} u_2 + \frac{B}{A}u_1 & \frac{A+B}{A}(u_1+u_2) & -\frac{2B}{A}u_3 \\ -\frac{B}{A}u_2 & -\frac{B}{A}u_1 & 0 \\ -u_3 & \frac{B}{A}u_3 & \frac{2B}{A}u_2 + \frac{A^2+B^2}{A^2}u_1 \end{pmatrix}.$$

5. Conclusions

In this work, we construct a metallic semi-Riemannian structure $(\Psi, \tilde{g})$ from the given almost paracontact metric structure (φ, ξ, η, g) on a $2n+1$ dimensional smooth manifold M. Then, we focus on para-Kenmotsu and para-Sasakian structures and show that the obtained φ-related structures are integrable and non-integrable, respectively. Furthermore, we study the curvature properties of these structures and give explicit examples.

Author contributions

Mehmet Solgun: supervision, conceptualization, writing-original draft, writing-review and editing, methodology, validation; Yasemin Karababa: writing-review and editing, writing-original draft, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interes in this paper.

References

1. V. W. de Spinadel, The family of metallic means, *Vis. Math*, **1** (1999), 1–16.
2. V. W. de Spinadel, The metallic means family and multifractal spectra, *Nonlinear Anal. Theory Method. Appl.*, **36** (1999), 721–745. https://doi.org/10.1016/S0362-546X(98)00123-0
3. C. E. Hretcanu, M. Crasmareanu, Metallic structures on Riemannian manifolds, *Rev. Un. Mat. Argentina*, **54** (2013), 15–27.
4. A. Gezer, Ç. Karaman, On metallic Riemannian structures, *Turk. J. Math.*, **39** (2015), 14. https://doi.org/10.3906/mat-1504-50
5. A. M. Blaga, A. Nannicini, On the geometry of metallic pseudo-Riemannian structures, *Rivista di Matematica della Universita di Parma*, **11** (2020), 69–87.

6. A. M. Blaga, A. Nannicini, On curvature tensors of Norden and metallic pseudo-Riemannian manifolds, *Complex manifolds*, **6** (2019), 150–159. https://doi.org/10.1515/coma-2019-0008
7. S. Kaneyuki, F. L. Williams, Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.*, **99** (1985), 173–187. https://doi.org/10.1017/S0027763000021565
8. S. Zamkovoy, Canonical connections on paracontact manifolds, *Ann. Glob. Anal. Geom.*, **36** (2009), 37–60. https://doi.org/10.1007/s10455-008-9147-3
9. N. Özdemir, Ş. Aktay, M. Solgun, Some results on almost paracontact paracomplex Riemannian manifolds, *AIMS Mathematics*, **10** (2025), 10764–10786. https://doi.org/10.3934/math.2025489
10. N. Özdemir, M. Solgun, Ş. Aktay, Almost para-contact metric structures on 5-dimensional nilpotent Lie algebras, *Fund. J. Math. Appl.*, **3** (2020), 175–184.
11. N. Özdemir, Ş. Aktay, M. Solgun, Almost paracontact structures obtained from $G^*_{2(2)}$ structures, *Turk. J. Math.*, **42** (2018), 3025–3033. https://doi.org/10.3906/mat-1706-10
12. M. Solgun, Some results on $\mathcal{D}$-homothetic deformation On almost paracontact metric manifolds, *Fund. J. Math. Appl.*, **4** (2021), 264–270.
13. N. Özdemir, Ş. Aktay, M. Solgun, Some results on almost paracontact paracomplex Riemannian manifolds, *AIMS Mathematics*, **10** (2025), 10764–10786.
14. N. Özdemir, M. Efe, Relationships between almost parahermitian and almost paracontact metric manifolds, *Filomat*, **39** (2025), 1247–1263. https://doi.org/10.2298/FIL2504247O
15. S. Zamkovoy, G. Nakova, The decomposition of almost paracontact metric manifolds in eleven classes revisited, *Journal of Geometry* **109** (2018), 18. https://doi.org/10.1007/s00022-018-0423-5
16. İ. K. Erken, On almost α-para-Kenmotsu manifolds satisfying certain conditions, *Commun. Fac. Sci. Univ.*, **68** (2019), 559–571.
17. M. Krishanu, U. C. De, On 3-dimensional α-para-Kenmotsu manifolds, *Konuralp J. Math.* **5** (2016), 11–23.
18. G. Beldjilali, Almost contact metric and metallic Riemannian structures, *Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat.*, **69** (2020), 1013–1024.
19. R. Ç. Akpınar, E. K. Kansu, Metallic deformation on para-Sasaki-like para-Norden manifold, *AIMS Mathematics*, **9** (2024), 19125–19136. https://doi.org/10.3934/math.2024932
20. S. Azami, Metallic structures on the tangent bundle of P-Sasakian manifolds, *Khayyam J. Math.*, **7** (2021), 298–309. https://doi.org/10.22034/KJM.2021.242375.1950
21. S. Zamkovoy, ParaSasakian manifolds with a constant paraholomorphic section curvature, 2008, arXiv: 0812.1676.
22. S. Zamkovoy, On para-Kenmotsu manifolds, 2017, arXiv: 1711.03008. https://doi.org/10.48550/arXiv.1711.03008
23. G. Ganchev, V. Mihova, Canonical connection and the canonical conformal group on an almost complex manifold with B-metric, *Ann. Univ. Sofia Fac. Math. Inform*, **81** (1987), 195–206.