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Research article

Hyers-Ulam stability of quadratic operators in locally convex cones

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Abstract: The stability problem in Ulam's sense has recently been explored in locally convex cone environments. In continuation of this research direction, our work examined the stability properties of the quadratic functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

in such structures. We presented novel stability theorems that offered enhanced comprehension of operator behavior when subjected to perturbations. These results advanced the theoretical framework of Hyers-Ulam stability within locally convex cones while elucidating distinctive characteristics of quadratic operators in this context. Our investigation both strengthened the mathematical underpinnings of stability theory and provided new perspectives on interactions between certain operators and locally convex spaces.

Keywords: quadratic functional equation; quadratic mapping; Hyers-Ulam stability; locally convex cone
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1. Introduction and preliminaries

The concept of locally convex cones was first introduced and further developed in [12, 17]. A cone is a mathematical structure defined by a set $\mathscr{P}$ equipped with two operations: addition, denoted as $(a, b) \mapsto a + b$, and scalar multiplication, denoted as $(\lambda, a) \mapsto \lambda a$, where λ is a nonnegative real number. The addition operation must satisfy the properties of associativity and commutativity, and there exists a neutral element $0 \in \mathscr{P}$ such that:

- $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathscr{P}$,

- $a + b = b + a$ for all $a, b \in \mathscr{P}$,
- $a + 0 = a$ for all $a \in \mathscr{P}$.

Scalar multiplication in $\mathscr{P}$ satisfies the usual algebraic properties. Specifically, for all $a, b \in \mathscr{P}$ and all nonnegative scalars $\lambda, \mu \geq 0$, we have:

Scalar multiplication must adhere to the standard associative and distributive rules, meaning:

- Associativity: $\lambda(\mu a) = (\lambda\mu)a$,
- Distributivity over scalars: $(\lambda + \mu)a = \lambda a + \mu a$,
- Distributivity over elements: $\lambda(a + b) = \lambda a + \lambda b$,
- Multiplicative identity: $1 \cdot a = a$,
- Zero scalar: $0 \cdot a = 0$.

This structure ensures that the set $\mathscr{P}$ behaves consistently under both addition and scalar multiplication, maintaining the necessary algebraic properties. Vector spaces defined over the field of real numbers naturally qualify as cones under this definition. However, cones differ from vector spaces in two key ways: first, they do not require the existence of additive inverses for their elements, and second, scalar multiplication is restricted to nonnegative real numbers. Unlike in vector spaces, the property $0 \cdot a = 0$ must be explicitly stated for cones, as it does not automatically follow from the other defining conditions. This distinction highlights the broader nature of cones compared to vector spaces, as they relax certain structural requirements while maintaining specific algebraic properties. This concept is illustrated by a straightforward example from [18]. Consider the set $\mathscr{P} = \{0, 1\}$ with addition defined as $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, and $1 + 1 = 1$. For all $a \in \mathscr{P}$ and $\lambda \geq 0$, scalar multiplication is defined as $\lambda \cdot a = a$. Clearly, all the previously mentioned axioms hold except for the last one, and in this case, $0 \cdot a = a$ for $a \in \{0, 1\}$. It can be readily demonstrated that the neutral element $0 \in \mathscr{P}$ for the addition operation is unique. The inclusion of the final axiom is crucial, as it ensures the property $\lambda \cdot 0 = 0$ holds for all $\lambda \geqslant 0$ (see [18, Proposition 1.1.1]).

A subcone $\mathscr{Q}$ of a cone $\mathscr{P}$ is a nonempty subset that is closed under addition and multiplication by nonnegative scalars. In other words, if a and b belong to $\mathscr{Q}$ and $\lambda \geq 0$, then $a + b$ and λa also belong to $\mathscr{Q}$.

It should be emphasized that the (algebraic) cancellation property, which states that $a + c = b + c$ implies $a = b$ for all $a, b, c \in \mathscr{P}$, is not assumed in general. This property holds exactly when the cone $\mathscr{P}$ can be embedded into a real vector space [18, Theorem 1.1.4].

Example 1.1. Let $\mathbb{R}$ denote the set of real numbers. In the extended set $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we adopt the standard algebraic operations, where $\mu + (+\infty) = +\infty$ for all $\mu \in \overline{\mathbb{R}}$, $0.(+\infty) = 0$ and $\lambda.(+\infty) = +\infty$ for all $\lambda > 0$. Under these operations, $\overline{\mathbb{R}}$ forms a cone. However, since $\lambda + (+\infty) = \mu + (+\infty)$ for any $\lambda, \mu \in \overline{\mathbb{R}}$, the cancellation law does not hold in $\overline{\mathbb{R}}$. Additionally, we use the notations $\overline{\mathbb{R}}_+ = \{\lambda \in \overline{\mathbb{R}} : \lambda \geqslant 0\}$ and $\mathbb{R}_+ = \{\lambda \in \mathbb{R} : \lambda \geqslant 0\}$ to denote the subcones of $\overline{\mathbb{R}}$ and $\mathbb{R}$, respectively.

Example 1.2. Let $\mathscr{P}$ be a cone and $\mathcal{X}$ a nonempty set. For functions from $\mathcal{X}$ into $\mathscr{P}$, addition and scalar multiplication are defined pointwise. Equipped with these operations, the set of all such functions itself, denoted by $\mathcal{F}(\mathcal{X}, \mathscr{P})$, forms a cone. The cancellation law applies to $\mathcal{F}(\mathcal{X}, \mathscr{P})$ if, and only if, it holds for the original cone $\mathscr{P}$.

A *preordered cone* is a cone $\mathscr{P}$ equipped with a reflexive and transitive relation $\leqslant$ which is compatible with both the addition and scalar multiplication operations. Specifically, for all $a, b, c \in \mathscr{P}$

and $\lambda \geqslant 0$, the following properties are satisfied:

$$a \leqslant b \implies a + c \leqslant b + c \quad \text{and} \quad \lambda a \leqslant \lambda b.$$

Note that the relation $\leqslant$ is not required to be antisymmetric, so $a \leqslant b$ and $b \leqslant a$ may hold without implying $a = b$. Since equality in $\mathscr{P}$ naturally defines an order, all results established for ordered cones are also valid for cones without a predefined order. Every ordered vector space inherently satisfies the conditions to be considered a preordered cone. As examples, the extended sets $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$, when endowed with the usual order and standard algebraic operations (including $0 \cdot (+\infty) = 0$), also form preordered cones. Let $(\mathscr{P}, \leqslant)$ be a preordered cone and $\mathcal{X}$ a nonempty set. Then, the set $\mathcal{F}(\mathcal{X}, \mathscr{P})$ of all functions from $\mathcal{X}$ into $\mathscr{P}$, equipped with the pointwise order, naturally inherits the structure of a preordered cone.

Any cone $\mathscr{P}$ can be equipped with a natural preorder, where $a \leqslant b$ holds if there exists some $c \in \mathscr{P}$ satisfying $a + c = b$. This preorder comes directly from the cone's structure and requires no further assumptions. Every subcone of a preordered cone $(\mathscr{P}, \leqslant)$ naturally inherits the structure of a preordered cone. Additionally, a form of weak order cancellation holds, as described below:

Proposition 1.3. [12] *Let $(\mathscr{P}, \leqslant)$ be a preordered cone. If $a + c \leqslant b + c$ for some $a, b, c \in \mathscr{P}$, then $a + \varepsilon c \leqslant b + \varepsilon c$ for all $\varepsilon > 0$.*

In a preordered cone $(\mathscr{P}, \leqslant)$, a subset E is referred to as decreasing (or increasing) when for $b \in \mathscr{P}$ and some $a \in E$, the condition $b \leqslant a$ (or $a \leqslant b$) implies that $b \in E$. Unions and intersections of arbitrary families of decreasing (or increasing) sets are again decreasing (or increasing). Furthermore, the complement $\mathscr{P} \setminus E$ of a decreasing subset E of $\mathscr{P}$ is increasing. To see why, suppose $b \leqslant c$ for $c \in \mathscr{P}$ and $b \in \mathscr{P} \setminus E$. If $c \notin \mathscr{P} \setminus E$, then $c \in E$, and since E is decreasing, we get $b \in E$, which contradicts $b \in \mathscr{P} \setminus E$. A similar argument shows that the complement $\mathscr{P} \setminus E$ of an increasing set $E \subseteq \mathscr{P}$ is decreasing.

An element $v \in \mathscr{P}$ is called an upper bound (or lower bound) for a subset E of a preordered cone $(\mathscr{P}, \leqslant)$ if $a \leqslant v$ (or $v \leqslant a$) for all $a \in E$. If such a bound v belongs to E, it is referred to as the greatest element (or smallest element) of E. However, since the preorder relation is not required to be antisymmetric, E may contain multiple greatest or smallest elements, meaning uniqueness is not guaranteed.

A subset E of a preordered cone $(\mathscr{P}, \leqslant)$ is said to be order convex if, for any $c \in \mathscr{P}$ and $a, b \in E$, the condition $a \leqslant c \leqslant b$ implies that $c \in E$. When an order convex set is multiplied by a positive scalar, the resulting set remains order convex. Additionally, the intersection of any collection of order convex sets is also order convex.

In a preordered cone $\mathscr{P}$, a subset $\mathcal{V}$ is referred to as an *abstract 0-neighborhood system* if it satisfies the following conditions:

(i) $0 < v$ for all $v \in \mathcal{V}$;
(ii) If $u, v \in \mathcal{V}$, then there exists an element $w \in \mathcal{V}$ such that $w \leqslant u$ and $w \leqslant v$;
(iii) If $u, v \in \mathcal{V}$ and $\lambda > 0$, then $u + v \in \mathcal{V}$ and $\lambda v \in \mathcal{V}$.

Each v in $\mathscr{V}$ specifies upper and lower neighborhoods for elements of $\mathscr{P}$ in the following way:

$$v(a) = \{b \in \mathscr{P} : b \leqslant a + v\} \quad \text{(upper neighborhood)},$$

$$(a)v = \{b \in \mathscr{P} : \ a \leqslant b + v\} \quad \text{(lower neighborhood)}.$$

The symmetric neighborhood of a is given by $v(a)v := v(a) \cap (a)v$. From these neighborhoods arise the upper, lower, and symmetric topologies on $\mathscr{P}$, respectively. Every upper neighborhood $v(a)$ is a decreasing convex subset of $\mathscr{P}$, while each lower neighborhood $(a)v$ is increasing convex. The symmetric neighborhoods $v(a)v$ are convex as well as order convex. This conclusion arises because $v(a)v$ is the intersection of a decreasing set and an increasing set. We also observe that the following relationships hold for all $a, b \in \mathscr{P}$ and $\lambda > 0$:

1. $\lambda v(a) = (\lambda v)(\lambda a)$,
2. $\lambda(a)v = (\lambda a)(\lambda v)$,
3. $\lambda v(a)v = (\lambda v)(\lambda a)(\lambda v)$,
4. $v(a) + b \subseteq v(a + b)$,
5. $(a)v + b \subseteq (a + b)v$,
6. $v(a)v + b \subseteq v(a + b)v$.

Each of the three topologies on $\mathscr{P}$ can rightly be described as locally convex. In the upper topology, any open subset of $\mathscr{P}$ is decreasing, as it is formed by unions of neighborhoods of its elements. Similarly, in the lower topology, all open subsets of $\mathscr{P}$ are increasing. Consequently, the complements of these open sets exhibit the opposite behavior: subsets closed in the upper topology are increasing, while those closed in the lower topology are decreasing. This duality arises from the relationship between open and closed sets in these topologies.

Consider a preordered cone $(\mathscr{P}, \leqslant)$ together with an abstract 0-neighborhood system $\mathscr{V} \subseteq \mathscr{P}$. The pair $(\mathscr{P}, \mathscr{V})$ is referred to as a *full locally convex cone*. Due to the inherent asymmetry of cones, asymmetric conditions naturally arise. Due to technical requirements, the elements of a full locally convex cone $(\mathscr{P}, \mathscr{V})$ are assumed to be bounded from below. This means that for every $a \in \mathscr{P}$ and $v \in \mathscr{V}$, there exists $\rho > 0$ such that $0 \leqslant a + \rho v$. Therefore, a full locally convex cone $(\mathscr{P}, \mathscr{V}, \leqslant)$ is a preordered cone $(\mathscr{P}, \leqslant)$ that contains an 0- neighborhood system $\mathscr{V}$ such that all its elements are bounded below. Any subcone of a full locally convex cone $(\mathscr{P}, \mathscr{V})$, even if it does not include the abstract 0-neighborhood system $\mathscr{V}$, is termed a *locally convex cone*. We say that an element $a \in \mathscr{P}$ is *upper bounded* if, for every $v \in \mathscr{V}$, one can find a positive scalar $\lambda > 0$ with $a \leqslant \lambda v$. The element $a \in \mathscr{P}$ is said to be *bounded* if it is both lower and upper bounded. Hence, in a full locally convex cone, an element is bounded if, and only if, it is upper bounded. By defining $\xi = \{\varepsilon > 0 : \ \varepsilon \in \mathbb{R}\}$, the pairs $(\overline{\mathbb{R}}, \xi)$ and $(\overline{\mathbb{R}}_+, \xi)$ form examples of full locally convex cones.

Let $(\mathscr{P}, \mathscr{V})$ be a locally convex cone, and let $a \in \mathscr{P}$. The closure of a is defined as the intersection of all its upper neighborhoods, that is,

$$\overline{a} := \cap\{v(a) : \ v \in \mathscr{V}\}.$$

It is straightforward to verify that $\bar{a}$ represents the closure of the singleton set $\{a\}$ under the lower topology [12, Corollary I.3.5]. A locally convex cone $(\mathscr{P}, \mathscr{V})$ is said to be *separated* if, for any $a, b \in \mathscr{P}$, the equality $\overline{a} = \overline{b}$ implies $a = b$. In other words, no two different elements of $\mathscr{P}$ share the same closure. It is commonly known that the locally convex cone $(\mathscr{P}, \mathscr{V})$ is separated exactly when the symmetric topology on $\mathscr{P}$ is Hausdorff (see [12, Proposition I.3.9]).

Let $(a_i)_{i \in I}$ be a net in $(\mathscr{P}, \mathscr{V})$. We say that $(a_i)_{i \in I}$ converges to an element $a \in \mathscr{P}$ with respect to the symmetric relative topology on $\mathscr{P}$ if, for every $v \in \mathscr{V}$, there exists an index $i_0 \in I$ such that $a_i \leqslant a + v$

for all $i \geqslant i_0$. The net $(a_i)_{i\in I}$ is termed a (symmetric) *Cauchy net* if, for each $v \in \mathscr{V}$, one can find an index $i_0 \in I$ such that $a_i \leqslant a_j + v$ for all $i, j \geqslant i_0$. Obviously, convergence of a net implies that it is also a Cauchy net. We say that a locally convex cone $(\mathscr{P}, \mathcal{V})$ is (symmetric) *complete* when every (symmetric) Cauchy net in $(\mathscr{P}, \mathscr{V})$ converges to some element of $\mathscr{P}$. A locally convex cone $(\mathscr{P}, \mathscr{V})$ is called a *uniformly convex cone* (*uc-cone*) if $\mathscr{V} = \{\lambda w : \lambda > 0\}$ for some $w \in \mathscr{V}$. In this case, the element w is referred to as the *generating element* of $\mathcal{V}$. Suppose $(\mathscr{P}, \mathscr{V})$ is a uc-cone and $\mathscr{P}$ also carries the structure of a real vector space. Then, under the symmetric topology induced by $(\mathscr{P}, \mathscr{V})$, $\mathscr{P}$ naturally acquires the structure of a semi-normed space. In the special case where $\mathscr{V} = \{\lambda w : \lambda > 0\}$, the semi-norm $q : \mathscr{P} \to [0, +\infty]$ is given by

$$q(a) = \inf\{\mu > 0 : \ \mu^{-1}a \in w(0)w\}, \quad a \in \mathscr{P}.$$

When the symmetric topology on $\mathscr{P}$ is Hausdorff, q defines a norm on $\mathscr{P}$ (see [4] for further details).

The study of functional equations and their stability has been a central topic in mathematical analysis for several decades. Among these, the quadratic functional equation, which arises naturally in various areas of mathematics and physics, has garnered significant attention.

Let $\mathscr{P}$ and $\mathscr{Q}$ be cones. A function $f : \mathscr{P} \to \mathscr{Q}$ is referred to as a quadratic mapping if it satisfies

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathscr{P}$ such that $x - y \in \mathscr{P}$. The study of functional equations has been a fundamental area of mathematical research, with applications in various fields such as physics, engineering, and economics. Among these, the quadratic functional equation holds significant importance due to its connection with quadratic forms, inner product spaces, and orthogonal additivity. The stability of functional equations, particularly in the sense of Hyers-Ulam, has been a major topic of investigation since the mid-20th century, leading to profound developments in both pure and applied mathematics. The quadratic functional equation was first systematically studied by Jordan and von Neumann [11] in the context of characterizing inner product spaces. They proved that a normed space satisfies the parallelogram law (and, hence, is an inner product space) if, and only if, its norm satisfies the quadratic functional equation. This established a deep link between quadratic functions and geometric structures.

The stability problem for functional equations originated from a question posed by Ulam [21] in 1940, concerning the approximation of homomorphisms. Specifically, Ulam asked: If a function approximately satisfies a functional equation, can it be approximated by an exact solution? In 1941, Hyers [10] provided a partial answer for additive Cauchy equations in Banach spaces, marking the birth of Hyers-Ulam stability theory.

Theorem 1.4. [10] *Let X be a normed space, Y a Banach space, and $f : X \to Y$ a mapping satisfying*

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in X$, where $\varepsilon > 0$ is a constant. Then, for each $x \in X$, the limit

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists, and the mapping $A : X \to Y$ is the unique additive function satisfying

$$\|f(x) - A(x)\| \leqslant \varepsilon$$

for all $x \in X$.

The stability theory of functional equations has witnessed significant developments through several key contributions. In 1950, Aoki [3] extended Hyers' theorem to additive mappings, marking an important generalization. Two decades later, Th. M. Rassias [16] achieved a crucial advancement by removing the boundedness restriction on the Cauchy difference for linear mappings. This work was further refined by Găvruţa [9] who introduced general control functions to bound the Cauchy difference. These theoretical breakthroughs have successfully extended the stability concept to various other functional equations. The stability of the quadratic functional equation was first investigated by Skof [20] and Cholewa [8], who proved that if a function f from an Abelian group $(G, +)$ to a Banach space Y satisfies the inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leqslant \varepsilon$$

for all $x, y \in G$ and some $\varepsilon \geqslant 0$, then there exists a unique quadratic mapping $Q : G \to Y$ such that $\|f(x) - Q(x)\| \leqslant \frac{\varepsilon}{2}$ for all $x \in G$.

The study of stability for functional equations has important applications in various fields, including dynamic systems, optimization theory, and mathematical modeling of physical and engineering processes. In recent years, the investigation of stability problems has been strongly connected with the analysis of broader classes of functional and differential equations, often motivated by practical applications in physics, finance, and engineering. For example, oscillatory properties of solutions to functional differential equations with noncanonical operators have been studied in [1], while stability and solution structures of adjoint nonlinear impulsive neutral mixed integral equations in dynamical systems have also been investigated [2]. Moreover, the Ulam–Hyers–Rassias stability for fractional stochastic impulsive differential equations driven by time-changed Brownian motion has been established, with applications to credit risk modeling [5]. Similarly, the Ulam–Hyers–Rassias stability of Hilfer fractional stochastic impulsive differential equations with nonlocal conditions has been examined, followed by applications in currency options pricing [6]. In addition, viscoelastic Kelvin–Voigt models have been analyzed in connection with Ulam–Hyers stability and T-controllability for coupled integro-fractional stochastic systems with integral boundary conditions [7]. Moreover, the concept of Ulam–Hyers stability has been extended to various types of integral and differential equations. For example, the Ulam–Hyers stability of incommensurate systems for weakly singular integral equations was investigated in [19], highlighting the broad applicability of stability theory beyond functional equations. These studies demonstrate that the techniques developed for functional equations can also inform stability analyses in more general settings, such as integral systems and dynamical models.

Understanding the behavior of quadratic operators in locally convex cones provides a rigorous framework to analyze perturbations and stability phenomena in ordered and convex environments, which can have implications in both theoretical and applied mathematics. The main contribution of this paper is the establishment of new Hyers–Ulam stability results for the quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

within the framework of locally convex cones. These structures extend locally convex spaces by incorporating order and convexity properties, allowing classical stability results to be generalized to ordered convex settings. Our approach examines the behavior of quadratic operators under perturbations in cone-related environments, thus not only generalizing previous results in Banach spaces but also providing a foundation for future studies on other functional equations, including Jensen-type, Drygas-type, and Popoviciu-type equations, in ordered and cone-normed spaces.

The primary motivation for this work stems from the need to understand the stability properties of quadratic operators in nonlinear and non-Archimedean settings. While the stability of quadratic mappings in Banach spaces and normed spaces has been extensively studied, the extension of these results to locally convex cones remains relatively unexplored. This paper aims to fill this gap by establishing Hyers-Ulam type stability results for quadratic operators in locally convex cones. Our approach leverages the structure of these cones, particularly their symmetric topology and the properties of bounded elements, to derive stability theorems.

Our results generalize and extend previous work on the stability of quadratic mappings, providing a unified framework for studying these operators in locally convex cones. The techniques employed in this paper are rooted in the theory of locally convex cones and functional analysis, and they offer a new perspective on the stability of quadratic operators in ordered and convex structures.

2. Main results

The following lemmas have essential roles in our main results.

Lemma 2.1. *Suppose* $(\mathscr{P}, \mathscr{V})$ *is a separated locally convex cone and let* $a, b \in \mathscr{P}$. *If, for every* $v \in \mathscr{V}$, *we have* $a \leqslant b + v$ *and* $b \leqslant a + v$, *then it follows that* $a = b$.

Proof. To establish that $a = b$, it is enough to show that $\overline{a} = \overline{b}$. Take any $x \in \overline{a}$. By definition, $x \leqslant a + \frac{1}{2}v$ for every $v \in \mathscr{V}$. Since $a \leqslant b + \frac{1}{2}v$, we then have $x \leqslant b + v$ for all $v \in \mathscr{V}$, which means that $x \in \overline{b}$. Hence, we conclude that $\overline{a} \subseteq \overline{b}$. By a similar argument, we can show that $\overline{b} \subseteq \overline{a}$, leading to $\overline{a} = \overline{b}$. Because $(\mathscr{P}, \mathscr{V})$ is a separated locally convex cone, we conclude that $a = b$. □

Corollary 2.2. *Suppose* $(\mathscr{P}, \mathscr{V})$ *is a separated locally convex cone and let* $a, b \in \mathscr{P}$. *If* $a + v = b + v$ *for each* $v \in \mathscr{V}$, *then* $a = b$.

Lemma 2.3. [14] *Let* $(\mathscr{P}, \mathscr{V})$ *be a locally convex cone, and let* $a \in \mathscr{P}$. *Suppose* $\{\lambda_n\}$ *is a sequence of nonnegative scalars such that* $\lim_n \lambda_n = 0$. *Then, a is bounded if, and only if,* $\lim_n \lambda_n a = 0$ *in the symmetric topology.*

Proposition 2.4. *Consider a cone* $\mathscr{E}$ *and a locally convex cone* $(\mathscr{P}, \mathscr{V})$. *Suppose a function* $f : \mathscr{E} \to \mathscr{P}$ *satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y), \quad x, y, x-y \in \mathscr{E} \tag{2.1}$$

and $f(0)$ *is bounded. Then,* f *is a quadratic mapping.*

Proof. Substituting $x = y = 0$ in (2.1), we obtain $2f(0) = f(0)$. Therefore, we deduce $f(0) = \frac{1}{2^n}f(0)$ for all $n \in \mathbb{N}$. Since $f(0)$ is bounded, by Lemma 2.3, we get $f(0) = 0$. Next, setting $y = 0$ in (2.1) and using $f(0) = 0$, we derive $4f\left(\frac{x}{2}\right) = f(x)$ for all $x \in \mathscr{E}$. Thus, (2.1) yields

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y, x-y \in \mathscr{E}.$$

□

Now, we present the following key results:

Theorem 2.5. *Let $(\mathscr{P}_1, \mathscr{V}_1)$ be a locally convex cone, and let $(\mathscr{P}_2, \mathscr{V}_2)$ be a separated full locally convex cone that is complete in the sense of the symmetric topology. Assume that a mapping $f : \mathscr{P}_1 \to \mathscr{P}_2$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) \in v(f(x) + f(y))v \tag{2.2}$$

for some bounded element $v \in \mathscr{V}_2$ and for all $x, y \in \mathscr{P}_1$ with $x - y \in \mathscr{P}_1$. If $f(0)$ is bounded, then there exists a unique quadratic function $Q : \mathscr{P}_1 \to \mathscr{P}_2$ and a positive real number γ such that

$$Q(x) \in (\gamma v)(f(x))(\gamma v), \quad x \in \mathscr{P}_1.$$

Proof. Let $v \in \mathscr{V}_2$ be a bounded element satisfying the condition in (2.2). By setting $y = 0$ in (2.2), we obtain

$$4f\left(\frac{x}{2}\right) \in v(f(x) + f(0))v$$

for all $x \in \mathscr{P}_1$. This implies the inequalities:

$$4f\left(\frac{x}{2}\right) \leqslant f(x) + f(0) + v, \tag{2.3}$$

$$f(x) + f(0) \leqslant 4f\left(\frac{x}{2}\right) + v. \tag{2.4}$$

Since $f(0)$ is bounded, there exists a positive real number $\lambda > 0$ such that

$$f(0) + \lambda v \geqslant 0 \quad \text{and} \quad f(0) \leqslant \lambda v. \tag{2.5}$$

Combining (2.3)–(2.5), we derive

$$4f\left(\frac{x}{2}\right) \leqslant f(x) + (\lambda + 1)v,$$

$$f(x) \leqslant 4f\left(\frac{x}{2}\right) + (\lambda + 1)v.$$

Hence

$$f(x) \leqslant \frac{f(2x)}{4} + \frac{\lambda+1}{4}v, \tag{2.6}$$

$$\frac{f(2x)}{4} \leqslant f(x) + \frac{\lambda+1}{4}v. \tag{2.7}$$

Using mathematical induction on n, we extend the result to the following inequalities:

$$f(x) \leqslant \frac{1}{4^n} f(2^n x) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right)(\lambda + 1)v, \tag{2.8}$$

$$\frac{1}{4^n} f(2^n x) \leqslant f(x) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right)(\lambda + 1)v \tag{2.9}$$

for all $x \in \mathscr{P}_1$ and all $n \in \mathbb{N}$. To establish (2.8), we first note that the inequality holds for $n = 1$ as a direct consequence of (2.6). Next, assume that (2.8) is valid for some n and all $x \in \mathscr{P}_1$. Then,

$$f(x) \leqslant \frac{f(2x)}{4} + \frac{\lambda+1}{4}v \quad \text{(by (2.6))}$$

$$\leqslant \frac{1}{4^{n+1}} f(2^{n+1}x) + \frac{1}{12}\left(1 - \frac{1}{4^n}\right)(\lambda+1)v + \frac{\lambda+1}{4}v \quad \text{(by (2.8))}$$
$$= \frac{1}{4^{n+1}} f(2^{n+1}x) + \frac{1}{3}\left(1 - \frac{1}{4^{n+1}}\right)(\lambda+1)v$$

for all $x \in \mathscr{P}_1$. This completes the proof of (2.8) for all $n \in \mathbb{N}$. To prove (2.9), we begin by observing that the inequality holds for $n = 1$ as a direct result of (2.7). Now, suppose that (2.9) is true for some n and all $x \in \mathscr{P}_1$. Starting from (2.9), we derive:

$$\frac{1}{4^{n+1}} f(2^{n+1}x) \leqslant \frac{f(2x)}{4} + \frac{1}{12}\left(1 - \frac{1}{4^n}\right)(\lambda+1)v \quad \text{(by (2.9))}$$
$$\leqslant f(x) + \frac{\lambda+1}{4}v + \frac{1}{12}\left(1 - \frac{1}{4^n}\right)(\lambda+1)v \quad \text{(by (2.7))}$$
$$= f(x) + \frac{1}{3}\left(1 - \frac{1}{4^{n+1}}\right)(\lambda+1)v$$

for all $x \in \mathscr{P}_1$. This establishes (2.9) for all $n \in \mathbb{N}$. Replacing x with $2^m x$ in (2.8) and (2.9) and multiplying both sides by $\frac{1}{4^m}$, we conclude

$$\frac{1}{4^m} f(2^m x) \leqslant \frac{1}{4^{n+m}} f(2^{n+m}x) + \frac{\lambda+1}{3\times 4^m}v, \tag{2.10}$$
$$\frac{1}{4^{n+m}} f(2^{n+m}x) \leqslant \frac{1}{4^m} f(2^m x) + \frac{\lambda+1}{3\times 4^m}v \tag{2.11}$$

for all $x \in \mathscr{P}_1$ and all $m, n \in \mathbb{N}$. Given that v is bounded, Lemma 2.3 ensures that the sequence $\left\{\frac{\lambda+1}{3\times 4^m}v\right\}_m$ converges to zero as $m \to +\infty$. Let $u \in \mathscr{V}_2$ an arbitrary element. Then, there exists a natural number $N \in \mathbb{N}$ such that

$$\frac{\lambda+1}{3\times 4^m}v \leqslant u, \quad m > N.$$

Using this result together with (2.10) and (2.11), we deduce:

$$\frac{1}{4^m} f(2^m x) \leqslant \frac{1}{4^{n+m}} f(2^{n+m}x) + u, \quad \frac{1}{4^{n+m}} f(2^{n+m}x) \leqslant \frac{1}{4^m} f(2^m x) + u$$

for all $x \in \mathscr{P}_1$ and all $m, n \in \mathbb{N}$ with $m > N$. This implies that the sequence $\left\{\frac{1}{4^n} f(2^n x)\right\}_n$ forms a Cauchy sequence in with respect to the symmetric topology. Because $(\mathscr{P}_2, \mathscr{V}_2)$ is a separated space, the symmetric topology is Hausdorff, ensuring that the limit of this sequence is uniquely determined. We now define the mapping $Q : \mathscr{P}_1 \to \mathscr{P}_2$ as follows:

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x), \quad x \in \mathscr{P}_1.$$

Thus, for $x \in \mathscr{P}_1$, there is $n \in \mathbb{N}$ such that

$$\frac{1}{4^n} f(2^n x) \leqslant Q(x) + \frac{v}{3}, \quad Q(x) \leqslant \frac{1}{4^n} f(2^n x) + \frac{v}{3}.$$

Using (2.8) and (2.9), we deduce

$$\begin{aligned} f(x) &\leqslant \frac{1}{4^n} f(2^n x) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right)(\lambda + 1)v \\ &\leqslant Q(x) + \frac{v}{3} + \frac{\lambda + 1}{3} v \\ &= Q(x) + \frac{\lambda + 2}{3} v \end{aligned}$$

and

$$\begin{aligned} Q(x) &\leqslant \frac{1}{4^n} f(2^n x) + \frac{v}{3} \\ &\leqslant f(x) + \frac{1}{3}\left(1 - \frac{1}{4^n}\right)(\lambda + 1)v + \frac{v}{3} \\ &\leqslant f(x) + \frac{\lambda + 2}{3} v. \end{aligned}$$

Therefore,

$$f(x) \leqslant Q(x) + \frac{\lambda + 2}{3} v, \quad Q(x) \leqslant f(x) + \frac{\lambda + 2}{3} v.$$

Hence, $Q(x) \in (\gamma v)(f(x))(\gamma v)$, where $\gamma = \frac{\lambda+2}{3}$. We now prove Q is a quadratic mapping on $\mathscr{P}_1$. Let $x, y \in \mathscr{P}_1$ such that $x - y \in \mathscr{P}_1$. Replacing x with $2^n x$ and y with $2^n y$ in (2.2), and multiplying both sides by $\frac{1}{4^n}$, we obtain

$$2\left[\frac{1}{4^n} f\left(2^n\left(\frac{x+y}{2}\right)\right) + \frac{1}{4^n} f\left(2^n\left(\frac{x-y}{2}\right)\right)\right] \leqslant \frac{1}{4^n} f(2^n x) + \frac{1}{4^n} f(2^n y) + \frac{1}{4^n} v, \tag{2.12}$$

$$\frac{1}{4^n} f(2^n x) + \frac{1}{4^n} f(2^n y) \leqslant 2\left[\frac{1}{4^n} f\left(2^n\left(\frac{x+y}{2}\right)\right) + \frac{1}{4^n} f\left(2^n\left(\frac{x-y}{2}\right)\right)\right] + \frac{1}{4^n} v \tag{2.13}$$

for all $n \in \mathbb{N}$. Let $u \in \mathscr{V}_2$. We can find $n \in \mathbb{N}$ such that

$$\frac{1}{4^n} v \leqslant u, \quad \frac{1}{4^n} f(2^n z) \leqslant Q(z) + u, \quad Q(z) \leqslant \frac{1}{4^n} f(2^n z) + u, \quad z \in \left\{x, y, \frac{x+y}{2}, \frac{x-y}{2}\right\}.$$

From (2.12) and (2.13), we deduce

$$\begin{aligned} 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) &\leqslant Q(x) + Q(y) + 7u, \\ Q(x) + Q(y) &\leqslant 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) + 7u. \end{aligned}$$

By Lemma 2.1, we infer that

$$2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) = Q(x) + Q(y), \quad x, y, x - y \in \mathscr{P}_1. \tag{2.14}$$

Since $f(0)$ is bounded, by Lemma 2.3, we have $Q(0) = 0$. Letting $y = 0$ in (2.14), we get $4Q\left(\frac{x}{2}\right) = Q(x)$ for all $x \in \mathscr{P}_1$. Thus, (2.14) implies

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad x, y, x - y \in \mathscr{P}_1.$$

To show the uniqueness of Q, let $q : \mathscr{P}_1 \to \mathscr{P}_2$ be another quadratic mapping satisfying $q(x) \in (\gamma v)(f(x))(\gamma v)$ for all $x \in \mathscr{P}_1$. Then,

$$f(x) \leqslant q(x) + \gamma v \quad \text{and} \quad q(x) \leqslant f(x) + \gamma v, \quad x \in \mathscr{P}_1.$$

Thus, by substituting x with $2^n x$, dividing both sides by 4^n, and applying the relation $q(2^n x) = 4^n q(x)$, we get

$$\frac{1}{4^n} f(2^n x) \leqslant q(x) + \frac{\gamma}{4^n} v \quad \text{and} \quad q(x) \leqslant \frac{1}{4^n} f(2^n x) + \frac{\gamma}{4^n} v, \quad x \in \mathscr{P}_1. \tag{2.15}$$

Let $u \in \mathscr{V}_2$ and $x \in \mathscr{P}_1$. We can find $n \in \mathbb{N}$ such that

$$\frac{\gamma}{4^n} v \leqslant u, \quad \frac{1}{4^n} f(2^n x) \leqslant Q(x) + u, \quad Q(x) \leqslant \frac{1}{4^n} f(2^n x) + u.$$

From (2.15), we deduce

$$Q(x) \leqslant q(x) + 2u \quad \text{and} \quad q(x) \leqslant Q(x) + 2u.$$

By Lemma 2.1, we infer that $Q(x) = q(x)$. This proves the uniqueness of Q. □

Corollary 2.6. *Consider a locally convex cone $(\mathscr{P}, \mathscr{V})$ and a mapping $f : \mathscr{P} \to (\overline{\mathbb{R}}, \xi)$ (or $f : \mathscr{P} \to (\overline{\mathbb{R}}_+, \xi)$) that satisfies the following condition:*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) \in \varepsilon\,(f(x) + f(y))\,\varepsilon$$

for some $\varepsilon > 0$ and for all $x, y \in \mathscr{P}$ with $x - y \in \mathscr{P}$, where $f(0) \neq +\infty$. Then, there exists a unique quadratic mapping $Q : \mathscr{P} \to \overline{\mathbb{R}}$ (or $Q : \mathscr{P} \to \overline{\mathbb{R}}_+$) such that

$$Q(x) \in \frac{\lambda+2}{3}\varepsilon(f(x))\frac{\lambda+2}{3}\varepsilon$$

holds for all $x \in \mathscr{P}$, with $\lambda > 0$ chosen such that $|f(0)| \leqslant \lambda\varepsilon$.

Remark 2.7. Let $(\mathscr{P}_1, \mathscr{V}_1)$ be a locally convex cone and $(\mathscr{P}_2, \mathscr{V}_2)$ a separated full uc-cone with the generating element v. Assume $(\mathscr{P}_2, \mathscr{V}_2)$ is complete under the symmetric topology. Suppose a mapping $f : \mathscr{P}_1 \to \mathscr{P}_2$ satisfies

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) \in v(f(x) + f(y))v$$

for all $x, y \in \mathscr{P}_1$ with $x - y \in \mathscr{P}_1$, where $f(0)$ is bounded. Then, by Theorem 2.5, there exists a unique quadratic mapping $Q : \mathscr{P}_1 \to \mathscr{P}_2$ and a positive real number γ such that

$$Q(x) \in \gamma v(f(x))\gamma v$$

for all $x \in \mathscr{P}_1$.

Corollary 2.8. *Consider a locally convex cone $(\mathscr{P}_1, \mathscr{V}_1)$ and a separated full uc-cone $(\mathscr{P}_2, \mathscr{V}_2)$, where $\mathscr{P}_2$ is additionally equipped with the structure of a real vector space. Suppose $(\mathscr{P}_2, \mathscr{V}_2)$ is complete* *with respect to its symmetric topology. If for some $\varepsilon > 0$, a given mapping $f : \mathscr{P}_1 \to \mathscr{P}_2$ satisfies the following condition:*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) \in \varepsilon v(f(x) + f(y))\varepsilon v$$

for all $x, y \in \mathscr{P}_1$ with $x - y \in \mathscr{P}_1$, where v denotes the generating element of $\mathscr{V}_2$, then there exists a unique quadratic mapping $Q : \mathscr{P}_1 \to \mathscr{P}_2$ satisfying

$$Q(x) \in \left(\frac{r\varepsilon}{3}v\right)\left(f(x) - \frac{1}{3}f(0)\right)\left(\frac{r\varepsilon}{3}v\right), \quad x \in \mathscr{P}_1$$

for all $r > 1$.

Proof. The symmetric topology on $\mathscr{P}_2$ is known to be Hausdorff (see [12], I.3.9), which implies that the pair $(\mathscr{P}_2, \mathscr{V}_2)$ becomes a Banach space when endowed with the norm ρ defined as:

$$\rho(a) = \inf\{\mu > 0 : \ \mu^{-1}a \in v(0)v\}, \quad a \in \mathscr{P}_2.$$

Furthermore, the vector space structure of $\mathscr{P}_2$ guarantees that all its elements are bounded. Consequently, the quantity $\rho(a)$ remains finite for every $a \in \mathscr{P}_2$. From our initial assumptions, we obtain

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \in (\varepsilon v)(0)(\varepsilon v)$$

for all $x, y \in \mathscr{P}_1$ with $x - y \in \mathscr{P}_1$. This relationship can be alternatively formulated as

$$\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \leqslant \varepsilon$$

for all $x, y \in \mathscr{P}_1$ with $x - y \in \mathscr{P}_1$. By employing Hyers' approach and taking $y = 0$, we obtain

$$\rho\left(4f\left(\frac{x}{2}\right) - f(x) - f(0)\right) \leqslant \varepsilon, \quad x \in \mathscr{P}_1.$$

Hence,

$$\rho\left(\frac{1}{4^{n+1}}f(2^{n+1}x) - \frac{1}{4^m}f(2^m x) + \sum_{k=m}^{n} \frac{1}{4^{k+1}}f(0)\right) \leqslant \sum_{k=m}^{n} \frac{\varepsilon}{4^{k+1}} \tag{2.16}$$

for all $x \in \mathscr{P}_1$ and $n \geqslant m \geqslant 0$. Based on (2.16), we observe that the sequence $\left\{\frac{1}{4^n}f(2^n x)\right\}_{n\geqslant 1}$ is Cauchy in the Banach space $(\mathscr{P}_2, \rho)$. Since the space $\mathscr{P}_2$ is complete with respect to the norm ρ, this sequence must converge. This allows us to introduce a well-defined mapping $Q : \mathscr{P}_1 \to \mathscr{P}_2$ through the pointwise limit:

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n}f(2^n x), \quad x \in \mathscr{P}_1.$$

Setting $m = 0$ and letting n approach infinity in (2.16) yields

$$\rho\left(Q(x) - f(x) + \frac{1}{3}f(0)\right) \leqslant \frac{\varepsilon}{3}, \quad x \in \mathscr{P}_1.$$

Consequently, for all $x \in \mathscr{P}_1$ and $r > 1$, we have $Q(x) - f(x) + \frac{1}{3}f(0) \in \frac{r\varepsilon}{3}(v(0)v)$. This implies that

$$Q(x) \in \left(\frac{r\varepsilon}{3}v\right)\left(f(x) - \frac{1}{3}f(0)\right)\left(\frac{r\varepsilon}{3}v\right), \quad x \in \mathscr{P}_1, \ r > 1.$$

□

Remark 2.9. The methods developed in this paper can also be applied to related notions of stability. For instance, by replacing the constant bound ε with a suitable control function, one may obtain corresponding results for Ulam-Hyers-Rassias stability.

3. Conclusions

In this paper, we have established new stability results for the quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

within the framework of locally convex cones. Our theorems extend existing approaches to Hyers-Ulam stability in such settings, revealing specific structural features of quadratic operators under perturbations. The findings contribute both to the deeper theoretical understanding of stability phenomena in locally convex spaces and to the broader study of functional equations in ordered topological structures. These results can also provide a basis for studying the stability of other functional equations, such as Jensen-type, Drygas-type, and Popoviciu-type equations. Future work in this direction looks especially promising in cone-related settings, including cone normed linear spaces, ordered Banach spaces, and locally convex cones. In these frameworks, one can further examine different forms of stability, such as Hyers-Ulam-Rassias stability, hyperstability, and asymptotic stability.

Use of Generative-AI tools declaration

The authors declare that the AI tool ChatGPT (OpenAI) was used solely to improve the English grammar and clarity of the text. No AI tool was employed to generate or verify any mathematical content or results.

Conflict of interest

The authors declare that they have no competing interests.

Author contributions

The first draft of the manuscript was prepared by Abbas Najati. Jae-Hyeong Bae and Jafar Mohammadpour contributed equally to the conception, design, and theoretical analysis of the study. Abbas Najati, as the corresponding author, also led the coordination of the study, supervised the research process, ensured the correctness of the results, and finalized the manuscript. All authors read and approved the final version of the manuscript.

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