Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 23151–23168.
DOI: 10.3934/math.20251027
Received: 03 May 2025
Revised: 13 September 2025
Accepted: 18 September 2025
Published: 13 October 2025

Research article

Soliton structures of a Dullin-Gottwald-Holm model for the shallow water

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Abstract: This work analyzes the recently introduced nonlinear dispersive shallow water wave equation, the Dullin-Gottwald-Holm (DGH) model. By taking into account a wider range of wave events that frequently occur in real shallow water systems, such as dispersive shocks, bores, and solitary waves, the advanced nonlinear dispersive shallow water model has been devised to predominate the classical models. This is accomplished by combining linear and nonlinear dispersion. The several traveling wave solutions of the governing model are systematically arranged using several well-known analytical methods. The main focus is to investigate how several important parameters affect the dynamics and shape of the resulting wave formations. As a result, many wave solutions, including solitary, shock, singular, shock-singular, hyperbolic, singular periodic, and rational waves, can be constructed by symbolic computations. The parameter regimes guarantee that a careful analysis determines the generated solutions' validity and physical significance. The interpretation of the results is improved by displaying the solutions across a variety of parameter values using contour graphs, 2D plots, and 3D surface representations. These results confirm the efficiency and dependability of the used computational paradigm, which has a great deal of promise for further studies looking at more intricate nonlinear processes.

Keywords: the Dullin-Gottwald-Holm equation; nonlinear dynamics; analytical techniques; solitons; water waves
Mathematics Subject Classification: 35Rxx, 35Qxx, 35Q51

1. Introduction

The wide range of applications of nonlinear partial differential equations (NPDEs) in science and engineering has garnered a lot of attention lately. In fields like fluid mechanics, plasma physics, optics, and ocean engineering, nonlinear evolution equations (NLEEs) are especially crucial for simulating a variety of physical phenomena. The search for analytical solutions to NLEEs is still ongoing, aided by sophisticated computational tools that make difficult and time-consuming computations more easier and reliable [1–3].

Many powerful and intriguing approaches to finding analytic solutions to nonlinear models have been presented in recent years, attracting the attention of famous scientists and engineers, such as the variational iteration method [4], the harmonic balance method [5], the the Riemann-theta function [6], the inverse scattering transform method [7], the Hirota bilinear method [8], the variable separation approach [9], the homogenous balance method [10], $(\frac{G'}{G})$ -expansion method [11], the sub-ODE method [12], the extended Kudryashov method [13], the simple equation method [14], the modified simple equation method [15], the extended simple equation method [16], the new extended $(\frac{G'}{G})$-expansion method [17], the Frobenius integrable decomposition [18], the exp-function method [19], the solitary wave ansatzes [20], the simplest equation method [21], the tanh-coth method [22], the $\exp(-\varphi(\xi))$-expansion method [23], the extended tanh-function method [24], the improve F-expansion method [25], the modified auxiliary equation mapping method [26], and the auxiliary equation method [27].

In order to develop the generalized Dullin-Gottwald-Holm equation, an integrable shallow water wave equation, Dullin, Gottwald, and Holm combined the linear dispersion from the Korteweg-de Vries equation with the nonlinear dispersion from the Camassa-Holm equation [28]. The generalized Dullin-Gottwald-Holm equation is an important NPDE that sheds light on power law nonlinearity and is relevant to integrability and conservation laws. Several proven analytical techniques are used in this study to analyze the DGH model [29], which is represented as follows:

$$\mathcal{U}_t + \alpha_1 \mathcal{U}_x + \alpha_2^2 (-(2\mathcal{U}_x \mathcal{U}_{xx} + \mathcal{U}_{xxt} + \mathcal{U}\mathcal{U}_{xxx})) + \alpha_3 \mathcal{U}_{xxx} + 3\mathcal{U}\mathcal{U}_x = 0, \quad t \geq 0, \tag{1.1}$$

where $\mathcal{U}(x,t)$ represents the fluid velocity of the above system in the spatial direction x. $\alpha_2^2(\alpha_2 > 0)$ while the ratio between α_1 and α_3 represents the squares of the length scales, whereas $\alpha_1 = \sqrt{gh}$ (while $\alpha_1 = 2\omega$) represents a linear speed for unperturbed water during equilibrium at spatial infinity.

In fluid mechanics, oceanography, and environmental engineering, understanding the dynamics of shallow water waves is essential considering its effects on coastal processes, tsunami modeling, river hydraulics, and estuary dynamics. Traditional models, such as the KdV and Camassa-Holm equations, can capture some aspects of shallow water behavior, but they often fail to account for both dispersive and nonlinear effects at higher levels of precision. The DGH equation, a sophisticated nonlinear dispersive shallow water wave model, was recently created to bridge the gap between existing integrable models. By combining both linear and nonlinear dispersion, the DGH model is capable of supporting a greater variety of wave events, including solitary waves, bores, and dispersive shocks, which are commonly observed in real shallow water systems. The DGH model has been the subject of numerous investigations in [30, 31] that investigated three types of the shallow water wave equations. The traveling wave solutions for a class of the dispersive systems with a parameter have been the main topic of [32, 33] that established the conservation laws for the DGH equation and its generalized form. In the meantime, [34] has discovered some periodic wave solutions for the DGH system, and [35] has put forth a simple sufficient condition on the initial data to guarantee that the solutions to the modified two-component DGH system that is weakly dissipative experience blow-up. The linked solutions have been seen to detonate within a specific time period, according to a new adequate condition on the original data that has been examined [36].

The wave structures obtained here are qualitatively consistent with shallow water phenomena found in controlled experimental settings and in real water bodies, such as estuaries, coastal shelves, and channels, despite the fact that this work concentrates on analytical and symbolic calculations. The DGH model appears to offer a mathematically sound framework for representing a wide variety of physically observed wave characteristics, such as both smooth and discontinuous waveforms, based on these comparisons. To the best of our knowledge, no other work has systematically derived and analyzed solitary, shock, singular, shock-singular, hyperbolic, singular periodic, and rational solutions for the DGH problem using the combined analytical and computational technique proposed in this research. Furthermore, unique contributions not previously recorded in the literature include the thorough parametric analysis and graphical representations (3D, 2D, and contour plots) to show how parameters affect wave profiles.

The structure of the study is as follows: A brief overview of the strategies employed is given in Section 2. The soliton solutions for the governing model are produced in Section 3. The stability analysis is covered in Section 4. Section 5 discusses the modulation instability. Section 6 explores the graphical depiction of various solutions in 3D, 2D, and contour graphs and includs a discussion of them. The conclusions of the paper are presented in section 7.

2. Methodology

Consider the NPDE in the form

$$\mathcal{F}\left(\mathcal{P}_t,\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_{tt},\mathcal{P}_{xx},\mathcal{P}_{xt},\mathcal{P}_{yy},\mathcal{P}_{xy},\cdots\right)=0, \tag{2.1}$$

where x, t, and y are the function of $\mathcal{P}$, and $\mathcal{F}$ represents the polynomial. Through the process of transforming the shape

$$\mathcal{P}(x,t)=\mathcal{U}(\eta),\ \ \eta=\alpha x-\rho t, \tag{2.2}$$

where the constants ρ and α are indicated. An ODE is obtained by using Eq (2.2) in Eq (2.1)

$$O(\mathcal{U},\mathcal{U}',\mathcal{U}'',\mathcal{U}''',\cdots)=0. \tag{2.3}$$

2.1. *The* $(\frac{1}{\mathcal{G}'})$ *expansion method*

The process's essential steps are listed below [37]:
Step 1: Consider the solution of (2.3) is of the form

$$\mathcal{U}(\eta)=\sum_{i=0}^{M}\alpha_i\left(\frac{1}{\mathcal{G}'(\eta)}\right)^i, \tag{2.4}$$

where $\mathcal{G}=\mathcal{G}(\eta)$ validates

$$\mathcal{G}''(\eta)+\varpi\mathcal{G}'(\eta)+\vartheta\ \ =\ \ 0, \tag{2.5}$$

while α_i, ϖ and ϑ are constants and M represents the homogenous number.
Step 2: The general solution of ODE (2.5)

$$\mathcal{G}(\eta) = -\frac{\vartheta\eta}{\varpi} + c_1 e^{-\varpi\eta} + c_2, \tag{2.6}$$

then

$$\frac{1}{\mathcal{G}'(\eta)} = \frac{\varpi}{-\vartheta + \varpi c_1(\cosh(\varpi\eta) - \sinh(\varpi\eta))}, \tag{2.7}$$

where c_1 and c_2 are arbitrary constants and will be computed later.
Step 3: Equating the coefficients yields a system of algebraic equations of $\left(\frac{1}{\mathcal{G}'}\right)$ to zero.
Step 4: In the final stage, evaluate the full set of equations by adding the quantities collected from the responses.

2.2. The $(\frac{\mathcal{G}'}{\mathcal{G}}, \frac{1}{\mathcal{G}})$ expansion method

The main steps in the process are listed below [38]:
Step 1: Consider the solution of (2.3) is of the form

$$\mathcal{U}(\eta) = \sum_{i=0}^{M} \alpha_i \varphi^i + \sum_{i=1}^{M} \beta_i \varphi^{i-1} \varrho, \tag{2.8}$$

where constants $\alpha_i, \beta_i, \varpi$, and ϑ will be found later. Examine the auxiliary equation of second order

$$\mathcal{G}''(\eta) + \varpi\mathcal{G}(\eta) = \vartheta, \tag{2.9}$$

where

$$\varphi' = \frac{\mathcal{G}'(\eta)}{\mathcal{G}(\eta)}, \quad \varrho' = \frac{1}{\mathcal{G}(\eta)}. \tag{2.10}$$

From (2.9) and (2.10), we have

$$\varphi' = -\varphi^2 + \vartheta\varrho - \varpi, \quad \varrho' = \varphi\varrho. \tag{2.11}$$

The following three separate subcases represent the general solution of (2.3): Case I: If $\varpi < 0$, accordingly, we acquire the solution as

$$\mathcal{G}(\eta) = \mathcal{A}\sinh(\sqrt{-\varpi}\eta) + \mathcal{B}\cosh(\sqrt{-\varpi}\eta) + \frac{\vartheta}{\varpi}, \tag{2.12}$$

and thus

$$\varrho^2 = -\frac{\varpi}{\varpi^2\sigma + \vartheta^2}\left(\varphi^2 - 2\vartheta\varrho + \varpi\right), \tag{2.13}$$

where $\sigma = \mathcal{A}^2 - \mathcal{B}^2$ and $\mathcal{A}$ and $\mathcal{B}$ are arbitrary constants.
Case II: If $\varpi > 0$, accordingly, we acquire the solution as

$$\mathcal{G}(\eta) = \mathcal{A}\cos(\sqrt{\varpi}\eta) + \mathcal{B}\sin(\sqrt{\varpi}\eta) + \frac{\vartheta}{\varpi}, \tag{2.14}$$

and thus

$$\varrho^2 = \frac{\varpi}{\varpi^2\sigma + \vartheta^2}\left(\varphi^2 - 2\vartheta\varrho + \varpi\right), \tag{2.15}$$

where $\sigma = \mathcal{A}^2 + \mathcal{B}^2$.
Case III: Consequently, if $\varpi = 0$, we obtain the solution as

$$\mathcal{G}(\eta) = \frac{1}{2}\vartheta\eta^2 + \mathcal{A}\eta + \mathcal{B}, \tag{2.16}$$

and thus

$$\varrho^2 = \frac{1}{\mathcal{A}^2 - 2\vartheta\mathcal{B}}\left(\varphi^2 - 2\vartheta\varrho\right). \tag{2.17}$$

Step 2: By substituting (2.8) into (2.3) using (2.11) and (2.13), a system of algebraic equations is developed by equating the coefficients of φ and ϱ to zero.
Step 3: In the last step, evaluate the complete set of equations by integrating the quantities obtained from the results.

3. Development of solitons

3.1. Applications of the $(\frac{1}{\mathcal{G}'})$-expansion method

Applying the transformation

$$\varphi(x,t) = \delta(\eta), \qquad \eta = x - \nu t, \tag{3.1}$$

the (1.1) will becomes

$$\alpha_2^2\nu\delta''' + \alpha_3\delta''' - \alpha_2^2\delta\delta''' + \alpha_1\delta' - \nu\delta' + 3\delta\delta' - 2\alpha_2^2\delta'\delta'' = 0, \tag{3.2}$$

and integration leads (3.2) to

$$\delta''\left(-\alpha_2^2\nu - \alpha_3 + \alpha_2^2\delta\right) - \frac{3\delta^2}{2} + \frac{1}{2}\alpha_2^2(\delta')^2 - \delta(\alpha_1 - \nu) = 0. \tag{3.3}$$

$m = 2$ in (3.3) is obtained by balancing the highest derivative with the highest degree of nonlinear term, suggesting that the solution takes the following form:

$$\delta(\eta) = a_0 + a_1\left(\frac{1}{\mathcal{G}'(\eta)}\right) + a_2\left(\frac{1}{\mathcal{G}'(\eta)}\right)^2, \tag{3.4}$$

where the values of the variables a_0, a_1, and a_2 must be determined. The system that results from collecting $G'(\eta)$ and substituting (3.4) with its first and second derivatives into (3.3) is as follows.

$$\begin{aligned}
\text{Constant} &: \quad -\alpha_1 a_0 + a_0\nu - \frac{1}{2}3a_0^2 = 0,\\
\frac{1}{\mathcal{G}'(\eta)} &: \quad a_1\alpha_2^2\varpi^2(-\nu) + a_0a_1\alpha_2^2\varpi^2 - a_1\alpha_3\varpi^2 - a_1\alpha_1 + a_1\nu - 3a_0a_1 = 0,
\end{aligned}$$

$$\frac{1}{\mathcal{G}'(\eta)^2} \ : \ -4a_2\alpha_2^2\varpi^2\nu + \frac{3}{2}a_1^2\alpha_2^2\varpi^2 + 4a_0a_2\alpha_2^2\varpi^2 - 4a_2\alpha_3\varpi^2 - 3a_1\alpha_2^2\varpi\vartheta\nu$$
$$+3a_0a_1\alpha_2^2\varpi\vartheta - 3a_1\alpha_3\varpi\vartheta - a_2\alpha_1 + a_2\nu - 3a_0a_2 - \frac{3a_1^2}{2} = 0,$$
$$\frac{1}{\mathcal{G}'(\eta)^3} \ : \ 7a_1a_2\alpha_2^2\varpi^2 - 10a_2\alpha_2^2\varpi\vartheta\nu + 4a_1^2\alpha_2^2\varpi\vartheta + 10a_0a_2\alpha_2^2\varpi\vartheta - 10a_2\alpha_3\varpi\vartheta$$
$$-2a_1\alpha_2^2\vartheta^2\nu + 2a_0a_1\alpha_2^2\vartheta^2 - 2a_1\alpha_3\vartheta^2 - 3a_1a_2 = 0,$$
$$\frac{1}{G'(\eta)^4} \ : \ 6a_2^2\alpha_2^2\varpi^2 + 17a_1a_2\alpha_2^2\varpi\vartheta - 6a_2\alpha_2^2\vartheta^2\nu + \frac{5}{2}a_1^2\alpha_2^2\vartheta^2$$
$$+6a_0a_2\alpha_2^2\vartheta^2 - 6a_2\alpha_3\vartheta^2 - \frac{3a_2^2}{2} = 0,$$
$$\frac{1}{\mathcal{G}'(\eta)^4} \ : \ 14a_2^2\alpha_2^2\varpi\vartheta + 10a_1a_2\alpha_2^2\vartheta^2 = 0.$$

The following cases arise by solving the above system of algebraic equations with the aid of Mathematica:

Case I. $a_0 = \frac{2}{3}(\nu - \alpha_1)$, $a_1 = 0$, $a_2 = \frac{6\alpha_2^2\vartheta^2\nu - 4\alpha_2^2\vartheta^2(\nu-\alpha_1) - \sqrt{\left(-6\alpha_2^2\vartheta^2\nu + 4\alpha_2^2\vartheta^2(\nu-\alpha_1) - 6\alpha_3\vartheta^2\right)^2 + 6\alpha_3\vartheta^2}}{3\left(4\alpha_2^2\varpi^2 - 1\right)}$.

By using values in Eq (3.4), we get

$$\delta(\eta) = \frac{1}{3}\left[\frac{2\varpi^2\left(-\sqrt{\vartheta^4\left(\alpha_2^2(2\alpha_1+\nu)+3\alpha_3\right)^2} + \alpha_2^2\vartheta^2(2\alpha_1+\nu) + 3\alpha_3\vartheta^2\right)}{\left(4\alpha_2^2\varpi^2-1\right)\left(A\varpi\sinh(\eta\varpi) - A\varpi\cosh(\eta\varpi) + \vartheta\right)^2} + 2(\nu-\alpha_1)\right], \tag{3.5}$$

and the reverse transmutation is used to obtain the solution

$$\mathcal{U}_1^I(x,t) = \frac{1}{3}\left[\frac{2\varpi^2\left(-\sqrt{\vartheta^4\left(\alpha_2^2(2\alpha_1+\nu)+3\alpha_3\right)^2} + \alpha_2^2\vartheta^2(2\alpha_1+\nu) + 3\alpha_3\vartheta^2\right)}{\left(4\alpha_2^2\varpi^2-1\right)\left(A\varpi\sinh(\varpi(x-\nu t)) - A\varpi\cosh(\varpi(x-\nu t)) + \vartheta\right)^2} + 2(\nu-\alpha_1)\right]. \tag{3.6}$$

Case II. $a_0 = \frac{2}{3}(\nu - \alpha_1)$, $a_1 = 0$, $a_2 = \frac{6\alpha_2^2\vartheta^2\nu - 4\alpha_2^2\vartheta^2(\nu-\alpha_1) + \sqrt{\left(-6\alpha_2^2\vartheta^2\nu + 4\alpha_2^2\vartheta^2(\nu-\alpha_1) - 6\alpha_3\vartheta^2\right)^2 + 6\alpha_3\vartheta^2}}{3\left(4\alpha_2^2\varpi^2 - 1\right)}$,

By using values in Eq (3.4), we get

$$\delta(\eta) = \frac{1}{3}\left[\frac{2\varpi^2\left(\sqrt{\vartheta^4\left(\alpha_2^2(2\alpha_1+\nu)+3\alpha_3\right)^2} + \alpha_2^2\vartheta^2(2\alpha_1+\nu) + 3\alpha_3\vartheta^2\right)}{\left(4\alpha_2^2\varpi^2-1\right)\left(A\varpi\sinh(\eta\varpi) - A\varpi\cosh(\eta\varpi) + \vartheta\right)^2} + 2(\nu-\alpha_1)\right], \tag{3.7}$$

and the reverse transmutation is used to obtain the solution

$$\mathcal{U}_2^I(x,t) = \frac{1}{3}\left[\frac{2\varpi^2\left(\sqrt{\vartheta^4\left(\alpha_2^2(2\alpha_1+\nu)+3\alpha_3\right)^2} + \alpha_2^2\vartheta^2(2\alpha_1+\nu) + 3\alpha_3\vartheta^2\right)}{\left(4\alpha_2^2\varpi^2-1\right)\left(A\varpi\sinh(\varpi(x-\nu t)) - A\varpi\cosh(\varpi(x-\nu t)) + \vartheta\right)^2} + 2(\nu-\alpha_1)\right]. \tag{3.8}$$

3.2. Applications of the $(\frac{\mathcal{G}'}{\mathcal{G}}, \frac{1}{\mathcal{G}})$-expansion method

Consider the solution of (3.3), is of the form:

$$\delta(\eta) = a_0 + a_1\varphi(\eta) + b_1\varrho(\eta) + a_2\varphi(\eta)^2 + b_2\varrho(\eta)^2, \tag{3.9}$$

where it is necessary to calculate the constants a_0, a_1 and a_2. The following system is obtained by substituting (3.9) and its first and second derivatives into (3.3) and gathering $\varphi(\eta), \varrho(\eta)$

$$\begin{aligned}
\text{constant} &: -2a_2\alpha_2^2\varpi^2\nu + \frac{1}{2}a_1^2\alpha_2^2\varpi^2 + 2a_0a_2\alpha_2^2\varpi^2 - 2a_2\alpha_3\varpi^2 \\
&\quad -a_0\alpha_1 + a_0\nu - \frac{3a_0^2}{2} = 0, \\
\varphi(\eta) &: 4a_1a_2\alpha_2^2\varpi^2 - 2a_1\alpha_2^2\varpi\nu + 2a_0a_1\alpha_2^2\varpi - 2a_1\alpha_3\varpi \\
&\quad -a_1\alpha_1 + a_1\nu - 3a_0a_1 = 0, \\
\varrho(\eta)\varphi(\eta) &: -8a_1a_2\alpha_2^2\varpi\vartheta + 3a_1\alpha_2^2\vartheta\nu - 3a_0a_1\alpha_2^2\vartheta + 3a_1\alpha_3\vartheta \\
&\quad +4a_1\alpha_2^2b_1\varpi - 3a_1b_1 = 0, \\
\varrho(\eta)^2\varphi(\eta) &: 4a_1a_2\alpha_2^2\vartheta^2 + 6a_1\alpha_2^2b_2\varpi - 5a_1\alpha_2^2b_1\vartheta - 3a_1b_2 = 0, \\
\varphi(\eta)^2 &: 4a_2^2\alpha_2^2\varpi^2 - 8a_2\alpha_2^2\varpi\nu + 3\alpha_2^2a_1^2\varpi + 8a_0a_2\alpha_2^2\varpi - 8a_2\alpha_3\varpi \\
&\quad -a_2\alpha_1 + a_2\nu - \frac{3a_1^2}{2} - 3a_0a_2 = 0, \\
\varrho(\eta)\varphi(\eta)^2 &: -8a_2^2\alpha_2^2\varpi\vartheta + 10a_2\alpha_2^2\vartheta\nu - 4a_1^2\alpha_2^2\vartheta - 10a_0a_2\alpha_2^2\vartheta + 10a_2\alpha_3\vartheta \\
&\quad +11a_2\alpha_2^2b_1\varpi + 2a_0\alpha_2^2b_1 - 3a_2b_1 - 2\alpha_2^2b_1\nu - 2\alpha_3b_1 = 0, \\
\varrho(\eta)^2\varphi(\eta)^2 &: 4a_2^2\alpha_2^2\vartheta^2 + 14a_2\alpha_2^2b_2\varpi - 13a_2\alpha_2^2b_1\vartheta + 6a_0\alpha_2^2b_2 \\
&\quad -3a_2b_2 - 6\alpha_2^2b_2\nu + \frac{5}{2}\alpha_2^2b_1^2 - 6\alpha_3b_2 = 0, \\
\varrho(\eta)^3\varphi(\eta)^2 &: 10\alpha_2^2b_1b_2 - 16a_2\alpha_2^2b_2\vartheta = 0, \\
\varphi(\eta)^3 &: 14a_1a_2\alpha_2^2\varpi - 2a_1\alpha_2^2\nu + 2a_0a_1\alpha_2^2 - 2a_1\alpha_3 - 3a_1a_2 = 0, \\
\varrho(\eta)\varphi(\eta)^3 &: 5a_1\alpha_2^2b_1 - 17a_1a_2\alpha_2^2\vartheta = 0, \\
\varphi(\eta)^4 &: 12\alpha_2^2a_2^2\varpi - 6\alpha_2^2a_2\nu + 6a_0\alpha_2^2a_2 - 6\alpha_3a_2 + \frac{5}{2}a_1^2\alpha_2^2 - \frac{3a_2^2}{2} = 0, \\
\varrho(\eta)\varphi(\eta)^4 &: 10a_2\alpha_2^2b_1 - 14a_2^2\alpha_2^2\vartheta = 0, \\
\varrho(\eta) &: 4a_2\alpha_2^2\varpi\vartheta\nu + a_1^2\alpha_2^2(-\varpi)\vartheta - 4a_0a_2\alpha_2^2\varpi\vartheta + 4a_2\alpha_3\varpi\vartheta + 2a_2\alpha_2^2b_1\varpi^2 \\
&\quad +a_0\alpha_2^2b_1\varpi - 3a_0b_1 - \alpha_2^2b_1\varpi\nu - \alpha_3b_1\varpi - \alpha_1b_1 + b_1\nu = 0, \\
\varrho(\eta)^2 &: -2a_2\alpha_2^2\vartheta^2\nu + \frac{1}{2}a_1^2\alpha_2^2\vartheta^2 + 2a_0a_2\alpha_2^2\vartheta^2 - 2a_2\alpha_3\vartheta^2 + 2a_2\alpha_2^2b_2\varpi^2 \\
&\quad -4a_2\alpha_2^2b_1\varpi\vartheta + 2a_0\alpha_2^2b_2\varpi - a_0\alpha_2^2b_1\vartheta - 3a_0b_2 - 2\alpha_2^2b_2\varpi\nu + \alpha_2^2b_1^2\varpi \\
&\quad -2\alpha_3b_2\varpi + \alpha_2^2b_1\vartheta\nu + \alpha_3b_1\vartheta - \alpha_1b_2 + b_2\nu - \frac{3b_1^2}{2} = 0, \\
\varrho(\eta)^3 &: -4a_2\alpha_2^2b_2\varpi\vartheta + 2a_2\alpha_2^2b_1\vartheta^2 - 2a_0\alpha_2^2b_2\vartheta + 3\alpha_2^2b_1b_2\varpi \\
&\quad +2\alpha_2^2b_2\vartheta\nu + \alpha_2^2b_1^2(-\vartheta) + 2\alpha_3b_2\vartheta - 3b_1b_2 = 0,
\end{aligned}$$

$$\varrho(\eta)^4 \quad : \quad 2a_2\alpha_2^2 b_2\vartheta^2 + 2\alpha_2^2 b_2^2\varpi - 3\alpha_2^2 b_1 b_2\vartheta - \frac{3b_2^2}{2} = 0.$$

For the solutions of this system we have the following situations.

Case I: $a_0 = \frac{7\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}-20\alpha_3}{20\alpha_2^2} - \alpha_1,\ a_1 = 0,\ b_1 = 0,\ b_2 = 0,$
$a_2 = 3\Big(-2\alpha_1\alpha_2^2 + \sqrt{10}\sqrt{\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_3\alpha_2^2 + \alpha_3^2} - 2\alpha_3\Big),$
$\nu = \frac{4\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}-25\alpha_3}{10\alpha_2^2} - \frac{3\alpha_1}{2},\ \varpi = \frac{1}{4\alpha_2^2},\ \vartheta = 0.$
Here, we have $\varpi > 0$ and for $\varpi > 0$, we have

$$\begin{aligned} \delta(\eta) &= -\alpha_1 + \frac{7\sqrt{10}\sqrt{\big(\alpha_1\alpha_2^2+\alpha_3\big)^2} - 20\alpha_3}{20\alpha_2^2} \\ &+15\Bigg[\Big(-2\alpha_1\alpha_2^2 + \sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2} - 2\alpha_3\Big) \\ &\Big(c_2\sinh\Big(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\Big) + c_1\cosh\Big(\frac{\eta}{2\alpha_2}\Big)\Big)^2 \\ &\Big(20\alpha_2^2\Big(c_1\sinh\Big(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\Big) + c_2\cosh\Big(\frac{\eta}{2\alpha_2}\Big)\Big)^2\Big)^{-1}\Bigg], \end{aligned} \tag{3.10}$$

and by taking the reverse transformation, we have the first solution of the form

$$\begin{aligned} \mathcal{U}_1^{II}(x,t) &= -\alpha_1 + \frac{7\sqrt{10}\sqrt{\big(\alpha_1\alpha_2^2+\alpha_3\big)^2} - 20\alpha_3}{20\alpha_2^2} \\ &+15\Bigg[\Big(-2\alpha_1\alpha_2^2 + \sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2} - 2\alpha_3\Big) \\ &\Big(c_2\sinh\Big(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\Big) + c_1\cosh\Big(\frac{(x-\nu t)}{2\alpha_2}\Big)\Big)^2 \\ &\Big(20\alpha_2^2\Big(c_1\sinh\Big(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\Big) + c_2\cosh\Big(\frac{(x-\nu t)}{2\alpha_2}\Big)\Big)^2\Big)^{-1}\Bigg]. \end{aligned} \tag{3.11}$$

Case II: $a_0 = \frac{-4\alpha_1\alpha_2^2+\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}-4\alpha_3}{4\alpha_2^2},\ a_1 = 0,\ b_1 = 0,\ b_2 = 0,$
$a_2 = 3\Big(-2\alpha_1\alpha_2^2 + \sqrt{10}\sqrt{\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_3\alpha_2^2 + \alpha_3^2} - 2\alpha_3\Big),$
$\nu = \frac{-3\alpha_1\alpha_2^2+\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}-5\alpha_3}{2\alpha_2^2},\ \vartheta = 0,\ \varpi = \frac{1}{4\alpha_2^2}.$
Again for $\varpi > 0$, we have

$$\delta(\eta) = \frac{-4\alpha_1\alpha_2^2 + \sqrt{10}\sqrt{\big(\alpha_1\alpha_2^2+\alpha_3\big)^2} - 4\alpha_3}{4\alpha_2^2}$$

$$+\Bigg[3\left(-2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}-2\alpha_3\right)$$
$$\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_1\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2$$
$$\left(4\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_2\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg], \tag{3.12}$$

and by taking the reverse transformation, we have the second solution of the form

$$\mathcal{U}_2^{II}(x,t) = \frac{-4\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}-4\alpha_3}{4\alpha_2^2}$$
$$+\Bigg[3\left(-2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}-2\alpha_3\right)$$
$$\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_1\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2$$
$$\left(4\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_2\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg]. \tag{3.13}$$

Case III: $a_0=-\frac{4\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+4\alpha_3}{4\alpha_2^2},\ a_1=0,\ b_1=0,\ b_2=0,$
$a_2=3\left(-2\alpha_1\alpha_2^2-\sqrt{10}\sqrt{\alpha_1^2\alpha_2^4+2\alpha_1\alpha_3\alpha_2^2+\alpha_3^2}-2\alpha_3\right),$
$\nu=-\frac{3\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+5\alpha_3}{2\alpha_2^2},\ \vartheta=0,\ \varpi=\frac{1}{4\alpha_2^2}.$
Again for $\varpi>0$, we have

$$\delta(\eta) = -\frac{4\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+4\alpha_3}{4\alpha_2^2}$$
$$-\Bigg[3\left(2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+2\alpha_3\right)$$
$$\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_1\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2$$
$$\left(4\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_2\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg], \tag{3.14}$$

and by taking the reverse transformation, we have the third solution of the form

$$\mathcal{U}_3^{II}(x,t) = -\frac{4\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+4\alpha_3}{4\alpha_2^2}$$

$$-\Bigg[3\left(2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+2\alpha_3\right)$$
$$\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_1\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2$$
$$\left(4\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_2\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg]. \tag{3.15}$$

Case IV: $a_0=-\frac{7\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}+20\alpha_3}{20\alpha_2^2}-\alpha_1,\ a_1=0,\ b_1=0,\ b_2=0,$
$a_2=3\left(-2\alpha_1\alpha_2^2-\sqrt{10}\sqrt{\alpha_1^2\alpha_2^4+2\alpha_1\alpha_3\alpha_2^2+\alpha_3^2}-2\alpha_3\right),$
$\nu=-\frac{15\alpha_1\alpha_2^2+4\sqrt{10}\sqrt{(\alpha_1\alpha_2^2+\alpha_3)^2}+25\alpha_3}{10\alpha_2^2},\ \varpi=\frac{1}{4\alpha_2^2},\ \vartheta=0.$
Again for $\varpi>0$, we have

$$\begin{aligned}\delta(\eta) &= -\alpha_1-\frac{7\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+20\alpha_3}{20\alpha_2^2}\\ &-\Bigg[15\left(2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+2\alpha_3\right)\\ &\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_1\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2\\ &\left(20\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}\eta\right)+c_2\cosh\left(\frac{\eta}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg],\end{aligned} \tag{3.16}$$

and by taking the reverse transformation, we have the fourth solution of the form

$$\begin{aligned}\mathcal{U}_4^{II}(x,t) &= -\alpha_1-\frac{7\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+20\alpha_3}{20\alpha_2^2}\\ &-\Bigg[15\left(2\alpha_1\alpha_2^2+\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2}+2\alpha_3\right)\\ &\left(c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_1\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2\\ &\left(20\alpha_2^2\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)+c_2\cosh\left(\frac{(x-\nu t)}{2\alpha_2}\right)\right)^2\right)^{-1}\Bigg].\end{aligned} \tag{3.17}$$

4. Stability analysis

The Hamilton structure's momentum in this section is found in Eq (1.1), and it can be expressed as

$$M=\frac{1}{2}\int_R \mathcal{U}^2\,d\eta, \tag{4.1}$$

where the electric field potential is represented by $\mathcal{U}$ and the momentum like conserved quantity by M. For soliton stability, the appropriate prerequisite is

$$\frac{\partial M}{\partial \nu} > 0, \tag{4.2}$$

where ν is the wavelength for wave speed $\eta = x - \nu t$. Integrating the traveling wave solution from Eq (3.17) into Eq (4.1) yields the following result

$$M = \frac{1}{2}\int_{-10}^{10}\left[\left\{3\left(-2\alpha_1\alpha_2^2 - \sqrt{10}\sqrt{\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_3\alpha_2^2 + \alpha_3^2} - 2\alpha_3\right)\right.\right.$$
$$\left.\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}c_2\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right) + \frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}c_1\cosh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)\right)^2\right\}$$
$$\left./\left(c_1\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\frac{1}{\alpha_2^2}}(x-\nu t)\right)\right)^2 - \frac{7\sqrt{10}\sqrt{\left(\alpha_1\alpha_2^2+\alpha_3\right)^2 + 20\alpha_3}}{20\alpha_2^2} - \alpha_1\right]^2 dx. \tag{4.3}$$

Applying the soliton stability condition found in Eq (4.2) establishes that the considered solution is stable. This means that as M increases, the wave number increases in a way that takes the system away from the intermediate state, implying stabilizing behavior.

5. Modulation instability

Nonlinear systems have both dispersive and unstable components, which can reduce the stability of the outputs. Instabilities in the stable solution of a novel PDE that meets the notion of optical transmission are known as modulating instabilities, and they are frequently brought on by a nonlinear element interacting through the dispersion principle. Continuous surface waves induce modulation instability (MI) in a nonlinear dispersion medium, resulting in intensity and phase self-modulation. The MI of Eq (1.1) is examined here. An illustration of an NPDE solution of the perturbation kind is

$$\mathcal{U}(x,t) = \phi\mathcal{L}(x,t) + \beta_0, \tag{5.1}$$

where $\phi << 1$ behaves as the perturbation parameter, $\mathcal{L}$ is the linearized part of the Eq (1.1) and indicates the impact term employed to determine the equilibrium deviation, and the constant β_0 represents the governing model in a steady state. After linearizing the system and including the Eq (5.1) into Eq (1.1), we obtain the following form

$$\phi\mathcal{L}_t + \alpha_1\phi\mathcal{L}_x + 3\beta_0\phi\mathcal{L}_x + \alpha_2^2(-\phi)\left(\mathcal{L}_{xxt} + \beta_0\mathcal{L}_{xxx}\right) + \alpha_3\phi\mathcal{L}_{xxx} = 0, \tag{5.2}$$

the assumption for the Eq (5.2) solution is

$$\mathcal{L}(x,t) = Pe^{i(\kappa x - \omega t)}, \tag{5.3}$$

where ω indicates the frequency, κ represents the wave number of the perturbation, and P represents the normalized wave number. Now, by integrating Eq (5.3) into Eq (5.2) and reducing the results

$$-i\kappa^3\left(\alpha_3-\alpha_2^2\beta_0\right)+i\kappa\left(\alpha_1+3\beta_0\right)+(-i\omega)\left(\alpha_2^2\kappa^2+1\right)=0, \tag{5.4}$$

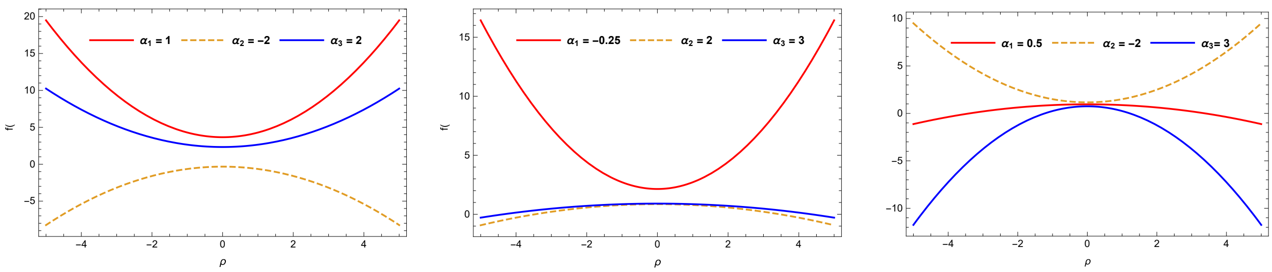
which gives

$$\omega=\frac{\alpha_2^2\beta_0\kappa^3-\alpha_3\kappa^3+\alpha_1\kappa+3\beta_0\kappa}{\alpha_2^2\kappa^2+1}, \tag{5.5}$$

According to Eq (5.5), the dispersion relation is dependent on a number of variables, such as the quantity of the waves, induced dispersing, the distribution of the collective speeds, and phase variations. If $\frac{\alpha_2^2\beta_0\kappa^3-\alpha_3\kappa^3+\alpha_1\kappa+3\beta_0\kappa}{\alpha_2^2\kappa^2+1}>0$ then we say that the ϕ is real for every normalized pulse number ω, and it illustrates the steady modulation for a small perturbation. When it comes to the modulations instability, the interaction $\frac{\alpha_2^2\beta_0\kappa^3-\alpha_3\kappa^3+\alpha_1\kappa+3\beta_0\kappa}{\alpha_2^2\kappa^2+1}<0$ for ϕ is imaginary, and the modulation rate grows exponentially. From this, the modulation instability gain $f(\omega)$ can be written as

$$f(\omega)\quad=2Im(\omega)=\quad\frac{\alpha_2^2\beta_0\kappa^3-\alpha_3\kappa^3+\alpha_1\kappa+3\beta_0\kappa}{\alpha_2^2\kappa^2+1}.$$

A number of factors, most notably the magnitude of the occurrence, cumulative velocity, and tuned down, affect modulation instabilities. Modulation instability fluctuation for changes in several parameters is shown in Figure 1.



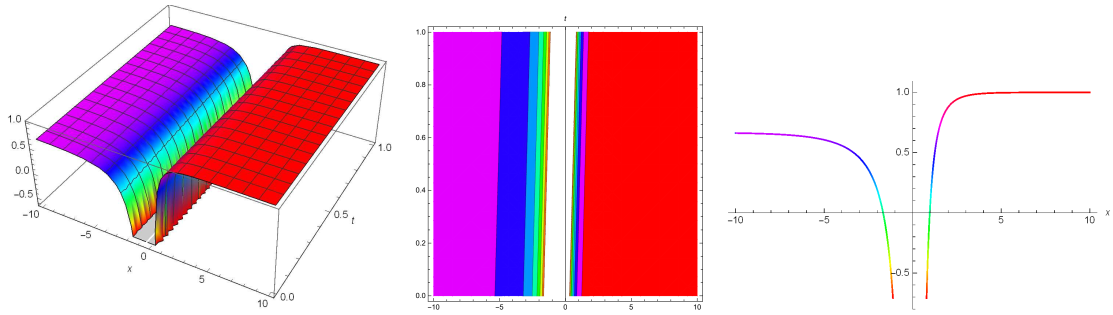
(a) $\alpha_1=-1,\ \alpha_2=2,\ \alpha_3=3,\ \beta_0=1.9$ (b) $\alpha_1=1,\ \alpha_2=-2,\ \alpha_3=3,\ \beta_0=0.5$ (c) $\alpha_1=1,\ \alpha_2=2,\ \alpha_3=-3,\ \beta_0=-2$

Figure 1. MI increases efficacy to various other factors

6. Discussion and results

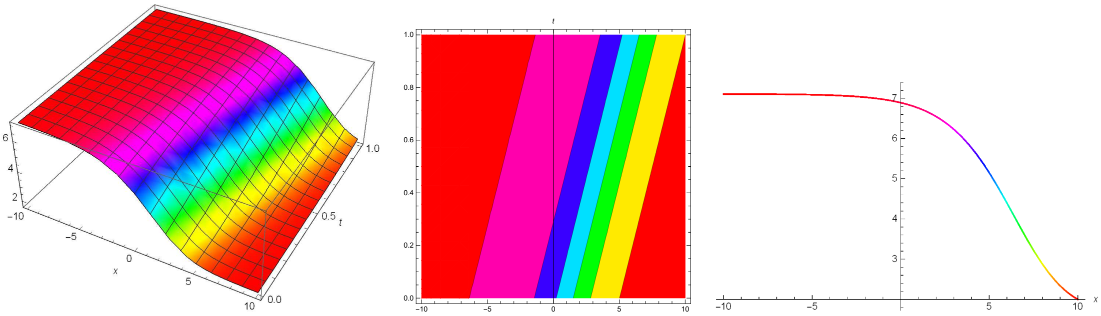
In this section, the physical representation of the DGH equation are achieved and some novel travelling wave solutions are successfully developed. This model demonstrates how waves propagate in shallow water. In order to illustrate the significance of nonlinear wave structures in everyday science and engineering, the 3D, 2D, and contour plots are constructed.

First, the $(\frac{1}{G'})$-expansion approach is utilized to describe various interesting wave structures. Figure 2 demonstrates a singular type dark soliton $\mathcal{U}_1^I(x,t)$ for (3.6) when $\alpha_1=2.5,\ \alpha_2=2,\ \alpha_3=3,\ A=2, \vartheta=4,\ \varpi=-0.5,\ \nu=5$. Figure 3 represents a solitary wave $\mathcal{U}_2^I(x,t)$ for (3.8) when $\alpha_1=1,\ \alpha_2=2,\ \alpha_3=-1,\ A=2,\ \vartheta=-0.8,\ \varpi=-0.5,\ \nu=0.5$.



(a) (b) (c)

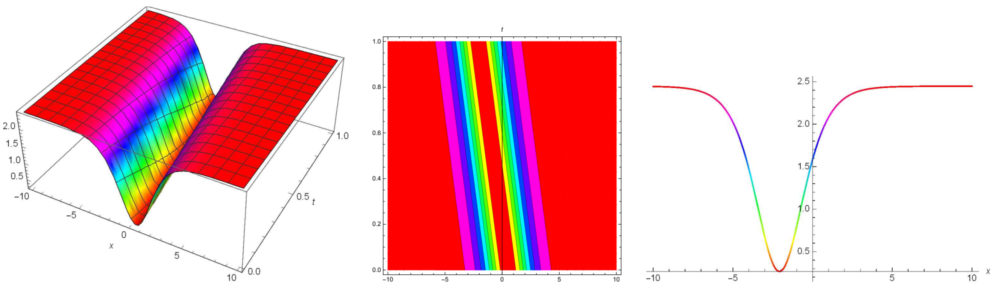
Figure 2. 3D, 2D, and contour representation of $\mathcal{U}_1^{I}(x,t)$ for (3.6) when $\alpha_1 = 2.5,\ \alpha_2 = 2,\ \alpha_3 = 3,\ A = 2, \vartheta = 4,\ \varpi = -0.5,\ \nu = 5$.



(a) (b) (c)

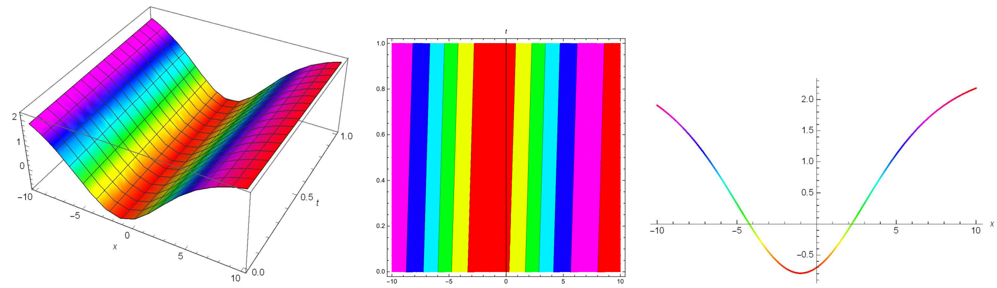
Figure 3. 3D, 2D, and contour representation of $\mathcal{U}_2^{I}(x,t)$ for (3.8) when$\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = -1,\ A = 2,\ \vartheta = -0.8,\ \varpi = -0.5,\ \nu = 0.5$.

Similarly, the $(\frac{\mathcal{G}'}{\mathcal{G}}, \frac{1}{\mathcal{G}})$-expansion technique is implemented to develop various interesting wave structures. Figure 4 illustrates a dark soliton solution $\mathcal{U}_1^{II}(x,t)$ for (3.11) when $\alpha_1 = 0.5,\ \alpha_2 = 1,\ \alpha_3 = 2,\ c_1 = -1,\ c_2 = 4$. Figure 5 visualizes a v-shaped solitary wave $\mathcal{U}_2^{II}(x,t)$ for (3.13) when $\alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = -2,\ c_1 = 1,\ c_2 = 4$. Figure 6 expresses a bright soliton $\mathcal{U}_3^{II}(x,t)$ for (3.15) when $\alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = 2,\ c_1 = -1,\ c_2 = 4$. Figure 7 visualizes a bell shaped bright structure $\mathcal{U}_4^{II}(x,t)$ for (3.17) when $\alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = 2,\ c_1 = -2,\ c_2 = 5$ respectively.



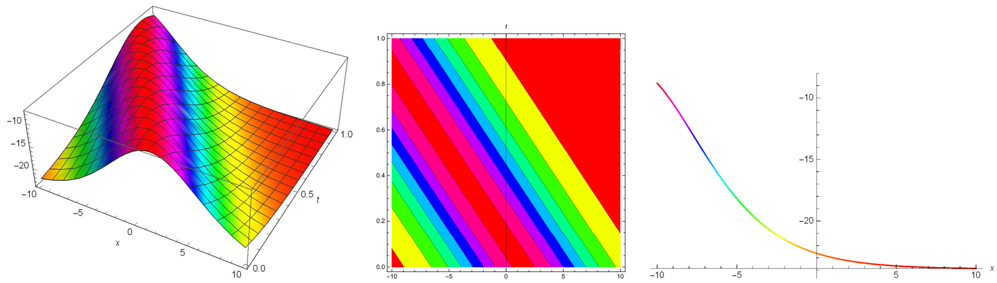
(a) (b) (c)

Figure 4. 3D, 2D, and contour representation of $\mathcal{U}_1^{II}(x,t)$ for (3.11) when $\alpha_1 = 0.5,\ \alpha_2 = 1,\ \alpha_3 = 2,\ c_1 = -1,\ c_2 = 4$.



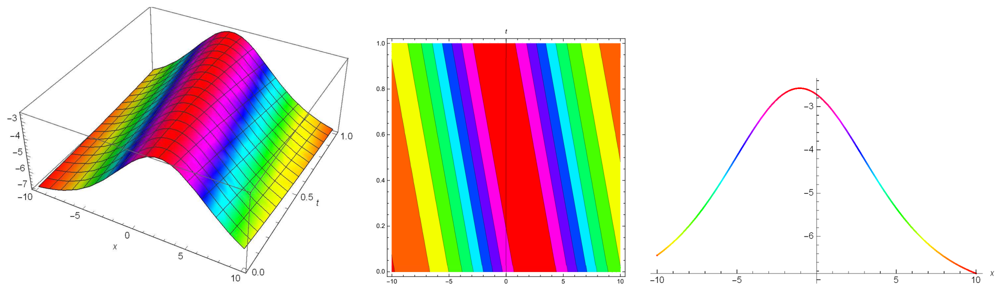
(a) (b) (c)

Figure 5. 3D, 2D, and contour representation of $\mathcal{U}_2^{II}(x,t)$ for (3.13) when$\alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = -2,\ c_1 = 1,\ c_2 = 4$.



(a) (b) (c)

Figure 6. 3D, 2D, and contour representation of $\mathcal{U}_3^{II}(x,t)$ for (3.15) when $\alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = 2,\ c_1 = -1,\ c_2 = 4$.



(a) (b) (c)

Figure 7. 3D, 2D, and contour representation of $\mathcal{U}_4^{II}(x,t)$ for (3.17) when $\alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = 2,\ c_1 = -2,\ c_2 = 5$.

The wave structures obtained here are qualitatively consistent with shallow water phenomena found in controlled experimental settings and in real water bodies, such as estuaries, coastal shelves, and channels, despite the fact that this work concentrates on analytical and symbolic calculations. The DGH model appears to offer a mathematically sound framework for representing a wide variety of physically observed wave characteristics, such as both smooth and discontinuous waveforms, based on these comparisons.

7. Conclusions

The nonlinear dispersive shallow water wave equation, often known as the DGH model, has been thoroughly examined analytically in this work. The behavior of shallow water waves is being analyzed by applying a few well-known analytical techniques to generate some innovative soliton solutions for the governing model. Numerous exact traveling wave solutions, such as solitary, shock, singular, shock-singular, hyperbolic, singular periodic, and rational forms, have been successfully identified by using well-established analytical methods supported by symbolic computing tools. The efficiency of the strategies employed shows that they are more reliable and effective ways to analyze a variety of applications in mathematical physics and engineering. The solutions exhibit a number of intriguing configurations, including bell-shaped, bright, dark, solo, and periodic wave solutions, as well as brilliant and dark soliton solutions.

With the help of Mathematica, the impact of the factors on the traveling wave forms is specifically investigated. Computer approaches are used to construct different wave shapes, including solitary, shock, singular, shock-singular, hyperbolic, singular periodic, and rational solutions. A wide range of wave behaviors dependent on parameter values has been revealed by a detailed examination of the influence of key factors on the shape and nature of the wave solutions. The 2D, 3D, and contour graphical representations corroborate the accuracy of the results produced and offer important insights into the dynamics of the solutions. The adaptability and resilience of the computational methods used are further demonstrated by these representations. Importantly, the study identifies specific parameter regimes that ensure the mathematical and physical correctness of the resulting wave structures (for additional details, see Figures 2–7).

We provide a new case study that has never been investigated previously, as far as we know, and will yield many new solutions for the NPDE that occurs in the present work. The results show that the strategies employed are more effective and capable than the traditional methods employed in previous research. The investigation is more significant because it is more reliable and important in explaining a variety of physical events. As a result, it is shown that the methods used are useful and applicable to a number of additional higher dimensional nonlinear evolution models in hydrodynamics, plasma, mathematics, and other ocean engineering and research fields. Overall, the results demonstrate how well the suggested analytical-computational framework explores and categorizes exact solutions of nonlinear dispersive wave models. Other sophisticated nonlinear systems in fluid dynamics and other fields can be analyzed using such strategies efficiently.

Furthermore, similar nonlinear dispersive equations that arise in other physical environments, such as plasma physics, nonlinear optics, and geophysical fluid dynamics, can be analyzed using the framework implemented in this paper. Another fascinating area that could be the fractional DGH model and its various forms. These directions have the potential to expand the applications of the governing model in both theoretical and applied disciplines and enhance our understanding of nonlinear wave dynamics.

Author contributions

Kalim U. Tariq: conceptualization, supervision, project administration. Mohammed Ahmed Alomair: software, validation, visualization, funding acquisition. Abdullah Mohammed Alomair:

methodology, resources, review and editing. Arslan Ahmed: Formal analysis and investigation, writing original draft.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

The Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant KFU252786].

Conflict of interest

The authors declare that they have no conflict of interest.

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