Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 23518–23533.
DOI: 10.3934/math.20251044
Received: 09 August 2025
Revised: 27 September 2025
Accepted: 13 October 2025
Published: 16 October 2025

Research article

Closed-form solutions of a new class of three-dimensional nonlinear difference equations

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Abstract: This paper explored a new three-dimensional nonlinear system of difference equations, capturing intricate dynamic interactions through advanced analytical and computational methods. By employing strategic transformations and analyzing the system's characteristic polynomial roots, we derived exact closed-form solutions for both distinct and repeated root scenarios. Numerical examples were used to validate the theoretical results and demonstrate that even small variations in initial conditions can induce markedly different dynamical behaviors, ranging from stable oscillations to divergent trajectories.

Keywords: nonlinear difference equations; three-dimensional systems; closed-form solutions; population dynamics
Mathematics Subject Classification: Primary 39A10, Secondary 40A05

1. Introduction

Difference equations have long attracted the interest of mathematicians due to their ability to describe and analyze the evolution of dynamical systems in diverse scientific fields, including population biology, physics, economics, systems theory, and time series analysis. Beyond these classical domains, recent studies have increasingly highlighted the growing importance of fractional and stochastic difference equations as powerful tools for understanding complex real-world phenomena. For instance, Alzeley (2024, [1]) and Ghezal et al. (2024, [2]) employed stochastic conditional variance models and Markovian threshold-switching models to capture nonlinear patterns in financial markets, while Ghezal (2023, [3]) introduced a spectral representation of bilinear processes with significant implications for econometric modeling. In the biological and neuroscientific context, Althagafi (2025, [4]) investigated the dynamics of difference systems to model neural interactions, and

Althagafi (2025, [5]) analyzed the stability of biological rhythms using three-dimensional difference equations. Collectively, these contributions illustrate the versatility of difference equation frameworks in linking rigorous mathematical theory with practical applications. The study of nonlinear difference equations is particularly important, as it provides deep insights into complex behaviors such as oscillations, stability, and attractors. Several research efforts have investigated the global behavior of rational systems, establishing explicit conditions that ensure stability and convergence (see [6–8]). Complementary studies have examined the periodic nature and dynamical expressions of solutions, as well as the role of forbidden sets in characterizing higher-order systems (see [9–11]). Together, these contributions establish a rigorous foundation for the qualitative study of rational systems of difference equations. Building on this foundation, the present work extends the analysis to encompass bilinear structures in higher dimensions, thereby capturing richer and more intricate interactions among the coefficients. More recent contributions in global stability analysis, coupled equilibrium points, and numerical simulations have further advanced the understanding of nonlinear rational systems (see [12–14]). Foundational works have also played a key role in shaping the field by extending rational difference systems to higher-order models and demonstrating their inherent complexity (see [15, 16]). In addition to their mathematical elegance, fractional/rational difference equations are particularly well-suited for modeling systems with nonlinear feedback and proportional interactions. Unlike linear models, they can effectively capture threshold effects, saturation phenomena, and the intricate interdependencies observed in biological, economic, and engineering systems. Their rational structure offers both analytical tractability and modeling flexibility, enabling researchers to derive explicit closed-form solutions while preserving the richness of the underlying nonlinear dynamics. This balance between solvability and the capacity to represent complex, realistic behaviors forms the core motivation for adopting rational difference equations in the present study. Among the systems that have drawn attention are bilinear difference equations, which, despite their nonlinear nature, allow for analytical treatment under certain conditions. These systems are instrumental in illustrating the dynamic interplay between variables and, with the appropriate parameter settings, can yield closed-form solutions that further enhance our understanding of nonlinear dynamics.

In recent years, bilinear difference equations have emerged as a central focus in the modeling of discrete-time systems with nonlinear interactions. The two-dimensional system of bilinear equations developed by Stević and Tollu [17] provides a foundational framework that balances solvability with nonlinear complexity. Earlier works by Stević ([18–20]) demonstrated that transformation techniques can systematically restructure polynomial nonlinear systems into analytically tractable forms, revealing key dynamical properties such as stability regions, oscillatory behavior, and long-term trends. These findings underscore the effectiveness of transformations in uncovering solvable structures within complex, interconnected systems and establish a methodological foundation for extending the analysis of bilinear systems to three-dimensional models comprising intrinsically interconnected sequences, as undertaken in the present study. This study represents a natural extension of previous research on bilinear systems. The findings of Stević and Tollu [17] demonstrated that two-dimensional structures can be systematically analyzed through suitable transformation techniques, yielding explicit solutions and revealing important properties such as stability and periodicity. However, while these contributions are significant, they remain confined to two-dimensional settings and therefore do not fully capture the dynamics of three-dimensional models, where multiple sequences interact simultaneously. Building on this foundation, the present work advances the analysis to a three-dimensional framework, deriving closed-form solutions.

Motivated by these advances, this paper extends the exploration of solvability by focusing on a specific two-dimensional bilinear system with multiple parameters that govern the interactions between three dynamic sequences. The system under investigation, defined by recursive relations, is structured as follows:

$$\begin{aligned} \forall m \geq 0, \quad Q_{m+1} &= Q_m \frac{\alpha S_m + \beta Q_{m-1}}{Q_{m-1}}, \\ R_{m+1} &= \frac{\gamma S_m + \delta Q_{m-1}}{Q_{m-1}}, \\ S_{m+1} &= Q_m \frac{\lambda R_m Q_{m-1} + \varepsilon Q_m}{\sigma R_m Q_{m-1} + \tau Q_m}, \end{aligned} \tag{1.a}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \tau, \sigma \in \mathbb{R}$, $\alpha^2+\beta^2 \neq 0$, $\gamma^2+\delta^2 \neq 0$, $\varepsilon^2+\lambda^2 \neq 0$, $\tau^2+\sigma^2 \neq 0$, and $Q_0, R_0, S_0, Q_{-1} \in \mathbb{R}$. In order to ensure that the system is well-defined, the sequences $(Q_m)_{m\geq-1}$, $(R_m)_{m\geq0}$, and $(S_m)_{m\geq0}$ are considered real-valued, i.e., $(Q_m)_{m\geq-1}$, $(R_m)_{m\geq0}$, $(S_m)_{m\geq0} \subset \mathbb{R}$, together with the requirement that all denominators remain nonzero throughout the iterations, i.e., $Q_{m-1} \neq 0$ and $\sigma R_m Q_{m-1} + \tau Q_m \neq 0$ for $m \geq 0$. These constraints on the parameters are crucial for preserving the nonlinear interactions and ensuring that the terms are well-defined throughout the iterations.

Our study aims to simplify the system's complexity and gain a clearer understanding of its properties by employing advanced transformation techniques. These transformations facilitate an analysis of the characteristic polynomial's roots, which serve as a foundation for determining cases of closed-form solutions. Ultimately, this work contributes to the theoretical development of nonlinear difference equations by establishing a comprehensive basis for the study of complex dynamics in multi-parameter systems. It is worth emphasizing that three-dimensional systems of difference equations have become an increasingly prominent topic in recent literature, owing to their ability to capture complex interactions that cannot be fully represented by one- or two-dimensional models. In recent years, significant contributions have been made, such as Stević's (2021, [21]) study on "Three-dimensional quasi-periodic systems solvable in closed form," which underscored the structural potential of this class of models, as well as Kara's works (2022, [14]; 2023, [22]) that explored general solutions and the dynamical behavior of three-dimensional systems with exponential structures or constant coefficients. Collectively, these studies have advanced our understanding of global behaviors, including stability, periodicity, and convergence, while also highlighting existing gaps, particularly the absence of a unified framework that integrates solvability with the structural richness of mutual interactions. Against this backdrop, the present research seeks to address this gap by extending the bilinear system to incorporate three intrinsically interconnected dynamical sequences, using advanced analytical transformations to derive explicit solutions under specific conditions. This study thus combines rigorous mathematical analysis with potential applications, offering a qualitative contribution to the ongoing discourse on three-dimensional systems of difference equations. In addition to the classical contributions on nonlinear and bilinear systems, recent studies have underscored the increasing importance of rational difference equations in capturing complex multidimensional dynamics. For instance, Zemmouri et al. (2024, [23]) investigated a three-dimensional system of $(m + 1)$ rational difference equations, offering valuable insights into higher-order interactions and stability properties. More recently, Kara and Yazlık (2025, [24]) examined the global dynamics of a system of difference equations, with particular emphasis on confinement and continuity conditions. These developments not only complement earlier findings but also highlight the active and ongoing progress in this area. Building upon and extending such contributions, the present study situates itself within this growing body of literature, offering new perspectives on solvability, explicit solutions, and dynamical behaviors.

The rest of the paper is organized as follows: Section 2 provides an overview of the bilinear system introduced by Stević and Tollu, detailing the system's structure and key properties. Section 3 delves into the conditions required for the system's solvability and presents closed-form solutions, exploring both cases of repeated and distinct roots. Finally, Section 4 concludes with insights into the broader implications of this work and potential directions for future research.

2. Foundational framework for the two-dimensional bilinear system of difference equations

We display the nonlinear two-dimensional system of difference equations introduced by Stević and Tollu [17], represented as:

$$\forall m \geq 0,\ \widehat{Q}_{m+1} = \frac{\alpha\widehat{S}_m + \beta}{\gamma\widehat{S}_m + \delta}, \quad \widehat{S}_{m+1} = \frac{\varepsilon\widehat{Q}_m + \lambda}{\tau\widehat{Q}_m + \sigma}, \tag{2.a}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \tau, \sigma \in \mathbb{R}$ and $Q_0, S_0 \in \mathbb{R}$. The sequences $(\widehat{Q}_m)_{m\geq 0}$ and $(\widehat{S}_m)_{m\geq 0}$ are considered real-valued, i.e., $(\widehat{Q}_m)_{m\geq 0}, (\widehat{S}_m)_{m\geq 0} \subset \mathbb{R}$, together with the requirement that all denominators remain nonzero throughout the iterations, i.e., $\gamma\widehat{S}_m + \delta \neq 0$ and $\tau\widehat{Q}_m + \sigma \neq 0$ for $m \geq 0$. Stević and Tollu demonstrated that this system is solvable and provided a comprehensive method for determining its general solution. Their work established that specific parameter conditions are essential for the system's solvability. For example, if $\gamma = 0$, the first equation simplifies, leading to a structure derived from a bilinear equation, solvable due to its form. Similarly, $\tau = 0$ represents a dual case. In our analysis, we assume $\gamma \neq 0$ and $\tau \neq 0$ to retain the system's nonlinear complexity and introduce additional conditions, namely $\gamma\widehat{S}_m + \delta \neq 0$ and $\tau\widehat{Q}_m + \sigma \neq 0$ for all m, ensuring that terms in the equations remain well-defined across iterations.

Lemma 2.1. *Let $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \tau, \sigma \in \mathbb{R}$, and assume $\gamma\left(\alpha^2 + \beta^2\right)\tau\left(\varepsilon^2 + \lambda^2\right) \neq 0$. Define μ_1 and μ_2 as the roots of the quadratic polynomial:*

$$L(\mu) = \mu^2 - (\alpha\varepsilon + \beta\tau + \gamma\lambda + \delta\sigma)\mu + (\beta\gamma - \alpha\delta)(\lambda\tau - \varepsilon\sigma).$$

Under these conditions, the system admits a closed-form solution, represented in two distinct cases based on the nature of μ_1 and μ_2.

i.* *Case 1: *$\mu_1 = \mu_2$. Here, $L(\mu)$ has a repeated root μ_1, and the general solution to the system is:*

$$\begin{aligned}\tau\widehat{Q}_{2m} + \sigma = &\left(\left(\left(\tau\left(\alpha\tfrac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma} + \beta\right) + \sigma\left(\gamma\tfrac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma} + \delta\right)\right) - \mu_1\right)m + \mu_1\right)\\ &\times\left(\tau\widehat{Q}_0 + \sigma\right)\left(\left(\left(\tau\widehat{Q}_0 + \sigma\right)\left(\gamma\tfrac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma} + \delta\right) - \mu_1\right)m + \mu_1\right)^{-1},\end{aligned}$$

$$\begin{aligned}\tau\widehat{Q}_{2m+1} + \sigma = &\,\mu_1\left(\left(\left(\tau\tfrac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta} + \sigma\right)\left(\gamma\widehat{S}_0 + \delta\right) - \mu_1\right)(m+1) + \mu_1\right)\left(\gamma\widehat{S}_0 + \delta\right)^{-1}\\ &\times\left(\left(\left(\gamma\left(\varepsilon\tfrac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta} + \lambda\right) + \left(\tau\tfrac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta} + \sigma\right)\delta\right) - \mu_1\right)m + \mu_1\right)^{-1},\end{aligned}$$

$$\gamma\widehat{S}_{2m}+\delta=\left(\gamma\widehat{S}_0+\delta\right)\left(\left(\left(\gamma\left(\varepsilon\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\lambda\right)+\delta\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\right)-\mu_1\right)m+\mu_1\right)$$
$$\times\left(\left(\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\left(\gamma\widehat{S}_0+\delta\right)-\mu_1\right)m+\mu_1\right)^{-1}, \tag{2.b}$$

$$\gamma\widehat{S}_{2m+1}+\delta=\mu_1\left(\left(\left(\tau\widehat{Q}_0+\sigma\right)\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)-\mu_1\right)(m+1)+\mu_1\right)\left(\tau\widehat{Q}_0+\sigma\right)^{-1}$$
$$\times\left(\left(\left(\tau\left(\alpha\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\beta\right)+\sigma\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)\right)-\mu_1\right)m+\mu_1\right)^{-1}. \tag{2.c}$$

ii. ***Case 2:*** *$\mu_1 \neq \mu_2$. For distinct roots, the solution form becomes more complex, depending on the values of μ_1 and μ_2, leading to either oscillatory or exponential behavior based on their magnitude and sign.*

$$\tau\widehat{Q}_{2m}+\sigma=\left(\left(\tau\left(\alpha\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\beta\right)+\sigma\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)$$
$$\times\left(\tau\widehat{Q}_0+\sigma\right)\left(\left(\tau\widehat{Q}_0+\sigma\right)\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)^{-1},$$

$$\tau\widehat{Q}_{2m+1}+\sigma=\left(\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\left(\gamma\widehat{S}_0+\delta\right)\left(\mu_1^{m+1}-\mu_2^{m+1}\right)+\mu_1\mu_2\left(\mu_2^m-\mu_1^m\right)\right)\left(\gamma\widehat{S}_0+\delta\right)^{-1}$$
$$\times\left(\left(\gamma\left(\varepsilon\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\lambda\right)+\delta\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)^{-1},$$

$$\gamma\widehat{S}_{2m}+\delta=\left(\left(\gamma\left(\varepsilon\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\lambda\right)+\delta\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)$$
$$\times\left(\gamma\widehat{S}_0+\delta\right)\left(\left(\tau\frac{\alpha\widehat{S}_0+\beta}{\gamma\widehat{S}_0+\delta}+\sigma\right)\left(\gamma\widehat{S}_0+\delta\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)^{-1}, \tag{2.d}$$

$$\gamma\widehat{S}_{2m+1}+\delta=\left(\left(\tau\widehat{Q}_0+\sigma\right)\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)\left(\mu_1^{m+1}-\mu_2^{m+1}\right)+\mu_1\mu_2\left(\mu_2^m-\mu_1^m\right)\right)\left(\tau\widehat{Q}_0+\sigma\right)^{-1}$$
$$\times\left(\left(\tau\left(\alpha\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\beta\right)+\sigma\left(\gamma\frac{\varepsilon\widehat{Q}_0+\lambda}{\tau\widehat{Q}_0+\sigma}+\delta\right)\right)(\mu_1^m-\mu_2^m)+\mu_1\mu_2\left(\mu_2^{m-1}-\mu_1^{m-1}\right)\right)^{-1}. \tag{2.e}$$

Proof. We follow the approach of Stević and Tollu [17], adapted directly to system (2.a). Consider the sequences $\left(\widehat{Q}_m, \widehat{S}_m\right)$ with parameters α, β, γ, δ, ε, λ, τ, σ. Define the linear forms $X_m := \tau\widehat{Q}_m + \sigma$, $Y_m := \gamma\widehat{S}_m + \delta$, under the assumptions $\tau \neq 0$, $\gamma \neq 0$, $X_m \neq 0$, and $Y_m \neq 0$ for all m. With these substitutions, the system can be reformulated and decomposed according to the parity of the index m (even/odd). By introducing suitable cumulative products and transforming the fractional relations, we obtain second-order linear recursions for these products. If the corresponding characteristic polynomial $L(.)$ has two distinct roots, the solution is expressed as a linear combination of powers μ_1^m, μ_2^m. In the case of a repeated root, polynomial terms in m appear, multiplied by the corresponding root power. Using these expressions, one recovers explicit formulas for $\tau\widehat{Q}_{2m}+\sigma$, $\tau\widehat{Q}_{2m+1}+\sigma$, $\gamma\widehat{S}_{2m}+\delta$, $\gamma\widehat{S}_{2m+1}+\delta$, as stated in the Lemma 2.1. The algebraic derivations are standard but lengthy; full details can be found in Stević and Tollu [17]. □

Remark 2.1. *The proposed model in (1.a) represents a qualitative enhancement over the framework of Stević and Tollu in (2.a). While their system is restricted to two coupled bilinear equations describing the interplay between only two variables, our formulation introduces a third variable that is dynamically and intrinsically intertwined with the others. This extension is not merely formal but instead reflects a more realistic architecture of reciprocal interactions, where the sequential feedback among the three variables generates novel dynamical features such as cyclic entanglement and multi-level interdependence. As a result, the model substantially broadens the landscape of dynamical analysis, allowing for the investigation of behaviors that remain inaccessible in the classical two-dimensional setting. In addition, the structural design of the proposed system admits reformulations and transformations that facilitate the derivation of closed-form solutions under suitable parameter conditions, thereby extending the methodological innovations of Stević and Tollu to a richer and more intricate class of nonlinear dynamics.*

Remark 2.2. *The transformation adopted in this study, which converts each state into two new ratios representing the relationships between the original variables at successive steps, does not alter the system's temporal evolution but instead reorganizes the information into a more tractable form. Provided that the denominators remain nonzero, each state of the transformed system corresponds uniquely to a state in the original system, allowing the original variables to be recovered from the new ratios in a one-to-one manner. Consequently, the qualitative properties of the system, such as periodic solutions, are preserved. Hence, the transformation can be regarded as a safe and reliable simplification tool for analyzing dynamic structures, with the important caveat that exceptional cases involving vanishing denominators or zero calibration coefficients must be explicitly identified and treated separately.*

Remark 2.3. *When specific algebraic relations among the parameters are satisfied, the system admits constant solutions. In particular, if $\beta\gamma = \alpha\delta$, then the sequences stabilize to constants:*

$$\forall m \geq 0,\ \widehat{Q}_m = \begin{cases} \dfrac{\alpha}{\gamma} & \text{if } \gamma \neq 0 \\ \dfrac{\beta}{\delta} & \text{if } \delta \neq 0 \end{cases} \quad , \quad \widehat{S}_m = \begin{cases} \dfrac{\alpha\varepsilon + \gamma\lambda}{\alpha\tau + \gamma\sigma} & \text{if } \gamma \neq 0 \\ \dfrac{\beta\varepsilon + \delta\lambda}{\beta\tau + \delta\sigma} & \text{if } \delta \neq 0 \end{cases} .$$

Similarly, if $\lambda\tau = \varepsilon\sigma$, another family of constant equilibria can be derived. These special parameter settings simplify the system considerably, as both sequences reduce to constant equilibria independent of the iteration index.

Remark 2.4. *To ensure that the characteristic polynomial admits two distinct roots, the following discriminant condition must be satisfied:*

$$\frac{(\alpha\varepsilon + \beta\tau + \gamma\lambda + \delta\sigma)^2}{(\beta\gamma - \alpha\delta)(\lambda\tau - \varepsilon\sigma)} \neq 4. \tag{2.f}$$

This requirement rules out the degenerate case where the two roots coincide, thereby preserving the generality of the closed-form solutions. By excluding the critical scenario of a vanishing discriminant, the system maintains the flexibility to exhibit distinct dynamical behaviors depending on the relative magnitudes and signs of the two roots.

3. Resolution of the system in (1.a)

In this section, we aim to systematically resolve the nonlinear difference equations presented in (1.a) under specified initial conditions. Consider the sequence (Q_m, R_m, S_m) representing a solution to this system. It is important to note that the solution may lose its definition if any initial values are zero, as this could disrupt the continuity of the equations. To ensure the system's consistency, the condition $Q_m \neq 0$ must hold for all values of m. This requirement guarantees that the solution remains well-defined and devoid of singularities. From this point onward, we adopt the assumption that $Q_{-k} \neq 0$ for $k \in \{0,1\}$. This assumption is essential for the continued existence of a coherent solution, as it preserves the system's structural integrity.

To facilitate analysis and gain a fresh perspective on system (1.a), we introduce a strategic change of variables. This transformation aims to simplify the interactions among variables and provide clearer insights into the system's nature. The variable transformation is defined as follows:

$$\widehat{Q}_m = \frac{Q_m}{R_m Q_{m-1}}, \ \widehat{S}_m = \frac{S_m}{Q_{m-1}} \text{ for } m \geq 0. \tag{3.a}$$

This change of variables reformulates the system into a more manageable form, which sets the stage for examining its core properties. As a result of this transformation, system (1.a) can now be re-expressed in a simplified version, referred to as (2.a).

By leveraging Lemma 2.1, we derive explicit, closed-form solutions for the original system (1.a). These solutions enable us to establish the following theorem.

Theorem 3.1. *Let $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda, \tau, \sigma \in \mathbb{R}$ be such that the condition $\gamma\left(\alpha^2+\beta^2\right)\sigma\left(\varepsilon^2+\lambda^2\right) \neq 0$ holds. Under these parameters, the system of difference equations presented in (1.a) admits a general closed-form solution. The solution to this system is expressed in two distinct cases, based on whether the roots μ_1 and μ_2 are identical or distinct.*

***i. Case** 1**: Identical roots:** If the assumption in (2.f) does not hold, the general solution becomes:*

$$Q_{2m+s} = Q_1 \prod_{k=1}^{m+s-1}\left(\alpha\widehat{S}_{2k}+\beta\right)\prod_{k=0}^{m-1}\left(\alpha\widehat{S}_{2k+1}+\beta\right), \ (Q_1 = Q_0\left(\alpha\widehat{S}_0+\beta\right)),$$

$$\begin{aligned} R_{2m+s} &= \gamma\widehat{S}_{2m+s-1}+\delta, \\ S_{2m+s} &= Q_{2m+s-1}\widehat{S}_{2m+s}, \end{aligned}$$

for $s \in \{0,1\}$ and $m \geq 1$. The sequences $\left(\widehat{S}_{2k}\right)$ and $\left(\widehat{S}_{2k+1}\right)$ are defined by (2.b) and (2.c), respectively.

***ii. Case** 2**: Distinct roots:** Assuming the condition specified in (2.f) holds, the general solution to the system is given by:*

$$Q_{2m+s} = Q_1 \prod_{k=1}^{m+s-1}\left(\alpha\widehat{S}_{2k}+\beta\right)\prod_{k=0}^{m-1}\left(\alpha\widehat{S}_{2k+1}+\beta\right),$$

$$\begin{aligned} R_{2m+s} &= \gamma\widehat{S}_{2m+s-1}+\delta, \\ S_{2m+s} &= Q_{2m+s-1}\widehat{S}_{2m+s}, \end{aligned}$$

for $s \in \{0,1\}$ and $m \geq 1$. The sequences $\left(\widehat{S}_{2k}\right)$ and $\left(\widehat{S}_{2k+1}\right)$ are defined by (2.d) and (2.e), respectively.

Proof. To establish the result, we begin by transforming system (1.a) using the change of variables (3.a) for $m \geq 0$. For the first component, we have

$$\widehat{Q}_{m+1} = \frac{Q_{m+1}}{R_{m+1}Q_m} = Q_m \frac{\alpha S_m + \beta Q_{m-1}}{Q_{m-1}} \frac{Q_{m-1}}{(\gamma S_m + \delta Q_{m-1})\, Q_m} = \frac{\alpha S_m + \beta Q_{m-1}}{\gamma S_m + \delta Q_{m-1}}.$$

Dividing the numerator and denominator by Q_{m-1}, we obtain the first equation of system (2.a). Similarly, for the second component we write

$$\widehat{S}_{m+1} = \frac{S_{m+1}}{Q_m} = Q_m \frac{\lambda R_m Q_{m-1} + \varepsilon Q_m}{(\sigma R_m Q_{m-1} + \tau Q_m)\, Q_m} = \frac{\lambda R_m Q_{m-1} + \varepsilon Q_m}{\sigma R_m Q_{m-1} + \tau Q_m}.$$

Dividing the numerator and denominator by $R_m Q_{m-1}$, we obtain the second equation of system (2.a). Thus, by rewriting system (1.a), we obtain:

$$\forall m \geq 0, \quad Q_{m+1} = Q_m \left(\alpha \frac{S_m}{Q_{m-1}} + \beta\right), R_{m+1} = \gamma \frac{S_m}{Q_{m-1}} + \delta.$$

For compactness, let us introduce the temporary ratio $\widehat{S}_m := \dfrac{S_m}{Q_{m-1}}$ for all m. With this notation, the recurrence reduces to $Q_{m+1} = Q_m \left(\alpha \widehat{S}_m + \beta\right)$. By iterating this identity from index 1 up to $m-2$, we obtain

$$\begin{aligned} Q_2 = &\quad Q_1 \left(\alpha \widehat{S}_1 + \beta\right) \\ Q_3 = &\quad Q_2 \left(\alpha \widehat{S}_2 + \beta\right) = Q_1 \left(\alpha \widehat{S}_1 + \beta\right)\left(\alpha \widehat{S}_2 + \beta\right) \\ \vdots \\ Q_{m-1} = &\quad Q_1 \prod_{k=1}^{m-2} \left(\alpha \widehat{S}_k + \beta\right). \end{aligned}$$

Hence, the general expression for Q_m is given by:

$$\forall m \geq 0,\; Q_m = Q_1 \prod_{k=1}^{m-1} \left(\alpha \widehat{S}_k + \beta\right). \tag{3.b}$$

Next, the second equation of the our system provides an expression for R_m:

$$\forall m \geq 0, \quad R_m = \gamma \widehat{S}_{m-1} + \delta.$$

Finally, we find that $S_m = Q_{m-1}\widehat{S}_m$ for $m \geq 0$ with Q_m defined by (3.b). Upon further decomposition, we obtain the following relationships: $S_{2m} = Q_{2m-1}\widehat{S}_{2m}$ and $S_{2m+1} = Q_{2m}\widehat{S}_{2m+1}$ for $m \geq 0$. These relationships emphasize the recursive structure of the model, in which each new term is systematically generated from its predecessor through interactions governed by the coefficients. This formulation not only captures the intricate interconnections between the sequences but also demonstrates how the applied transformations preserve both internal consistency and the system's inherent nonlinearity. □

Remark 3.1. *The above change of variables (3.a) is not merely a technical step but a fundamental simplification that reformulates the nonlinear interactions of the system (1.a) into a bilinear form. This transformation effectively removes higher-order nonlinearities, yielding a more tractable structure while preserving the essential dependencies among* (Q_m, R_m, S_m). *Importantly, it enables the characteristic polynomial to arise naturally, thereby providing a direct route to obtaining closed-form solutions and distinguishing between oscillatory and stable dynamical behaviors.*

Example 3.1. *Consider the system described in (1.a). To explore its behavior, we choose specific values for the parameters and initial conditions as follows:*

- $\alpha = 0.6, \beta = -0.45, \gamma = 1.1, \delta = 0.8, \lambda = -0.9, \varepsilon = 0.75, \sigma = 0.7, \tau = -0.5.$
- *Initial conditions:* $Q_1 = -0.6$, $R_1 = 0.3$, $S_1 = 0.5$, *and* $Q_0 = 0.7$.

With these values, we iteratively compute the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ to observe how each sequence evolves under the system's equations. To illustrate the system's behavior over time, we generate plots of $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ across multiple iterations. This graphical analysis is shown in Figure 1.

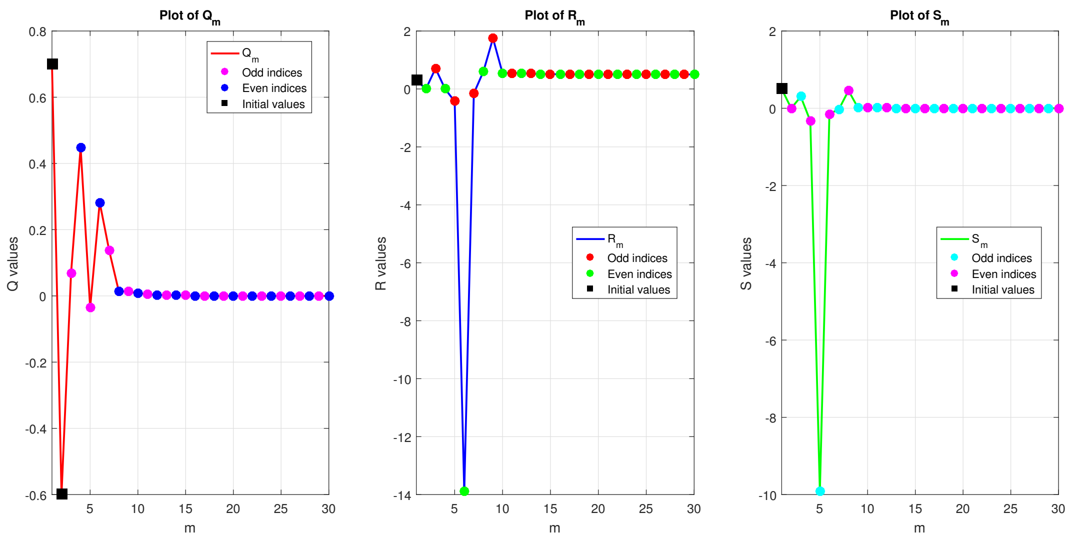

Figure 1. Characterization of oscillatory convergence in Q_m, R_m, and S_m across 30 iterations, using parameters $\alpha = 0.6$, $\beta = -0.45$, $\gamma = 1.1$, $\delta = 0.8$, $\lambda = -0.9$, $\varepsilon = 0.75$, $\sigma = 0.7$, and $\tau = -0.5$, and initial conditions $Q_1 = -0.6$, $R_1 = 0.3$, $S_1 = 0.5$, and $Q_0 = 0.7$.

Figure 1 shows the run evolution of the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ for the parameter set $\alpha = 0.6$, $\beta = -0.45$, $\gamma = 1.1$, $\delta = 0.8$, $\lambda = -0.9$, $\varepsilon = 0.75$, $\sigma = 0.7$, and $\tau = -0.5$, and initial conditions $Q_1 = -0.6$, $R_1 = 0.3$, $S_1 = 0.5$, and $Q_0 = 0.7$. Numerical values in the tail (see Table 1 below for $m = 480, \ldots, 500$) indicate that Q_m and S_m decay exponentially to zero (typical per-step multiplicative factor ≈ 0.667), whereas R_m converges to 0.5. These observations are consistent with local asymptotic stability of the computed equilibrium: The dominant linear factors governing the late-time dynamics have modulus strictly less than one, causing rapid attenuation of Q and S and stabilization of R. To complement the graphical analysis, we report below a numerical table of the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ over the interval $m = 480$ to $m = 500$, in order to highlight their asymptotic behavior at large iterations.

Table 1. Numerical values of Q_m, R_m, and S_m for iterations $m = 480$–500.

m	480	481	482	483	484	485	486	487	488	489	490
Q_m	$1.5e-85$	$1.0e-85$	$6.7e-86$	$4.5e-86$	$3.0e-86$	$2.0e-86$	$1.3e-86$	$8.9e-87$	$5.9e-87$	$3.9e-87$	$2.6e-87$
R_m	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
S_m	$2.3e-85$	$1.5e-85$	$1.0e-85$	$6.7e-86$	$4.5e-86$	$3.0e-86$	$2.0e-86$	$1.3e-86$	$8.9e-87$	$5.9e-87$	$3.9e-87$
m	491	492	493	494	495	496	497	498	499	500	–
Q_m	$1.7e-87$	$1.2e-87$	$7.8e-88$	$5.2e-88$	$3.5e-88$	$2.3e-88$	$1.5e-88$	$1.0e-88$	$6.8e-89$	$4.6e-89$	–
R_m	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	–
S_m	$2.6e-87$	$1.7e-87$	$1.2e-87$	$7.8e-88$	$5.2e-88$	$3.5e-88$	$2.3e-88$	$1.5e-88$	$1.0e-88$	$6.8e-89$	–

Example 3.2. *Consider the system described in (1.a). To explore its behavior, we choose specific values for the parameters and initial conditions as follows:*

- $\alpha = 1.5, \beta = -1.0, \gamma = 1.8, \delta = 1.2, \lambda = -1.5, \varepsilon = 1.0, \sigma = 1.0,$ *and* $\tau = -0.8$.
- *Initial conditions:* $Q_1 = -0.8$, $R_1 = 0.5$, $S_1 = 0.4$, *and* $Q_0 = 1.0$.

With these values, we iteratively compute the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ to observe how each sequence evolves under the system's equations. To illustrate the system's behavior over time, we generate plots of $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ across multiple iterations. This graphical analysis is shown in Figure 2.

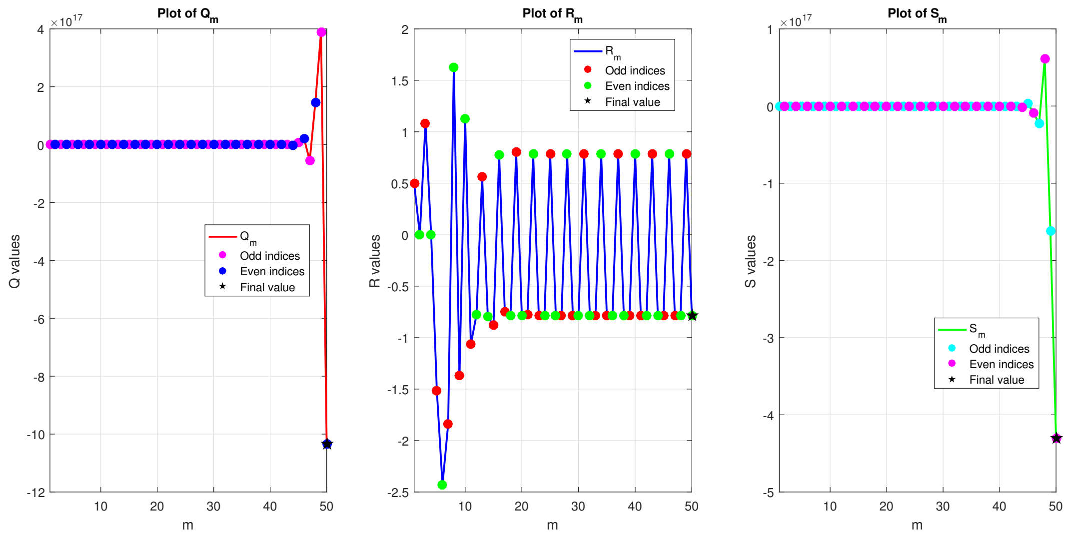

Figure 2. Exploration of divergent dynamic patterns in Q_m, R_m, and S_m across 50 iterations, using parameters $\alpha = 1.5$, $\beta = -1.0$, $\gamma = 1.8$, $\delta = 1.2$, $\lambda = -1.5, \varepsilon = 1.0$, $\sigma = 1.0$, and $\tau = -0.8$, and initial conditions $Q_1 = -0.8$, $R_1 = 0.5$, $S_1 = 0.4$, and $Q_0 = 1.0$.

Figure 2, obtained under different initial conditions and parameter values, $\alpha = 1.5$, $\beta = -1.0$, $\gamma = 1.8$, $\delta = 1.2$, $\lambda = -1.5$, $\varepsilon = 1.0$, $\sigma = 1.0$, $\tau = -0.8$, reveals a marked qualitative transformation in the system's behavior. Specifically, the oscillation amplitudes of Q_m and R_m expand significantly, while the fluctuations of S_m deepen toward negative values over 50 iterations. From an analytical perspective, this transition suggests that the system has crossed a stability boundary in parameter space. Possible bifurcation scenarios include a flip (period-doubling) bifurcation when an eigenvalue crosses -1, or a Neimark–Sacker bifurcation when a complex-conjugate pair approaches the unit circle. Both cases represent gateways to more intricate behaviors, such as quasi-periodicity or even chaotic regimes. The increasing amplitude of R_m, accompanied by its alternation between positive and negative values, points to a reinforcing feedback mechanism that amplifies oscillations rather than damping them. In contrast, the pronounced negative drift of S_m suggests either a departure from the equilibrium of the studied field or the emergence of a strongly negative attractor. To enable a deeper assessment of the system's dynamics and to verify or challenge the descriptive interpretations previously presented, we extended the numerical analysis along three complementary directions: (a) performing a wide parametric sweep of the key parameter α, (b) constructing a bifurcation diagram with respect to the same parameter to identify splitting points and the onset of higher-order or quasi-periodic cycles, and (c) testing sensitivity to initial conditions by comparing trajectories that start from closely related initial values. These three analyses are synthesized in Figure 3, which provides a comprehensive view of how the system behaves.

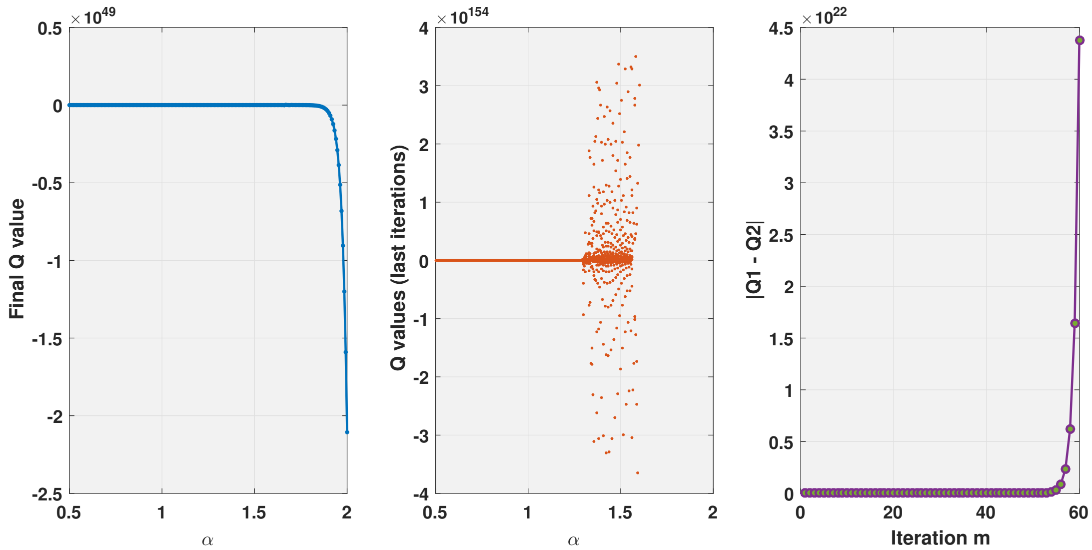

Figure 3. Integrated numerical analysis. Left: parameter sweep; middle: bifurcation diagram; right: sensitivity to initial conditions.

Figure 3 presents a comprehensive numerical overview of the system's behavior along three axes: parameter sweep (left), bifurcation diagram (middle), and sensitivity analysis (right). The parametric scan of α shows that the final values of Q_m remain stable within certain ranges but undergo abrupt deviations when crossing stability boundaries, signaling sharp dynamical transitions. The bifurcation diagram illustrates evolution from simple equilibrium states to multi-period cycles, eventually progressing toward complex dynamics, including period-doubling cascades and quasi-periodic or chaotic regimes. Sensitivity analysis reveals that even small perturbations in the initial conditions lead to increasing divergence in Q_m, consistent with the presence of a positive Lyapunov exponent in certain parameter regions. Overall, Figure 3 demonstrates that the model is not confined to static or periodic behaviors but also exhibits rich transitional dynamics, reflecting a delicate balance between stability and instability. The numerical values reported in Table 2 below provide further evidence of the long-term behavior described in Figure 2. In particular, Q_m and R_m oscillations amplify significantly, while S_m shows small but persistent negative drift, consistent with a bifurcation scenario.

Table 2. Numerical values of Q_m, R_m, and S_m between iterations $m = 480$ and $m = 500$, illustrating the onset of amplified oscillations and stability loss.

m	480	481	482	483	484	485	486	487	488	489	490
Q_m	0.0337	0.0152	0.0188	0.0274	0.0407	0.0606	0.0903	0.1335	0.1886	0.2281	0.1816
R_m	0.9478	0.7719	0.7137	0.7141	0.7989	0.9330	1.1136	1.3339	1.5468	1.6321	1.4512
S_m	−0.0033	0.0146	0.0197	0.0290	0.0440	0.0662	0.0972	0.1337	0.1578	0.1320	0.0465
m	491	492	493	494	495	496	497	498	499	500	–
Q_m	0.0601	0.0169	0.0168	0.0240	0.0356	0.0529	0.0788	0.1171	0.1692	0.2202	–
R_m	1.0685	0.8001	0.7447	0.7019	0.7635	0.8822	1.0467	1.2559	1.4808	1.6275	–
S_m	−0.0070	0.0119	0.0177	0.0252	0.0382	0.0577	0.0856	0.1213	0.1531	0.1487	–

Example 3.3. *Consider the system described in (1.a). To explore its behavior, we choose specific values for the parameters and initial conditions as follows:*

- $\alpha = \beta = \gamma = -\delta = \varepsilon = \sigma = \tau = 1$, *and* $\lambda = 3$.
- *Initial conditions:* $Q_1 = 0.6, R_1 = 0.5, S_1 = 0.3,$ *and* $Q_0 = 0.8$.

With these values, we iteratively compute the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ to observe how each sequence evolves under the system's equations. To illustrate the system's behavior over time, we generate plots of $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ across multiple iterations. This graphical analysis is shown in Figure 4.

Figure 4, obtained for the parameter choice $\alpha = \beta = \gamma = -\delta = \varepsilon = \sigma = \tau = 1$, $\lambda = 3$ with initial conditions $Q_1 = 0.6$, $R_1 = 0.5$, $S_1 = 0.3$, and $Q_0 = 0.8$, illustrates the evolution of the sequences $\{Q_m\}$, $\{R_m\}$, and $\{S_m\}$ over 30 iterations. What is striking in this figure is the remarkable stability of all three sequences, in contrast with the more irregular patterns observed in the previous two cases. Left panel $\{Q_m\}$: The trajectory exhibits small initial oscillations that quickly diminish, after which the sequence remains confined within a narrow, stable range. Middle panel $\{R_m\}$: The variable R_m rapidly approaches a nearly constant value after only a few steps. This rapid stabilization reflects strong damping in its recurrence relation, which suppresses transient deviations and causes the sequence to cluster around an internal attractor. Right panel $\{S_m\}$: The sequence shows an even stronger form of stability. Its values remain confined within a narrow band close to zero, with only negligible fluctuations. This behavior reveals that the nonlinear terms in the defining recurrence are finely balanced, preventing any uncontrolled growth and locking the dynamics into a stable equilibrium regime. Figure 4 confirms complete stability: All three variables converge toward constant states, with transient oscillations vanishing in finite time. The chosen parameters thus define a stable dynamical regime that suppresses bifurcations and higher-order cycles. Finally, numerical experiments for longer horizons (e.g., $m = 500$) confirm that the system remains stable without divergence or oscillatory amplification, establishing full asymptotic stability over the long term.

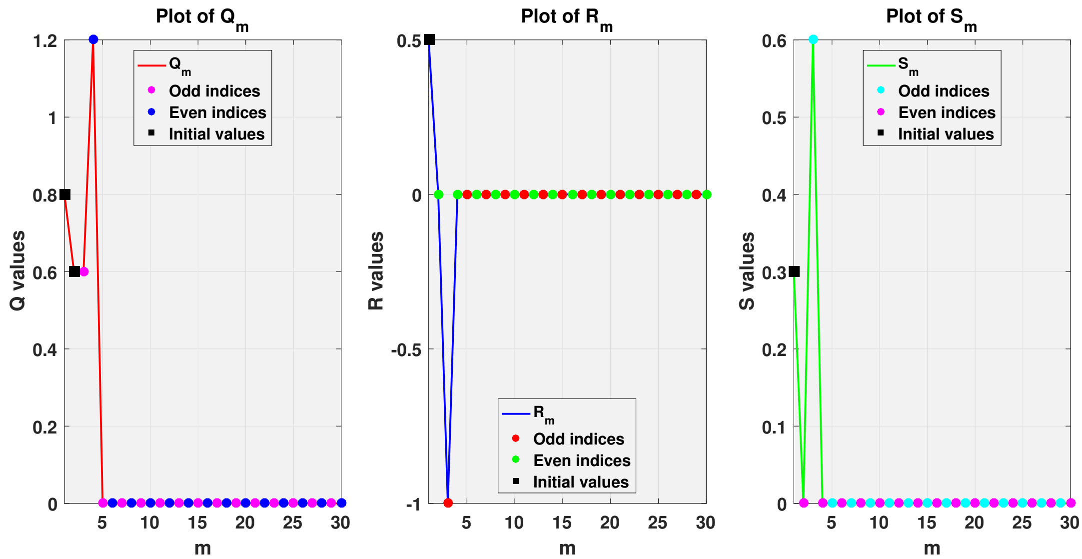

Figure 4. Exploration of divergent dynamic patterns in Q_m, R_m, and S_m across 30 iterations, using parameters $\alpha = \beta = \gamma = -\delta = \varepsilon = \sigma = \tau = 1$, $\lambda = 3$ and initial conditions $Q_1 = 0.6$, $R_1 = 0.5$, $S_1 = 0.3$, and $Q_0 = 0.8$.

Remark 3.2. *It is worth emphasizing that the presented examples correspond to two fundamentally different scenarios regarding the nature of the characteristic roots. Examples 3.1 and 3.2 illustrate the case of distinct roots, where the sequences exhibit qualitatively varying dynamics over time, including oscillatory and transitional behaviors. In contrast, Example 3.3 was constructed with coefficients that deliberately satisfy the condition of equal roots, which explains the highly stable and regular patterns observed in the corresponding graphs, even over a large number of iterations. The examples thus provide a clear and comprehensive illustration of the system's dynamics, highlighting how the distinction between equal and distinct roots governs the long-term behavior of the solutions.*

4. Conclusions

This paper presents a comprehensive framework for analyzing a nonlinear three-dimensional system of difference equations. The main contribution of this study lies in establishing explicit closed-form solutions for the system under clearly defined parameter conditions, distinguishing between cases of repeated and distinct characteristic roots. These results not only confirm the system's solvability but also reveal how parameter constraints fundamentally shape the stability and long-term behavior of the sequences. In particular, we show that the transformation of the original system into a bilinear form provides a powerful tool for simplifying its nonlinear structure and uncovering oscillatory and bifurcating dynamics. Collectively, these findings demonstrate not only the theoretical solvability of the three-dimensional system but also provide a detailed map of its dynamic regimes, offering insights into oscillatory, stable, and bifurcating behaviors under varying parameter settings. Numerical examples confirm the theoretical findings, highlighting the system's sensitivity to initial conditions and parameter variations. For instance, small changes in growth rates or interaction coefficients can transform the dynamics from stable oscillations to divergence, underlining the potential for bifurcation phenomena.

Future directions for research include extending the analysis to higher-dimensional systems of difference equations; incorporating stochastic perturbations, time delays, or external forcing terms to capture more realistic scenarios; integrating the proposed framework with data-driven approaches, such as machine learning, for parameter estimation and empirical validation; and developing control strategies to stabilize oscillatory or chaotic regimes. In addition, exploring interdisciplinary applications, such as neural networks, population dynamics, and economic models, would broaden the impact of this framework and demonstrate its relevance across scientific domains. Overall, this paper not only offers rigorous mathematical contributions but also establishes a solid foundation for further exploration of the rich dynamics exhibited by nonlinear difference systems.

Author contributions

Ahmed Ghezal: Software, validation, resources, data curation, writing – original draft preparation, supervision, project administration; Najmeddine Attia: Methodology, validation, formal analysis, investigation, visualization, writing – original draft preparation, supervision, project administration, writing review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU253650].

Conflict of interest

The authors declare no conflicts of interest in this paper.

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