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Research article

Endomorphic GE-derivations

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Abstract: Using the binary operation "¶" on a GE-algebra X given by $\text{¶}\,(x, y) = (y * x) * x$ and the GE-endomorphism $\Omega : X \to X$, the notion of $\Omega_{(l,r)}$-endomorphic (resp., $\Omega_{(r,l)}$-endomorphic) GE-derivation is introduced, and several properties are investigated. Also, examples that illustrate these are provided. Conditions under which $\Omega_{(l,r)}$-endomorphic GE-derivations or $\Omega_{(l,r)}$-endomorphic GE-derivations to satisfy certain equalities and inequalities are studied. We explored the conditions under which f becomes order preserving when f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. The f-kernel and Ω-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation turns out to be GE-subalgebras. It is observed that the Ω-kernel of f is a GE-filter of X. The condition under which the f-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation becomes a GE-filter is explored.

Keywords: $\Omega_{(l,r)}$-endomorphic GE-derivation; $\Omega_{(r,l)}$-endomorphic GE-derivation; Endomorphism; GE-filter; Kernel
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1. Introduction

In the 1950s, Hilbert algebras were introduced by L. Henkin and T. Skolem as a means to investigate non-classical logics, particularly intuitionistic logic. As demonstrated by A. Diego, these algebras belong to the category of locally finite varieties, a fact highlighted in [5]. Over time, a community of scholars developed the theory of Hilbert algebras, as evidenced by works such as [3, 4, 6]. Within the realm of mathematics, the study of derivations holds a significant place in the theory of algebraic structures. This branch evolved from the principles of Galois theory and the theory of invariants. K.

H. Kim et al. extended the concept of derivations to BE-algebras, delving into properties in [10]. C. Jana et al. [7] introduced the notion of left-right (respectively, right-left) derivation, f-derivation, and generalized derivation of KUS-algebras, and their properties are established. In the broader scope of algebraic structures, the process of generalization is of utmost importance. The introduction of GE-algebras, proposed by R. K. Bandaru et al. as an extension of Hilbert algebras, marked a significant step in this direction. This advancement led to the examination of various properties, as explored in [1]. The evolution of GE-algebras was greatly influenced by filter theory. In light of this, R. K. Bandaru et al. introduced the concept of belligerent GE-filters in GE-algebras, closely investigating its attributes as documented in [2]. Rezaei et al. [11] introduced the concept of prominent GE-filters in GE-algebras. Building upon the foundation laid by Y. B. Jun et al., the concepts of ξ-inside GE-derivation and ξ-outside GE-derivation are introduced and their properties are studied. The authors established prerequisites for a self-map on a GE-algebra to qualify as both a ξ-inside and ξ-outside GE-derivation. The conditions for an order-preserving ξ-inside GE-derivation and a ξ-outside GE-derivation were thoroughly explored, as detailed in [8].

In this paper, we introduce the notion of $\Omega_{(l,r)}$-endomorphic (resp., $\Omega_{(r,l)}$-endomorphic) GE-derivation using the binary operation "$\P$" on a GE-algebra X given by $\P\,(x,y) = (y * x) * x$ and the GE-endomorphism $\Omega : X \to X$ and investigate several properties. We study the conditions under which $\Omega_{(l,r)}$-endomorphic GE-derivations or $\Omega_{(l,r)}$-endomorphic GE-derivations to satisfy certain equalities and inequalities. We explore the conditions under which f becomes order preserving when f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. We observe that the f-kernel of f and the Ω-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation are GE-subalgebras. Also, we observe that the Ω-kernel of f is a GE-filter of X, but the f-kernel of f is not a GE-filter of X. Finally, we explore the condition under which the f-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation becomes a GE-filter.

2. Preliminaries

Definition 2.1 ([1]). A *GE-algebra* is a non-empty set X with a constant "1" and a binary operation "$*$" satisfying the following axioms:

(GE1) $\mathfrak{u} * \mathfrak{u} = 1$,

(GE2) $1 * \mathfrak{u} = \mathfrak{u}$,

(GE3) $\mathfrak{u} * (\mathfrak{v} * \mathfrak{w}) = \mathfrak{u} * (\mathfrak{v} * (\mathfrak{u} * \mathfrak{w}))$

for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,(\mathfrak{u} \leq \mathfrak{v} \;\Leftrightarrow\; \mathfrak{u} * \mathfrak{v} = 1)\,. \tag{2.1}$$

Definition 2.2 ([1,2]). A GE-algebra X is said to be

- *Transitive* if it satisfies:

$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} * \mathfrak{v} \leq (\mathfrak{w} * \mathfrak{u}) * (\mathfrak{w} * \mathfrak{v}))\,. \tag{2.2}$$

- *Commutative* if it satisfies:

$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,((\mathfrak{u} * \mathfrak{v}) * \mathfrak{v} = (\mathfrak{v} * \mathfrak{u}) * \mathfrak{u})\,. \tag{2.3}$$

Proposition 2.3 ([1]). *Every GE-algebra X satisfies the following items:*

$$(\forall \mathfrak{u} \in X)\,(\mathfrak{u} * 1 = 1)\,. \tag{2.4}$$
$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,(\mathfrak{u} * (\mathfrak{u} * \mathfrak{v}) = \mathfrak{u} * \mathfrak{v})\,. \tag{2.5}$$
$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,(\mathfrak{u} \le \mathfrak{v} * \mathfrak{u})\,. \tag{2.6}$$
$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} * (\mathfrak{v} * \mathfrak{w}) \le \mathfrak{v} * (\mathfrak{u} * \mathfrak{w}))\,. \tag{2.7}$$
$$(\forall \mathfrak{u} \in X)\,(1 \le \mathfrak{u} \;\Rightarrow\; \mathfrak{u} = 1)\,. \tag{2.8}$$
$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,(\mathfrak{u} \le (\mathfrak{v} * \mathfrak{u}) * \mathfrak{u})\,. \tag{2.9}$$
$$(\forall \mathfrak{u}, \mathfrak{v} \in X)\,(\mathfrak{u} \le (\mathfrak{u} * \mathfrak{v}) * \mathfrak{v})\,. \tag{2.10}$$
$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} \le \mathfrak{v} * \mathfrak{w} \;\Leftrightarrow\; \mathfrak{v} \le \mathfrak{u} * \mathfrak{w})\,. \tag{2.11}$$

If X is transitive, then

$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} \le \mathfrak{v} \;\Rightarrow\; \mathfrak{w} * \mathfrak{u} \le \mathfrak{w} * \mathfrak{v},\; \mathfrak{v} * \mathfrak{w} \le \mathfrak{u} * \mathfrak{w})\,. \tag{2.12}$$
$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} * \mathfrak{v} \le (\mathfrak{v} * \mathfrak{w}) * (\mathfrak{u} * \mathfrak{w}))\,. \tag{2.13}$$
$$(\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in X)\,(\mathfrak{u} \le \mathfrak{v},\; \mathfrak{v} \le \mathfrak{w} \;\Rightarrow\; \mathfrak{u} \le \mathfrak{w})\,. \tag{2.14}$$

Definition 2.4 ([1]). A non-empty subset S of X is called a *GE-subalgebra* of X if it satisfies:

$$(\forall x, y \in X)(x, y \in S \;\Rightarrow\; x * y \in S). \tag{2.15}$$

Definition 2.5 ([1]). A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.16}$$
$$(\forall x, y \in X)(x * y \in F,\; x \in F \;\Rightarrow\; y \in F). \tag{2.17}$$

3. Endomorphic GE-derivations

In what follows, given a self-mapping f on a GE-algebra X, the image of $x \in X$ under f is denoted by f_x for the convenience, and let X denote a GE-algebra unless otherwise specified.

A self mapping $\Omega : X \to X$ is called a *GE-endomorphism* if $\Omega_{x*y} = \Omega_x * \Omega_y$ for all $x, y \in X$.

It is clear that if Ω is a GE-endomorphism, then $\Omega_1 = 1$.

We define a binary operation "$\Lsh$" on X as follows:

$$\Lsh : X \times X \to X,\; (x, y) \mapsto (y * x) * x. \tag{3.1}$$

Lemma 3.1. *The binary operations "$\Lsh$" on a GE-algebra X satisfies:*

$$(\forall \mathfrak{u} \in X)(x \Lsh 1 = 1 = 1 \Lsh x), \tag{3.2}$$
$$(\forall \mathfrak{u} \in X)(x \Lsh x = x). \tag{3.3}$$

Proof. Straightforward. □

Using the binary operation "$\urcorner$" and the GE-endomorphism $\Omega : X \to X$, we will define endomorphic GE-derivations on X and study its properties.

Definition 3.2. A mapping $f : X \to X$ is called an $\Omega_{(l,r)}$*-endomorphic GE-derivation* on X if there exists a GE-endomorphism $\Omega : X \to X$ satisfying the following condition:

$$(\forall x, y \in X)(f_{x*y} = (\Omega_x * f_y) \urcorner (f_x * \Omega_y)). \tag{3.4}$$

Definition 3.3. A mapping $f : X \to X$ is called an $\Omega_{(r,l)}$*-endomorphic GE-derivation* on X if there exists a GE-endomorphism $\Omega : X \to X$ satisfying the following condition:

$$(\forall x, y \in X)(f_{x*y} = (f_x * \Omega_y) \urcorner (\Omega_x * f_y)). \tag{3.5}$$

Remark 3.4. It is clear that if X is a commutative GE-algebra, then the two concepts of $\Omega_{(l,r)}$-endomorphic GE-derivation and $\Omega_{(r,l)}$-endomorphic GE-derivation are consistent.

Example 3.5. (i) Let $X = \{1, \top_1, \top_2, \top_3, \top_4\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	$\top_1$	$\top_2$	$\top_3$	$\top_4$
1	1	$\top_1$	$\top_2$	$\top_3$	$\top_4$
$\top_1$	1	1	$\top_2$	$\top_4$	$\top_4$
$\top_2$	1	1	1	$\top_3$	$\top_3$
$\top_3$	1	$\top_1$	$\top_2$	1	1
$\top_4$	1	1	$\top_2$	1	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top_1, \top_3, \top_4\}, \\ \top_2 & \text{if } x = \top_2. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top_1\}, \\ \top_2 & \text{if } x = \top_2, \\ \top_3 & \text{if } x \in \{\top_3, \top_4\}. \end{cases}$$

Then, f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X.

(ii) Let $X = \{1, \top_1, \top_2, \top_3, \top_4\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	$\top_1$	$\top_2$	$\top_3$	$\top_4$
1	1	$\top_1$	$\top_2$	$\top_3$	$\top_4$
$\top_1$	1	1	$\top_2$	$\top_3$	1
$\top_2$	1	$\top_4$	1	1	$\top_4$
$\top_3$	1	$\top_1$	1	1	$\top_1$
$\top_4$	1	1	$\top_2$	$\top_3$	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \text{ㅜ}_1, \text{ㅜ}_3, \text{ㅜ}_4\}, \\ \text{ㅜ}_2 & \text{if } x = \text{ㅜ}_2. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \text{ㅜ}_1, \text{ㅜ}_4\}, \\ \text{ㅜ}_2 & \text{if } x = \text{ㅜ}_2, \\ \text{ㅜ}_3 & \text{if } x = \text{ㅜ}_3. \end{cases}$$

Then, f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X.

(iii) Let $X = \{1, \text{ㅜ}_1, \text{ㅜ}_2, \text{ㅜ}_3, \text{ㅜ}_4\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	ㅜ_1	ㅜ_2	ㅜ_3	ㅜ_4
1	1	ㅜ_1	ㅜ_2	ㅜ_3	ㅜ_4
ㅜ_1	1	1	ㅜ_2	ㅜ_3	ㅜ_3
ㅜ_2	1	1	1	ㅜ_4	ㅜ_4
ㅜ_3	1	ㅜ_1	1	1	1
ㅜ_4	1	ㅜ_1	1	1	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \text{ㅜ}_1\}, \\ \text{ㅜ}_1 & \text{if } x = \text{ㅜ}_2, \\ \text{ㅜ}_4 & \text{if } x \in \{\text{ㅜ}_3, \text{ㅜ}_4\}. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \text{ㅜ}_1\}, \\ \text{ㅜ}_2 & \text{if } x = \text{ㅜ}_2, \\ \text{ㅜ}_4 & \text{if } x \in \{\text{ㅜ}_3, \text{ㅜ}_4\}. \end{cases}$$

Then, f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. But, it is not an $\Omega_{(r,l)}$-endomorphic GE-derivation on X since

$$\begin{aligned} (f_{\text{ㅜ}_1} * \Omega_{\text{ㅜ}_2}) \Lsh (\Omega_{\text{ㅜ}_1} * f_{\text{ㅜ}_2}) &= ((\Omega_{\text{ㅜ}_1} * f_{\text{ㅜ}_2}) * (f_{\text{ㅜ}_1} * \Omega_{\text{ㅜ}_2})) * (f_{\text{ㅜ}_1} * \Omega_{\text{ㅜ}_2}) \\ &= ((1 * \text{ㅜ}_1) * (1 * \text{ㅜ}_2)) * (1 * \text{ㅜ}_2) \\ &= (\text{ㅜ}_1 * \text{ㅜ}_2) * \text{ㅜ}_2 = \text{ㅜ}_2 * \text{ㅜ}_2 = 1 \\ &\neq \text{ㅜ}_1 = f_{\text{ㅜ}_2} = f_{\text{ㅜ}_1 * \text{ㅜ}_2}. \end{aligned}$$

(iv) Let $X = \{1, \text{ㅜ}_1, \text{ㅜ}_2, \text{ㅜ}_3, \text{ㅜ}_4, \text{ㅜ}_5\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	ㅜ_1	ㅜ_2	ㅜ_3	ㅜ_4	ㅜ_5
1	1	ㅜ_1	ㅜ_2	ㅜ_3	ㅜ_4	ㅜ_5
ㅜ_1	1	1	ㅜ_2	ㅜ_5	ㅜ_4	ㅜ_5
ㅜ_2	1	ㅜ_1	1	ㅜ_3	ㅜ_3	ㅜ_3
ㅜ_3	1	ㅜ_1	ㅜ_2	1	ㅜ_2	1
ㅜ_4	1	ㅜ_1	1	1	1	1
ㅜ_5	1	ㅜ_1	ㅜ_2	1	ㅜ_2	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top_3, \top_5\}, \\ \top_1 & \text{if } x = \top_1, \\ \top_3 & \text{if } x \in \{\top_2, \top_4\}. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top_3, \top_5\}, \\ \top_1 & \text{if } x = \top_1, \\ \top_4 & \text{if } x \in \{\top_2, \top_4\}. \end{cases}$$

Then, f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. But, it is not an $\Omega_{(l,r)}$-endomorphic GE-derivation on X:

$$\begin{aligned} f_{\top_1 * \top_2} &= f_{\top_2} = \top_3 \neq \top_5 = 1 * \top_5 = (\top_4 * \top_5) * \top_5 \\ &= (\top_1 * \top_4) * (\top_1 * \top_3)) * (\top_1 * \top_3) \\ &= ((f_{\top_1} * \Omega_{\top_2}) * (\Omega_{\top_1} * f_{\top_2})) * (\Omega_{\top_1} * f_{\top_2}) \\ &= (\Omega_{\top_1} * f_{\top_2}) \Lsh (f_{\top_1} * \Omega_{\top_2}). \end{aligned}$$

Proposition 3.6. *If $f : X \to X$ is a GE-endomorphism, then it is both an $f_{(r,l)}$-endomorphic GE-derivation and an $f_{(l,r)}$-endomorphic GE-derivation X.*

Proof. If $f : X \to X$ is a GE-endomorphism, then

$$f_{x*y} = f_x * f_y = (f_x * f_y) \Lsh (f_x * f_y)$$

for all $x, y \in X$. This completes the proof. □

Proposition 3.7. *If $f : X \to X$ is an $\Omega_{(r,l)}$-endomorphic GE-derivation or an $\Omega_{(l,r)}$-endomorphic GE-derivation on X, then $f_1 = 1$.*

Proof. Assume that $f : X \to X$ is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. Then,

$$f_1 = f_{x*1} = (f_x * \Omega_1) \Lsh (\Omega_x * f_1) = (f_x * 1) \Lsh (\Omega_x * f_1) = 1 \Lsh (\Omega_x * f_1) = 1$$

by (2.4) and Lemma 3.1. If $f : X \to X$ is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X, then

$$f_1 = f_{x*1} = (\Omega_x * f_1) \Lsh (f_x * \Omega_1) = (\Omega_x * f_1) \Lsh (f_x * 1) = (\Omega_x * f_1) \Lsh 1 = 1$$

by (2.4) and Lemma 3.1. □

Proposition 3.8. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then $f_{1 \Lsh x} = 1 = f_{x \Lsh 1}$ for all $x \in X$.*

Proof. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation, then

$$\begin{aligned} f_{1 \Lsh x} = f_{(x*1)*1} &= (\Omega_{x*1} * f_1) \Lsh (f_{x*1} * \Omega_1) \\ &= (\Omega_1 * f_1) \Lsh (f_1 * 1) = (1 * f_1) \Lsh 1 = 1 \end{aligned}$$

for all $x \in X$ by (2.4) and Lemma 3.1. Suppose that f is an $\Omega_{(r,l)}$-endomorphic GE-derivation. Then,

$$\begin{aligned} f_{1 \Lsh x} &= f_{(x*1)*1} = (f_{x*1} * \Omega_1) \Lsh (\Omega_{x*1} * f_1) \\ &= (f_1 * \Omega_1) \Lsh (\Omega_1 * f_1) = (f_1 * 1) \Lsh (1 * f_1) \\ &= 1 \Lsh f_1 = 1 \end{aligned}$$

for all $x \in X$ by (GE2), (2.4), and Lemma 3.1. Similarly, we can show that $1 = f_{x \Lsh 1}$ for all $x \in X$. □

Lemma 3.9. *Every $\Omega_{(r,l)}$-endomorphic GE-derivation f on X satisfies:*

$$(\forall x \in X)\ (f_x = \Omega_x \Lsh f_x)). \tag{3.6}$$

Proof. If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then

$$f_x = f_{1*x} = (f_1 * \Omega_x) \Lsh (\Omega_1 * f_x) = (1 * \Omega_x) \Lsh (1 * f_x) = \Omega_x \Lsh f_x$$

for all $x \in X$ by (GE2) and Proposition 3.7. □

The Eq (3.6) is not valid if f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. In fact, the $\Omega_{(l,r)}$-endomorphic GE-derivation f in Example 3.5(iii) does not satisfy (3.6) since

$$f_{\mathsf{T2}} = \mathsf{T1} \neq 1 = \mathsf{T2} * \mathsf{T2} = (\mathsf{T1} * \mathsf{T2}) * \mathsf{T2} = (f_{\mathsf{T2}} * \Omega_{\mathsf{T2}}) * \Omega_{\mathsf{T2}} = \Omega_{\mathsf{T2}} \Lsh f_{\mathsf{T2}}.$$

Proposition 3.10. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then*

$$(\forall x \in X)(f_x = f_x \Lsh \Omega_x). \tag{3.7}$$

Proof. Assume that f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. Using (GE2) and Proposition 3.7 induces

$$f_x = f_{1*x} = (\Omega_1 * f_x) \Lsh (f_1 * \Omega_x) = (1 * f_x) \Lsh (1 * \Omega_x) = f_x \Lsh \Omega_x$$

for all $x \in X$. If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then

$$\begin{aligned} \Omega_x * f_x &\overset{(3.6)}{=} \Omega_x * (\Omega_x \Lsh f_x) \overset{(3.1)}{=} \Omega_x * ((f_x * \Omega_x) * \Omega_x) \\ &\overset{(GE3)}{=} \Omega_x * ((f_x * \Omega_x) * (\Omega_x * \Omega_x)) \\ &\overset{(GE1)}{=} \Omega_x * ((f_x * \Omega_x) * 1) \\ &\overset{(2.4)}{=} \Omega_x * 1 \overset{(2.4)}{=} 1. \end{aligned}$$

It follows from (GE2) that $f_x = 1 * f_x = (\Omega_x * f_x) * f_x = f_x \Lsh \Omega_x$ for all $x \in X$. □

Proposition 3.11. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation f on X, then the following equation is valid:*

$$(\forall x \in X)(f_{\Omega_x * f_x} = 1). \tag{3.8}$$

Proof. Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. Then,

$$\begin{aligned} f_{\Omega_x * f_x} &\overset{(3.6)}{=} f_{\Omega_x*(\Omega_x \Lsh f_x)} \overset{(3.1)}{=} f_{\Omega_x*((f_x*\Omega_x)*\Omega_x)} \\ &\overset{(GE3)}{=} f_{\Omega_x*((f_x*\Omega_x)*(\Omega_x*\Omega_x))} \\ &\overset{(GE1)}{=} f_{\Omega_x*((f_x*\Omega_x)*1)} \\ &\overset{(2.4)}{=} f_{\Omega_x*1} \overset{(2.4)}{=} f_1 = 1 \end{aligned}$$

for all $x \in X$. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X, then

$$\begin{aligned} f_{\Omega_x * f_x} &\overset{(3.7)}{=} f_{\Omega_x*(f_x \Lsh \Omega_x)} \overset{(3.1)}{=} f_{\Omega_x*((\Omega_x*f_x)*f_x)} \\ &\overset{(GE3)}{=} f_{\Omega_x*((\Omega_x*f_x)*(\Omega_x*f_x))} \\ &\overset{(GE1)}{=} f_{\Omega_x*1} \overset{(2.4)}{=} f_1 = 1 \end{aligned}$$

for all $x \in X$. □

Proposition 3.12. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then the following assertion is valid:*

$$(\forall x \in X)(\Omega_x \leq f_x). \tag{3.9}$$

Proof. Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. For every $x \in X$, we have

$$\begin{aligned} \Omega_x * f_x &\overset{(3.7)}{=} \Omega_x * (f_x \Lsh \Omega_x) \overset{(3.1)}{=} \Omega_x * ((\Omega_x * f_x) * f_x) \\ &\overset{(GE3)}{=} \Omega_x * ((\Omega_x * f_x) * (\Omega_x * f_x)) \\ &\overset{(GE1)}{=} \Omega_x * 1 \overset{(2.4)}{=} 1, \end{aligned}$$

and so (3.9) is valid. Assume that f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. Then,

$$\begin{aligned} \Omega_x * f_x &\overset{(3.6)}{=} \Omega_x * (\Omega_x \Lsh f_x) \overset{(3.1)}{=} \Omega_x * ((f_x * \Omega_x) * \Omega_x) \\ &\overset{(GE3)}{=} \Omega_x * ((f_x * \Omega_x) * (\Omega_x * \Omega_x)) \\ &\overset{(GE1)}{=} \Omega_x * ((f_x * \Omega_x) * 1) \\ &\overset{(2.4)}{=} \Omega_x * 1 \overset{(2.4)}{=} 1 \end{aligned}$$

for all $x, y \in X$. Thus, (3.9) is valid. □

Proposition 3.13. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on a transitive GE-algebra X, then the following assertion is valid:*

$$(\forall x, y \in X)(f_x * \Omega_y \leq \Omega_x * f_y). \tag{3.10}$$

Proof. Suppose that X is a transitive GE-algebra, and let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. Then, the combination of (2.12) and (3.9) leads to the following assertion:

$$f_x * \Omega_y \leq \Omega_x * \Omega_y \leq \Omega_x * f_y$$

and thus $f_x * \Omega_y \leq \Omega_x * f_y$ for all $x, y \in X$ by (2.14). □

The following example shows that (3.10) is not valid in Proposition 3.13 if the condition "X is transitive" is omitted.

Example 3.14. Let $X = \{1, \mathsf{T1}, \mathsf{T2}, \mathsf{T3}, \mathsf{T4}\}$ be a set with a binary operation " $*$ " given in the following table:

$*$	1	$\mathsf{T1}$	$\mathsf{T2}$	$\mathsf{T3}$	$\mathsf{T4}$
1	1	$\mathsf{T1}$	$\mathsf{T2}$	$\mathsf{T3}$	$\mathsf{T4}$
$\mathsf{T1}$	1	1	1	$\mathsf{T3}$	1
$\mathsf{T2}$	1	$\mathsf{T1}$	1	1	$\mathsf{T1}$
$\mathsf{T3}$	1	$\mathsf{T4}$	$\mathsf{T2}$	1	$\mathsf{T4}$
$\mathsf{T4}$	1	1	1	1	1

Then, X is a GE-algebra which is not transitive. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \mathsf{T3}\}, \\ \mathsf{T1} & \text{if } x \in \{\mathsf{T1}, \mathsf{T4}\}, \\ \mathsf{T3} & \text{if } x = \mathsf{T2}. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \mathsf{T3}\}, \\ \mathsf{T1} & \text{if } x \in \{\mathsf{T1}, \mathsf{T4}\}, \\ \mathsf{T2} & \text{if } x = \mathsf{T2}. \end{cases}$$

Then, f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. But, f does not satisfy (3.10) since $(f_{\mathsf{T1}} * \Omega_{\mathsf{T2}}) * (\Omega_{\mathsf{T1}} * f_{\mathsf{T2}}) = (\mathsf{T1} * \mathsf{T2}) * (\mathsf{T1} * \mathsf{T3}) = 1 * \mathsf{T3} = \mathsf{T3} \neq 1$, that is, $f_{\mathsf{T1}} * \Omega_{\mathsf{T2}} \not\leq \Omega_{\mathsf{T1}} * f_{\mathsf{T2}}$.

Example 3.15. Let $X = \{1, \mathsf{T1}, \mathsf{T2}, \mathsf{T3}, \mathsf{T4}, \mathsf{T5}\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	$\mathsf{T1}$	$\mathsf{T2}$	$\mathsf{T3}$	$\mathsf{T4}$	$\mathsf{T5}$
1	1	$\mathsf{T1}$	$\mathsf{T2}$	$\mathsf{T3}$	$\mathsf{T4}$	$\mathsf{T5}$
$\mathsf{T1}$	1	1	$\mathsf{T5}$	1	1	$\mathsf{T5}$
$\mathsf{T2}$	1	1	1	1	1	1
$\mathsf{T3}$	1	$\mathsf{T4}$	1	1	$\mathsf{T4}$	1
$\mathsf{T4}$	1	$\mathsf{T3}$	1	$\mathsf{T3}$	1	1
$\mathsf{T5}$	1	1	1	1	1	1

Then, X is a GE-algebra which is not transitive. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top_4, \top_5\}, \\ \top_3 & \text{if } x \in \{\top_1, \top_3\}, \\ \top_2 & \text{if } x = \top_2. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ \top_1 & \text{if } x = \top_1, \\ \top_2 & \text{if } x = \top_2, \\ \top_3 & \text{if } x = \top_3, \\ \top_4 & \text{if } x = \top_4, \\ \top_5 & \text{if } x = \top_5. \end{cases}$$

Then, f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. But, f does not satisfy (3.10) since $(f_{\top_1} * \Omega_{\top_2}) * (\Omega_{\top_1} * f_{\top_2}) = (\top_3 * \top_2) * (\top_1 * \top_2) = 1 * \top_5 = \top_5 \neq 1$, that is, $f_{\top_1} * \Omega_{\top_2} \not\leq \Omega_{\top_1} * f_{\top_2}$.

Let f and Ω be self-maps on X, and consider the following equality:

$$(\forall x, y \in X)(f_{x*y} = \Omega_x * f_y). \tag{3.11}$$

Question 3.16. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then does Eq (3.11) work?

The answer to Question 3.16 is negative and confirmed in the following examples.

Example 3.17. (i) In Example 3.14, we can observe that X is a GE-algebra which is not commutative. Also, the $\Omega_{(l,r)}$-endomorphic GE-derivation f does not satisfy (3.11) since $f_{\top_1 * \top_2} = f_1 = 1 \neq \top_3 = \top_1 * \top_3 = \Omega_{\top_1} * f_{\top_2}$.

(ii) In Example 3.15, we can observe that X is a GE-algebra which is not commutative. Also, $\Omega_{(r,l)}$-endomorphic GE-derivation f does not satisfy (3.11) since $f_{\top_1 * \top_2} = f_{\top_5} = 1 \neq \top_5 = \top_1 * \top_2 = \Omega_{\top_1} * f_{\top_2}$.

We explore conditions under which the answer to Question 3.16 will be positive.

Theorem 3.18. *If X is a commutative GE-algebra, then every $\Omega_{(r,l)}$-endomorphic GE-derivation f on X satisfies Eq* (3.11).

Proof. Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation on a commutative GE-algebra X. Since X is commutative, it is also transitive (see [9]). Hence,

$$\begin{aligned} f_{x*y} &\overset{(3.5)}{=} (f_x * \Omega_y) \Lsh (\Omega_x * f_y) \overset{(3.1)}{=} ((\Omega_x * f_y) * (f_x * \Omega_y)) * (f_x * \Omega_y) \\ &\overset{(2.3)}{=} ((f_x * \Omega_y) * (\Omega_x * f_y)) * (\Omega_x * f_y) \\ &\overset{(3.10)}{=} 1 * (\Omega_x * f_y) \overset{(GE2)}{=} \Omega_x * f_y \end{aligned}$$

for all $x, y \in X$. □

Based on Remark 3.4, the following is the corollary of Theorem 3.18.

Corollary 3.19. *If X is a commutative GE-algebra, then every $\Omega_{(l,r)}$-endomorphic GE-derivation f on X satisfies equality* (3.11).

Question 3.20. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then are the two self-maps f and Ω consistent?

The answer to Question 3.20 is negative and confirmed in the following example.

Example 3.21. (i) If we take the $\Omega_{(l,r)}$-endomorphic GE-derivation f in Example 3.5(i), then $f_{\top 3} = 1 \neq \top 3 = \Omega_{\top 3}$.

(ii) If we take the $\Omega_{(r,l)}$-endomorphic GE-derivation f in Example 3.5(ii), then $f_{\top 3} = 1 \neq \top 3 = \Omega_{\top 3}$.

Given two self-maps f and Ω on X, consider the following equation:

$$(\forall x, y \in X)(f_{x*y} = f_x * \Omega_y). \tag{3.12}$$

If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(r,l)}$-endomorphic GE-derivation on a commutative GE-algebra X, then f may not satisfy (3.12).

Example 3.22. Let $X = \{1, \top 1, \top 2, \top 3, \top 4\}$ be a set with a binary operation " $*$ " given in the following table:

$*$	1	$\top 1$	$\top 2$	$\top 3$	$\top 4$
1	1	$\top 1$	$\top 2$	$\top 3$	$\top 4$
$\top 1$	1	1	$\top 2$	$\top 3$	$\top 4$
$\top 2$	1	$\top 1$	1	$\top 3$	$\top 4$
$\top 3$	1	$\top 1$	$\top 2$	1	$\top 4$
$\top 4$	1	$\top 1$	$\top 2$	$\top 3$	1

Then, X is a commutative GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top 1, \top 2\}, \\ \top 2 & \text{if } x = \top 3, \\ \top 4 & \text{if } x = \top 4. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ \top 1 & \text{if } x = \top 1, \\ \top 3 & \text{if } x = \top 2, \\ \top 2 & \text{if } x = \top 3, \\ \top 4 & \text{if } x = \top 4. \end{cases}$$

Then, f is both an $\Omega_{(l,r)}$-endomorphic GE-derivation and an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. But, f does not satisfy (3.12) since $f_{\top 1 * \top 1} = f_1 = 1 \neq \top 1 = 1 * \top 1 = f_{\top 1} * \Omega_{\top 1}$.

The following example shows that there is an $\Omega_{(l,r)}$-endomorphic GE-derivation f or $\Omega_{(r,l)}$-endomorphic GE-derivation f on X that would not normally establish Eq (3.12).

Example 3.23. (i) The $\Omega_{(l,r)}$-endomorphic GE-derivation f in Example 3.5(i) does not satisfy the Eq (3.12) since

$$f_{\top 1 * \top 3} = f_{\top 4} = 1 \neq \top 3 = 1 * \top 3 = f_{\top 1} * \Omega_{\top 3}.$$

(ii) The $\Omega_{(r,l)}$-endomorphic GE-derivation f in Example 3.5(ii) does not satisfy the Eq (3.12) since

$$f_{\top 1 * \top 3} = f_{\top 3} = 1 \neq \top 3 = 1 * \top 3 = f_{\top 1} * \Omega_{\top 3}.$$

We investigate the conditions under which two self-maps f and Ω match in X.

Theorem 3.24. *If an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation f on X satisfies Eq* (3.12)*, then f matches Ω.*

Proof. Assume that f satisfies Eq (3.12). Then,

$$f_x \overset{(GE2)}{=} f_{1*x} \overset{(3.12)}{=} f_1 * \Omega_x \overset{\text{Proposition 3.7}}{=} 1 * \Omega_x \overset{(GE2)}{=} \Omega_x$$

for all $x \in X$. Hence, f matches Ω. □

If an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation f on X satisfies the Eq (3.11), then f may not match Ω.

Example 3.25. Let $X = \{1, \top 1, \top 2, \top 3, \top 4\}$ be a set with a binary operation " $*$ " given in the following table:

$*$	1	$\top 1$	$\top 2$	$\top 3$	$\top 4$
1	1	$\top 1$	$\top 2$	$\top 3$	$\top 4$
$\top 1$	1	1	1	$\top 3$	$\top 3$
$\top 2$	1	1	1	$\top 4$	$\top 4$
$\top 3$	1	$\top 1$	$\top 2$	1	1
$\top 4$	1	$\top 1$	$\top 2$	1	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top 1, \top 3, \top 4\}, \\ \top 1 & \text{if } x = \top 2. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top 3, \top 4\}, \\ \top 1 & \text{if } x \in \{\top 1, \top 2\}. \end{cases}$$

Then, f is both an $\Omega_{(l,r)}$-endomorphic GE-derivation and an $\Omega_{(r,l)}$-endomorphic GE-derivation on X satisfying (3.11). But, f does not match with Ω since $f_{\top 1} = 1 \neq \top 1 = \Omega_{\top 1}$.

Question 3.26. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then does the following equation work?

$$(\forall x, y \in X)(\Omega_x * f_y = f_x * \Omega_y). \tag{3.13}$$

The answer to Question 3.26 is negative and confirmed in the following examples.

Example 3.27. (i) The $\Omega_{(l,r)}$-endomorphic GE-derivation f in Example 3.5(i) does not satisfy Eq (3.13) since

$$\Omega_{\top 2} * f_{\top 3} = \top 2 * 1 = 1 \neq \top 3 = \top 2 * \top 3 = f_{\top 2} * \Omega_{\top 3}.$$

(ii) The $\Omega_{(r,l)}$-endomorphic GE-derivation f in Example 3.5(ii) does not satisfy Eq (3.13) since

$$\Omega_{\top 4} * f_{\top 3} = 1 * 1 = 1 \neq \top 3 = 1 * \top 3 = f_{\top 4} * \Omega_{\top 3}.$$

Lemma 3.28. *Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. If it satisfies* (3.13)*, then Eq* (3.12) *is valid.*

Proof. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X satisfying (3.13), then

$$\begin{aligned} f_{x*y} &\overset{(3.4)}{=} (\Omega_x * f_y) \Lsh (f_x * \Omega_y) \\ &\overset{(3.1)}{=} ((f_x * \Omega_y) * (\Omega_x * f_y)) * (\Omega_x * f_y) \\ &\overset{(3.13)}{=} ((f_x * \Omega_y) * (f_x * \Omega_y)) * (f_x * \Omega_y) \\ &\overset{(GE1)}{=} 1 * (f_x * \Omega_y) \overset{(GE2)}{=} f_x * \Omega_y \end{aligned}$$

for all $x, y \in X$. Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation on X satisfying (3.13). Then,

$$\begin{aligned} f_{x*y} &\overset{(3.5)}{=} (f_x * \Omega_y) \Lsh (\Omega_x * f_y) \\ &\overset{(3.1)}{=} ((\Omega_x * f_y) * (f_x * \Omega_y)) * (f_x * \Omega_y) \\ &\overset{(3.13)}{=} ((f_x * \Omega_y) * (f_x * \Omega_y)) * (f_x * \Omega_y) \\ &\overset{(GE1)}{=} 1 * (f_x * \Omega_y) \overset{(GE2)}{=} f_x * \Omega_y \end{aligned}$$

for all $x, y \in X$. □

Corollary 3.29. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then f matches Ω if and only if Eq* (3.13) *holds.*

Proof. Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. Suppose f matches Ω and $x, y \in X$. Then, $f_x = \Omega_x$ for all $x \in X$, and hence $f_x * \Omega_y = \Omega_x * f_y$. Conversely, assume that Eq (3.13) holds. Let $x \in X$. Then, $f_x = 1 * f_x = \Omega_1 * f_x = f_1 * \Omega_x = 1 * \Omega_x = \Omega_x$, which is true for all $x \in X$. Hence, f matches Ω. □

Question 3.30. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then is f order preserving?

The answer to Question 3.30 is negative and confirmed in the following examples.

Example 3.31. (i) From Example 3.5(iii), the map f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. We can observe that $\top 4 \leq \top 2$ and $f_{\top 4} * f_{\top 2} = \top 4 * \top 1 = \top 1 \neq 1$, i.e., $f_{\top 4} \nleq f_{\top 2}$. Hence, f is not order preserving.

(ii) From Example 3.5(ii), the map f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. We can observe that $\tau_3 \leq \tau_2$ and $f_{\tau_3} * f_{\tau_2} = 1 * \tau_2 = \tau_2 \neq 1$, that is, $f_{\tau_3} \nleq f_{\tau_2}$. Hence, f is not order preserving.

(iii) Let $X = \{1, \tau_1, \tau_2, \tau_3, \tau_4\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	τ_1	τ_2	τ_3	τ_4
1	1	τ_1	τ_2	τ_3	τ_4
τ_1	1	1	1	τ_4	τ_4
τ_2	1	1	1	τ_3	τ_3
τ_3	1	1	1	1	1
τ_4	1	τ_1	1	1	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \tau_1\}, \\ \tau_1 & \text{if } x = \tau_2, \\ \tau_3 & \text{if } x = \tau_3, \\ \tau_4 & \text{if } x = \tau_4. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ \tau_1 & \text{if } x = \tau_1, \\ \tau_2 & \text{if } x = \tau_2, \\ \tau_3 & \text{if } x = \tau_3, \\ \tau_4 & \text{if } x = \tau_4. \end{cases}$$

Then, f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. We can observe that $\tau_1 \leq \tau_2$ and $f_{\tau_1} * f_{\tau_2} = 1 * \tau_1 = \tau_1 \neq 1$, i.e., $f_{\tau_1} \nleq f_{\tau_2}$. Hence, f is not order preserving.

Now we explore the conditions under which f becomes order preserving when f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X.

Theorem 3.32. *Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation or an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. If X is transitive and f satisfies:*

$$(\forall x, y \in X)(f_x \Lsh f_y \leq f_{x \Lsh y}), \tag{3.14}$$

then f is order preserving.

Proof. Let X be a transitive GE-algebra and let $x, y \in X$ be such that $x \leq y$. Then, $y \Lsh x = (x * y) * y = 1 * y = y$. Assume that f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X satisfying (3.14). Then,

$$f_x \overset{(2.10)}{\leq} (f_x * f_y) * f_y = f_y \Lsh f_x \overset{(3.14)}{\leq} f_{y \Lsh x} = f_y$$

Hence, f is order preserving. Similarly, if f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X satisfying (3.14), then f is order preserving. □

Corollary 3.33. *Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation or an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. If X is commutative and f satisfies* (3.14)*, then f is order preserving.*

In general, an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation f on X does not satisfy (3.14) as seen in the following example.

Example 3.34. (i) Let $X = \{1, \mathsf{T}1, \mathsf{T}2, \mathsf{T}3, \mathsf{T}4, \mathsf{T}5\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	$\mathsf{T}1$	$\mathsf{T}2$	$\mathsf{T}3$	$\mathsf{T}4$	$\mathsf{T}5$
1	1	$\mathsf{T}1$	$\mathsf{T}2$	$\mathsf{T}3$	$\mathsf{T}4$	$\mathsf{T}5$
$\mathsf{T}1$	1	1	$\mathsf{T}2$	$\mathsf{T}3$	1	$\mathsf{T}3$
$\mathsf{T}2$	1	1	1	$\mathsf{T}5$	1	$\mathsf{T}5$
$\mathsf{T}3$	1	$\mathsf{T}1$	$\mathsf{T}2$	1	1	1
$\mathsf{T}4$	1	$\mathsf{T}1$	$\mathsf{T}2$	1	1	1
$\mathsf{T}5$	1	$\mathsf{T}1$	$\mathsf{T}2$	1	1	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \mathsf{T}3, \mathsf{T}4, \mathsf{T}5\}, \\ \mathsf{T}3 & \text{if } x = \mathsf{T}1, \\ \mathsf{T}1 & \text{if } x = \mathsf{T}2. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \mathsf{T}3, \mathsf{T}4, \mathsf{T}5\}, \\ \mathsf{T}4 & \text{if } x = \mathsf{T}1, \\ \mathsf{T}2 & \text{if } x = \mathsf{T}2. \end{cases}$$

Then, f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. The $\Omega_{(l,r)}$-endomorphic GE-derivation f does not satisfy (3.14), since

$$\begin{aligned} &((f_{\mathsf{T}2} * f_{\mathsf{T}1}) * f_{\mathsf{T}1}) * f_{(\mathsf{T}2 * \mathsf{T}1) * \mathsf{T}1} = ((\mathsf{T}1 * \mathsf{T}3) * \mathsf{T}3) * f_{1 * \mathsf{T}1} \\ &= (\mathsf{T}3 * \mathsf{T}3) * f_{\mathsf{T}1} = 1 * \mathsf{T}1 = \mathsf{T}1 \neq 1, \end{aligned}$$

that is, $f_{\mathsf{T}1} \Lsh f_{\mathsf{T}2} \not\leq f_{\mathsf{T}1 \Lsh \mathsf{T}2}$.

(ii) Let $X = \{1, \mathsf{T}1, \mathsf{T}2, \mathsf{T}3, \mathsf{T}4, \mathsf{T}5\}$ be a set with a binary operation "$*$" given in the following table:

$*$	1	$\mathsf{T}1$	$\mathsf{T}2$	$\mathsf{T}3$	$\mathsf{T}4$	$\mathsf{T}5$
1	1	$\mathsf{T}1$	$\mathsf{T}2$	$\mathsf{T}3$	$\mathsf{T}4$	$\mathsf{T}5$
$\mathsf{T}1$	1	1	$\mathsf{T}2$	$\mathsf{T}5$	$\mathsf{T}4$	$\mathsf{T}5$
$\mathsf{T}2$	1	1	1	$\mathsf{T}3$	$\mathsf{T}3$	$\mathsf{T}3$
$\mathsf{T}3$	1	$\mathsf{T}1$	$\mathsf{T}2$	1	$\mathsf{T}2$	1
$\mathsf{T}4$	1	1	1	1	1	1
$\mathsf{T}5$	1	$\mathsf{T}1$	$\mathsf{T}2$	1	$\mathsf{T}2$	1

Then, X is a GE-algebra. Define the self-maps:

$$f : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \mathsf{T}3, \mathsf{T}5\}, \\ \mathsf{T}1 & \text{if } x = \mathsf{T}1, \\ \mathsf{T}3 & \text{if } x \in \{\mathsf{T}2, \mathsf{T}4\}. \end{cases}$$

and

$$\Omega : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, \top 3, \top 5\}, \\ \top 1 & \text{if } x = \top 1, \\ \top 4 & \text{if } x \in \{\top 2, \top 4\}. \end{cases}$$

Then, f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. The $\Omega_{(r,l)}$-endomorphic GE-derivation f does not satisfy (3.14), since

$$\begin{aligned} &((f_{\top 2} * f_{\top 1}) * f_{\top 1}) * f_{(\top 2 * \top 1) * \top 1} = ((\top 3 * \top 1) * \top 1) * f_{1 * \top 1} \\ &= (\top 1 * \top 1) * f_{\top 1} = 1 * \top 1 = \top 1 \neq 1, \end{aligned}$$

that is, $f_{\top 1} \Lsh f_{\top 2} \nleq f_{\top 1 \Lsh \top 2}$.

Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation or an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. We consider the following set:

$$\Omega_f(X) := \{x \in X \mid f_x = \Omega_x\}. \tag{3.15}$$

Theorem 3.35. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then the set $\Omega_f(X)$ is a GE-subalgebra of X and $1 \in \Omega_f(X)$.*

Proof. Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. If $x, y \in \Omega_f(X)$, then $f_x = \Omega_x$ and $f_y = \Omega_y$. Hence,

$$\begin{aligned} f_{x*y} &\overset{(3.4)}{=} (\Omega_x * f_y) \Lsh (f_x * \Omega_y) \\ &\overset{(3.1)}{=} ((f_x * \Omega_y) * (\Omega_x * f_y)) * (\Omega_x * f_y) \\ &= ((\Omega_x * f_y) * (\Omega_x * f_y)) * (\Omega_x * f_y) \\ &\overset{(GE1)}{=} 1 * (\Omega_x * \Omega_y) \\ &\overset{(GE2)}{=} \Omega_x * \Omega_y = \Omega_{x*y}, \end{aligned}$$

and so $x * y \in \Omega_f(X)$. Hence, $\Omega_f(X)$ is a GE-subalgebra of X. Similarly, if f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then $\Omega_f(X)$ is a GE-subalgebra of X. It is clear that $1 \in \Omega_f(X)$. □

Proposition 3.36. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then the set $\Omega_f(X)$ is closed under the operation "$\Lsh$".*

Proof. Let $x, y \in \Omega_f(X)$. Then, $f_x = \Omega_x$ and $f_y = \Omega_y$. Assume that f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. Then,

$$\begin{aligned} f_{x \Lsh y} = f_{(y*x)*x} &\overset{(3.4)}{=} (\Omega_{y*x} * f_x) \Lsh (f_{y*x} * \Omega_x) \\ &\overset{(3.4)}{=} (\Omega_{y*x} * f_x) \Lsh (((\Omega_y * f_x) \Lsh (f_y * \Omega_x)) * \Omega_x) \\ &= (\Omega_{y*x} * \Omega_x) \Lsh (((\Omega_y * \Omega_x) \Lsh (\Omega_y * \Omega_x)) * \Omega_x) \\ &\overset{(3.3)}{=} (\Omega_{y*x} * \Omega_x) \Lsh ((\Omega_y * \Omega_x) * \Omega_x) \end{aligned}$$

$$= (\Omega_{y*x} * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * \Omega_x)$$
$$\overset{(3.3)}{=} \Omega_{y*x} * \Omega_x = \Omega_{(y*x)*x} = \Omega_{x\text{⸿}y},$$

and so $x \,\text{⸿}\, y \in \Omega_f(X)$. This shows that $\Omega_f(X)$ is closed under the operation "⸿". If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then

$$\begin{aligned}
f_{x\text{⸿}y} = f_{(y*x)*x} &\overset{(3.5)}{=} (f_{y*x} * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * f_x)\\
&\overset{(3.5)}{=} (((f_y * \Omega_x) \,\text{⸿}\, (\Omega_y * f_x)) * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * f_x)\\
&= (((\Omega_y * \Omega_x) \,\text{⸿}\, (\Omega_y * \Omega_x)) * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * \Omega_x)\\
&\overset{(3.3)}{=} ((\Omega_y * \Omega_x) * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * \Omega_x)\\
&= (\Omega_{y*x} * \Omega_x) \,\text{⸿}\, (\Omega_{y*x} * \Omega_x)\\
&\overset{(3.3)}{=} \Omega_{y*x} * \Omega_x = \Omega_{(y*x)*x} = \Omega_{x\text{⸿}y},
\end{aligned}$$

and so $x \,\text{⸿}\, y \in \Omega_f(X)$. This shows that $\Omega_f(X)$ is closed under the operation "⸿". □

Let f be an $\Omega_{(r,l)}$-endomorphic GE-derivation or an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. We consider the following sets:

$$\ker_X(f) := \{x \in X \mid f_x = 1\}, \tag{3.16}$$
$$\ker_X(\Omega) := \{x \in X \mid \Omega_x = 1\} \tag{3.17}$$

which is called the *f-kernel* of f and the *Ω-kernel* of f, respectively, in X.

Theorem 3.37. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then its f-kernel and its Ω-kernel are GE-subalgebras of X and* $1 \in \ker_X(f) \cap \ker_X(\Omega)$.

Proof. Let $x, y \in \ker_X(f)$. Then, $f_x = 1$ and $f_y = 1$. Assume that f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X. Then,

$$\begin{aligned}
f_{x*y} &\overset{(3.4)}{=} (\Omega_x * f_y) \,\text{⸿}\, (f_x * \Omega_y) = (\Omega_x * 1) \,\text{⸿}\, (1 * \Omega_y)\\
&\overset{(2.4)\&(GE2)}{=} 1 \,\text{⸿}\, \Omega_y \overset{(3.2)}{=} 1,
\end{aligned}$$

and so $x * y \in \ker_X(f)$. Hence, $\ker_X(f)$ is a GE-subalgebra of X. If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then

$$\begin{aligned}
f_{x*y} &\overset{(3.5)}{=} (f_x * \Omega_y) \,\text{⸿}\, (\Omega_x * f_y) = (1 * \Omega_y) \,\text{⸿}\, (\Omega_x * 1)\\
&\overset{(2.4)\&(GE2)}{=} \Omega_y \,\text{⸿}\, 1 \overset{(3.2)}{=} 1,
\end{aligned}$$

and so $x * y \in \ker_X(f)$. Hence, $\ker_X(f)$ is a GE-subalgebra of X. If $x, y \in \ker_X(\Omega)$, then $\Omega_x = 1$ and $\Omega_y = 1$. Since Ω is a GE-endomorphism, it follows that $\Omega_{x*y} = \Omega_x * \Omega_y = 1 * 1 = 1$ and $\Omega_1 = 1$. Thus, $x * y \in \ker_X(\Omega)$ and $1 \in \ker_X(\Omega)$. Hence, $\ker_X(\Omega)$ is a GE-subalgebra of X. It is clear that $1 \in \ker_X(f)$ by Proposition 3.7. Therefore, $1 \in \ker_X(f) \cap \ker_X(\Omega)$. □

The example below illustrates Theorem 3.37.

Example 3.38. (i) In Example 3.5(i), we can observe that $\ker_X(f) = \{1, \mathsf{T}_1, \mathsf{T}_3, \mathsf{T}_4\}$ and $\ker_X(\Omega) = \{1, \mathsf{T}_1\}$ are GE-subalgebras of X, and $1 \in \ker_X(f) \cap \ker_X(\Omega)$.

(ii) In Example 3.5(ii), we can observe that $\ker_X(f) = \{1, \mathsf{T}_1, \mathsf{T}_3, \mathsf{T}_4\}$ and $\ker_X(\Omega) = \{1, \mathsf{T}_1, \mathsf{T}_4\}$ are GE-subalgebras of X, and $1 \in \ker_X(f) \cap \ker_X(\Omega)$.

Proposition 3.39. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then its f-kernel satisfies:*

$$(\forall x, y \in X)(x \in \ker_X(f) \;\Rightarrow\; y * x \in \ker_X(f),\; x ⁋ y \in \ker_X(f)). \tag{3.18}$$

Proof. Let $x, y \in X$ be such that $x \in \ker_X(f)$. Then, $f_x = 1$. If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on X, then

$$f_{y*x} \overset{(3.4)}{=} (\Omega_y * f_x) ⁋ (f_y * \Omega_x) = (\Omega_y * 1) ⁋ (f_y * \Omega_x) \overset{(2.4)}{=} 1 ⁋ (f_y * \Omega_x) \overset{(3.2)}{=} 1$$

and

$$\begin{aligned} f_{x⁋y} &\overset{(3.1)}{=} f_{(y*x)*x} \overset{(3.4)}{=} (\Omega_{y*x} * f_x) ⁋ (f_{y*x} * \Omega_x) \\ &\overset{(3.4)}{=} (\Omega_{y*x} * f_x) ⁋ (((\Omega_y * f_x) ⁋ (f_y * \Omega_x)) * \Omega_x) \\ &= (\Omega_{y*x} * 1) ⁋ (((\Omega_y * 1) ⁋ (f_y * \Omega_x)) * \Omega_x) \\ &\overset{(2.4)}{=} 1 ⁋ ((1 ⁋ (f_y * \Omega_x)) * \Omega_x) \overset{(3.2)}{=} 1 \end{aligned}$$

If f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then

$$f_{y*x} \overset{(3.5)}{=} (f_y * \Omega_x) ⁋ (\Omega_y * f_x) = (f_y * \Omega_x) ⁋ (\Omega_y * 1) \overset{(2.4)}{=} (f_y * \Omega_x) ⁋ 1 \overset{(3.2)}{=} 1$$

and

$$\begin{aligned} f_{x⁋y} &\overset{(3.1)}{=} f_{(y*x)*x} \overset{(3.5)}{=} (f_{y*x} * \Omega_x) ⁋ (\Omega_{y*x} * f_x) \\ &= (f_{y*x} * \Omega_x) ⁋ (\Omega_{y*x} * 1) \overset{(2.4)}{=} (f_{y*x} * \Omega_x) ⁋ 1 \overset{(3.2)}{=} 1. \end{aligned}$$

Hence, $y * x \in \ker_X(f)$ and $x ⁋ y \in \ker_X(f)$. □

For any $\Omega_{(l,r)}$-endomorphic GE-derivation or $\Omega_{(r,l)}$-endomorphic GE-derivation f on X, its f-kernel does not satisfy the following assertions:

$$(\forall x, y \in X)(x \in \ker_X(f) \Rightarrow\; x * y \in \ker_X(f)), \tag{3.19}$$

$$(\forall x, y \in X)(x \in \ker_X(f) \Rightarrow\; y ⁋ x \in \ker_X(f)). \tag{3.20}$$

In fact, in Example 3.31(iii), we can observe that $\ker_X(f) = \{1, \mathsf{T}_1\}$. But, it does not satisfy (3.19) and (3.20) since $\mathsf{T}_1 * \mathsf{T}_3 = \mathsf{T}_4 \notin \ker_X(f)$ and

$$\mathsf{T}_2 ⁋ \mathsf{T}_1 = (\mathsf{T}_1 * \mathsf{T}_2) * \mathsf{T}_2 = 1 * \mathsf{T}_2 = \mathsf{T}_2 \notin \ker_X(f).$$

Also, in Example 3.5(ii), we can observe that $\ker_X(f) = \{1, \mathsf{T}_1, \mathsf{T}_3, \mathsf{T}_4\}$. But, it does not satisfy (3.19) and (3.20) since $\mathsf{T}_1 * \mathsf{T}_2 = \mathsf{T}_2 \notin \ker_X(f)$ and

$$\mathsf{T}_2 ⁋ \mathsf{T}_3 = (\mathsf{T}_3 * \mathsf{T}_2) * \mathsf{T}_2 = 1 * \mathsf{T}_2 = \mathsf{T}_2 \notin \ker_X(f).$$

Proposition 3.40. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then its Ω-kernel satisfies:*

$$(\forall x, y \in X)\left(x \in \ker_X(\Omega) \Rightarrow \begin{cases} y * x \in \ker_X(\Omega) \\ x \Lsh y, y \Lsh x \in \ker_X(\Omega) \end{cases} \right). \tag{3.21}$$

Proof. Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. For every $x, y \in X$, if $x \in \ker_X(\Omega)$, then $\Omega_x = 1$. Hence,

$$\Omega_{y*x} = \Omega_y * \Omega_x = \Omega_y * 1 = 1,$$

$\Omega_{x \Lsh y} = \Omega_{(y*x)*x} = \Omega_{y*x} * \Omega_x = (\Omega_y * \Omega_x) * \Omega_x = (\Omega_y * 1) * 1 = 1$ and

$$\Omega_{y \Lsh x} = \Omega_{(x*y)*y} = \Omega_{x*y} * \Omega_y = (\Omega_x * \Omega_y) * \Omega_y = (1 * \Omega_y) * \Omega_y = \Omega_y * \Omega_y = 1.$$

Therefore, $y * x \in \ker_X(\Omega)$ and $x \Lsh y$, $y \Lsh x \in \ker_X(\Omega)$. □

For any $\Omega_{(l,r)}$-endomorphic GE-derivation or $\Omega_{(r,l)}$-endomorphic GE-derivation f on X, its Ω-kernel does not satisfy the following assertions:

$$(\forall x, y \in X)(x \in \ker_X(\Omega) \Rightarrow x * y \in \ker_X(\Omega)), \tag{3.22}$$

In fact, in Example 3.5(i), we can observe that $\ker_X(\Omega) = \{1, \top_1\}$. But, it does not satisfy (3.22) since $\top_1 * \top_2 = \top_2 \notin \ker_X(\Omega)$. Also, in Example 3.5(ii), we can observe that $\ker_X(\Omega) = \{1, \top_1, \top_4\}$. But, it does not satisfy (3.22) since

$$\top_1 * \top_2 = \top_2 \notin \ker_X(\Omega).$$

Proposition 3.41. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then it satisfies:*

$$(\forall x, y \in X)(x \leq y,\ x \in \ker_X(\Omega) \Rightarrow y \in \ker_X(\Omega)). \tag{3.23}$$

Proof. Let $x, y \in X$ be such that $x \leq y$ and $x \in \ker_X(\Omega)$. Hence, $\Omega_{x*y} = 1$ by $\Omega_x = 1$, so that $\Omega_x * \Omega_y = 1$. Hence, $\Omega_y = 1 * \Omega_y = \Omega_x * \Omega_y = 1$. Therefore, $y \in \ker_X(\Omega)$. □

Remark 3.42. In an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation f on X, the following does not apply:

$$(\forall x, y \in X)(x \leq y,\ x \in \ker_X(f) \Rightarrow y \in \ker_X(f)). \tag{3.24}$$

In fact, in Example 3.31(iii), we can observe that $\ker_X(f) = \{1, \top_1\}$. But, (3.24) is not valid since $\top_1 * \top_2 = 1$, i.e., $\top_1 \leq \top_2$) and $\top_1 \in \ker_X(f)$, but $\top_2 \notin \ker_X(f)$. Also, in Example 3.5(ii), we can observe that $\ker_X(f) = \{1, \top_1, \top_3, \top_4\}$. But, (3.24) is not valid since $\top_3 * \top_2 = 1$, i.e., $\top_3 \leq \top_2$ and $\top_3 \in \ker_X(f)$, but $\top_2 \notin \ker_X(f)$.

Proposition 3.43. *Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. If X is commutative, then it satisfies* (3.24).

Proof. Let $x, y \in X$ be such that $x \leq y$. Then, $x * y = 1$. Assume that f is an $\Omega_{(l,r)}$-endomorphic GE-derivation on a commutative GE-algebra X. If $x \in \ker_X(f)$, then $f_x = 1$, and so

$$\begin{aligned} f_y &\overset{(GE2)}{=} f_{1*y} = f_{(x*y)*y} \overset{(2.3)}{=} f_{(y*x)*x} \\ &\overset{(3.4)}{=} (\Omega_{y*x} * f_x) \Lsh (f_{y*x} * \Omega_x) = (\Omega_{y*x} * 1) \Lsh (f_{y*x} * \Omega_x) \\ &\overset{(2.4)}{=} 1 \Lsh (f_{y*x} * \Omega_x) \overset{(3.2)}{=} 1 \end{aligned}$$

which shows that $y \in \ker_X(f)$. Suppose that f is an $\Omega_{(r,l)}$-endomorphic GE-derivation on a commutative GE-algebra X. If $x \in \ker_X(f)$, then $f_x = 1$, and so

$$\begin{aligned} f_y &\overset{(GE2)}{=} f_{1*y} = f_{(x*y)*y} \overset{(2.3)}{=} f_{(y*x)*x} \\ &\overset{(3.4)}{=} (f_{y*x} * \Omega_x) \Lsh (\Omega_{y*x} * f_x) = (f_{y*x} * \Omega_x) \Lsh (\Omega_{y*x} * 1) \\ &\overset{(2.4)}{=} (f_{y*x} * \Omega_x) \Lsh 1 \overset{(3.2)}{=} 1 \end{aligned}$$

which shows that $y \in \ker_X(f)$. □

If X satisfies (3.24), then X may not be commutative. From Example 3.34(i), we can observe that $\ker_X(f) = \{1, \top_3, \top_4, \top_5\}$ satisfies (3.24). But, X is not commutative. Also, from Example 3.34(ii), we can observe that $\ker_X(f) = \{1, \top_3, \top_5\}$ satisfies (3.24). But, X is not commutative since $(\top_1 * \top_2) * \top_2 = \top_2 * \top_2 = 1 \neq \top_1 = 1 * \top_1 = (\top_2 * \top_1) * \top_1$.

Corollary 3.44. *If f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X, then its Ω-kernel is a GE-filter of X.*

Proof. It is clear that $1 \in \ker_X(\Omega)$. Let $x, y \in X$ be such that $x \in \ker_X(\Omega)$ and $x * y \in \ker_X(\Omega)$. Then, $\Omega_x = 1$ and $\Omega_{x*y} = 1$, and so $1 = \Omega_{x*y} = \Omega_x * \Omega_y = 1 * \Omega_y = \Omega_y$, that is, $y \in \ker_X(\Omega)$. Therefore, $\ker_X(\Omega)$ is a GE-filter of X. □

We know from Remark 3.42 that the f-kernel is not a GE-filter of X for every $\Omega_{(l,r)}$-endomorphic GE-derivation or $\Omega_{(r,l)}$-endomorphic GE-derivation f on X. Finally, we find a condition for the f-kernel to be a GE-filter of X.

Theorem 3.45. *Let f be an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. If f is a GE-endomorphism on X, then its f-kernel is a GE-filter of X.*

Proof. Assume that f is a GE-endomorphism on X. It is clear that $1 \in \ker_X(f)$. Let $x, y \in X$ be such that $x \in \ker_X(f)$ and $x * y \in \ker_X(f)$. Then, $f_x = 1$ and $f_{x*y} = 1$. Hence, $1 = f_{x*y} = f_x * f_y = 1 * f_y = f_y$, and thus $y \in \ker_X(f)$. Therefore, $\ker_X(f)$ is a GE-filter of X. □

4. Conclusions

The concept of derivation is commonly used in a variety of contexts, including mathematics, linguistics, physics, and chemistry, as it represents a source or the process of obtaining something from a source. It is a well-known fact that the concept of derivation is mainly addressed in calculus in the field of mathematics. With the aim of addressing the concept of derivation in GE-algebra, one of the logical algebras, we have introduced the notion of $\Omega_{(l,r)}$-endomorphic (resp., $\Omega_{(r,l)}$-endomorphic) GE-derivation using the binary operation "$\daleth$" on a GE-algebra X given by $\daleth\,(x, y) = (y * x) * x$ and the GE-endomorphism $\Omega : X \to X$, and investigated several properties. We have studied the conditions under which $\Omega_{(l,r)}$-endomorphic GE-derivations or $\Omega_{(l,r)}$-endomorphic GE-derivations to satisfy certain equalities and inequalities. We have explored the conditions under which f becomes order preserving when f is an $\Omega_{(l,r)}$-endomorphic GE-derivation or an $\Omega_{(r,l)}$-endomorphic GE-derivation on X. We have observed that the f-kernel of f and the Ω-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation are GE-subalgebras. Also, we have observed that the Ω-kernel of f is a GE-filter of X, but the f-kernel of f is not a GE-filter of X. Finally, we have explored the condition under which the f-kernel of f formed by the $\Omega_{(r,l)}$-endomorphic GE-derivation or $\Omega_{(l,r)}$-endomorphic GE-derivation becomes a GE-filter.

With the results and ideas obtained in this paper in the background, we will attempt to develop various forms of derivations on GE-algebras, and we also plan to study the concept of derivations in various forms of logical algebra.

Author contributions

Young Bae Jun: Conceptualization, Methodology, Validation, Writing—original draft, Writing—review and editing; Ravikumar Bandaru: Conceptualization, Methodology, Validation, Writing—original draft, Writing—review and editing; Amal S. Alali: Conceptualization, Methodology, Validation, Writing—review and editing, Funding. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Young Bae Jun is the Guest Editor of special issue "General Algebraic Structures and Fuzzy Algebras" for AIMS Mathematics. Young Bae Jun was not involved in the editorial review and the decision to publish this article. The authors declare that they have no conflicts of interest.

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