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Research article

On the deep holes of a class of Cauchy codes

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Abstract: Suppose $\mathbb{F}_q$ is a finite field having an odd characteristic. Let $D = \{x_1, \cdots, x_{n-1}, \infty\}$, where $\{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$. Assume that k is an integer with $2 \le k < n$. We show that $u(D)$ is a deep hole of Cauchy code $C(D, k)$ if $u(x) = \lambda(x-\delta)^{q-2} + \nu x^{k-1} + f_{\le k-2}(x)$, where $\lambda \in \mathbb{F}_q^*$, $\delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}$,$\nu \in \mathbb{F}_q$ and $f_{\le k-2}(x) \in \mathbb{F}_q[x]$ of a degree that does not exceed $k-2$. This expands the result shown in our previous paper. In particular, we also show that the received word $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{F}_q^n$ is a deep hole of $C(D, k)$ if and only if the Lagrange interpolation polynomial of the first $n-1$ components of $\boldsymbol{u}$ is $\lambda(x-\delta)^{q-2} + u_n x^{k-1} + u_{\le k-2}(x)$, where $\lambda \in \mathbb{F}_q^*$, $\delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}$, and $u_{\le k-2}(x)$ is a polynomial over $\mathbb{F}_q$ whose degree does not exceed $k-2$, if $\frac{q-1}{2} \le k < n \le q-2$.

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1. Introduction and primary results

Let $\mathbb{F}_q$ be a finite field. Suppose that n, k are positive integers with $k < n$. Assume $D = \{x_1, \cdots, x_n\} \subseteq \mathbb{F}_q \bigcup\{\infty\}$. Now, the *Cauchy codes* $C(D, k)$ with a length n and dimension k is defined as

$$C(D, k) := \{(f(x_1), \ldots, f(x_n)) \in \mathbb{F}_q^n \mid f(x) \in \mathbb{F}_q[x], \deg f(x) \le k-1\},$$

with $f(\infty) = c_{k-1}(f)$, where $c_{k-1}(f)$ is the coefficient of x^{k-1} in $f(x)$. Note that the Cauchy codes are called Reed-Solomon codes, singly-extended Reed-Solomon codes, doubly-extended Reed-Solomon codes (also known as projective Reed-Solomon codes), when $D = \mathbb{F}_q^*, \mathbb{F}_q, \mathbb{F}_q \bigcup\{\infty\}$.

We employ $\#(A)$ to represent the cardinality of any finite set A. Let $\boldsymbol{u} = (u_1, \ldots, u_n), \boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$.

$$d(\boldsymbol{u}, \boldsymbol{v}) := \#\{1 \le i \le n \mid u_i \ne v_i, u_i \in \mathbb{F}_q, v_i \in \mathbb{F}_q\}$$

is named the *Hamming distance* of $\boldsymbol{u}$ and $\boldsymbol{v}$. For a $[n, k]$ linear code C, we have

$$d(C) := \min\{d(x, y) \mid x \in C, y \in C, x \neq y\}$$

is referred to as the *minimum distance* of C. If $d(C) = n - k + 1$, then the $[n, k, d]$ linear code C is named the *maximum distance separable* (MDS) code. For the received word $\boldsymbol{u} \in \mathbb{F}_q^n$

$$d(\boldsymbol{u}, C) := \min_{v \in C}\{d(\boldsymbol{u}, \mathbf{v})\}$$

is also called the *error distance* with the code C. We write

$$\rho(C) := \max_{\boldsymbol{u} \in \mathbb{F}_q^n}\{d(\boldsymbol{u}, C)\},$$

then $\rho(C)$ is called the *covering radius* of C. If $d(\boldsymbol{u}, C) = \rho(C)$, then the received word $\boldsymbol{u}$ is called a *deep hole* of C.

In the decoding progress of a Cauchy code $C(D, k)$, let $\boldsymbol{u} = (u_1, \ldots, u_n)$ be a received word of $\mathbb{F}_q^n$. The *Lagrange interpolation polynomial* (abbreviated as LIP) $u(x)$ of the first $n - 1$ components of $\boldsymbol{u}$ is defined as

$$u(x) := \sum_{i=1}^{n-1} u_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \frac{x - x_j}{x_i - x_j} \in \mathbb{F}_q[x],$$

where $u(x_i) = u_i$ for $1 \leq i \leq n - 1$.

Let $\boldsymbol{u}$ be a received word of $\mathbb{F}_q^n$. In the maximum likelihood decoding progress [1], we decode $\boldsymbol{u}$ as $\boldsymbol{v}$ when $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{u}, C)$ with $\boldsymbol{v} \in C$. If $d(\boldsymbol{u}, C) \leq n - \sqrt{nk}$, Guruswami and Sudan [2] presented a polynomial-time list decoding algorithm for decoding $\boldsymbol{u}$. Otherwise, Guruswami and Vardy [3] proved that the maximum-likelihood decoding of Reed-Solomon codes is NP-hard. In [4], Dür obtained the covering radius of Cauchy codes $C(D, k)$ in certain cases. If $D \subseteq \mathbb{F}_q$, a bound of $d(\boldsymbol{u}, C(D, k))$ has also been obtained by Li and Wan [5]. That is

$$n - \deg u(x) \leq d(\boldsymbol{u}, C(D, k)) \leq n - k = \rho(C(D, k)).$$

Obviously, we assume $\deg u(x) = k$, then the received word $\boldsymbol{u}$ is a deep hole of $C(D, k)$. Wu and Hong [6] discovered a new deep hole of $C(\mathbb{F}_q^*, k)$ with its LIP being $ax^{q-2} + f_{\leq k-1}(x)$, where $a \in \mathbb{F}_q^*$, $f_{\leq k-1}(x) \in \mathbb{F}_q[x]$ whose degree does not exceed $k - 1$. In 2016, Hong and Wu [7] generalized the result above to $C(D, k)$. More results on the deep holes of Cauchy codes can also be found in [1, 8].

Reed-Solomon codes, as a class of Cauchy code, have many applications in communication. New MDS codes were built by making use of Reed-Solomon codes in [9–12]. Qiang, Wei, and Hong [13] obtained two characterizations for an almost MDS code to become a near MDS code, and then employed these criteria to prove the validity of the conjecture presented by Zhou.

In the following parts of this paper, let $D = \{x_1, \cdots, x_{n-1}, \infty\}$ with $\{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$. Given a polynomial $f(x)$ of $\mathbb{F}_q[x]$, we write

$$f(D) := (f(x_1), \cdots, f(x_{n-1}), f(\infty)),$$

with $f(\infty) = c_{k-1}(f(x))$.

Zhang and Wan [14] demonstrated that the received word $f(\mathbb{F}_q \bigcup\{\infty\})$ is a deep hole of $C(\mathbb{F}_q \bigcup\{\infty\}, k)$, when $\deg f(x) = k$. Xu [15] also obtained some new findings on deep holes of Cauchy codes. Zhang, Wan, and Kaipa [16] constructed some new deep holes of Cauchy codes $C(\mathbb{F}_q \bigcup\{\infty\}, k)$ with fixed redundancy.

In the current paper, we intend to present characterizations for the received words to be deep holes of the Cauchy code $C(D, k)$, where $D = \{x_1, \cdots, x_{n-1}, \infty\}, \{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$. Next, we present the main results of this article.

Theorem 1.1. *Let $\mathbb{F}_q$ be a finite field with an odd prime characteristic. Suppose $D = \{x_1, \cdots, x_{n-1}, \infty\}, \{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$, and k be an integer such that $2 \le k < n$. Assume that*

$$u(x) = \lambda(x-\delta)^{q-2} + \nu x^{k-1} + f_{\le k-2}(x),$$

where $\lambda \in \mathbb{F}_q^*, \delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}, \nu \in \mathbb{F}_q$, and $f_{\le k-2}(x) \in \mathbb{F}_q[x]$ with its degree not exceeding $k-2$. Then the received word $u(D)$ is a deep hole of $C(D, k)$.

Remark. Letting $\delta = 0$ in Theorem 1.1, gives us the main result of [15].

Now by assuming $\frac{q-1}{2} \le k < n \le q-2$, we are able to classify the deep holes of $C(D, k)$ as follows.

Theorem 1.2. *Let $\mathbb{F}_q$ be a finite field with an odd prime characteristic. Assume that $D = \{x_1, \cdots, x_{n-1}, \infty\}, \{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$, and k is an integer with $\frac{q-1}{2} \le k < n \le q-2$. Suppose that $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{F}_q^n$, and $u(x)$ is the LIP of the first $n-1$ components of $\boldsymbol{u}$. Then the received word $\boldsymbol{u}$ is a deep hole of $C(D, k)$ if and only if*

$$u(x) = \lambda(x-\delta)^{q-2} + u_n x^{k-1} + u_{\le k-2}(x),$$

where $\lambda \in \mathbb{F}_q^*, \delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}$ and $u_{\le k-2}(x)$ is a polynomial over $\mathbb{F}_q$ whose degree does not exceed $k-2$.

From Theorems 1.1 and 1.2, we are able to infer a result regarding the deep holes of $C(D, k)$ as follows.

Corollary 1.3. *Let $\mathbb{F}_q$ be the finite field with an odd prime characteristic. Assume $D = \{x_1, \cdots, x_{n-1}, \infty\}, \{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$, and let k be an integer such that $2 \le k < n$. Suppose that*

$$u(x) = \lambda(x-\delta)^{q-2} + f_{\le k-1}(x),$$

where $\lambda \in \mathbb{F}_q^*, 0 \ne \delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}$, $f_{\le k-1}(x) \in \mathbb{F}_q[x]$ with $\deg(f_{\le k-1}(x)) \le k-1$. If $k \equiv 0 \pmod p$, then the received word $u(D)$ is a deep hole of $C(D, k)$. Furthermore, if $\frac{q-1}{2} \le k \le q-2$, then the received word $u(D)$ is a deep hole of $C(D, k)$ if and only if $k \equiv 0 \pmod p$.

MDS codes and a Vandemonde determinant are utilized to prove Theorems 1.1 and 1.2. A crucial element in these proofs is the so-called Dür's theorem on the covering radius of $C(D, k)$(see Lemma 2.3 below). Another significant element is the relationship between projective Reed-Solomon codes and Reed-Solomon codes.

We arranged the remaining part of this paper as follows. In Section 2, we present some lemmas required in the proofs of Theorems 1.1 and 1.2. In Section 3, we employ the lemmas in Section 2 to provide the proof of Theorem 1.1. In Section 4, by using the results given in Section 2, we prove Theorem 1.2 and Corollary 1.3. Finally, we offer the conclusion of our work in Section 5.

2. Some auxiliary lemmas

In this part, we present some preliminary results for the proof of the main theorem.

As is widely known, the generator matrix of a linear code is a collection of bases that acts as a linear space. Given $D = \{x_1, \cdots, x_{n-1}, \infty\}$, then a generator matrix of $C(D, k)$ is the following $k \times n$ matrix:

$$\begin{pmatrix} 1 & \dots & 1 & 0 \\ x_1 & \dots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \dots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \dots & x_{n-1}^{k-1} & 1 \end{pmatrix}_{k\times n}. \tag{2.1}$$

We will use this generator matrix of $C(D, k)$ in the following portion of this paper.

We first give a result to decide an MDS code as follows.

Proposition 2.1. [17] *Let G be a generator matrix of the $[n, k]$ linear code C. Then C is an* MDS *code if and only if any k different columns of G are linearly independent over the finite field $\mathbb{F}_q$.*

In the following, we point out that Cauchy code is an MDS code.

Lemma 2.2. [14] *Let $D = \{x_1, \cdots, x_{n-1}, \infty\}$, where $\{x_1, \cdots, x_{n-1}\}$ is a proper subset of $\mathbb{F}_q$. Then $C(D, k)$ is an $[n, k]$* MDS *code over $\mathbb{F}_q$.*

The following result connects the covering radius to the minimum distance of $C(D, k)$ and will be used in the remaining part of this paper.

Lemma 2.3. [4] *Let $D = \{x_1, \cdots, x_{n-1}, \infty\}$ with $\{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$. Thus*

$$\rho(C(D, k)) = d(C(D, k)) - 1.$$

We now give a well-known criterion to determine whether a received word is a deep hole of MDS code or not.

Lemma 2.4. [15] *Let G serve as a generator matrix of an* MDS *code C over the finite field $\mathbb{F}_q$. Suppose that the covering radius is equal to $d(C) - 1$, then the received word $\boldsymbol{u} \in \mathbb{F}_q^n$ is a deep hole of C if and only if the matrix $\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix}$ also generates another* MDS *code.*

Remark. By Lemmas 2.2, 2.3, and 2.4, $\boldsymbol{u}$ is a deep hole of $C(D, k)$ if and only if the matrix $\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix}$ generates an MDS code over $\mathbb{F}_q$.

Subsequently, we present a lemma to decide whether a received word is a deep hole of $C(D, k)$ or not.

Lemma 2.5. [16] *Let $\mathbb{F}_q$ be of an odd characteristic and $D \subset \mathbb{F}_q$. Suppose $\frac{q-1}{2} \le k \le q-2$ with k being an integer. Assume that $u(x)$ is the LIP of $\boldsymbol{u}$. Thus $\boldsymbol{u}$ is a deep hole of $C(D, k)$ if and only if $u(x)$ is equal to $\lambda x^k + f_{\le k-1}(x)$ or $\lambda(x-\delta)^{q-2} + f_{\le k-1}(x)$, where $\lambda \in \mathbb{F}_q^*, \delta \in \mathbb{F}_q \setminus D, f_{\le k-1}(x) \in \mathbb{F}_q[x]$ whose degree is no greater than $k - 1$.*

Meanwhile, we have the following relation for two received words to be deep holes.

Lemma 2.6. *Let $D = \{x_1, \cdots, x_{n-1}, \infty\}$ with $\{x_1, \cdots, x_{n-1}\} \subsetneq \mathbb{F}_q$. Assume*

$$\boldsymbol{u} = (u_1, \cdots, u_{n-1}, u_n) \in \mathbb{F}_q^n \text{ and } \boldsymbol{v} = (v_1, \cdots, v_{n-1}, v_n) \in \mathbb{F}_q^n$$

be two received words. Let $u(x)$ and $v(x)$ be, respectively, the LIP of the first $n-1$ components of $\boldsymbol{u}$ and $\boldsymbol{v}$. Suppose that $u(x) = \lambda v(x) + f_{\leq k-2}(x)$ and $u_n = \lambda v_n$, where $\lambda \in \mathbb{F}_q^*, f_{\leq k-2}(x) \in \mathbb{F}_q[x]$ with $\deg(f_{\leq k-2}(x)) \leq k-2$*, then $\boldsymbol{u}$ is a deep hole of $C(D,k)$ if and only if $\boldsymbol{v}$ is a deep hole of $C(D,k)$.*

Proof. By (2.1), one can get

$$\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_1 & \ldots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \ldots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \ldots & x_{n-1}^{k-1} & 1 \\ u_1 & \ldots & u_{n-1} & u_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_1 & \ldots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \ldots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \ldots & x_{n-1}^{k-1} & 1 \\ u(x_1) & \ldots & u(x_{n-1}) & u_n \end{pmatrix}.$$

Since $u(x) = \lambda v(x) + f_{\leq k-2}(x)$ and $u_n = \lambda v_n$, where $\lambda \in \mathbb{F}_q^*$, $f_{\leq k-2}(x) \in \mathbb{F}_q[x]$ with $\deg(f_{\leq k-2}(x)) \leq k-2$. One can immediately obtain

$$\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_1 & \ldots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \ldots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \ldots & x_n^{k-1} & 1 \\ \lambda v(x_1) + f_{\leq k-2}(x_1) & \ldots & \lambda v(x_{n-1}) + f_{\leq k-2}(x_{n-1}) & \lambda v_n \end{pmatrix}.$$

By using the elementary row transformations of $\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix}$, we have

$$\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix} \sim \begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_1 & \ldots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \ldots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \ldots & x_{n-1}^{k-1} & 1 \\ \lambda v(x_1) & \ldots & \lambda v(x_{n-1}) & \lambda v_n \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_1 & \ldots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \ldots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \ldots & x_{n-1}^{k-1} & 1 \\ v(x_1) & \ldots & v(x_{n-1}) & v_n \end{pmatrix} = \begin{pmatrix} G \\ \boldsymbol{v} \end{pmatrix}.$$

Hence the two matrixes $\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix}$ and $\begin{pmatrix} G \\ \boldsymbol{v} \end{pmatrix}$ are equivalent. Therefore, Proposition 2.1 and Lemma 2.4 indicate that $\boldsymbol{u}$ is a deep hole of $C(D, k)$ if and only if $\boldsymbol{v}$ is a deep hole of $C(D, k)$. □

In the following, we present a t-sum generators result in the finite field of odd characteristics which is proved by Roth and Lempel.

Lemma 2.7. [18] *Let $\mathbb{F}_q$ be of an odd characteristic. Suppose that k is an integer such that $\frac{q-1}{2} \le k \le q-2$ and let $S \subset \mathbb{F}_q$ with $\#(S) \ge k+1$. Then any element of $\mathbb{F}_q$ can be expressed as a sum of k distinct elements in S.*

Now we give an interesting division relationship on binomial coefficients as follows.

Lemma 2.8. *Let q be a power of the odd prime p. Suppose that t is an integer such that $2 \le t \le q-1$. Then the binomial coefficient $\binom{q-2}{t-1}$ is divisible by p if and only if t is divisible by p.*

Proof. Denote $v_p(x)$ as the p-adic valuation of x. Notice that

$$\binom{q-2}{t-1} = (q-t)\prod_{i=2}^{t-1}\frac{q-i}{i}.$$

Hence

$$\begin{aligned} v_p\Big(\binom{q-2}{t-1}\Big) &= \sum_{i=2}^{t-1} v_p\Big(\frac{q-i}{i}\Big) + v_p(q-t) \\ &= \sum_{i=2}^{t-1}(v_p(q-i) - v_p(i)) + v_p(q-t). \end{aligned} \tag{2.2}$$

Since q is a power of p and $2 \le i \le t-1 \le q-2 < q$, one can get

$$v_p(q-i) = v_p(i). \tag{2.3}$$

Moreover, (2.2) and (2.3) indicate that

$$v_p\Big(\binom{q-2}{t-1}\Big) = v_p(q-t) = v_p(t),$$

as needed. So $p \mid \binom{q-2}{t-1}$ if and only if $p \mid t$. □

3. Proof of Theorem 1.1

We use the lemmas presented in Section 2 to prove Theorem 1.1 in the next section.

Proof of Theorem 1.1. Assume

$$u(x) = \lambda(x-\delta)^{q-2} + \nu x^{k-1} + f_{\le k-2}(x),$$

where $\lambda \in \mathbb{F}_q^*, \delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}, \nu \in \mathbb{F}_q, f_{\le k-2}(x) \in \mathbb{F}_q[x]$ and $\deg(f_{\le k-2}(x)) \le k-2$. Denote $v_\delta(x)$ as

$$v_\delta(x) := (x-\delta)^{q-2} + \lambda^{-1}\nu x^{k-1},$$

and define a received word $\boldsymbol{v}_\delta$ associated with $v_\delta(x)$ as

$$\boldsymbol{v}_\delta := (v_\delta(x_1), \cdots, v_\delta(x_{n-1}), \lambda^{-1}v).$$

One can immediately get

$$u(x) = \lambda v_\delta(x) + f_{\le k-2}(x),$$

and also

$$v = \lambda(\lambda^{-1}v).$$

It follows from Lemma 2.6 that the received word $u(D)$ is a deep hole of $C(D,k)$ if and only if $\boldsymbol{v}_\delta$ is a deep hole of $C(D,k)$.

Since $\delta \in \mathbb{F}_q \setminus \{x_1, \cdots, x_{n-1}\}$, we have $(x_i-\delta)^{q-2} = (x_i-\delta)^{-1}, 1 \le i \le n-1$. Hence (2.1) tells us that

$$\begin{pmatrix} G \\ \boldsymbol{v}_\delta \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 & 0 \\ x_1 & \dots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \dots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \dots & x_{n-1}^{k-1} & 1 \\ (x_1-\delta)^{-1} + \lambda^{-1}v x_1^{k-1} & \dots & (x_{n-1}-\delta)^{-1} + \lambda^{-1}v x_{n-1}^{k-1} & \lambda^{-1}v \end{pmatrix}$$
$$:=(\hat{G}_1, \dots, \hat{G}_n). \tag{3.1}$$

On the other hand, Lemma 2.4 and Proposition 2.1 indicate that $\boldsymbol{v}_\delta = (v_\delta(x_1), \cdots, v_\delta(x_{n-1}), \lambda^{-1}v)$ is a deep hole of $C(D,k)$ if and only if

$$\det(\hat{G}_{j_1}, \dots, \hat{G}_{j_{k+1}}) \ne 0,$$

for any $k+1$ integers $j_1, \dots, j_{k+1}$ such that $1 \le j_1 < \cdots < j_{k+1} \le n$. Hence, we have to consider the following two cases.

Case 1. $j_{k+1} \ne n$. Then $k+1 \le j_{k+1} \le n-1$, and by (3.1), one has

$$(\hat{G}_{j_1}, \dots, \hat{G}_{j_{k+1}}) = \begin{pmatrix} 1 & \dots & 1 \\ x_{j_1} & \dots & x_{j_{k+1}} \\ \vdots & \vdots & \vdots \\ x_{j_1}^{k-1} & \dots & x_{j_{k+1}}^{k-1} \\ (x_{j_1}-\delta)^{-1} + \lambda^{-1}v x_{j_1}^{k-1} & \dots & (x_{j_{k+1}}-\delta)^{-1} + \lambda^{-1}v x_{j_{k+1}}^{k-1} \end{pmatrix}.$$

So one can see that

$$\det(\hat{G}_{j_1}, \dots, \hat{G}_{j_{k+1}}) = \left(\prod_{i=1}^{k+1} (x_{j_i}-\delta)^{-1}\right) \det \begin{pmatrix} x_{j_1}-\delta & \dots & x_{j_{k+1}}-\delta \\ x_{j_1}(x_{j_1}-\delta) & \dots & x_{j_{k+1}}(x_{j_{k+1}}-\delta) \\ \vdots & \vdots & \vdots \\ x_{j_1}^{k-1}(x_{j_1}-\delta) & \dots & x_{j_{k+1}}^{k-1}(x_{j_{k+1}}-\delta) \\ 1 & \dots & 1 \end{pmatrix}.$$

By using the row operation of the determinant, one can deduce that

$$\det(\hat{G}_{j_1},\ldots,\hat{G}_{j_{k+1}}) =\left(\prod_{i=1}^{k+1}(x_{j_i}-\delta)^{-1}\right)\det\begin{pmatrix} x_{j_1} & \ldots & x_{j_{k+1}} \\ \vdots & \vdots & \vdots \\ x_{j_1}^k & \ldots & x_{j_{k+1}}^k \\ 1 & \ldots & 1 \end{pmatrix}$$
$$=(-1)^k\left(\prod_{i=1}^{k+1}(x_{j_i}-\delta)^{-1}\right)\det\begin{pmatrix} 1 & \ldots & 1 \\ x_{j_1} & \ldots & x_{j_{k+1}} \\ \vdots & \vdots & \vdots \\ x_{j_1}^k & \ldots & x_{j_{k+1}}^k \end{pmatrix}$$
$$=(-1)^k\left(\prod_{i=1}^{k+1}(x_{j_i}-\delta)^{-1}\right)\prod_{1\le s<t\le k+1}(x_{j_t}-x_{j_s})\neq 0 \tag{3.2}$$

since $x_{j_1},\ldots,x_{j_{k+1}}$ are different from each other.

Case 2. Here, $j_{k+1}=n$. Then $1\le j_1<\cdots<j_k\le n-1$. So (3.1) gives us

$$\det(\hat{G}_{j_1},\ldots,\hat{G}_{j_k},\hat{G}_{j_n})$$
$$=\det\begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_{j_1} & \ldots & x_{j_k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{j_1}^{k-2} & \ldots & x_{j_k}^{k-2} & 0 \\ x_{j_1}^{k-1} & \ldots & x_{j_k}^{k-1} & 1 \\ (x_{j_1}-\delta)^{-1}+\lambda^{-1}\nu x_{j_1}^{k-1} & \ldots & (x_{j_k}-\delta)^{-1}+\lambda^{-1}\nu x_{j_k}^{k-1} & \lambda^{-1}\nu \end{pmatrix}$$
$$=-\det\begin{pmatrix} 1 & \ldots & 1 \\ x_{j_1} & \cdots & x_{j_k} \\ \vdots & \vdots & \vdots \\ x_{j_1}^{k-2} & \ldots & x_{j_k}^{k-2} \\ (x_{j_1}-\delta)^{-1} & \ldots & (x_{j_k}-\delta)^{-1} \end{pmatrix}$$
$$=-\left(\prod_{i=1}^{k}(x_{j_i}-\delta)^{-1}\right)\det\begin{pmatrix} x_{j_1} & \ldots & x_{j_k} \\ \vdots & \vdots & \vdots \\ x_{j_1}^{k-1} & \ldots & x_{j_k}^{k-1} \\ 1 & \ldots & 1 \end{pmatrix}$$
$$=(-1)^k\left(\prod_{i=1}^{k}(x_{j_i}-\delta)^{-1}\right)\prod_{1\le s<t\le k}(x_{j_t}-x_{j_s})\neq 0 \tag{3.3}$$

since $x_{j_1},\ldots,x_{j_k}$ are k pairwise distinct elements of D.

For any $k+1$ integers $j_1,\ldots,j_{k+1}$ with $1\le j_1<\cdots<j_{k+1}\le n$, (3.2) and (3.3) imply that $\det(\hat{G}_{j_1},\ldots,\hat{G}_{j_{k+1}})\neq 0$ holds. Thus the received word $\boldsymbol{v}_\delta$ is a deep hole of $C(D,k)$. Then Lemma 2.6 tells us that $u(D)$ is a deep hole of $C(D,k)$.

The proof of Theorem 1.1 is completed. □

4. Proofs of Theorem 1.2 and Corollary 1.3

In the present section, the proofs of Theorem 1.2 and Corollary 1.3 are given. The proof of Theorem 1.2 is given first as follows.

Proof of Theorem 1.2. First of all, letting $v = u_n$ in Theorem 1.1, the sufficiency of Theorem 1.2 can be immediately obtained.

Now we intend to present the necessity part. Let $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{F}_q^n$ be a deep hole of $C(D, k)$. Assume that the LIP of the initial $n-1$ components of $\boldsymbol{u}$ is $w(x)$. Hence one can derive

$$\boldsymbol{u} = (w(x_1), \cdots, w(x_{n-1}), u_n) := (\boldsymbol{w}, u_n). \tag{4.1}$$

Then (2.1) tells us that

$$\begin{aligned}\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix} =& \begin{pmatrix} 1 & \dots & 1 & 0 \\ x_1 & \dots & x_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{k-2} & \dots & x_{n-1}^{k-2} & 0 \\ x_1^{k-1} & \dots & x_{n-1}^{k-1} & 1 \\ u_1 & \dots & u_{n-1} & u_n \end{pmatrix} \\ :=&(\hat{G}_1, \dots, \hat{G}_n). \end{aligned} \tag{4.2}$$

Meanwhile, Lemmas 2.2 and 2.3 inform us that $C(D, k)$ is an MDS code with the covering radius $d-1$. Since $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{F}_q^n$ is a deep hole of $C(D, k)$, by Proposition 2.1 and Lemma 2.4, one can conclude that any $k+1$ columns of the matrix $\begin{pmatrix} G \\ \boldsymbol{u} \end{pmatrix}$ in (4.2) are linearly independent over a finite field $\mathbb{F}_q$. It infers that any $k+1$ columns of $(\hat{G}_1, \dots, \hat{G}_{n-1})$ must be linearly independent. By Proposition 2.1 and Lemma 2.4, $\boldsymbol{w} = (w_1, \cdots, w_{n-1})$ is a deep hole of $C(D, k)$ with $D \subset \mathbb{F}_q$. Therefore, Lemma 2.5 tells us that $w(x)$ equals $\lambda x^k + w_{\le k-1}(x)$ or $\lambda(x-\delta)^{q-2} + w_{\le k-1}(x)$, where $w(x)$ is the LIP of $\boldsymbol{w}$ with $\lambda \in \mathbb{F}_q^*, \delta \in \mathbb{F}_q \backslash D$, and $w_{\le k-1}(x)$ is a polynomial over $\mathbb{F}_q$ whose degree is no more than $k-1$. Therefore, the two cases of LIP on $\boldsymbol{w}$ need to be considered.

Case 1. Here, $w(x) = \lambda x^k + w_{k-1}x^{k-1} + w_{\le k-2}(x)$. By (4.1) and (4.2), it can be inferred that

$$\begin{aligned} &\det(\hat{G}_{j_1}, \dots, \hat{G}_{j_k}, \hat{G}_n) \\ =& \det \begin{pmatrix} 1 & \dots & 1 & 0 \\ x_{j_1} & \dots & x_{j_k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{j_1}^{k-2} & \dots & x_{j_k}^{k-2} & 0 \\ x_{j_1}^{k-1} & \dots & x_{j_k}^{k-1} & 1 \\ \lambda x_{j_1}^k + w_{k-1}x_{j_1}^{k-1} + w_{\le k-2}(x_{j_1}) & \dots & \lambda x_{j_k}^k + w_{k-1}x_{j_k}^{k-1} + w_{\le k-2}(x_{j_k}) & u_n \end{pmatrix} \\ =&\left(u_n - \left(\lambda \sum_{i=1}^{k} x_{j_i} + w_{k-1}\right)\right) \prod_{1 \le s < t \le k} (x_{j_t} - x_{j_s}). \end{aligned} \tag{4.3}$$

For $\frac{q-1}{2} \leq k \leq q-2$, according to Lemma 2.7, one can obtain the equation

$$u_n - w_{k-1} - \lambda \sum_{i=1}^{k} x_{j_i} = 0,$$

namely, the equation

$$\sum_{i=1}^{k} x_{j_i} = \lambda^{-1}(u_n - w_{k-1}) \in \mathbb{F}_q$$

has solutions over $\{x_1, \cdots, x_{n-1}\} \subset \mathbb{F}_q$.

From (4.3), one can know that $\{\beta_{j_1}, \ldots, \beta_{j_k}\} \subset \{x_1, \cdots, x_{n-1}\}$ exist such that

$$\det(\hat{G}_{j_1}, \ldots, \hat{G}_{j_k}, \hat{G}_n) = 0.$$

Thus, Proposition 2.1 and Lemma 2.4 tell us that $\boldsymbol{u} = (w, u_n)$ is not a deep hole of $C(D, k)$.

Case 2. Here, $w(x) = \lambda(x-\delta)^{q-2} + w_{k-1}x^{k-1} + w_{\leq k-2}(x)$. From (4.1) and (4.2), one has

$$\begin{aligned}\det(\hat{G}_{j_1}, \ldots, \hat{G}_{j_k}, \hat{G}_n) &= \det\begin{pmatrix} 1 & \ldots & 1 & 0 \\ x_{j_1} & \ldots & x_{j_k} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_{j_1}^{k-2} & \ldots & x_{j_k}^{k-2} & 0 \\ x_{j_1}^{k-1} & \ldots & x_{j_k}^{k-1} & 1 \\ w(x_{j_1}) & \ldots & w(x_{j_k}) & u_n \end{pmatrix} \\ &= \left(u_n - w_{k-1} + (-1)^k \lambda \prod_{i=1}^{k} (x_{j_i} - \delta)^{-1}\right) \prod_{1 \leq s < t \leq k} (x_{j_t} - x_{j_s}). \end{aligned} \tag{4.4}$$

Also noticed that

$$u_n - w_{k-1} + (-1)^k \lambda \prod_{i=1}^{k} (x_{j_i} - \delta)^{-1} = 0,$$

if and only if

$$\prod_{i=1}^{k} (x_{j_i} - \delta)^{-1} = (-1)^{k-1} \lambda^{-1} (u_n - w_{k-1}).$$

By (4.4), one has

$$\det(\hat{G}_{j_1}, \ldots, \hat{G}_{j_k}, \hat{G}_n) \neq 0$$

if and only if for any $\{x_{j_1}, \ldots, x_{j_k}\} \subset \{x_1, \cdots, x_{n-1}\}$,

$$0 \neq \prod_{i=1}^{k} (x_{j_i} - \delta)^{-1} \neq (-1)^{k-1} \lambda^{-1} (u_n - w_{k-1}).$$

If and only if

$$u_n = w_{k-1}.$$

Now, as $\boldsymbol{u} = (u_1, \cdots, u_n)$ is a deep hole of $C(D, k)$, Cases 1 and 2 indicate that $w(x)$ is equal to $\lambda(x-\delta)^{q-2} + u_n x^{k-1} + w_{\leq k-2}(x)$. Thus, the necessity aspect is demonstrated.

The proof of Theorem 1.2 is completed. □

We can now use Theorems 1.1 and 1.2 to prove Corollary 1.3.

Proof of Corollary 1.3. Since $u(x) = \lambda(x-\delta)^{q-2} + f_{\leq k-1}(x)$, and also $p|k$ with $2 \leq k \leq q-2$, Lemma 2.8 shows us that

$$\begin{aligned} c_{k-1}(u(x)) =&\lambda\binom{q-2}{k-1}(-\delta)^{q-1-k} + c_{k-1}(f_{\leq k-1}(x)) \\ =&c_{k-1}(f_{\leq k-1}(x)). \end{aligned}$$

Hence Theorem 1.1 tells us that

$$u(D) = (u(x_1), \cdots, u(x_{n-1}), c_{k-1}(f_{\leq k-1}(x))) = (u(x_1), \cdots, u(x_{n-1}), c_{k-1}(u(x)))$$

is a deep hole of $C(D, k)$.

Furthermore, for $\frac{q-1}{2} \leq k \leq q-2$,

$$u(D) = (u(x_1), \cdots, u(x_{n-1}), c_{k-1}(u(x)))$$

is a deep hole of $C(D, k)$ if and only if $c_{k-1}(u(x)) = c_{k-1}(f_{\leq k-1}(x))$ by Theorem 1.2, and if and only if $\lambda\binom{q-2}{k-1}(-\delta)^{q-1-k} = 0$ with $\lambda \in \mathbb{F}_q^*, 0 \neq \delta \in \mathbb{F}_q\backslash\{x_1, \cdots, x_{n-1}\}$, if and only if $p|\binom{q-2}{k-1}$. According to Lemma 2.8, which holds if and only if $p|k$.

This ends the proof of Corollary 1.3. □

5. Conclusions

Let $D = \{x_1, \cdots, x_{n-1}, \infty\}$, where $\{x_1, \cdots, x_{n-1}\}$ is a proper subset of $\mathbb{F}_q$. In the current paper, new deep holes of $C(D, k)$ were obtained. Specifically, when $\frac{q-1}{2} \leq k \leq q-2$, we also fully determined the deep holes of a certain class of $C(D, k)$.

Author contributions

Xiaofan Xu: Mathematical proof, writing-original draft; Zongbing Lin: Writing-original draft, modification of the paper. We also agreed to publish in this version.

Use of Generative-AI tools declaration

The artificial intelligence tools have not been utilized in this article.

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Conflict of interest

We declare that we have no conflict of interest.

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