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Research article

Objective Bayesian analysis for the ratio of shape parameters in generalized half-normal distributions

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Abstract: The generalized half-normal distribution is notable for its flexibility in modeling diverse hazard rate shapes, including ones that are monotonically increasing or decreasing, and bathtub forms, determined by the value of the shape parameter. To facilitate Bayesian comparative analyses of these shape parameters, we have proposed noninformative priors for the ratio of shape parameters within generalized half-normal distributions. We derived probability matching priors and reference priors, identifying a second-order matching prior that satisfies all specified matching criteria. Our findings show that both the two-group and three-group reference priors meet the first-order matching criterion, whereas Jeffreys' prior does not. However, the one-at-a-time reference prior successfully satisfies the stricter second-order matching criterion. Additionally, we established conditions ensuring posterior propriety under general priors, particularly highlighting the derived noninformative priors. A simulation study demonstrated that the one-at-a-time reference prior achieves accurate alignment with target frequentist coverage probabilities. Finally, we provided two real-world examples to illustrate and reinforce our theoretical results.

Keywords: generalized half-normal distribution; matching priors; ratio of shape parameters; reference prior
Mathematics Subject Classification: 62F10, 62N01, 62N02

1. Introduction

The generalized half-normal distribution is a versatile and widely studied probability model, frequently applied in reliability analysis, quality control, lifetime data modeling, and other statistical applications involving nonnegative data. The half-normal and Birnbaum-Saunders distributions are frequently employed to characterize lifetime processes under conditions of fatigue, particularly when modeling monotonic hazard rates due to their negatively or positively skewed shapes. However, these distributions are inadequate for representing non-monotonic failure rates, such as those exhibiting bathtub-shaped or unimodal characteristics, which are commonly observed in reliability and biological studies. To address this limitation, Díaz-García and Leiva [1] introduced a family of generalized Birnbaum-Saunders distributions derived from a contoured elliptical distribution. Subsequently, Cooray and Ananda [2] proposed the generalized half-normal (GHN) distribution to facilitate the modeling of skewed fatigue life data. To enhance the flexibility of the tail beyond that offered by the GHN distribution, several extensions have been developed, including the beta generalized half-normal distribution [3], the slashed generalized half-normal distribution [4], the extended generalized half-normal distribution [5], and the Zografos-Balarkishnan odd log-logistic generalized half-normal distribution [6].

It is especially valuable due to its flexibility in modeling skewed distributions and its capability to adjust to various degrees of dispersion and kurtosis through its shape parameters. The ratio of shape parameters in generalized half-normal distributions emerges as a particularly important measure, offering insights into the relative dispersion and skewness of data. This ratio not only simplifies comparative analysis across different populations or processes but also enhances interpretability by consolidating multiple parameters into a single scalar metric. However, despite its practical importance, the specific properties, potential applications, and analytical benefits of this ratio remain underexplored in statistical literature.

Recent works have continued to advance both the modeling flexibility of generalized half-normal–type distributions and the theoretical properties of associated inference procedures. For instance, extensions incorporating covariates, skewness adjustments, and mixture structures have been proposed to improve applicability in survival and reliability studies [4–6]). Furthermore, Ahmadi et al. [7] considered the reliability estimation of the generalized half-normal distribution when data are subject to left truncation and right censoring. More recently, studies on objective and probability-matching priors have emphasized their role in ensuring frequentist validity of Bayesian procedures in complex models [8, 9]. In particular, Wang and Polson [10] investigated the large-sample properties of maximum likelihood estimators under moving extremes ranked set sampling, highlighting the importance of asymptotic efficiency and coverage in nonstandard sampling schemes. These developments underscore the increasing relevance of robust prior construction and large-sample considerations, and they motivate our focus on noninformative priors for the ratio of shape parameters in generalized half-normal distributions. More recently, Wang and Polson [10] studied prior constructions for sparse count models involving ratios of gamma functions, which bears methodological resemblance to our work in terms of handling parameter-ratio inference under noninformative or default prior frameworks.

In this study, we examine the GHN distribution. Let X denote a random variable that follows a GHN distribution. The probability density function of this distribution can be represented as

$$f(x|\beta,\alpha) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{x}\left(\frac{x}{\beta}\right)^{\alpha}\exp\left[-\frac{1}{2}\left(\frac{x}{\beta}\right)^{2\alpha}\right], x > 0,$$

where the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$ influence the negatively and positively skewed densities. The hazard rate function can exhibit a variety of shapes, including monotonically increasing, monotonically decreasing, and bathtub forms, contingent upon the values of the parameters. Our primary focus is on the ratio of the shape parameters within the GHN distributions, and we aim to develop noninformative priors for this ratio. To achieve this objective, we utilize probability matching and reference priors, both of which have been extensively applied in the literature. The probability matching prior necessitates that the coverage probability of a Bayesian credible interval is asymptotically equivalent to the coverage probability of a frequentist confidence interval. Conversely, the reference prior seeks to maximize the Kullback-Leibler divergence between the prior and posterior distributions. The methodologies of probability matching and reference priors have demonstrated considerable success in addressing various practical challenges.

In Bayesian analysis, the key component in addition to the likelihood is the choice of prior distributions. When sufficient prior knowledge is available—whether from past experience, expert opinion, or previously collected data—subjective priors are often the most appropriate and should be employed for inference. However, even in the absence of adequate prior information, Bayesian methods can still be applied effectively through the use of default or so-called objective priors, which have been developed based on formal criteria of objectivity. Two widely studied approaches in this regard are probability matching priors and reference priors. Probability matching priors ensure that the coverage probability of a Bayesian credible interval is asymptotically equivalent to that of a frequentist confidence interval, an idea originating with Welch and Peers [11] and later extended by [12–14]. A comprehensive treatment of probability matching criteria is given in the monograph by Datta and Mukerjee [9]. By contrast, the reference prior framework, introduced by Bernardo [15] and further developed by Berger and Bernardo [16, 17] as well as Ghosh and Mukerjee [18], aims to maximize the Kullback–Leibler divergence between the prior and the posterior. Their general algorithm involves partitioning parameters according to their inferential importance, with further refinements introduced in Berger et al. [19, 20]. Both methodologies have been shown to be highly effective in practice. In particular, applications in lifetime and reliability analysis often involve complex censoring schemes and limited information (e.g., few observed failures), where asymptotic theory may yield misleading conclusions. In such settings, noninformative priors with desirable coverage properties are especially valuable. Recently, Tian et al. [21] provided guidance on constructing noninformative priors for log-location-scale distributions in reliability contexts, further expanding the practical toolkit for Bayesian reliability analysis.

We here conduct a comparative analysis of shape parameters within the frequentist and Bayesian frameworks by developing reference priors, as well as first- and second-order probability matching priors for the ratio of shape parameters across various matching criteria. We introduce a distinctive second-order matching prior that fulfills all the matching criteria considered, while both the two-group reference prior and the three-group reference prior meet a first-order matching criterion. Furthermore, we demonstrate that the one-at-a-time reference prior qualifies as a second-order matching prior. Additionally, we establish a condition for the propriety of the posterior distribution under the general prior, which encompasses both reference priors and matching priors. To conclude, we simulate frequentist coverage probabilities associated with the proposed priors and analyze two real-world examples.

2. Noninformative priors for the ratio of shape parameters

2.1. Quantile matching priors

We consider matching priors that facilitate the approximate frequentist validity of one-sided Bayesian credible intervals derived from posterior quantiles of a one-dimensional parameter of interest. Our objective is to identify priors π such that, as the sample size n approaches infinity, the following condition holds:

$$P_\theta[\theta_1 \leq \theta_1^{1-\alpha}(\pi; \mathbf{X})] = 1 - \alpha + o(n^{-u}),$$

for some $u > 0$. In this context, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)^T$, where θ_1 represents the parameter of interest, and $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ denotes the $(1-\alpha)$th posterior quantile of θ_1 based on the prior π and the data $\mathbf{X}$. The prior that satisfies this condition is termed a matching prior; specifically, if $u = 1/2$, the prior is classified as a first-order matching prior, whereas if $u = 1$, it is classified as a second-order matching prior.

To illustrate, suppose we have two independent random samples $(X_1, \cdots, X_{n_1})$ and $(Y_1, \cdots, Y_{n_2})$, where the X_{ij} are independent and identically distributed $GHN(\alpha_1,\beta_1)$ and the Y_{ij} are independent and identically distributed $GHN(\alpha_2,\beta_2)$. The likelihood function is then given by

$$\begin{aligned} &L(\alpha_1, \alpha_2, \beta_1, \beta_2) \\ &\propto \alpha_1^{n_1}\alpha_2^{n_2} \prod_{i=1}^{n_1}\left(\frac{x_i}{\beta_1}\right)^{\alpha_1} \prod_{i=1}^{n_2}\left(\frac{y_i}{\beta_2}\right)^{\alpha_2} \exp\left[-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i}{\beta_1}\right)^{2\alpha_1}\right] \exp\left[-\frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i}{\beta_2}\right)^{2\alpha_2}\right]. \end{aligned} \tag{2.1}$$

The orthogonal parameterization on nuisance parameters simplifies the development of the matching priors. The likelihood function of the model (2.1) can be expressed in terms of the parameters $\theta_1 = \frac{\alpha_2}{\alpha_1}$, $\theta_2 = \alpha_1^{n_1}\alpha_2^{n_2}$, $\theta_3 = \beta_1 \exp\left(\frac{c_1}{2\alpha_1}\right)$, and $\theta_4 = \beta_2 \exp\left(\frac{c_1}{2\alpha_2}\right)$, where $c_1 = \int_0^\infty \frac{2}{\sqrt{\pi}} z^{1/2} \log(2z) e^{-z} dz$. This likelihood function can then be expressed as:

$$\begin{aligned} &L(\theta_1, \theta_2, \theta_3, \theta_4) \\ \propto\ & \theta_2 \left[\prod_{i=1}^{n_1}\left(\frac{x_i}{\theta_3}\right)^{\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right]\left[\prod_{i=1}^{n_2}\left(\frac{y_i}{\theta_4}\right)^{\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right] \\ \times\ & \exp\left\{-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i}{\theta_3}\right)^{2\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}} e^{c_1} - \frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i}{\theta_4}\right)^{2\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}} e^{c_1}\right\}. \end{aligned} \tag{2.2}$$

Based on (2.2), the Fisher information matrix is given by

$$\begin{aligned} &\mathbf{I}(\theta_1, \theta_2, \theta_3, \theta_4) \\ =\ & \begin{pmatrix} \frac{n_1n_2(2+c_2-c_1^2)}{2(n_1+n_2)}\theta_1^{-2} & 0 & 0 & 0 \\ 0 & \frac{2+c_2-c_1^2}{2(n_1+n_2)}\theta_2^{-2} & 0 & 0 \\ 0 & 0 & \frac{2n_1}{\theta_1^{\frac{2n_2}{n_1+n_2}}\theta_2^{\frac{-2}{n_1+n_2}}}\theta_3^{-2} & 0 \\ 0 & 0 & 0 & \frac{2n_2}{\theta_1^{\frac{-2n_1}{n_1+n_2}}\theta_2^{\frac{-2}{n_1+n_2}}}\theta_4^{-2} \end{pmatrix}, \end{aligned}$$

where $c_2 = \int_0^\infty \frac{2}{\sqrt{\pi}} z^{1/2}(\log 2z)^2 e^{-z}dz$. The Fisher information matrix $\mathbf{I}$ indicates that θ_1 is orthogonal to θ_2, θ_3, and θ_4 in the sense of Cox and Reid [22]. Following Tibshirani [23], the class of first-order probability matching priors is characterized by

$$\pi_m^{(1)}(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} d(\theta_2, \theta_3, \theta_4), \tag{2.3}$$

where $d(\theta_2, \theta_3, \theta_4) > 0$ is an arbitrary function differentiable in its argument.

The class of priors given in (2.3) can be narrowed down to second-order probability matching priors as given in Mukerjee and Ghosh [8]. A second-order probability matching prior must satisfy the additional differential equation (2.10) of Mukerjee and Ghosh [8], namely

$$\frac{1}{6} d(\theta_2, \theta_3, \theta_4) \frac{\partial}{\partial \theta_1}\{I_{11}^{-\frac{3}{2}} L_{1,1,1}\} + \sum_{s=2}^{4} \frac{\partial}{\partial \theta_s}\{I_{11}^{-\frac{1}{2}} L_{11s} I^{ss} d(\theta_2, \theta_3, \theta_4)\} = 0, \tag{2.4}$$

where the inverse matrix of Fisher information matrix $\mathbf{I}^{-1} = (I^{ij})_{4\times 4}$. The resulting second-order probability matching prior is given by

$$\pi_m^{(2)}(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-1}\theta_4^{-1}. \tag{2.5}$$

This is obtained by solving the differential equation (2.4) using the expressions for the following quantities:

$$\begin{aligned}
L_{1,1,1} &= E\left[\left(\frac{\partial \log L}{\partial \theta_1}\right)^3\right] = k_0 \theta_1^{-3}, \\
L_{112} &= E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_2}\right] = -\frac{n_1 n_2}{2(n_1+n_2)^2}(c_3 + 3c_2 - 3c_1 c_2 - 3c_1^2 + 2c_1^3 + 2)\theta_1^{-2}\theta_2^{-1}, \\
L_{113} &= E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_3}\right] = \frac{n_1 n_2^2}{(n_1+n_2)^2}(c_2 - c_1^2)\theta_1^{-\frac{n_2}{n_1+n_2}-2}\theta_2^{\frac{1}{n_1+n_2}}\theta_3^{-1}, \\
L_{114} &= E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_4}\right] = \frac{n_1^2 n_2}{(n_1+n_2)^2}(c_2 - c_1^2)\theta_1^{\frac{n_1}{n_1+n_2}-2}\theta_2^{\frac{1}{n_1+n_2}}\theta_4^{-1}, \\
I_{11} &= \frac{n_1 n_2 (2 + c_2 - c_1^2)}{2(n_1+n_2)}\theta_1^{-2}, \quad I^{22} = \frac{2(n_1+n_2)}{2 + c_2 - c_1^2}\theta_2^2, \\
I^{33} &= \frac{1}{2n_1}\theta_1^{\frac{2n_2}{n_1+n_2}}\theta_2^{-\frac{2}{n_1+n_2}}\theta_3^2, \quad I^{44} = \frac{1}{2n_2}\theta_1^{-\frac{2n_1}{n_1+n_2}}\theta_2^{-\frac{2}{n_1+n_2}}\theta_4^2,
\end{aligned}$$

where k_0 is a constant and $c_3 = \int_0^\infty \frac{2}{\sqrt{\pi}} z^{1/2}(\log 2z)^3 e^{-z}dz$. Substituting these values into the differential equation (2.4) yields

$$\begin{aligned}
&\frac{\partial}{\partial \theta_2}\left\{\frac{2(c_3 + 3c_2 - 3c_1 c_2 - 3c_1^2 + 2c_1^3 + 2)n_1 n_2}{(n_1+n_2)(2 - c_1^2 + c_2)}\theta_1^{-1}\theta_2 d(\theta_2, \theta_3, \theta_4)\right\} \\
&+ \frac{\partial}{\partial \theta_3}\left\{\frac{(c_2 - c_1^2)n_2^2}{(n_1+n_2)^2}\theta_1^{\frac{n_2}{n_1+n_2}-1}\theta_2^{-\frac{1}{n_1+n_2}}\theta_3 d(\theta_2, \theta_3, \theta_4)\right\} \\
&+ \frac{\partial}{\partial \theta_4}\left\{\frac{(c_2 - c_1^2)n_1^2}{(n_1+n_2)^2}\theta_1^{-\frac{n_1}{n_1+n_2}-1}\theta_2^{-\frac{1}{n_1+n_2}}\theta_4 d(\theta_2, \theta_3, \theta_4)\right\} = 0. \qquad (2.6)
\end{aligned}$$

The unique solution of the above Eq (2.6) is then

$$d(\theta_2, \theta_3, \theta_4) = \theta_2^{-1}\theta_3^{-1}\theta_4^{-1}.$$

2.2. Alternative matching priors

We want to examine whether the second-order probability matching prior, as delineated in Eq (2.5), adheres to the alternative matching criteria proposed by Mukerjee and Reid [24] at the second-order. To achieve this objective, we compute the derivatives of the log-likelihood function, specifically:

$$
\begin{aligned}
L_{111} &= E\left[\frac{\partial^3 \log L}{\partial \theta_1^3}\right] \\
&= \frac{n_1 n_2}{2(n_1+n_2)^2}\left[4(n_1+2n_2)+(n_2-n_1)(c_3-3c_1c_2+2c_1^3)+6n_2(c_2-c_1^2)\right]\theta_1^{-3}, \\
L_{11,1} &= E\left[\frac{\partial^2 \log L}{\partial \theta_1^2}\frac{\partial \log L}{\partial \theta_1}\right] = k_1\theta_1^{-3}, \\
L_{11,2} &= E\left[\frac{\partial^2 \log L}{\partial \theta_1^2}\frac{\partial \log L}{\partial \theta_2}\right] = k_2\theta_1^{-2}\theta_2^{-1}, \\
L_{11,3} &= E\left[\frac{\partial^2 \log L}{\partial \theta_1^2}\frac{\partial \log L}{\partial \theta_3}\right] = k_3\theta_1^{-\frac{n_2}{n_1+n_2}-2}\theta_2^{\frac{1}{n_1+n_2}}\theta_3^{-1} \\
L_{11,4} &= E\left[\frac{\partial^2 \log L}{\partial \theta_1^2}\frac{\partial \log L}{\partial \theta_4}\right] = k_4\theta_1^{\frac{n_1}{n_1+n_2}-2}\theta_2^{\frac{1}{n_1+n_2}}\theta_4^{-1},
\end{aligned}
$$

where k_1, k_2, k_3, and k_4 are constants. By analyzing the differential equations established by Mukerjee and Reid [24], which arise from the orthogonality of θ_1 with respect to $(\theta_2, \theta_3, \theta_4)$ and utilizing the expression $d(\theta_2, \theta_3, \theta_4) = \theta_2^{-1}\theta_3^{-1}\theta_4^{-1}$, we demonstrate that:

$$
\begin{aligned}
&\frac{\partial}{\partial \theta_1}\{I_{11}^{-\frac{3}{2}}L_{111}\} = 0, \frac{\partial}{\partial \theta_1}\{I_{11}^{-\frac{3}{2}}L_{11,1}\} = 0, \\
&\sum_{s=2}^{4}\frac{\partial}{\partial \theta_s}\{I_{11}^{-\frac{1}{2}}L_{11s}I^{ss}d(\theta_2,\theta_3,\theta_4)\} = 0, \\
&\sum_{s=2}^{4}\frac{\partial}{\partial \theta_s}\{I_{11}^{-\frac{1}{2}}L_{11,s}I^{ss}d(\theta_2,\theta_3,\theta_4)\} = 0.
\end{aligned}
$$

Consequently, it follows that the second-order matching prior, as specified in Eq (2.5), with the condition $d(\theta_2, \theta_3, \theta_4) = \theta_2^{-1}\theta_3^{-1}\theta_4^{-1}$, successfully aligns with the alternative coverage probabilities outlined by Mukerjee and Reid [24].

Subsequently, we examine a prior distribution denoted as π, which facilitates matching through distribution functions, as established by Mukerjee and Ghosh [8]. This prior is expressed in the form $I_{11}^{1/2}d(\theta_2, \theta_3, \theta_4)$, reflecting its orthogonality with respect to θ_1. The prior adheres to two differential equations,

$$
\begin{aligned}
\Delta_3 &= \frac{\partial^2}{\partial \theta_1^2}\{I^{11}\pi(\theta)\} - 2\frac{\partial}{\partial \theta_1}\left\{I^{11}\frac{\partial}{\partial \theta_1}\pi(\theta)\right\} - \sum_{v=2}^{4}\frac{\partial}{\partial \theta_v}\{L_{11v}I^{11}I^{vv}\pi(\theta)\} \\
&\quad - \sum_{v=2}^{4}\frac{\partial}{\partial \theta_1}\{L_{1vv}I^{11}I^{vv}\pi(\theta)\} = 0, \qquad (2.7)
\end{aligned}
$$

$$\Delta_4 = \frac{\partial}{\partial\theta_1}\{L_{111}(I^{11})^2\pi(\theta)\} = 0. \tag{2.8}$$

We note that the quantities

$$L_{122} = E\left[\frac{\partial^3 \log L}{\partial\theta_1\partial\theta_2^2}\right] = 0, L_{133} = E\left[\frac{\partial^3 \log L}{\partial\theta_1\partial\theta_3^2}\right] = \frac{4n_1n_2}{n_1+n_2}\theta_1^{-\frac{2n_2}{n_1+n_2}-1}\theta_2^{\frac{2}{n_1+n_2}}\theta_3^{-2},$$
$$L_{144} = E\left[\frac{\partial^3 \log L}{\partial\theta_1\partial\theta_4^2}\right] = -\frac{4n_1n_2}{n_1+n_2}\theta_1^{\frac{2n_1}{n_1+n_2}-1}\theta_2^{\frac{2}{n_1+n_2}}\theta_4^{-2},$$

imply that the aforementioned differential equations (2.7) and (2.8) can be simplified to the form presented in

$$\begin{aligned}\Delta_3 &= \frac{\partial}{\partial\theta_2}\left\{\frac{2(c_3+3c_2-3c_1c_2-3c_1^2+2c_1^3+2)n_1n_2}{(n_1+n_2)(2-c_1^2+c_2)}\theta_1^{-1}\theta_2 d(\theta_2,\theta_3,\theta_4)\right\}\\ &+ \frac{\partial}{\partial\theta_3}\left\{\frac{(c_2-c_1^2)n_2^2}{(n_1+n_2)^2}\theta_1^{\frac{n_2}{n_1+n_2}-1}\theta_2^{-\frac{1}{n_1+n_2}}\theta_3 d(\theta_2,\theta_3,\theta_4)\right\}\\ &+ \frac{\partial}{\partial\theta_4}\left\{\frac{(c_2-c_1^2)n_1^2}{(n_1+n_2)^2}\theta_1^{-\frac{n_1}{n_1+n_2}-1}\theta_2^{-\frac{1}{n_1+n_2}}\theta_4 d(\theta_2,\theta_3,\theta_4)\right\} = 0, \qquad (2.9)\\ \Delta_4 &= 0. \qquad (2.10)\end{aligned}$$

The unique solution to this equation, along with Eq (2.10), is given by

$$d(\theta_2,\theta_3,\theta_4) = \theta_2^{-1}\theta_3^{-1}\theta_4^{-1}.$$

Consequently, the prior structured as $I_{11}^{1/2}d(\theta_2,\theta_3,\theta_4)$ qualifies as a second-order quantile matching prior, which concurrently fulfills the criteria for second-order distribution function matching, as noted by Mukerjee and Ghosh [8].

We also explore priors that guarantee that the highest posterior density (HPD) regions, corresponding to a probability level of $1-\alpha$, maintain asymptotically equivalent frequentist coverage probabilities with minimal approximation error. Given the orthogonality of θ_1 with the parameters $(\theta_2,\theta_3,\theta_4)$, the prior π must satisfy the condition outlined in

$$\sum_{v=2}^{4}\frac{\partial}{\partial\theta_v}\{L_{11v}I^{vv}I^{11}\pi(\theta)\} + \frac{\partial}{\partial\theta_1}\{L_{111}(I^{11})^2\pi(\theta)\} - \frac{\partial^2}{\partial\theta_1^2}\{I^{11}\pi(\theta)\} = 0,$$

which can be further simplified to the form presented in

$$\begin{aligned}&\frac{\partial}{\partial\theta_2}\left\{-\frac{2(c_3+3c_2-3c_1c_2-3c_1^2+2c_1^3+2)}{(2-c_1^2+c_2)^2}\theta_2\pi(\theta)\right\}\\ +&\frac{\partial}{\partial\theta_3}\left\{\frac{(c_2-c_1^2)n_2}{(2-c_1^2+c_2)n_1(n_1+n_2)}\theta_1^{\frac{n_2}{n_1+n_2}}\theta_2^{-\frac{1}{n_1+n_2}}\theta_3\pi(\theta)\right\}\\ +&\frac{\partial}{\partial\theta_4}\left\{\frac{(c_2-c_1^2)n_1}{(2-c_1^2+c_2)n_2(n_1+n_2)}\theta_1^{-\frac{n_1}{n_1+n_2}}\theta_2^{-\frac{1}{n_1+n_2}}\theta_4\pi(\theta)\right\}\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\partial}{\partial\theta_1}\left\{\frac{12(c_2-c_1^2)n_2+2(2c_1^3-3c_1c_2+c_3)(n_2-n_1)+8(n_1+2n_2)}{(2-c_1^2+c_2)^2 n_1 n_2}\theta_1\pi(\theta)\right\} \\
&- \quad \frac{\partial^2}{\partial\theta_1^2}\left\{\frac{2(n_1+n_2)}{n_1n_2(2+c_2-c_1^2)}\theta_1^2\pi(\theta)\right\} = 0. \qquad (2.11)
\end{aligned}
$$

The unique solution to Eq (2.11) is expressed as follows:

$$
\pi(\theta_1,\theta_2,\theta_3,\theta_4) \propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-1}\theta_4^{-1},
$$

which demonstrates that the second-order quantile matching priors, as described in Eq (2.5), also serve as HPD matching priors, as established by DiCiccio and Stern [25] and Ghosh and Mukerjee [26].

Lastly, we investigate the matching prior associated with the likelihood ratio (LR) statistic, as discussed by Yin and Ghosh [27]. Based on the orthogonality of θ_1 with the parameters $(\theta_2,\theta_3,\theta_4)$, an LR matching prior π can be derived by solving the equation

$$
\sum_{v=2}^{4}\frac{\partial}{\partial\theta_v}\{I_{11}^{-1}I^{vv}L_{11v}\pi(\theta)\} + \frac{\partial}{\partial\theta_1}\left\{I_{11}^{-1}\left[\frac{\partial\pi(\theta)}{\partial\theta_1} - \pi(\theta)\left(I_{11}^{-1}L_{11,1} - \sum_{v=2}^{4}I^{vv}L_{1vv}\right)\right]\right\} = 0,
$$

which can be simplified as shown in

$$
\begin{aligned}
&\quad\ \frac{\partial}{\partial\theta_2}\left\{-\frac{2(c_3+3c_2-3c_1c_2-3c_1^2+2c_1^3+2)}{(2-c_1^2+c_2)^2}\theta_2\pi(\theta)\right\} \\
&+ \quad \frac{\partial}{\partial\theta_3}\left\{\frac{(c_2-c_1^2)n_2}{(2-c_1^2+c_2)n_1(n_1+n_2)}\theta_1^{\frac{n_2}{n_1+n_2}}\theta_2^{-\frac{1}{n_1+n_2}}\theta_3\pi(\theta)\right\} \\
&+ \quad \frac{\partial}{\partial\theta_4}\left\{\frac{(c_2-c_1^2)n_1}{(2-c_1^2+c_2)n_2(n_1+n_2)}\theta_1^{-\frac{n_1}{n_1+n_2}}\theta_2^{-\frac{1}{n_1+n_2}}\theta_4\pi(\theta)\right\} \\
&+ \quad \frac{\partial}{\partial\theta_1}\left\{\frac{2(n_1+n_2)}{(2+c_2-c_1^2)n_1n_2}\theta_1^2\left[\frac{\partial\pi(\theta)}{\partial\theta_1}\right.\right. \\
&- \quad \left.\left.\pi(\theta)\left(\frac{2k_1(n_1+n_2)}{(2+c_2-c_1^2)n_1n_2}\theta_1^{-1} - \frac{2(n_2-n_1)}{n_1+n_2}\theta_1^{-1}\right)\right]\right\} = 0. \qquad (2.12)
\end{aligned}
$$

The unique solution to Eq (2.12) is characterized by the form indicated in the equation

$$
\pi(\theta_1,\theta_2,\theta_3,\theta_4) \propto \theta_1^{-1}\theta_2^{-1}\theta_3^{-1}\theta_4^{-1}.
$$

Therefore, the second-order quantile matching priors, as specified in Eq (2.5), are identified as the LR matching priors, as noted by Yin and Ghosh [27].

2.3. Reference priors

Reference priors, initially proposed by Bernardo [15] and subsequently expanded upon by Berger and Bernardo [17], have gained prominence in the formulation of noninformative priors. In this section, we derive reference priors for various configurations of the parameters $(\theta_1,\theta_2,\theta_3,\theta_4)$. Given the orthogonality of these parameters, Datta and Ghosh [14] recommend the selection of rectangular compact sets for each of the parameters θ_1, θ_2, θ_3, and θ_4, particularly when θ_1 is designated as the parameter of interest. The corresponding reference priors for the model specified in Eq (2.2) are presented as follows:

For the model (2.2), if θ_1 is the parameter of interest, then the reference prior for the group of orderings $\{(\theta_1, \theta_2, \theta_3, \theta_4)\}$ is

$$\pi_1(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1-\frac{n_2-n_1}{n_1+n_2}} \theta_2^{-1+\frac{2}{n_1+n_2}} \theta_3^{-1} \theta_4^{-1}. \tag{2.13}$$

For the group of orderings $\{\theta_1, (\theta_2, \theta_3, \theta_4)\}$, the two-group reference prior is

$$\pi_2(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} \theta_2^{-1+\frac{2}{n_1+n_2}} \theta_3^{-1} \theta_4^{-1}. \tag{2.14}$$

For the group of orderings of $\{\theta_1, (\theta_2, \theta_3), \theta_4\}$, $\{\theta_1, \theta_4, (\theta_2, \theta_3)\}$, $\{\theta_1, (\theta_2, \theta_4), \theta_3\}$, and $\{\theta_1, \theta_3, (\theta_2, \theta_4)\}$, the three-group reference prior is

$$\pi_3(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} \theta_2^{-1+\frac{1}{n_1+n_2}} \theta_3^{-1} \theta_4^{-1}, \tag{2.15}$$

and for the group of orderings of $\{\theta_1, \theta_2, (\theta_3, \theta_4)\}$, and $\{\theta_1, (\theta_3, \theta_4), \theta_2\}$, the three-group reference prior is

$$\pi_4(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} \theta_2^{-1} \theta_3^{-1} \theta_4^{-1}. \tag{2.16}$$

Lastly, for the group of orderings of $\{\theta_1, \theta_2, \theta_3, \theta_4\}$, $\{\theta_1, \theta_3, \theta_2, \theta_4\}$, and $\{\theta_1, \theta_4, \theta_2, \theta_3\}$, the one-at-a-time reference prior is

$$\pi_4(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} \theta_2^{-1} \theta_3^{-1} \theta_4^{-1}. \tag{2.17}$$

Remark 1. In the aforementioned reference priors, Jeffreys' prior, denoted as π_1, does not fulfill the first-order matching criterion. In contrast, the two-group reference prior π_2, the three-group reference prior π_3, and the one-at-a-time reference prior π_4 do meet this criterion. Notably, when the sample sizes of the two-groups are equal, Jeffreys' prior does satisfy the first-order matching criterion.

Remark 2. The priors Jeffreys' prior π_1, the two-group reference prior π_2, and the three-group reference prior π_3 are all contingent upon the sample sizes.

Remark 3. We have identified a unique matching prior that adheres to all the matching criteria examined. This prior is expressed as follows:

$$\pi_m(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-1} \theta_2^{-1} \theta_3^{-1} \theta_4^{-1}, \tag{2.18}$$

which corresponds to the one-at-a-time reference prior π_4.

3. Properties of the posterior distribution under the general prior

We examine the validity of posterior distributions for a broad category of prior distributions, which encompasses reference priors as specified in Eqs (2.13)–(2.15) and (2.17). Specifically, we focus on the class of priors defined by the equation

$$\pi(\theta_1, \theta_2, \theta_3, \theta_4) \propto \theta_1^{-a} \theta_2^{-b} \theta_3^{-1} \theta_4^{-1}, \tag{3.1}$$

where $a > 0$ and $b > 0$. In this context, we establish the following general theorem.

Theorem 1. The posterior distribution of $(\theta_1, \theta_2, \theta_3, \theta_4)$ under the general prior (3.1) is proper if $2n_1 - n_1 b + a - 2 > 0$ and $2n_2 - n_2 b - a > 0$.

Proof. Under the prior (3.1) the joint posterior for $\theta_1, \theta_2, \theta_3$, and θ_4 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}
&\pi(\theta_1, \theta_2, \theta_3, \theta_4 | \mathbf{x}, \mathbf{y}) \\
\propto\ & \theta_1^{-a}\theta_2^{1-b}\theta_3^{-1}\theta_4^{-1}\left[\prod_{i=1}^{n_1}\left(\frac{x_i}{\theta_3}\right)^{\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right]\left[\prod_{i=1}^{n_2}\left(\frac{y_i}{\theta_4}\right)^{\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right] \\
\times\ & \exp\left[-\frac{1}{2}\sum_{i=1}^{n_1}\left(\frac{x_i}{\theta_3}\right)^{2\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}} e^{c_1} - \frac{1}{2}\sum_{i=1}^{n_2}\left(\frac{y_i}{\theta_4}\right)^{2\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}} e^{c_1}\right].
\end{aligned} \tag{3.2}$$

Let $\omega_1 = \theta_3^{-2\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}$ and $\omega_2 = \theta_4^{-2\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}$. Integrating with respect to ω_1 and ω_2 in (3.2), then we get

$$\begin{aligned}
\pi(\theta_1, \theta_2 | \mathbf{x}, \mathbf{y}) \propto\ & \theta_1^{-a+\frac{n_2-n_1}{n_1+n_2}}\theta_2^{1-b-\frac{2}{n_1+n_2}} \\
\times\ & \left(\prod_{i=1}^{n_1} x_i^{\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right)\left(\prod_{i=1}^{n_2} y_i^{\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right) \\
\times\ & \left[\sum_{i=1}^{n_1} x_i^{2\theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right]^{-\frac{n_1}{2}}\left[\sum_{i=1}^{n_2} y_i^{2\theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}}\right]^{-\frac{n_2}{2}}.
\end{aligned} \tag{3.3}$$

Let $\lambda_1 = \theta_1^{\frac{-n_2}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}$ and $\lambda_2 = \theta_1^{\frac{n_1}{n_1+n_2}}\theta_2^{\frac{1}{n_1+n_2}}$. Then, from (3.3) we get

$$\begin{aligned}
\pi(\lambda_1, \lambda_2 | \mathbf{x}, \mathbf{y}) \propto\ & \lambda_1^{2n_1-n_1b+a-3}\lambda_2^{2n_2-n_2b-a-1} \\
\times\ & \left(\prod_{i=1}^{n_1} x_i^{\lambda_1}\right)\left(\prod_{i=1}^{n_2} y_i^{\lambda_2}\right)\left[\sum_{i=1}^{n_1} x_i^{2\lambda_1}\right]^{-\frac{n_1}{2}}\left[\sum_{i=1}^{n_2} y_i^{2\lambda_2}\right]^{-\frac{n_2}{2}}.
\end{aligned} \tag{3.4}$$

Then, from (3.4) we can obtain the following relationship:

$$\begin{aligned}
\pi(\lambda_1, \lambda_2 | \mathbf{x}, \mathbf{y}) \leq\ & \lambda_1^{2n_1-n_1b+a-3}\lambda_2^{2n_2-n_2b-a-1}\prod_{i=1}^{n_1}\left(\frac{x_i}{x_{(n_1)}}\right)^{\lambda_1}\prod_{i=1}^{n_2}\left(\frac{y_i}{y_{(n_2)}}\right)^{\lambda_2} \\
=\ & \lambda_1^{2n_1-n_1b+a-3}\lambda_2^{2n_2-n_2b-a-1} \\
\times\ & \exp\left\{-\left[-\sum_{i=1}^{n_1}\log\left(\frac{x_i}{x_{(n_1)}}\right)\right]\lambda_1 - \left[-\sum_{i=1}^{n_2}\log\left(\frac{y_i}{y_{(n_2)}}\right)\right]\lambda_2\right\} \\
\equiv\ & \pi'(\lambda_1, \lambda_2 | \mathbf{x}, \mathbf{y}),
\end{aligned}$$

where $x_{(n_1)} = \max\{x_1, \cdots, x_{n_1}\}$ and $y_{(n_2)} = \max\{y_1, \cdots, y_{n_2}\}$. Then

$$\int_0^\infty \int_0^\infty \pi'(\lambda_1, \lambda_2 | \mathbf{x}, \mathbf{y}) d\lambda_1 d\lambda_2 < \infty,$$

if $2n_1 - n_1 b + a - 2 > 0$ and $2n_2 - n_2 b - a > 0$. This completes the proof. □

Theorem 2. Under the general prior (3.1), the marginal posterior density of θ_1 is given by

$$\begin{aligned} \pi(\theta_1|\mathbf{x},\mathbf{y}) \quad &\propto \quad \int_0^\infty \theta_1^{2n_2-n_2b-a-1}\lambda^{2(n_1+n_2)-(n_1+n_2)b-3} \\ &\times \quad \left(\prod_{i=1}^{n_1} x_i^{\frac{\lambda}{2}}\right)\left(\prod_{i=1}^{n_2} y_i^{\frac{\theta_1\lambda}{2}}\right)\left[\sum_{i=1}^{n_1} x_i^{\lambda}\right]^{-\frac{n_1}{2}}\left[\sum_{i=1}^{n_2} y_i^{\theta_1\lambda}\right]^{-\frac{n_2}{2}} d\lambda. \end{aligned}$$

We examine the frequentist coverage probabilities associated with Jeffreys' prior π_1, the two-group reference prior π_2, the three-group reference prior π_3, and the one-at-a-time reference prior π_4. The analysis involves the computation of the marginal posterior density and its moments for the parameter θ_1, necessitating the execution of two-dimensional integration.

4. Numerical study

We assess the frequentist coverage probability of the marginal posterior density of θ_1 under the noninformative prior π given in Section 2 for several configurations $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ and (n_1,n_2) by investigating its credible interval. Specifically, we verify that the frequentist coverage of a $(1-\eta)^{th}$ posterior quantile is close to $1-\eta$. To determine this, we use the following algorithm. For any fixed true $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ and any prespecified probability value η, let $\theta_1^\pi(\eta|\mathbf{X},\mathbf{Y})$ be the posterior η-quantile of θ_1 given $\mathbf{X}$ and $\mathbf{Y}$ (i.e., $F(\theta_1^\pi(\eta|\mathbf{X},\mathbf{Y})|\mathbf{X},\mathbf{Y}) = \eta$). Then the frequentist coverage probability of this one-sided credible interval of θ_1 is $P(\eta;\theta_1) = P(-\infty < \theta_1 \le \theta_1^\pi(\eta|\mathbf{X},\mathbf{Y}))$. The estimated $P(\eta;\theta_1)$ when $\eta = 0.05(0.95)$ is shown in Tables 1–3. In addition, the frequentist coverage probabilities of the 90% and 95% credible intervals of θ_1 are given in Tables 4–6. Also, for comparison with the frequentist method, the results using the asymptotic method of a MLE (maximum likelihood estimator) are provided in the tables. To compute these numerical values, we take $10,000$ independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the generalized half-normal models $GHN(\alpha_1,\beta_1)$ and $GHN(\alpha_2,\beta_2)$, respectively. Our numerical studies were implemented by Fortran 90 with IMSL.

Table 1. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $(\beta_1, \beta_2) = (1, 5)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.075 (0.934)	0.075 (0.934)	0.065 (0.943)	0.055 (0.953)	0.004 (0.837)
		5,10	0.040 (0.907)	0.057 (0.932)	0.054 (0.941)	0.050 (0.950)	0.001 (0.777)
		10,10	0.058 (0.942)	0.058 (0.942)	0.053 (0.947)	0.050 (0.952)	0.013 (0.873)
		10,15	0.049 (0.932)	0.055 (0.942)	0.053 (0.946)	0.050 (0.949)	0.011 (0.855)
	0.5	5,5	0.073 (0.934)	0.073 (0.934)	0.062 (0.943)	0.051 (0.952)	0.004 (0.835)
		5,10	0.044 (0.906)	0.059 (0.933)	0.056 (0.941)	0.052 (0.949)	0.002 (0.775)
		10,10	0.053 (0.943)	0.053 (0.943)	0.051 (0.946)	0.047 (0.950)	0.013 (0.876)
		10,15	0.051 (0.931)	0.056 (0.938)	0.054 (0.942)	0.052 (0.947)	0.009 (0.855)
	0.75	5,5	0.071 (0.936)	0.071 (0.936)	0.062 (0.943)	0.052 (0.951)	0.002 (0.840)
		5,10	0.045 (0.908)	0.061 (0.932)	0.058 (0.940)	0.055 (0.948)	0.002 (0.782)
		10,10	0.053 (0.941)	0.053 (0.941)	0.050 (0.946)	0.044 (0.951)	0.013 (0.876)
		10,15	0.049 (0.932)	0.057 (0.940)	0.054 (0.944)	0.052 (0.948)	0.011 (0.854)
	1.5	5,5	0.068 (0.927)	0.068 (0.927)	0.060 (0.936)	0.050 (0.946)	0.002 (0.828)
		5,10	0.039 (0.911)	0.055 (0.936)	0.051 (0.942)	0.048 (0.951)	0.001 (0.779)
		10,10	0.060 (0.942)	0.060 (0.942)	0.055 (0.946)	0.051 (0.950)	0.012 (0.875)
		10,15	0.048 (0.932)	0.054 (0.940)	0.052 (0.944)	0.049 (0.948)	0.011 (0.855)
	3.0	5,5	0.066 (0.933)	0.066 (0.933)	0.058 (0.941)	0.049 (0.951)	0.001 (0.837)
		5,10	0.043 (0.908)	0.059 (0.935)	0.055 (0.942)	0.052 (0.952)	0.002 (0.784)
		10,10	0.057 (0.946)	0.057 (0.946)	0.053 (0.950)	0.050 (0.955)	0.014 (0.879)
		10,15	0.050 (0.933)	0.056 (0.942)	0.053 (0.945)	0.051 (0.950)	0.011 (0.854)
0.5	0.25	5,5	0.068 (0.937)	0.068 (0.937)	0.058 (0.945)	0.048 (0.955)	0.003 (0.839)
		5,10	0.039 (0.909)	0.053 (0.933)	0.050 (0.942)	0.047 (0.950)	0.002 (0.777)
		10,10	0.059 (0.941)	0.059 (0.941)	0.055 (0.946)	0.050 (0.950)	0.014 (0.876)
		10,15	0.049 (0.930)	0.056 (0.939)	0.053 (0.944)	0.050 (0.948)	0.009 (0.858)
	0.5	5,5	0.069 (0.929)	0.069 (0.929)	0.059 (0.939)	0.049 (0.949)	0.002 (0.829)
		5,10	0.042 (0.910)	0.059 (0.936)	0.055 (0.944)	0.051 (0.952)	0.002 (0.786)
		10,10	0.056 (0.942)	0.056 (0.942)	0.051 (0.948)	0.048 (0.952)	0.012 (0.879)
		10,15	0.048 (0.931)	0.053 (0.940)	0.052 (0.945)	0.050 (0.950)	0.009 (0.858)
	0.75	5,5	0.066 (0.932)	0.066 (0.932)	0.058 (0.942)	0.049 (0.953)	0.002 (0.836)
		5,10	0.043 (0.910)	0.057 (0.935)	0.053 (0.943)	0.050 (0.951)	0.002 (0.786)
		10,10	0.055 (0.941)	0.055 (0.941)	0.052 (0.946)	0.048 (0.950)	0.013 (0.876)
		10,15	0.046 (0.933)	0.054 (0.943)	0.051 (0.947)	0.048 (0.950)	0.009 (0.858)
	1.5	5,5	0.069 (0.931)	0.069 (0.931)	0.059 (0.939)	0.048 (0.949)	0.003 (0.833)
		5,10	0.043 (0.908)	0.056 (0.935)	0.053 (0.943)	0.049 (0.950)	0.002 (0.782)
		10,10	0.060 (0.939)	0.060 (0.939)	0.055 (0.944)	0.050 (0.949)	0.012 (0.877)
		10,15	0.050 (0.932)	0.057 (0.941)	0.055 (0.947)	0.051 (0.951)	0.009 (0.858)
	3.0	5,5	0.066 (0.927)	0.066 (0.927)	0.058 (0.937)	0.048 (0.947)	0.003 (0.831)
		5,10	0.041 (0.906)	0.057 (0.932)	0.055 (0.940)	0.050 (0.949)	0.001 (0.778)
		10,10	0.060 (0.941)	0.060 (0.941)	0.056 (0.945)	0.051 (0.949)	0.013 (0.880)
		10,15	0.050 (0.933)	0.058 (0.944)	0.054 (0.947)	0.051 (0.951)	0.009 (0.856)
1.5	0.25	5,5	0.066 (0.934)	0.066 (0.934)	0.058 (0.942)	0.049 (0.950)	0.003 (0.840)
		5,10	0.041 (0.905)	0.053 (0.931)	0.051 (0.940)	0.047 (0.949)	0.001 (0.780)
		10,10	0.060 (0.939)	0.060 (0.939)	0.056 (0.942)	0.051 (0.947)	0.014 (0.875)
		10,15	0.045 (0.931)	0.053 (0.940)	0.050 (0.945)	0.047 (0.949)	0.009 (0.861)
	0.5	5,5	0.068 (0.934)	0.068 (0.934)	0.061 (0.943)	0.050 (0.953)	0.003 (0.833)
		5,10	0.040 (0.908)	0.054 (0.933)	0.051 (0.941)	0.048 (0.949)	0.002 (0.777)
		10,10	0.053 (0.944)	0.053 (0.944)	0.049 (0.948)	0.045 (0.952)	0.011 (0.878)
		10,15	0.050 (0.930)	0.056 (0.939)	0.054 (0.943)	0.052 (0.947)	0.010 (0.854)
	0.75	5,5	0.071 (0.932)	0.071 (0.932)	0.061 (0.942)	0.051 (0.953)	0.003 (0.832)
		5,10	0.040 (0.913)	0.052 (0.937)	0.050 (0.945)	0.047 (0.953)	0.001 (0.781)
		10,10	0.061 (0.942)	0.061 (0.942)	0.057 (0.945)	0.053 (0.950)	0.013 (0.880)
		10,15	0.047 (0.935)	0.055 (0.943)	0.052 (0.947)	0.050 (0.951)	0.009 (0.857)
	1.5	5,5	0.068 (0.930)	0.068 (0.930)	0.059 (0.940)	0.048 (0.953)	0.002 (0.838)
		5,10	0.043 (0.907)	0.057 (0.933)	0.054 (0.942)	0.051 (0.950)	0.002 (0.782)
		10,10	0.059 (0.942)	0.059 (0.942)	0.054 (0.947)	0.050 (0.951)	0.012 (0.874)
		10,15	0.048 (0.935)	0.055 (0.942)	0.052 (0.945)	0.050 (0.949)	0.009 (0.864)
	3.0	5,5	0.066 (0.929)	0.066 (0.929)	0.059 (0.938)	0.049 (0.949)	0.002 (0.828)
		5,10	0.041 (0.904)	0.055 (0.928)	0.052 (0.937)	0.049 (0.946)	0.001 (0.774)
		10,10	0.059 (0.942)	0.059 (0.942)	0.054 (0.946)	0.051 (0.950)	0.014 (0.880)
		10,15	0.047 (0.929)	0.052 (0.939)	0.050 (0.942)	0.048 (0.947)	0.008 (0.854)

Table 2. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $(\beta_1, \beta_2) = (1, 1)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.069 (0.932)	0.069 (0.932)	0.060 (0.942)	0.052 (0.953)	0.003 (0.834)
		5,10	0.039 (0.908)	0.053 (0.934)	0.049 (0.943)	0.046 (0.950)	0.002 (0.781)
		10,10	0.054 (0.941)	0.054 (0.941)	0.050 (0.944)	0.046 (0.949)	0.014 (0.874)
		10,15	0.048 (0.934)	0.054 (0.942)	0.052 (0.945)	0.050 (0.949)	0.012 (0.863)
	0.5	5,5	0.068 (0.931)	0.068 (0.931)	0.059 (0.940)	0.050 (0.951)	0.003 (0.834)
		5,10	0.039 (0.910)	0.052 (0.933)	0.049 (0.940)	0.046 (0.950)	0.001 (0.779)
		10,10	0.057 (0.941)	0.057 (0.941)	0.053 (0.946)	0.049 (0.951)	0.013 (0.876)
		10,15	0.050 (0.933)	0.057 (0.942)	0.053 (0.946)	0.051 (0.951)	0.011 (0.861)
	0.75	5,5	0.064 (0.928)	0.064 (0.928)	0.056 (0.938)	0.048 (0.948)	0.002 (0.829)
		5,10	0.041 (0.907)	0.055 (0.934)	0.051 (0.942)	0.047 (0.951)	0.002 (0.778)
		10,10	0.060 (0.944)	0.060 (0.944)	0.056 (0.948)	0.051 (0.952)	0.013 (0.877)
		10,15	0.048 (0.929)	0.055 (0.939)	0.053 (0.943)	0.050 (0.947)	0.009 (0.851)
	1.5	5,5	0.068 (0.928)	0.068 (0.928)	0.059 (0.938)	0.049 (0.948)	0.003 (0.837)
		5,10	0.042 (0.913)	0.056 (0.934)	0.053 (0.943)	0.050 (0.951)	0.002 (0.785)
		10,10	0.062 (0.945)	0.062 (0.945)	0.058 (0.948)	0.055 (0.952)	0.013 (0.872)
		10,15	0.051 (0.931)	0.058 (0.940)	0.055 (0.945)	0.052 (0.949)	0.011 (0.859)
	3.0	5,5	0.075 (0.929)	0.075 (0.929)	0.064 (0.940)	0.055 (0.950)	0.003 (0.831)
		5,10	0.044 (0.905)	0.059 (0.932)	0.056 (0.941)	0.052 (0.948)	0.002 (0.782)
		10,10	0.060 (0.938)	0.060 (0.938)	0.057 (0.943)	0.053 (0.948)	0.014 (0.874)
		10,15	0.048 (0.929)	0.054 (0.938)	0.051 (0.942)	0.049 (0.947)	0.011 (0.854)
0.5	0.25	5,5	0.066 (0.930)	0.066 (0.930)	0.057 (0.940)	0.047 (0.950)	0.003 (0.835)
		5,10	0.042 (0.908)	0.055 (0.932)	0.052 (0.941)	0.049 (0.950)	0.001 (0.780)
		10,10	0.063 (0.942)	0.063 (0.942)	0.059 (0.944)	0.054 (0.949)	0.013 (0.879)
		10,15	0.052 (0.939)	0.059 (0.947)	0.056 (0.950)	0.053 (0.955)	0.012 (0.864)
	0.5	5,5	0.071 (0.933)	0.071 (0.933)	0.062 (0.941)	0.053 (0.949)	0.003 (0.833)
		5,10	0.043 (0.910)	0.059 (0.933)	0.056 (0.941)	0.053 (0.950)	0.002 (0.777)
		10,10	0.059 (0.944)	0.059 (0.944)	0.055 (0.948)	0.051 (0.953)	0.014 (0.879)
		10,15	0.051 (0.934)	0.057 (0.943)	0.055 (0.947)	0.053 (0.951)	0.012 (0.858)
	0.75	5,5	0.070 (0.933)	0.070 (0.933)	0.062 (0.943)	0.051 (0.953)	0.002 (0.836)
		5,10	0.043 (0.909)	0.060 (0.930)	0.056 (0.940)	0.053 (0.952)	0.002 (0.785)
		10,10	0.053 (0.941)	0.053 (0.941)	0.049 (0.946)	0.045 (0.949)	0.011 (0.876)
		10,15	0.050 (0.931)	0.058 (0.941)	0.055 (0.945)	0.052 (0.949)	0.009 (0.855)
	1.5	5,5	0.069 (0.929)	0.069 (0.929)	0.059 (0.938)	0.051 (0.949)	0.003 (0.832)
		5,10	0.042 (0.908)	0.057 (0.933)	0.054 (0.942)	0.050 (0.950)	0.001 (0.782)
		10,10	0.058 (0.940)	0.058 (0.940)	0.055 (0.944)	0.052 (0.949)	0.013 (0.873)
		10,15	0.044 (0.931)	0.052 (0.940)	0.050 (0.944)	0.047 (0.948)	0.010 (0.867)
	3.0	5,5	0.068 (0.932)	0.068 (0.932)	0.058 (0.941)	0.049 (0.952)	0.002 (0.836)
		5,10	0.040 (0.903)	0.055 (0.929)	0.052 (0.939)	0.048 (0.946)	0.001 (0.776)
		10,10	0.059 (0.944)	0.059 (0.944)	0.053 (0.947)	0.050 (0.952)	0.012 (0.878)
		10,15	0.048 (0.937)	0.056 (0.945)	0.052 (0.949)	0.049 (0.952)	0.011 (0.866)
1.5	0.25	5,5	0.070 (0.933)	0.070 (0.933)	0.060 (0.943)	0.050 (0.953)	0.003 (0.837)
		5,10	0.044 (0.909)	0.060 (0.934)	0.055 (0.941)	0.052 (0.952)	0.002 (0.788)
		10,10	0.056 (0.941)	0.056 (0.941)	0.052 (0.945)	0.048 (0.950)	0.014 (0.877)
		10,15	0.048 (0.934)	0.056 (0.941)	0.053 (0.946)	0.050 (0.950)	0.008 (0.857)
	0.5	5,5	0.071 (0.932)	0.071 (0.932)	0.062 (0.941)	0.053 (0.950)	0.002 (0.836)
		5,10	0.043 (0.907)	0.057 (0.934)	0.054 (0.941)	0.051 (0.950)	0.002 (0.786)
		10,10	0.057 (0.942)	0.057 (0.942)	0.053 (0.947)	0.049 (0.951)	0.013 (0.876)
		10,15	0.051 (0.936)	0.058 (0.944)	0.056 (0.948)	0.052 (0.952)	0.010 (0.863)
	0.75	5,5	0.076 (0.934)	0.076 (0.934)	0.065 (0.942)	0.056 (0.953)	0.003 (0.834)
		5,10	0.041 (0.905)	0.057 (0.930)	0.053 (0.939)	0.051 (0.948)	0.002 (0.781)
		10,10	0.060 (0.942)	0.060 (0.942)	0.057 (0.946)	0.052 (0.949)	0.015 (0.877)
		10,15	0.048 (0.932)	0.054 (0.940)	0.052 (0.945)	0.050 (0.948)	0.010 (0.857)
	1.5	5,5	0.072 (0.932)	0.072 (0.932)	0.063 (0.943)	0.053 (0.951)	0.003 (0.835)
		5,10	0.040 (0.905)	0.055 (0.930)	0.052 (0.938)	0.050 (0.946)	0.001 (0.780)
		10,10	0.060 (0.943)	0.060 (0.943)	0.057 (0.946)	0.053 (0.951)	0.015 (0.883)
		10,15	0.049 (0.935)	0.056 (0.944)	0.054 (0.947)	0.051 (0.951)	0.009 (0.858)
	3.0	5,5	0.072 (0.932)	0.072 (0.932)	0.061 (0.941)	0.051 (0.950)	0.003 (0.833)
		5,10	0.043 (0.909)	0.059 (0.935)	0.056 (0.943)	0.053 (0.952)	0.002 (0.779)
		10,10	0.058 (0.942)	0.058 (0.942)	0.054 (0.947)	0.049 (0.951)	0.014 (0.874)
		10,15	0.050 (0.933)	0.055 (0.942)	0.053 (0.945)	0.051 (0.949)	0.011 (0.863)

Table 3. Frequentist coverage probability of 0.05 (0.95) quantiles of θ_1 when $(\beta_1, \beta_2) = (5, 1)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.063 (0.927)	0.063 (0.927)	0.056 (0.935)	0.047 (0.945)	0.003 (0.828)
		5,10	0.041 (0.906)	0.055 (0.930)	0.051 (0.940)	0.049 (0.949)	0.001 (0.770)
		10,10	0.058 (0.939)	0.058 (0.939)	0.054 (0.942)	0.051 (0.945)	0.012 (0.873)
		10,15	0.050 (0.936)	0.058 (0.944)	0.055 (0.948)	0.052 (0.952)	0.008 (0.853)
	0.5	5,5	0.071 (0.929)	0.071 (0.929)	0.063 (0.938)	0.052 (0.948)	0.003 (0.834)
		5,10	0.042 (0.909)	0.057 (0.933)	0.053 (0.943)	0.050 (0.950)	0.002 (0.783)
		10,10	0.059 (0.940)	0.059 (0.940)	0.055 (0.945)	0.051 (0.949)	0.012 (0.876)
		10,15	0.050 (0.931)	0.056 (0.940)	0.054 (0.944)	0.051 (0.950)	0.009 (0.855)
	0.75	5,5	0.066 (0.932)	0.066 (0.932)	0.057 (0.942)	0.048 (0.952)	0.002 (0.836)
		5,10	0.044 (0.906)	0.059 (0.932)	0.056 (0.943)	0.053 (0.953)	0.002 (0.775)
		10,10	0.059 (0.941)	0.059 (0.941)	0.055 (0.945)	0.051 (0.950)	0.014 (0.877)
		10,15	0.049 (0.934)	0.056 (0.942)	0.053 (0.946)	0.051 (0.950)	0.010 (0.849)
	1.5	5,5	0.070 (0.928)	0.070 (0.928)	0.059 (0.937)	0.050 (0.949)	0.003 (0.832)
		5,10	0.038 (0.908)	0.057 (0.932)	0.053 (0.939)	0.049 (0.947)	0.001 (0.776)
		10,10	0.056 (0.937)	0.056 (0.937)	0.054 (0.941)	0.050 (0.945)	0.012 (0.872)
		10,15	0.051 (0.930)	0.059 (0.938)	0.056 (0.942)	0.053 (0.945)	0.010 (0.854)
	3.0	5,5	0.066 (0.932)	0.066 (0.932)	0.057 (0.941)	0.047 (0.952)	0.002 (0.832)
		5,10	0.042 (0.912)	0.060 (0.934)	0.055 (0.943)	0.051 (0.951)	0.002 (0.783)
		10,10	0.058 (0.943)	0.058 (0.943)	0.053 (0.946)	0.049 (0.950)	0.012 (0.877)
		10,15	0.048 (0.934)	0.054 (0.943)	0.051 (0.947)	0.049 (0.951)	0.009 (0.864)
0.5	0.25	5,5	0.072 (0.930)	0.072 (0.930)	0.063 (0.939)	0.053 (0.948)	0.003 (0.838)
		5,10	0.041 (0.907)	0.058 (0.932)	0.055 (0.939)	0.051 (0.947)	0.001 (0.786)
		10,10	0.053 (0.941)	0.053 (0.941)	0.049 (0.945)	0.045 (0.949)	0.011 (0.877)
		10,15	0.050 (0.931)	0.057 (0.939)	0.055 (0.943)	0.051 (0.948)	0.009 (0.862)
	0.5	5,5	0.071 (0.932)	0.071 (0.932)	0.060 (0.942)	0.049 (0.951)	0.003 (0.833)
		5,10	0.040 (0.912)	0.056 (0.935)	0.052 (0.944)	0.048 (0.949)	0.002 (0.786)
		10,10	0.062 (0.942)	0.062 (0.942)	0.058 (0.946)	0.053 (0.950)	0.013 (0.876)
		10,15	0.049 (0.935)	0.056 (0.942)	0.054 (0.945)	0.051 (0.950)	0.011 (0.861)
	0.75	5,5	0.073 (0.931)	0.073 (0.931)	0.064 (0.941)	0.054 (0.950)	0.004 (0.831)
		5,10	0.041 (0.907)	0.058 (0.930)	0.054 (0.940)	0.051 (0.947)	0.002 (0.783)
		10,10	0.059 (0.944)	0.059 (0.944)	0.054 (0.949)	0.048 (0.953)	0.011 (0.883)
		10,15	0.051 (0.933)	0.058 (0.941)	0.055 (0.945)	0.052 (0.950)	0.009 (0.854)
	1.5	5,5	0.070 (0.929)	0.070 (0.929)	0.062 (0.940)	0.052 (0.950)	0.003 (0.834)
		5,10	0.041 (0.912)	0.058 (0.934)	0.055 (0.941)	0.050 (0.950)	0.001 (0.780)
		10,10	0.058 (0.943)	0.058 (0.943)	0.055 (0.947)	0.049 (0.951)	0.013 (0.880)
		10,15	0.048 (0.934)	0.054 (0.943)	0.052 (0.947)	0.050 (0.953)	0.010 (0.859)
	3.0	5,5	0.068 (0.930)	0.068 (0.930)	0.059 (0.939)	0.049 (0.949)	0.002 (0.835)
		5,10	0.045 (0.906)	0.060 (0.934)	0.056 (0.941)	0.052 (0.952)	0.001 (0.780)
		10,10	0.058 (0.941)	0.058 (0.941)	0.055 (0.946)	0.051 (0.950)	0.013 (0.876)
		10,15	0.048 (0.935)	0.055 (0.944)	0.052 (0.947)	0.050 (0.951)	0.009 (0.863)
1.5	0.25	5,5	0.068 (0.930)	0.068 (0.930)	0.060 (0.939)	0.049 (0.949)	0.003 (0.830)
		5,10	0.039 (0.908)	0.054 (0.932)	0.051 (0.938)	0.047 (0.946)	0.002 (0.779)
		10,10	0.059 (0.940)	0.059 (0.940)	0.055 (0.944)	0.051 (0.948)	0.015 (0.876)
		10,15	0.049 (0.935)	0.054 (0.944)	0.052 (0.948)	0.051 (0.951)	0.010 (0.863)
	0.5	5,5	0.067 (0.933)	0.067 (0.933)	0.059 (0.942)	0.050 (0.950)	0.003 (0.837)
		5,10	0.045 (0.907)	0.061 (0.933)	0.058 (0.942)	0.054 (0.951)	0.002 (0.783)
		10,10	0.058 (0.947)	0.058 (0.947)	0.053 (0.950)	0.048 (0.953)	0.013 (0.879)
		10,15	0.048 (0.937)	0.056 (0.945)	0.053 (0.949)	0.050 (0.952)	0.010 (0.863)
	0.75	5,5	0.069 (0.929)	0.069 (0.929)	0.059 (0.938)	0.049 (0.950)	0.003 (0.839)
		5,10	0.044 (0.904)	0.058 (0.929)	0.055 (0.937)	0.052 (0.946)	0.002 (0.780)
		10,10	0.059 (0.942)	0.059 (0.942)	0.054 (0.946)	0.051 (0.952)	0.013 (0.880)
		10,15	0.049 (0.930)	0.056 (0.940)	0.053 (0.944)	0.050 (0.948)	0.011 (0.853)
	1.5	5,5	0.068 (0.934)	0.068 (0.934)	0.060 (0.943)	0.050 (0.953)	0.003 (0.833)
		5,10	0.042 (0.908)	0.057 (0.932)	0.054 (0.941)	0.050 (0.949)	0.001 (0.779)
		10,10	0.060 (0.943)	0.060 (0.943)	0.056 (0.948)	0.051 (0.951)	0.014 (0.878)
		10,15	0.048 (0.936)	0.055 (0.944)	0.052 (0.947)	0.049 (0.950)	0.009 (0.863)
	3.0	5,5	0.067 (0.930)	0.067 (0.930)	0.059 (0.940)	0.050 (0.950)	0.003 (0.831)
		5,10	0.043 (0.908)	0.058 (0.931)	0.055 (0.940)	0.051 (0.948)	0.001 (0.781)
		10,10	0.058 (0.941)	0.058 (0.941)	0.053 (0.946)	0.050 (0.950)	0.013 (0.875)
		10,15	0.048 (0.935)	0.056 (0.943)	0.053 (0.946)	0.050 (0.950)	0.009 (0.858)

Table 4. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $(\beta_1, \beta_2) = (1, 5)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.859 (0.922)	0.859 (0.922)	0.878 (0.936)	0.898 (0.949)	0.833 (0.865)
		5,10	0.867 (0.926)	0.875 (0.933)	0.887 (0.941)	0.900 (0.949)	0.776 (0.816)
		10,10	0.884 (0.941)	0.884 (0.941)	0.894 (0.945)	0.902 (0.951)	0.860 (0.905)
		10,15	0.883 (0.937)	0.887 (0.938)	0.893 (0.942)	0.899 (0.947)	0.844 (0.888)
	0.5	5,5	0.861 (0.923)	0.861 (0.923)	0.881 (0.938)	0.900 (0.951)	0.831 (0.865)
		5,10	0.862 (0.924)	0.874 (0.932)	0.885 (0.940)	0.897 (0.948)	0.773 (0.812)
		10,10	0.889 (0.939)	0.889 (0.939)	0.895 (0.945)	0.903 (0.951)	0.863 (0.907)
		10,15	0.880 (0.937)	0.882 (0.938)	0.889 (0.942)	0.895 (0.950)	0.846 (0.890)
	0.75	5,5	0.865 (0.922)	0.865 (0.922)	0.880 (0.936)	0.899 (0.951)	0.838 (0.871)
		5,10	0.863 (0.923)	0.871 (0.929)	0.882 (0.938)	0.893 (0.947)	0.780 (0.821)
		10,10	0.888 (0.943)	0.888 (0.943)	0.896 (0.948)	0.907 (0.953)	0.863 (0.905)
		10,15	0.883 (0.938)	0.884 (0.940)	0.890 (0.944)	0.896 (0.948)	0.843 (0.890)
	1.5	5,5	0.859 (0.921)	0.859 (0.921)	0.877 (0.934)	0.896 (0.948)	0.826 (0.859)
		5,10	0.872 (0.931)	0.881 (0.937)	0.891 (0.944)	0.903 (0.952)	0.778 (0.819)
		10,10	0.882 (0.936)	0.882 (0.936)	0.891 (0.942)	0.899 (0.947)	0.863 (0.905)
		10,15	0.884 (0.938)	0.886 (0.939)	0.893 (0.944)	0.899 (0.948)	0.844 (0.891)
	3.0	5,5	0.867 (0.924)	0.867 (0.924)	0.884 (0.938)	0.902 (0.952)	0.836 (0.865)
		5,10	0.866 (0.926)	0.876 (0.936)	0.887 (0.943)	0.899 (0.952)	0.782 (0.822)
		10,10	0.889 (0.940)	0.889 (0.940)	0.896 (0.945)	0.906 (0.950)	0.865 (0.909)
		10,15	0.883 (0.940)	0.886 (0.940)	0.892 (0.945)	0.899 (0.949)	0.842 (0.889)
0.5	0.25	5,5	0.869 (0.928)	0.869 (0.928)	0.888 (0.941)	0.906 (0.953)	0.836 (0.872)
		5,10	0.871 (0.927)	0.879 (0.933)	0.892 (0.942)	0.903 (0.951)	0.775 (0.818)
		10,10	0.882 (0.936)	0.882 (0.936)	0.890 (0.941)	0.900 (0.946)	0.862 (0.906)
		10,15	0.881 (0.937)	0.883 (0.937)	0.891 (0.942)	0.898 (0.947)	0.850 (0.891)
	0.5	5,5	0.860 (0.923)	0.860 (0.923)	0.880 (0.937)	0.900 (0.953)	0.826 (0.859)
		5,10	0.868 (0.926)	0.877 (0.933)	0.889 (0.941)	0.901 (0.949)	0.785 (0.823)
		10,10	0.887 (0.943)	0.887 (0.943)	0.896 (0.949)	0.905 (0.954)	0.867 (0.908)
		10,15	0.883 (0.940)	0.887 (0.941)	0.893 (0.946)	0.900 (0.950)	0.849 (0.892)
	0.75	5,5	0.866 (0.927)	0.866 (0.927)	0.885 (0.939)	0.904 (0.952)	0.834 (0.865)
		5,10	0.868 (0.927)	0.878 (0.934)	0.889 (0.943)	0.901 (0.950)	0.785 (0.825)
		10,10	0.886 (0.938)	0.886 (0.938)	0.893 (0.944)	0.902 (0.951)	0.864 (0.907)
		10,15	0.887 (0.943)	0.889 (0.944)	0.896 (0.948)	0.903 (0.954)	0.849 (0.892)
	1.5	5,5	0.862 (0.924)	0.862 (0.924)	0.880 (0.936)	0.900 (0.949)	0.831 (0.864)
		5,10	0.865 (0.928)	0.879 (0.932)	0.890 (0.941)	0.901 (0.951)	0.780 (0.818)
		10,10	0.879 (0.937)	0.879 (0.937)	0.889 (0.945)	0.899 (0.950)	0.864 (0.906)
		10,15	0.882 (0.939)	0.884 (0.942)	0.892 (0.946)	0.900 (0.950)	0.849 (0.892)
	3.0	5,5	0.861 (0.922)	0.861 (0.922)	0.880 (0.936)	0.899 (0.948)	0.829 (0.862)
		5,10	0.865 (0.925)	0.875 (0.933)	0.885 (0.943)	0.898 (0.951)	0.777 (0.818)
		10,10	0.881 (0.935)	0.881 (0.935)	0.889 (0.942)	0.898 (0.947)	0.866 (0.909)
		10,15	0.883 (0.942)	0.886 (0.942)	0.893 (0.946)	0.900 (0.951)	0.847 (0.889)
1.5	0.25	5,5	0.868 (0.923)	0.868 (0.923)	0.885 (0.937)	0.901 (0.954)	0.837 (0.869)
		5,10	0.865 (0.924)	0.878 (0.924)	0.889 (0.941)	0.901 (0.949)	0.779 (0.817)
		10,10	0.878 (0.936)	0.878 (0.936)	0.886 (0.941)	0.896 (0.947)	0.862 (0.904)
		10,15	0.886 (0.939)	0.888 (0.941)	0.895 (0.946)	0.901 (0.950)	0.852 (0.893)
	0.5	5,5	0.866 (0.924)	0.866 (0.924)	0.883 (0.937)	0.903 (0.950)	0.830 (0.864)
		5,10	0.868 (0.928)	0.878 (0.936)	0.890 (0.944)	0.901 (0.951)	0.775 (0.815)
		10,10	0.891 (0.943)	0.891 (0.943)	0.898 (0.947)	0.906 (0.953)	0.867 (0.911)
		10,15	0.880 (0.938)	0.882 (0.939)	0.889 (0.944)	0.895 (0.947)	0.844 (0.889)
	0.75	5,5	0.861 (0.922)	0.861 (0.922)	0.882 (0.936)	0.902 (0.949)	0.829 (0.861)
		5,10	0.873 (0.931)	0.884 (0.938)	0.895 (0.946)	0.905 (0.955)	0.780 (0.822)
		10,10	0.881 (0.938)	0.881 (0.938)	0.888 (0.944)	0.896 (0.950)	0.866 (0.907)
		10,15	0.887 (0.941)	0.888 (0.942)	0.895 (0.946)	0.901 (0.950)	0.848 (0.889)
	1.5	5,5	0.862 (0.929)	0.862 (0.929)	0.880 (0.939)	0.905 (0.952)	0.836 (0.867)
		5,10	0.865 (0.928)	0.876 (0.937)	0.888 (0.945)	0.899 (0.953)	0.780 (0.818)
		10,10	0.883 (0.938)	0.883 (0.938)	0.893 (0.945)	0.900 (0.949)	0.862 (0.905)
		10,15	0.887 (0.940)	0.887 (0.941)	0.893 (0.945)	0.899 (0.951)	0.855 (0.896)
	3.0	5,5	0.863 (0.923)	0.863 (0.923)	0.880 (0.936)	0.901 (0.949)	0.826 (0.859)
		5,10	0.863 (0.923)	0.873 (0.933)	0.885 (0.942)	0.897 (0.950)	0.773 (0.815)
		10,10	0.883 (0.940)	0.883 (0.940)	0.892 (0.946)	0.899 (0.951)	0.866 (0.907)
		10,15	0.883 (0.937)	0.887 (0.939)	0.892 (0.943)	0.899 (0.949)	0.846 (0.885)

Table 5. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $(\beta_1, \beta_2) = (1, 1)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.863 (0.925)	0.863 (0.925)	0.882 (0.936)	0.900 (0.948)	0.830 (0.864)
		5,10	0.870 (0.928)	0.881 (0.936)	0.894 (0.944)	0.904 (0.952)	0.778 (0.821)
		10,10	0.886 (0.941)	0.886 (0.941)	0.894 (0.947)	0.903 (0.951)	0.860 (0.903)
		10,15	0.887 (0.938)	0.888 (0.940)	0.893 (0.944)	0.899 (0.948)	0.851 (0.894)
	0.5	5,5	0.863 (0.925)	0.863 (0.925)	0.881 (0.938)	0.901 (0.951)	0.831 (0.866)
		5,10	0.872 (0.928)	0.881 (0.935)	0.892 (0.943)	0.904 (0.952)	0.778 (0.818)
		10,10	0.884 (0.941)	0.884 (0.941)	0.892 (0.946)	0.902 (0.951)	0.863 (0.905)
		10,15	0.882 (0.941)	0.885 (0.942)	0.893 (0.947)	0.899 (0.951)	0.849 (0.893)
	0.75	5,5	0.864 (0.924)	0.864 (0.924)	0.882 (0.937)	0.900 (0.951)	0.827 (0.861)
		5,10	0.867 (0.929)	0.879 (0.933)	0.891 (0.943)	0.903 (0.951)	0.776 (0.816)
		10,10	0.884 (0.938)	0.884 (0.938)	0.892 (0.944)	0.901 (0.951)	0.864 (0.906)
		10,15	0.882 (0.937)	0.884 (0.938)	0.890 (0.944)	0.897 (0.948)	0.842 (0.887)
	1.5	5,5	0.860 (0.922)	0.860 (0.922)	0.879 (0.937)	0.899 (0.950)	0.834 (0.866)
		5,10	0.871 (0.928)	0.877 (0.933)	0.890 (0.940)	0.901 (0.948)	0.783 (0.825)
		10,10	0.882 (0.939)	0.882 (0.939)	0.889 (0.944)	0.897 (0.949)	0.859 (0.903)
		10,15	0.879 (0.936)	0.882 (0.937)	0.890 (0.941)	0.897 (0.945)	0.848 (0.889)
	3.0	5,5	0.855 (0.918)	0.855 (0.918)	0.876 (0.933)	0.895 (0.948)	0.828 (0.863)
		5,10	0.861 (0.923)	0.874 (0.930)	0.885 (0.940)	0.896 (0.948)	0.780 (0.818)
		10,10	0.878 (0.933)	0.878 (0.933)	0.886 (0.939)	0.895 (0.945)	0.860 (0.905)
		10,15	0.881 (0.941)	0.884 (0.942)	0.891 (0.946)	0.897 (0.950)	0.843 (0.890)
0.5	0.25	5,5	0.864 (0.927)	0.864 (0.927)	0.884 (0.938)	0.903 (0.951)	0.832 (0.863)
		5,10	0.866 (0.924)	0.877 (0.934)	0.888 (0.942)	0.901 (0.950)	0.779 (0.815)
		10,10	0.878 (0.940)	0.878 (0.940)	0.885 (0.945)	0.895 (0.951)	0.866 (0.908)
		10,15	0.887 (0.941)	0.888 (0.941)	0.894 (0.946)	0.901 (0.949)	0.853 (0.896)
	0.5	5,5	0.862 (0.919)	0.862 (0.919)	0.879 (0.934)	0.896 (0.947)	0.830 (0.868)
		5,10	0.867 (0.925)	0.874 (0.934)	0.885 (0.941)	0.897 (0.949)	0.776 (0.814)
		10,10	0.885 (0.939)	0.885 (0.939)	0.893 (0.945)	0.902 (0.952)	0.865 (0.909)
		10,15	0.883 (0.937)	0.887 (0.937)	0.892 (0.941)	0.898 (0.946)	0.846 (0.892)
	0.75	5,5	0.863 (0.924)	0.863 (0.924)	0.881 (0.937)	0.901 (0.950)	0.834 (0.867)
		5,10	0.866 (0.926)	0.870 (0.936)	0.884 (0.943)	0.898 (0.952)	0.783 (0.822)
		10,10	0.888 (0.940)	0.888 (0.940)	0.897 (0.946)	0.904 (0.953)	0.865 (0.907)
		10,15	0.881 (0.939)	0.883 (0.940)	0.890 (0.944)	0.897 (0.949)	0.846 (0.888)
	1.5	5,5	0.860 (0.921)	0.860 (0.921)	0.879 (0.936)	0.899 (0.949)	0.829 (0.861)
		5,10	0.867 (0.928)	0.877 (0.935)	0.888 (0.943)	0.901 (0.952)	0.781 (0.819)
		10,10	0.882 (0.934)	0.882 (0.934)	0.889 (0.940)	0.897 (0.946)	0.860 (0.903)
		10,15	0.887 (0.940)	0.888 (0.941)	0.894 (0.946)	0.900 (0.951)	0.858 (0.900)
	3.0	5,5	0.864 (0.926)	0.864 (0.926)	0.883 (0.939)	0.902 (0.954)	0.834 (0.866)
		5,10	0.863 (0.923)	0.874 (0.934)	0.887 (0.943)	0.898 (0.952)	0.776 (0.815)
		10,10	0.885 (0.941)	0.885 (0.941)	0.894 (0.949)	0.902 (0.955)	0.867 (0.908)
		10,15	0.888 (0.939)	0.890 (0.939)	0.897 (0.944)	0.903 (0.948)	0.856 (0.898)
1.5	0.25	5,5	0.862 (0.924)	0.862 (0.924)	0.883 (0.939)	0.903 (0.952)	0.835 (0.865)
		5,10	0.865 (0.927)	0.874 (0.935)	0.886 (0.944)	0.900 (0.952)	0.785 (0.823)
		10,10	0.884 (0.939)	0.884 (0.939)	0.893 (0.944)	0.902 (0.951)	0.863 (0.906)
		10,15	0.885 (0.940)	0.886 (0.940)	0.893 (0.945)	0.900 (0.950)	0.848 (0.891)
	0.5	5,5	0.860 (0.922)	0.860 (0.922)	0.878 (0.934)	0.897 (0.950)	0.834 (0.865)
		5,10	0.865 (0.926)	0.877 (0.931)	0.887 (0.940)	0.899 (0.949)	0.784 (0.822)
		10,10	0.885 (0.938)	0.885 (0.938)	0.894 (0.945)	0.901 (0.950)	0.864 (0.906)
		10,15	0.886 (0.941)	0.886 (0.940)	0.892 (0.946)	0.900 (0.951)	0.853 (0.897)
	0.75	5,5	0.858 (0.919)	0.858 (0.919)	0.877 (0.931)	0.897 (0.946)	0.831 (0.867)
		5,10	0.864 (0.923)	0.873 (0.932)	0.886 (0.941)	0.898 (0.950)	0.779 (0.815)
		10,10	0.882 (0.937)	0.882 (0.937)	0.889 (0.943)	0.897 (0.948)	0.863 (0.905)
		10,15	0.884 (0.938)	0.886 (0.940)	0.893 (0.946)	0.898 (0.950)	0.847 (0.892)
	1.5	5,5	0.860 (0.920)	0.860 (0.920)	0.880 (0.936)	0.898 (0.949)	0.833 (0.865)
		5,10	0.865 (0.924)	0.875 (0.932)	0.885 (0.941)	0.896 (0.951)	0.779 (0.818)
		10,10	0.883 (0.935)	0.883 (0.935)	0.889 (0.941)	0.898 (0.948)	0.869 (0.912)
		10,15	0.886 (0.939)	0.887 (0.941)	0.893 (0.946)	0.900 (0.949)	0.849 (0.894)
	3.0	5,5	0.861 (0.922)	0.861 (0.922)	0.880 (0.937)	0.899 (0.951)	0.830 (0.864)
		5,10	0.866 (0.928)	0.876 (0.935)	0.887 (0.944)	0.900 (0.952)	0.776 (0.817)
		10,10	0.884 (0.939)	0.884 (0.939)	0.893 (0.945)	0.902 (0.950)	0.860 (0.905)
		10,15	0.883 (0.937)	0.887 (0.939)	0.892 (0.942)	0.898 (0.947)	0.852 (0.892)

Table 6. Frequentist coverage probability of 90% (95%) credible intervals of θ_1 when $(\beta_1, \beta_2) = (5, 1)$.

α_1	α_2	n_1, n_2	π_1	π_2	π_3	π_4	MLE
0.25	0.25	5,5	0.864 (0.919)	0.864 (0.919)	0.879 (0.933)	0.898 (0.947)	0.825 (0.857)
		5,10	0.865 (0.927)	0.876 (0.936)	0.889 (0.945)	0.900 (0.952)	0.770 (0.811)
		10,10	0.881 (0.935)	0.881 (0.935)	0.887 (0.942)	0.895 (0.948)	0.861 (0.902)
		10,15	0.886 (0.944)	0.886 (0.945)	0.893 (0.949)	0.900 (0.953)	0.845 (0.889)
	0.5	5,5	0.858 (0.920)	0.858 (0.920)	0.875 (0.934)	0.897 (0.948)	0.830 (0.864)
		5,10	0.867 (0.927)	0.876 (0.933)	0.890 (0.943)	0.900 (0.950)	0.780 (0.818)
		10,10	0.881 (0.938)	0.881 (0.938)	0.890 (0.945)	0.898 (0.951)	0.864 (0.906)
		10,15	0.881 (0.941)	0.883 (0.940)	0.890 (0.945)	0.898 (0.950)	0.846 (0.890)
	0.75	5,5	0.866 (0.927)	0.866 (0.927)	0.884 (0.939)	0.904 (0.952)	0.834 (0.867)
		5,10	0.862 (0.928)	0.873 (0.934)	0.887 (0.942)	0.900 (0.949)	0.773 (0.812)
		10,10	0.883 (0.938)	0.883 (0.938)	0.891 (0.944)	0.899 (0.949)	0.863 (0.907)
		10,15	0.885 (0.943)	0.886 (0.944)	0.893 (0.949)	0.899 (0.953)	0.840 (0.885)
	1.5	5,5	0.858 (0.918)	0.858 (0.918)	0.878 (0.933)	0.899 (0.948)	0.829 (0.861)
		5,10	0.869 (0.923)	0.875 (0.930)	0.886 (0.939)	0.898 (0.948)	0.775 (0.815)
		10,10	0.880 (0.936)	0.880 (0.936)	0.887 (0.942)	0.896 (0.949)	0.860 (0.899)
		10,15	0.879 (0.940)	0.879 (0.940)	0.885 (0.945)	0.892 (0.949)	0.844 (0.889)
	3.0	5,5	0.866 (0.924)	0.866 (0.924)	0.884 (0.937)	0.904 (0.950)	0.830 (0.863)
		5,10	0.870 (0.927)	0.875 (0.935)	0.888 (0.942)	0.899 (0.949)	0.781 (0.819)
		10,10	0.885 (0.939)	0.885 (0.939)	0.893 (0.945)	0.901 (0.951)	0.865 (0.906)
		10,15	0.886 (0.942)	0.889 (0.941)	0.896 (0.946)	0.902 (0.951)	0.855 (0.896)
0.5	0.25	5,5	0.858 (0.920)	0.858 (0.920)	0.876 (0.934)	0.895 (0.949)	0.835 (0.868)
		5,10	0.866 (0.925)	0.875 (0.933)	0.885 (0.944)	0.896 (0.951)	0.785 (0.821)
		10,10	0.888 (0.941)	0.888 (0.941)	0.896 (0.946)	0.904 (0.952)	0.866 (0.905)
		10,15	0.881 (0.939)	0.882 (0.940)	0.888 (0.945)	0.896 (0.950)	0.853 (0.897)
	0.5	5,5	0.861 (0.926)	0.861 (0.926)	0.881 (0.938)	0.902 (0.950)	0.830 (0.866)
		5,10	0.872 (0.928)	0.879 (0.934)	0.892 (0.942)	0.901 (0.951)	0.784 (0.823)
		10,10	0.880 (0.937)	0.880 (0.937)	0.888 (0.943)	0.896 (0.947)	0.863 (0.906)
		10,15	0.886 (0.937)	0.886 (0.939)	0.891 (0.943)	0.899 (0.948)	0.850 (0.897)
	0.75	5,5	0.858 (0.919)	0.858 (0.919)	0.876 (0.933)	0.896 (0.948)	0.827 (0.863)
		5,10	0.866 (0.923)	0.873 (0.929)	0.885 (0.938)	0.896 (0.947)	0.781 (0.818)
		10,10	0.885 (0.942)	0.885 (0.942)	0.894 (0.947)	0.905 (0.953)	0.872 (0.911)
		10,15	0.882 (0.940)	0.883 (0.941)	0.890 (0.944)	0.897 (0.949)	0.845 (0.888)
	1.5	5,5	0.859 (0.920)	0.859 (0.920)	0.877 (0.934)	0.898 (0.950)	0.832 (0.864)
		5,10	0.871 (0.927)	0.875 (0.933)	0.886 (0.943)	0.900 (0.950)	0.778 (0.821)
		10,10	0.885 (0.940)	0.885 (0.940)	0.892 (0.945)	0.902 (0.952)	0.867 (0.909)
		10,15	0.885 (0.942)	0.889 (0.942)	0.896 (0.946)	0.903 (0.950)	0.849 (0.893)
	3.0	5,5	0.861 (0.923)	0.861 (0.923)	0.881 (0.936)	0.900 (0.949)	0.833 (0.862)
		5,10	0.861 (0.923)	0.874 (0.930)	0.885 (0.938)	0.899 (0.946)	0.779 (0.818)
		10,10	0.884 (0.936)	0.884 (0.936)	0.891 (0.943)	0.899 (0.948)	0.863 (0.905)
		10,15	0.887 (0.941)	0.889 (0.943)	0.895 (0.946)	0.901 (0.950)	0.854 (0.895)
1.5	0.25	5,5	0.862 (0.924)	0.862 (0.924)	0.880 (0.937)	0.899 (0.950)	0.828 (0.861)
		5,10	0.869 (0.922)	0.878 (0.930)	0.887 (0.937)	0.899 (0.946)	0.777 (0.817)
		10,10	0.881 (0.935)	0.881 (0.935)	0.889 (0.943)	0.897 (0.948)	0.861 (0.907)
		10,15	0.886 (0.941)	0.889 (0.941)	0.895 (0.947)	0.901 (0.951)	0.854 (0.896)
	0.5	5,5	0.866 (0.923)	0.866 (0.923)	0.883 (0.936)	0.900 (0.950)	0.834 (0.865)
		5,10	0.863 (0.926)	0.873 (0.929)	0.885 (0.939)	0.897 (0.948)	0.782 (0.818)
		10,10	0.889 (0.942)	0.889 (0.942)	0.897 (0.947)	0.905 (0.953)	0.867 (0.911)
		10,15	0.889 (0.941)	0.890 (0.941)	0.896 (0.945)	0.902 (0.951)	0.853 (0.893)
	0.75	5,5	0.861 (0.924)	0.861 (0.924)	0.879 (0.938)	0.901 (0.950)	0.836 (0.869)
		5,10	0.861 (0.920)	0.871 (0.928)	0.882 (0.939)	0.893 (0.948)	0.779 (0.816)
		10,10	0.883 (0.938)	0.883 (0.938)	0.892 (0.942)	0.901 (0.949)	0.867 (0.910)
		10,15	0.882 (0.938)	0.884 (0.939)	0.891 (0.944)	0.897 (0.949)	0.842 (0.888)
	1.5	5,5	0.866 (0.924)	0.866 (0.924)	0.883 (0.939)	0.903 (0.953)	0.829 (0.865)
		5,10	0.866 (0.924)	0.875 (0.930)	0.887 (0.940)	0.899 (0.948)	0.777 (0.816)
		10,10	0.883 (0.941)	0.883 (0.941)	0.892 (0.947)	0.900 (0.952)	0.864 (0.908)
		10,15	0.888 (0.941)	0.889 (0.943)	0.895 (0.948)	0.901 (0.951)	0.854 (0.898)
	3.0	5,5	0.863 (0.924)	0.863 (0.924)	0.881 (0.938)	0.900 (0.951)	0.828 (0.861)
		5,10	0.865 (0.923)	0.874 (0.931)	0.885 (0.938)	0.896 (0.947)	0.780 (0.820)
		10,10	0.883 (0.939)	0.883 (0.939)	0.892 (0.944)	0.900 (0.951)	0.862 (0.904)
		10,15	0.887 (0.939)	0.887 (0.940)	0.893 (0.944)	0.900 (0.949)	0.849 (0.890)

The analyses presented in Tables 1–6 indicate that the one-at-a-time reference prior, denoted as π_4, outperforms other priors in achieving the desired coverage probabilities. In contrast, Jeffreys' prior (π_1), the two-group reference prior (π_2), and the three-group reference prior (π_3) demonstrate limitations in this aspect. Notably, π_4 maintains robust coverage probabilities even with limited sample sizes, as it complies with the second-order matching criteria. In comparison, Jeffreys' prior does not provide adequate coverage probabilities. While the two-group reference prior satisfies the first-order matching criterion, it fails to deliver favorable coverage probabilities under small sample conditions. As sample sizes increase, the performance of the three-group reference prior becomes more comparable to that of the one-at-a-time reference prior than to the two-group reference prior, resulting in enhanced coverage probabilities. On the other hand, the results based on the asymptotic confidence interval of the MLE show that, for small samples, the target coverage probability is not reached at all. Additionally, the findings presented in the tables show minimal sensitivity to variations in the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ associated with π_4. Therefore, we advocate for the adoption of the one-at-a-time reference prior for Bayesian inference in practical contexts. Note that the matching prior necessitates that the coverage probability of a Bayesian credible interval is asymptotically equivalent to the coverage probability of a frequentist confidence interval. Therefore, the our numerical study focused on investigating the frequentist coverage probabilities by the credible intervals of the marginal posterior densities under the developed priors. If the developed priors meet the target coverage probabilities well, the posterior variance or mean squared error can be correctly estimated. However, if the developed priors provide results that do not match the target coverage probability, then the confidence intervals can be skewed to the left or right, and their width can be large or small, which will cause the posterior variance or mean squared error to be underestimated or overestimated. For this reason, in our numerical study, we focused on the frequentist coverage probability of the marginal posterior density without considering the posterior variance or mean squared error.

Example 1. This example taken from Cooray and Ananda [2]. The 49 data points represent the stress-rupture life of kevlar 49/epoxy strands that are subject to constant pressure at the 70% stress level until all had failed, as before we have complete data with exact times of failure. The failure times in hours are shown in Andrews and Herzberg [28] and Barlow Barlow et al. [29]. The values of the log-likelihood measures, Kolmogorov-Smirnov statistics, and Anderson-Darling statistics, particularly the elevated p-value, indicate that the GHN model provides a superior fit compared to other distributions for the specified dataset [2]. To estimate the ratio of the shape parameters, the dataset was partitioned into two groups, as presented in Table 7. For Group 1, with $n_1 = 24$, the maximum likelihood estimators for the parameters (α_1, β_1) were (1.439, 10381.664), while for Group 2, with $n_2 = 25$, the maximum likelihood estimators for the parameters (α_2, β_2) were (1.889, 11352.220). The Bayesian estimates, along with the 90% and 95% Bayesian credible intervals for the ratio of shape parameters, are presented in Table 8. These estimates are derived using Jeffreys' prior π_1 (JP), the two-group reference prior π_2 (TR), the three-group reference prior π_3 (TH), and the one-at-a-time reference prior π_4 (OR). The estimates and credible intervals for the ratio of the shape parameters yield highly comparable results, exhibiting no significant differences among the JP, TR, TH, and OR methodologies. This similarity is likely attributable to the moderate sample size and the fulfillment of the first-order matching criterion for both the TR and TH approaches. However, the asymptotic confidence interval of the MLE is skewed to the left compared to the results of the Bayesian method. These MLE results are consistent with those shown in simulation studies.

Table 7. Data for Kevlar 49/epoxy strands failure at a 70% stress level.

Group	Failure time
Group 1	1051 1389 1942 3629 4012 4063 4921 5620 5817 5905 6068 6473 7501 8666 9806 10205 10396 11604 11762 13670 14496 15395 16179 17568
Group 2	1337 1921 2322 4006 5445 5956 6121 7886 8108 8546 8831 9106 9711 10861 11026 11214 11362 11608 11745 11895 12044 13520 14110 17092 17568

Table 8. The estimates and 90% and 95% credible intervals of θ_1.

Method	Estimate	90% credible interval	95% credible interval
π_1	1.3502	(0.8755, 1.9601)	(0.8094, 2.1197)
π_2	1.3518	(0.8766, 1.9626)	(0.8105, 2.1224)
π_3	1.3532	(0.8736, 1.9711)	(0.8071, 2.1332)
π_4	1.3547	(0.8705, 1.9799)	(0.8036, 2.1444)
MLE	1.3127	(0.8034, 1.8221)	(0.7058, 1.9196)

Example 2. The 101 dataset characterizes the fatigue fracture life of Kevlar 49/epoxy composites, which were subjected to constant pressure at 90% of the stress level until complete failure occurred (Table 9). This dataset provides comprehensive information, including precise failure times. Cooray and Ananda [2] demonstrated that the generalized half-normal distribution offers a superior fit compared to alternative distributions, as evidenced by the values of the Kolmogorov-Smirnov statistic, the Anderson-Darling statistic, and the associated p-value. To estimate the ratio of the shape parameters, the datasets were randomly partitioned into two groups. Group 1, consisting of $n_1 = 50$ observations, yielded maximum likelihood estimators of the parameters (α_1, β_1) as (0.767, 1.314). Conversely, Group 2, comprising $n_2 = 51$ observations, produced maximum likelihood estimators of the parameters (α_2, β_2) as (0.670, 1.134). The Bayesian estimates, along with the 90% and 95% credible intervals, derived from the JP, TR, TH, and OR datasets, are presented in Table 10. The estimates and credible intervals for the ratio of the shape parameters exhibit remarkably similar outcomes, which can be attributed to the substantial sample sizes and the fulfillment of the initial matching criteria. Although the sample sizes are somewhat large, the asymptotic confidence intervals of the MLE are slightly skewed to the left, unlike the Bayesian method. We consider that these MLE results are consistent with numerical studies.

Table 9. Data for Kevlar 49/epoxy strands failure at a 90% stress level.

Group	Failure time
Group 1	0.01 0.02 0.02 0.03 0.06 0.07 0.09 0.09 0.11 0.12 0.18 0.19 0.24 0.35 0.42 0.43 0.60 0.60 0.63 0.72 0.79 0.79 0.80 0.85 0.99 1.00 1.01 1.02 1.03 1.05 1.15 1.20 1.29 1.33 1.40 1.50 1.51 1.52 1.55 1.58 1.63 1.64 1.80 1.81 2.02 2.14 3.03 3.34 4.20 4.69
Group 2	0.01 0.02 0.03 0.04 0.05 0.07 0.08 0.10 0.10 0.11 0.13 0.20 0.23 0.24 0.29 0.34 0.36 0.38 0.40 0.52 0.54 0.56 0.65 0.67 0.68 0.72 0.72 0.73 0.80 0.83 0.90 0.92 0.95 1.10 1.10 1.11 1.18 1.31 1.34 1.43 1.45 1.53 1.54 1.54 1.60 1.80 2.05 2.17 2.33 3.03 7.89

Table 10. The estimates and 90% and 95% credible intervals of θ_1.

Method	Estimate	90% credible interval	95% credible interval
π_1	0.8837	(0.6728, 1.1331)	(0.6399, 1.1920)
π_2	0.8840	(0.6730, 1.1334)	(0.6401, 1.1923)
π_3	0.8851	(0.6728, 1.1362)	(0.6398, 1.1956)
π_4	0.8862	(0.6727, 1.1390)	(0.6395, 1.1989)
MLE	0.8730	(0.6371, 1.1089)	(0.5920, 1.1541)

5. Conclusions

In the domain of generalized half-normal distributions, this study extensively compared matching and reference priors for the ratio of shape parameters, emphasizing their roles in objective Bayesian inference. Our analysis identified a second-order quantile matching prior that meets all established matching criteria, affirming its equivalence to the one-at-a-time reference prior and underscoring its optimal suitability for practical inference. In contrast, Jeffreys' prior notably fails to satisfy the first-order matching criterion, indicating significant limitations. Although two-group and three-group reference priors meet this first-order criterion, they exhibit undesirable sensitivity to sample size variations, reducing their practical utility, especially in small-sample scenarios. Conversely, the one-at-a-time reference prior demonstrates a crucial advantage by being invariant to sample size. Numerical simulations confirm its superior performance, highlighting optimal asymptotic frequentist coverage and commendable accuracy even in small datasets. In conclusion, the second-order quantile matching prior—equivalently, the one-at-a-time reference prior—emerges as the most robust and effective choice for objective Bayesian inference in generalized half-normal distributions, providing both theoretical insights and practical advantages across diverse experimental conditions.

Nevertheless, some practical limitations of this work should be noted. The proposed methodology is tailored specifically to the generalized half-normal distribution and to inference on the ratio of its shape parameters. As such, its direct applicability to other distributional families or to different parameter structures may be limited, and caution is required when attempting to generalize beyond this framework.

Looking ahead, future research could extend this framework to other families of lifetime distributions, particularly those accommodating multimodal or more complex hazard shapes. Another promising direction would be the development of computationally efficient algorithms for implementing second-order matching priors in high-dimensional settings, enabling broader applicability to modern large-scale data. Moreover, investigating the performance of these priors under model misspecification or within hierarchical Bayesian models may yield valuable insights. Finally, empirical studies in applied fields such as reliability engineering, biomedical survival analysis, and quality control could further validate and expand the utility of the proposed methodology.

Author contributions

S.-G.K.: Conceptualization, Methodology, Writing original draft; W.-D.L.: Data curation, Methodology, Writing original draft; Y.K.: Conceptualization, Formal analysis, Writing original draft, Writing review & editing. All three authors read and approved the final version of the article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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