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Research article

Oscillation criteria for mixed neutral differential equations

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Abstract: In this study, we aim to contribute to the increasing interest in functional differential equations by obtaining new theorems for the oscillation of second-order neutral differential equations of mixed type in a non-canonical form. The results obtained here improve and extend those reported in the literature. The applicability of the results is illustrated by several examples.

Keywords: neutral differential equations of mixed type; second-order; non-canonical form
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1. Introduction

This paper deals with the oscillatory behavior of mixed neutral differential equations

$$\left(r(s)\left(\left(y(s)+\psi(s)\,y(\lambda(s))\right)'\right)^{\alpha}\right)' + \int_{a_1}^{a_2} \eta(s,\varrho)F\left(y(\delta(s,\varrho))\right)\mathrm{d}\varrho = 0, \tag{1.1}$$

where $s \geq s_0$. Throughout this study, we will assume:

(M1) $\alpha > 1$ is the quotient of odd positive integers;

(M2) $r \in C^1([s_0,\infty),(0,\infty))$, $r' \geq 0$, $\eta \in C([s_0,\infty)\times[a_1,a_2],\mathbb{R})$, $\eta(s,\varrho) \geq 0$ for $s \geq s_0$ and $\varrho \in [a_1,a_2]$, $\psi \in C([s_0,\infty),(0,1))$, $\inf_{s\geq s_0}\psi(s) \neq 0$, ψ,η are not identically zero for large s, and under the non-canonical form, that is,

$$\int_{s_0}^{\infty} \frac{1}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta < \infty; \tag{1.2}$$

(M3) $\lambda \in C([s_0, \infty), (0, \infty))$, $\lambda(s) \leq s$, and $\lim_{s\to\infty} \lambda(s) = \infty$;

(M4) $\delta \in C([s_0, \infty) \times [a_1, a_2], \mathbb{R})$, $\delta(s, \varrho) \geq s$ for $s \geq s_0$ and $\varrho \in [a_1, a_2]$, δ has nonnegative partial derivatives, and $\lim_{s\to\infty} \delta(s, \varrho) = \infty$;

(M5) $F \in C(\mathbb{R}, \mathbb{R})$ such that $|F(y)| \geq k|y^{\alpha}|$ for $y \neq 0$, where k is a constant, $k > 0$.

By a solution of (1.1), we mean any differentiable function $\vartheta(s) = y(s) + \psi(s) y(\lambda(s))$ which does not vanish eventually such that $r(s)((y(s) + \psi(s) y(\lambda(s)))')^{\alpha}$ is differentiable, satisfying (1.1) for sufficiently large s.

As is customary, a solution $y(s)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Neutral differential equations play an important role in applications of real life, for instance, in applied mathematics [1, 2], ecology [3], and engineering [4].

In dynamical models, oscillation and retarded/advanced effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [5, 6]. The oscillation theory of differential equations is an important branch of performance of differential equations, which is widely used in the natural sciences and engineering. Therefore, the vibration performance of different parts has attracted people's attention, and a lot of research work has been done in the area of oscillatory behavior in various classes of differential equations. For further exploration of this topic, we recommend referring to the articles cited as [7–12] and the related references mentioned within those works.

Due to the fact that many phenomena are influenced not only by the present conditions but also by their past states, by considering delay differential equations, we can gain insight into the intrinsic nature of phenomena and predict their future evolution. Hence, studying delay differential equations holds significant theoretical and practical importance.

The literature provides a comprehensive discussion of the oscillation and asymptotic behavior of solutions for different classes of delay and advanced differential equations, see [13–26], where the researchers have made significant contributions to this field, employing various mathematical techniques.

Extensive efforts have been dedicated to the advancement of the oscillation theory pertaining to second-order delay equations and advanced equations. Numerous notable contributions have been made in this area, see [27–37]. In the case of second-order delay equations, Dzurina and Jadlovska [27, 28] investigated the second-order differential equations

$$(r(s)(y'(s))^{\alpha})' + \eta(s) y^{\alpha}(\delta(s)) = 0, \tag{1.3}$$

in the non-canonical case (1.2), where $\delta(s) \leq s$. The authors prove that Eq (1.3) is oscillatory using one condition.

Chatzarakis and Jadlovska [32] presented some sufficient conditions for the oscillation of (1.3) in the canonical case

$$\int_{s_0}^{\infty} \frac{1}{r^{1/\alpha}(\zeta)} \mathrm{d}\zeta = \infty, \tag{1.4}$$

where $\delta(s) \leq s$.

Research papers [33–35] presented different results in the study of the oscillation of equation

$$(r(s)((y(s)+\psi(s)y(\lambda(s)))')^{\alpha})'+\eta(s)y^{\alpha}(\delta(s))=0,$$

in the canonical case (1.4), where $\delta(s)\le s$.

In the case of advanced equations of the second-order, Jadlovska [36] and Chatzarakis et al. [37] established new oscillation results of (1.3) in the non-canonical case (1.2), where $\delta(s)\ge s$.

The investigation of oscillation behavior in solutions of delay differential equations has received considerable attention. However, when it comes to the study of mixed differential equations, the available results are relatively scarce, see, for example, [38–43].

Qi and Yu [38] and Zhang et al. [39] studied the differential equation

$$\begin{aligned}&(r(s)((y(s)+\psi(s)y(s-\delta_1)+\psi_1(s)y(s+\delta_2))')^{\alpha})'\\&+\int_{a_1}^{a_2}\eta_1(s,\varrho)y^{\alpha}(s-\varrho)\,\mathrm{d}\varrho+\int_{a_1}^{a_2}\eta(s,\varrho)y^{\alpha}(s+\varrho)\,\mathrm{d}\varrho=0,\end{aligned}\tag{1.5}$$

where $\delta_1,\ \delta_2\ge 0$, $\psi_1\in C([s_0,\infty),(0,1))$, and $\eta_1\in C([s_0,\infty)\times[a_1,a_2],\mathbb{R})$. By using the Riccati transformation technique, they obtained some sufficient conditions for oscillation of (1.5), under the conditions (1.2) and (1.4).

Using new monotonic properties, Shi and Bai [40] studied the oscillatory behavior of solutions to a second-order nonlinear differential equation with mixed neutral terms

$$(r(s)((y(s)+\psi_1(s)y(\lambda_1(s))+\psi_2(s)y(\lambda_2(s)))')^{\alpha})'+\eta(s)y^{\beta}(\delta(s))=0,$$

where $\lambda_1(s)\le s,\ \lambda_2(s)\ge s,\ \delta(s)\ge s$, and α and β are ratios of two positive odd integers. They introduced new conditions for the oscillation under condition (1.2). In the following, we describe one of the results obtained in [40] for the convenience of the reader.

Theorem 1.1. *Let $\alpha=\beta$. Assume that (1.2) and*

$$\int_{s_0}^{\infty}\eta(\zeta)\left(1-\psi_1(\delta(\zeta))-\psi_2(\delta(\zeta))\frac{R(\lambda_2(\delta(\zeta)))}{R(\delta(\zeta))}\right)^{\alpha}\mathrm{d}\zeta=\infty\tag{1.6}$$

hold. If

$$\limsup_{s\to\infty}\pi^{\alpha}(s)\int_{s_1}^{s}\eta(\zeta)\left(1-\psi_1(\delta(\zeta))\frac{\pi(\lambda_1(\delta(\zeta)))}{\pi(\delta(\zeta))}-\psi_2(\delta(\zeta))\right)^{\alpha}\frac{\pi^{\alpha}(\delta(\zeta))}{\pi^{\alpha}(\zeta)}\mathrm{d}\zeta>1,\tag{1.7}$$

then (1.1) is oscillatory, where $R(s)=\int_{s_0}^{s}r^{-1/\alpha}(\zeta)\,\mathrm{d}\zeta$.

Using generalized Riccati substitution, Moaaz et al. [43] studied the nonlinear differential equation with mixed neutral terms

$$(r(s)((y(s)+\psi_1(s)y(\lambda_1(s))+\psi_2(s)y(\lambda_2(s)))')^{\alpha})'+\eta_1(s)y^{\alpha}(\delta_1(s))+\eta_2(s)y^{\alpha}(\delta_2(s))=0,\tag{1.8}$$

where $\psi_1,\ \psi_2,\ \eta_1,\ \eta_2\in C([s_0,\infty),[0,\infty))$, $\lambda_1,\ \lambda_2,\ \delta_1,\ \delta_2\in C([s_0,\infty),\mathbb{R})$, $\lambda_1(s)\le s,\ \lambda_2(s)\ge s,\ \delta_1(s)\le s$, and $\delta_2(s)\ge s$. The authors obtained new oscillation criteria that guarantee the oscillation of the studied equation under condition (1.2). In the following, we describe one of the results obtained in [43] for the convenience of the reader.

Theorem 1.2. *Suppose that $H^{*}(s) \geq G^{*}(s) > 0$. If*

$$\limsup_{s\to\infty} \pi^{\alpha}(\delta_2(s), \infty) \int_{s_1}^{s} G^{*}(\zeta)\, \mathrm{d}\zeta > 1, \tag{1.9}$$

then all solutions of (1.1) are oscillatory, where

$$\pi(s, \infty) = \int_{s}^{\infty} \frac{1}{r^{1/\alpha}(\zeta)} \mathrm{d}\zeta,$$

$$H^{*}(s) = \eta_1(s) B_{*}^{\alpha}(\delta_1(\zeta)) + \eta_2(s) B_{*}^{\alpha}(\delta_2(\zeta)),$$

$$G^{*}(s) = \eta_1(s)(B^{*}(\delta_1(\zeta)))^{\alpha} + \eta_2(s)(B^{*}(\delta_2(\zeta)))^{\alpha},$$

$$B^{*}(s) = 1 - \psi_1(s) \frac{\pi(\lambda_1(s), \infty)}{\pi(s, \infty)} - \psi_2(s)$$

and

$$B_{*}(s) = 1 - \psi_1(s) - \psi_2(s) \frac{\pi(s_1, \lambda_2(s))}{\pi(s_1, s)}, \text{ for } s \geq s_1 \geq s_0.$$

It is noted that most of the works concerned with studying the oscillatory behavior of mixed differential equations were in the legal case $\int_{s_0}^{\infty} 1/r^{1/\alpha}(\zeta)\, \mathrm{d}\zeta = \infty$. Therefore, the aim of this research was to study the oscillatory behavior of mixed differential equations in the non-canonical case $\int_{s_0}^{\infty} 1/r^{1/\alpha}(\zeta)\, \mathrm{d}\zeta < \infty$, as well as, finding new oscillation criteria that improve and extend some of the results in previous studies. To illustrate the applicability of our results, several examples are presented.

2. Main results

In the following, we present some notations that will be used in the rest of the paper:

$$\vartheta(s) := y(s) + \psi(s) y(\lambda(s))$$

and

$$\pi(s) = \int_{s}^{\infty} \frac{1}{r^{1/\alpha}(\zeta)} \mathrm{d}\zeta.$$

First, we present the following useful lemmas, which will be used later in the proofs of our results.

Lemma 2.1. *Let $y(s)$ be a positive solution of (1.1). If (1.2) holds and*

$$\int_{s_0}^{\infty} \left(\int_{a_1}^{a_2} \eta(\zeta, \varrho)\, \mathrm{d}\varrho \right) \mathrm{d}\zeta = \infty, \tag{2.1}$$

then,

$$\vartheta(s) > 0,\ \vartheta'(s) < 0,\ (r(s)(\vartheta'(s))^{\alpha})' \leq 0. \tag{2.2}$$

Proof. Suppose that (1.1) has a positive solution $y(s)$ on $[s_0, \infty)$. Obviously, $y(s) > 0$, $y(\lambda(s)) > 0$, and $y(\delta(s)) > 0$ for $s \geq s_1 \geq s_0$. In view of (1.1), we have

$$(r(s)(\vartheta'(s))^{\alpha})' = -\int_{a_1}^{a_2} \eta(s,\varrho) F(y(\delta(s,\varrho)))\,\mathrm{d}\varrho \leq 0. \tag{2.3}$$

Thus, $r(s)(\vartheta'(s))^{\alpha}$ is decreasing, and so either $\vartheta'(s) < 0$ or $\vartheta'(s) > 0$. Suppose that $\vartheta'(s) > 0$. Then,

$$y(s) = \vartheta(s) - \psi(s)\, y(\lambda(s)) \geq \vartheta(s) - \psi(s)\,\vartheta(\lambda(s)) \geq \vartheta(s)(1 - \psi(s)),$$

and so

$$F(y(\delta(s,\varrho))) \geq k y^{\alpha}(\delta(s,\varrho)) \geq k\vartheta^{\alpha}(\delta(s,\varrho))(1 - \psi(\delta(s,\varrho)))^{\alpha}, \tag{2.4}$$

and using (2.4) and (2.3), we have

$$(r(s)(\vartheta'(s))^{\alpha})' \leq -k\int_{a_1}^{a_2} \eta(s,\varrho)\,\vartheta^{\alpha}(\delta(s,\varrho))(1 - \psi(\delta(s,\varrho)))^{\alpha}\,\mathrm{d}\varrho.$$

Since $\delta(s,\varrho)$ is nondecreasing with respect to ϱ, we find $\delta(s,\varrho) \geq \delta(s,a_1)$ for $\varrho \in (a_1, a_2)$, and so

$$(r(s)(\vartheta'(s))^{\alpha})' \leq -k\vartheta^{\alpha}(\delta(s,a_1)) \int_{a_1}^{a_2} \eta(s,\varrho)(1 - \psi(\delta(s,\varrho)))^{\alpha}\,\mathrm{d}\varrho. \tag{2.5}$$

Define $\omega(s)$ by

$$\omega(s) = \frac{r(s)(\vartheta'(s))^{\alpha}}{\vartheta^{\alpha}(\delta(s,a_1))}. \tag{2.6}$$

Then, $\omega(s) > 0$. Differentiating (2.6) and using (2.5), we find

$$\begin{aligned}
\omega'(s) &= \frac{(r(s)(\vartheta'(s))^{\alpha})'}{\vartheta^{\alpha}(\delta(s,a_1))} - \frac{\alpha r(s)(\vartheta'(s))^{\alpha}\,\vartheta^{\alpha-1}(\delta(s,a_1))\,\vartheta'(\delta(s,a_1))\,\delta'(s,a_1)}{\vartheta^{2\alpha}(\delta(s,a_1))} \\
&\leq -k\int_{a_1}^{a_2} \eta(s,\varrho)(1 - \psi(\delta(s,\varrho)))^{\alpha}\,\mathrm{d}\varrho - \frac{\alpha r(s)(\vartheta'(s))^{\alpha}\,\vartheta^{\alpha-1}(\delta(s,a_1))\,\vartheta'(\delta(s,a_1))\,\delta'(s,a_1)}{\vartheta^{2\alpha}(\delta(s,a_1))} \\
&\leq -k\int_{a_1}^{a_2} \eta(s,\varrho)(1 - \psi(\delta(s,\varrho)))^{\alpha}\,\mathrm{d}\varrho - \frac{\alpha\vartheta'(\delta(s,a_1))\,\delta'(s,a_1)}{\vartheta(\delta(s,a_1))}\omega(s) \\
&\leq -k\int_{a_1}^{a_2} \eta(s,\varrho)(1 - \psi(\delta(s,\varrho)))^{\alpha}\,\mathrm{d}\varrho.
\end{aligned} \tag{2.7}$$

Integrating (2.7), we get

$$\begin{aligned}
\omega(s) &\leq \omega(s_2) - k\int_{s_2}^{s}\left(\int_{a_1}^{a_2} \eta(\zeta,\varrho)(1 - \psi(\delta(\zeta,\varrho)))^{\alpha}\,\mathrm{d}\varrho\right)\mathrm{d}\zeta \\
&\leq \omega(s_2) - k \inf_{s\geq s_2}\inf_{a_2\geq a_1}(1 - \psi(\delta(s,a_2)))^{\alpha}\int_{s_2}^{s}\left(\int_{a_1}^{a_2} \eta(\zeta,\varrho)\,\mathrm{d}\varrho\right)\mathrm{d}\zeta.
\end{aligned} \tag{2.8}$$

From (2.8) and (2.1), we see that $\omega(s) \to -\infty$ as $s \to \infty$, a contradiction. Thus, $\vartheta'(s) > 0$ is impossible. Hence, $\vartheta(s)$ satisfies (2.2). □

Lemma 2.2. *Let $y(s)$ be a positive solution of (1.1). If (2.2) holds, then*

$$\left(\frac{\vartheta(s)}{\pi(s)}\right)' \geq 0, \tag{2.9}$$

for $s \geq s_1 \geq s_0$.

Proof. Suppose that (1.1) has a positive solution $y(s)$ on $[s_0, \infty)$. Obviously, $y(s) > 0$, $y(\lambda(s)) > 0$, and $y(\delta(s)) > 0$ for $s \geq s_1 \geq s_0$. Since (2.2) is satisfied, it follows from the monotonicity of $r^{1/\alpha}(s)\vartheta'(s)$ that

$$\begin{aligned}\vartheta(s) &\geq -\int_s^\infty \frac{r^{1/\alpha}(\zeta)\vartheta'(\zeta)}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta \geq -r^{1/\alpha}(s)\vartheta'(s)\int_s^\infty \frac{1}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta \\ &\geq -r^{1/\alpha}(s)\vartheta'(s)\pi(s),\end{aligned} \tag{2.10}$$

that is,

$$\vartheta(s) + r^{1/\alpha}(s)\vartheta'(s)\pi(s) \geq 0. \tag{2.11}$$

Now,

$$\left(\frac{\vartheta(s)}{\pi(s)}\right)' = \frac{\pi(s)\vartheta'(s) - \vartheta(s)\pi'(s)}{\pi^2(s)}. \tag{2.12}$$

From (2.11) and (2.12), we conclude that

$$\left(\frac{\vartheta(s)}{\pi(s)}\right)' = \frac{r^{1/\alpha}(s)\pi(s)\vartheta'(s) + \vartheta(s)}{r^{1/\alpha}(s)\pi^2(s)} \geq 0.$$

The proof is complete. □

Next, we introduce the oscillation criteria for (1.1).

Theorem 2.1. *Assume that (1.2) holds. If*

$$0 < 1 - \psi(\delta(s, a_2))\frac{\pi(\lambda(\delta(s, a_2)))}{\pi(\delta(s, a_2))} < 1, \quad \inf_{s\geq s_1}\inf_{a_2\geq a_1}\left(1 - \psi(\delta(s, a_2))\frac{\pi(\lambda(\delta(s, a_2)))}{\pi(\delta(s, a_2))}\right) > 0 \tag{2.13}$$

and

$$\int_{s_0}^\infty \left(\frac{1}{r^{1/\alpha}(u)}\left(\int_{s_0}^u \left(\int_{a_1}^{a_2} \eta(\zeta, \varrho)\pi^\alpha(\delta(\zeta, \varrho))\,\mathrm{d}\varrho\right)\mathrm{d}\zeta\right)^{1/\alpha}\right)\mathrm{d}u = \infty, \tag{2.14}$$

then (1.1) is oscillatory.

Proof. Suppose that (1.1) has a positive solution $y(s)$ on $[s_0, \infty)$. Obviously, $y(s) > 0$, $y(\lambda(s)) > 0$, and $y(\delta(s)) > 0$ for $s \geq s_1 \geq s_0$. It is known that (2.1) is necessary for (2.14) to be valid. In fact, since the function

$$\int_{s_0}^\infty \left(\int_{a_1}^{a_2} \eta(\zeta, \varrho)\pi^\alpha(\delta(\varrho))\,\mathrm{d}\varrho\right)\mathrm{d}\zeta$$

is unbounded due to (1.2) and $\pi'(s) < 0$, (2.1) must hold. From Lemma 2.1, we find that $\vartheta(s)$ satisfies (2.2) for $s \geq s_1$, and by using (2.9) in Lemma 2.2, we see that there is $c > 0$ such that

$$\frac{\vartheta(s)}{\pi(s)} \geq c \tag{2.15}$$

and

$$\begin{aligned} y(s) &= \vartheta(s) - \psi(s)\, y(\lambda(s)) \geq \vartheta(s) - \psi(s)\,\vartheta(\lambda(s)) \\ &\geq \vartheta(s) - \psi(s)\,\frac{\pi(\lambda(s))\,\vartheta(s)}{\pi(s)} = \vartheta(s)\left(1 - \psi(s)\,\frac{\pi(\lambda(s))}{\pi(s)}\right), \end{aligned}$$

and so

$$F(y(\delta(s,\varrho))) \geq k y^{\alpha}(\delta(s,\varrho)) \geq k\vartheta^{\alpha}(\delta(s,\varrho))\left(1 - \psi(\delta(s,\varrho))\,\frac{\pi(\lambda(\delta(s,\varrho)))}{\pi(\delta(s,\varrho))}\right)^{\alpha}. \tag{2.16}$$

Using (2.16) and (1.1), we have

$$\left(r(s)\left(\vartheta'(s)\right)^{\alpha}\right)' \leq -k\int_{a_1}^{a_2} \eta(s,\varrho)\,\vartheta^{\alpha}(\delta(s,\varrho))\left(1 - \psi(\delta(s,\varrho))\,\frac{\pi(\lambda(\delta(s,\varrho)))}{\pi(\delta(s,\varrho))}\right)^{\alpha} \mathrm{d}\varrho. \tag{2.17}$$

From (2.15) and (2.17), we see that

$$\left(r(s)\left(\vartheta'(s)\right)^{\alpha}\right)' \leq -k\int_{a_1}^{a_2} \eta(s,\varrho)\, c^{\alpha}\pi^{\alpha}(\delta(s,\varrho))\left(1 - \psi(\delta(s,\varrho))\,\frac{\pi(\lambda(\delta(s,\varrho)))}{\pi(\delta(s,\varrho))}\right)^{\alpha} \mathrm{d}\varrho. \tag{2.18}$$

Integrating (2.18), we get

$$r(s)\left(\vartheta'(s)\right)^{\alpha} \leq -c^{\alpha}k\int_{s_1}^{s}\left(\int_{a_1}^{a_2} \eta(\zeta,\varrho)\,\pi^{\alpha}(\delta(\zeta,\varrho))\left(1 - \psi(\delta(\zeta,\varrho))\,\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha} \mathrm{d}\varrho\right)\mathrm{d}\zeta,$$

that is,

$$\vartheta'(s) \leq -\frac{ck^{1/\alpha}}{r^{1/\alpha}(s)}\left(\int_{s_1}^{s}\left(\int_{a_1}^{a_2} \eta(\zeta,\varrho)\,\pi^{\alpha}(\delta(\zeta,\varrho))\left(1 - \psi(\delta(\zeta,\varrho))\,\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha} \mathrm{d}\varrho\right)\mathrm{d}\zeta\right)^{1/\alpha}. \tag{2.19}$$

Integrating (2.19), and using (2.13) and (2.14), we find

$$\begin{aligned} \vartheta(s) &\leq \vartheta(s_1) - \int_{s_1}^{s} \frac{ck^{1/\alpha}}{r^{1/\alpha}(u)}\left(\int_{s_1}^{u}\left(\int_{a_1}^{a_2} \frac{\eta(\zeta,\varrho)\left(1 - \psi(\delta(\zeta,\varrho))\,\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha}}{\pi^{-\alpha}(\delta(\zeta,\varrho))}\mathrm{d}\varrho\right)\mathrm{d}\zeta\right)^{1/\alpha}\mathrm{d}u \\ &\leq -ck^{1/\alpha}\inf_{s\geq s_1}\inf_{a_2\geq a_1}\left(1 - \psi(\delta(s,a_2))\,\frac{\pi(\lambda(\delta(s,a_2)))}{\pi(\delta(s,a_2))}\right)\int_{s_1}^{s}\frac{\left(\int_{s_1}^{u}\int_{a_1}^{a_2}\frac{\eta(\zeta,\varrho)}{\pi^{-\alpha}(\delta(\zeta,\varrho))}\mathrm{d}\varrho\mathrm{d}\zeta\right)^{1/\alpha}}{r^{1/\alpha}(u)}\mathrm{d}u + \vartheta(s_1), \end{aligned}$$

from (2.13) and (2.14), we see that $\vartheta(s) \to -\infty$ as $s \to \infty$, a contradiction. □

Theorem 2.2. *Assume that (1.2), (2.1) and (2.13) hold. If*

$$\int_{s_0}^{\infty}\left(k\pi^{\alpha}(\delta(\zeta,a_2))\int_{a_1}^{a_2}\eta(\zeta,\varrho)\left(1 - \psi(\delta(\zeta,\varrho))\,\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha}\mathrm{d}\varrho - \frac{\alpha^{\alpha+1}r^{-1/\alpha}(\zeta)}{(\alpha+1)^{\alpha+1}\pi(\zeta)}\right)\mathrm{d}\zeta = \infty, \tag{2.20}$$

then (1.1) is oscillatory.

Proof. Suppose that (1.1) has a positive solution $y(s)$ on $[s_0,\infty)$. Obviously, $y(s)>0$, $y(\lambda(s))>0$, and $y(\delta(s))>0$ for $s\geq s_1\geq s_0$. From Lemma 2.1, we see that $\vartheta(s)$ satisfies (2.2). Define $\phi(s)$ by

$$\phi(s)=\frac{r(s)(\vartheta'(s))^{\alpha}}{\vartheta^{\alpha}(s)}. \tag{2.21}$$

Differentiating on both sides of (2.21), we obtain

$$\phi'(s)=\frac{(r(s)(\vartheta'(s))^{\alpha})'}{\vartheta^{\alpha}(s)}-\frac{\alpha r(s)(\vartheta'(s))^{\alpha+1}}{\vartheta^{\alpha+1}(s)}. \tag{2.22}$$

Using (2.17) and (2.22), we find

$$\phi'(s)\leq\frac{-k\int_{a_1}^{a_2}\eta(s,\varrho)\vartheta^{\alpha}(\delta(s,\varrho))\left(1-\psi(\delta(s,\varrho))\frac{\pi(\lambda(\delta(s,\varrho)))}{\pi(\delta(s,\varrho))}\right)^{\alpha}\mathrm{d}\varrho}{\vartheta^{\alpha}(s)}-\frac{\alpha r(s)(\vartheta'(s))^{\alpha+1}}{\vartheta^{\alpha+1}(s)}.$$

By using (2.9) in Lemma 2.2 and (2.21), we conclude that

$$\phi'(s)\leq-k\frac{\pi^{\alpha}(\delta(s,a_2))}{\pi^{\alpha}(s)}\int_{a_1}^{a_2}\eta(s,\varrho)\left(1-\psi(\delta(s,\varrho))\frac{\pi(\lambda(\delta(s,\varrho)))}{\pi(\delta(s,\varrho))}\right)^{\alpha}\mathrm{d}\varrho-\alpha r(s)\frac{\phi^{(\alpha+1)/\alpha}(s)}{r^{(\alpha+1)/\alpha}(s)}. \tag{2.23}$$

Multiplying (2.23) by $\pi^{\alpha}(s)$ and integrating the resulting inequality, we find

$$\begin{aligned}\frac{\phi(s)}{\pi^{-\alpha}(s)}-\frac{\phi(s_1)}{\pi^{-\alpha}(s_1)} \leq\; & -k\int_{s_1}^{s}\pi^{\alpha}(\delta(\zeta,a_2))\int_{a_1}^{a_2}\eta(\zeta,\varrho)\left(1-\psi(\delta(\zeta,\varrho))\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha}\mathrm{d}\varrho\mathrm{d}\zeta\\ & -\int_{s_1}^{s}\alpha\pi^{\alpha}(\zeta)\frac{\phi^{(\alpha+1)/\alpha}(\zeta)}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta-\int_{s_1}^{s}\frac{\alpha\pi^{\alpha-1}(\zeta)\phi(\zeta)}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta. \end{aligned}\tag{2.24}$$

Using the inequality

$$-B\Omega+A\Omega^{(\alpha+1)/\alpha}\geq\frac{-\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\quad A,B>0, \tag{2.25}$$

where

$$A=\frac{\alpha\pi^{\alpha}(\zeta)}{r^{1/\alpha}(\zeta)},\ B=\frac{\alpha\pi^{\alpha-1}(\zeta)}{r^{1/\alpha}(\zeta)}\text{ and }\Omega=-\phi(s),$$

(2.24) becomes

$$\begin{aligned}\frac{\phi(s)}{\pi^{-\alpha}(s)}-\frac{\phi(s_1)}{\pi^{-\alpha}(s_1)} \leq\; & -k\int_{s_1}^{s}\pi^{\alpha}(\delta(\zeta,a_2))\int_{a_1}^{a_2}\eta(\zeta,\varrho)\left(1-\psi(\delta(\zeta,\varrho))\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha}\mathrm{d}\varrho\mathrm{d}\zeta\\ & +\int_{s_1}^{s}\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{r^{1/\alpha}(\zeta)\pi(\zeta)}\mathrm{d}\zeta. \end{aligned}\tag{2.26}$$

In view of (2.10) and (2.21), we have

$$1\geq-\frac{r(s)(\vartheta'(s))^{\alpha}\pi^{\alpha}(s)}{\vartheta^{\alpha}(s)}=-\phi(s)\pi^{\alpha}(s). \tag{2.27}$$

From (2.26) and (2.27), we obtain

$$1+\pi^{\alpha}(s_1)\phi(s_1) \geq \int_{s_1}^{s} k\pi^{\alpha}(\delta(\zeta,a_2))\int_{a_1}^{a_2}\eta(\zeta,\varrho)\left(1-\psi(\delta(\zeta,\varrho))\frac{\pi(\lambda(\delta(\zeta,\varrho)))}{\pi(\delta(\zeta,\varrho))}\right)^{\alpha}\mathrm{d}\varrho\mathrm{d}\zeta - \int_{s_1}^{s}\frac{\alpha^{\alpha+1}r^{-1/\alpha}(\zeta)}{(\alpha+1)^{\alpha+1}\pi(\zeta)}\mathrm{d}\zeta,$$

a contradiction. □

The following examples illustrate the applicability of our main results.

Example 2.1. *Consider the following equation:*

$$\left(s^2\left(\left(y(s)+\frac{1}{9}y\left(\frac{s}{2}\right)\right)'\right)^{5/3}\right)' + \int_{a_1}^{a_2}\eta_0 s^3 y^{5/3}(5s)\,\mathrm{d}\varrho = 0. \tag{2.28}$$

We note that $\alpha = 5/3 > 1$, $r(s) = s^2$, $\psi(s) = 1/9$, $\lambda(s) = s/2$, $\eta(s,\varrho) = \eta_0 s^3$, *and* $\delta(s,\varrho) = 5s$. *Then, it is not difficult to see that*

$$\pi(s) = \int_s^{\infty}\frac{1}{r^{1/\alpha}(\zeta)}\mathrm{d}\zeta = \int_s^{\infty}\frac{1}{(\zeta^2)^{3/5}(\zeta)}\mathrm{d}\zeta = \int_s^{\infty}\zeta^{-6/5}\mathrm{d}\zeta = \frac{5}{s^{1/5}},$$

$$0 < 1-\psi(\delta(s,a_2))\frac{\pi(\lambda(\delta(s,a_2)))}{\pi(\delta(s,a_2))} = 1-\left(\frac{1}{9}\right)2^{1/5} < 1$$

and

$$\begin{aligned}
&\int_{s_0}^{\infty}\left(\frac{1}{r^{1/\alpha}(u)}\left(\int_{s_0}^{u}\left(\int_{a_1}^{a_2}\eta(\zeta,\varrho)\pi^{\alpha}(\delta(\zeta,\varrho))\mathrm{d}\varrho\right)\mathrm{d}\zeta\right)^{1/\alpha}\right)\mathrm{d}u\\
&= \int_{s_0}^{\infty}\left(\frac{1}{(u^2)^{3/5}}\left(\int_{s_0}^{u}\left(\eta_0\zeta^3\frac{5^{5/3}}{\left((5\zeta)^{1/5}\right)^{5/3}}(a_2-a_1)\right)\mathrm{d}\zeta\right)^{3/5}\right)\mathrm{d}u\\
&= \int_{s_0}^{\infty}\left(\frac{1}{u^{6/5}}\left(\eta_0(a_2-a_1)5^{4/3}\int_{s_0}^{u}\zeta^{8/3}\mathrm{d}\zeta\right)^{3/5}\right)\mathrm{d}u\\
&= \left(\eta_0(a_2-a_1)5^{4/3}\right)^{3/5}\left(\frac{3}{11}\right)^{3/5}\int_{s_0}^{\infty}\left(\frac{1}{u^{6/5}}u^{33/15}\right)\mathrm{d}u = \infty.
\end{aligned}$$

From Theorem 2.1, we note that (2.28) is oscillatory.

Example 2.2. *Consider the following equation:*

$$\left(s^9\left(\left(y(s)+\frac{1}{20}y\left(\frac{s}{4}\right)\right)'\right)^{3}\right)' + \int_{1}^{2}\eta_0 s^5\varrho y^{3}(4s)\,\mathrm{d}\varrho = 0. \tag{2.29}$$

We note that $a_2 = 2$, $a_1 = 1$, $\psi(s) = 1/20$, $\alpha = 3 > 1$, $r(s) = s^9$, $\lambda(s) = s/4$, $\eta(s,\varrho) = \eta_0 s^5\varrho$, *and* $\delta(s,\varrho) = 4s$. *Then, it is not difficult to see that*

$$\pi(s) = \int_s^\infty \frac{1}{r^{1/\alpha}(\zeta)} d\zeta = \int_s^\infty \zeta^{-3} d\zeta = \frac{1}{2s^2},$$

$$\int_{s_0}^\infty \left(\int_{a_1}^{a_2} \eta(\zeta, \varrho) d\varrho \right) d\zeta = \frac{3\eta_0}{2} \int_{s_0}^\infty \zeta^5 d\zeta = \infty,$$

$$0 < 1 - \psi(\delta(s, a_2)) \frac{\pi(\lambda(\delta(s, a_2)))}{\pi(\delta(s, a_2))} = 1 - \frac{16}{20} < 1$$

and

$$\int_{s_0}^\infty \left(k\pi^\alpha (\delta(\zeta, a_2)) \int_{a_1}^{a_2} \eta(\zeta, \varrho) \left(1 - \psi(\delta(\zeta, \varrho)) \frac{\pi(\lambda(\delta(\zeta, \varrho)))}{\pi(\delta(\zeta, \varrho))} \right)^\alpha d\varrho - \frac{\alpha^{\alpha+1} r^{-1/\alpha}(\zeta)}{(\alpha+1)^{\alpha+1} \pi(\zeta)} \right) d\zeta$$
$$= \int_{s_0}^\infty \left(\pi^3 (4\zeta) \int_{a_1}^{a_2} \eta_0 \zeta^5 \varrho \left(1 - \frac{1}{20} \frac{2(4\zeta)^2}{2(\zeta)^2} \right)^3 d\varrho - \frac{3^4 \left(2\zeta^2\right)}{4^4 (\zeta^9)^{1/3}} \right) d\zeta$$
$$= \int_{s_0}^\infty \left(\frac{3(0.2)^3 \eta_0}{2^4 (4^6) \zeta} - \frac{3^4 (2)}{(4^4) \zeta} \right) d\zeta.$$

Consequently, we conclude that condition (2.20) is satisfied if

$$\eta_0 > \frac{\left(3^3\right)\left(2^5\right)\left(4^2\right)}{(0.2)^3} = 1.728 \times 10^6. \tag{2.30}$$

From Theorem 2.2, we find that (2.29) is oscillatory if (2.30) holds.

Remark 2.1. *If we assume that $\psi_2(s) = 0$ and $\eta_1(s) = 0$ in Eq (1.8), then it follows that*

$$\left(r(s)\left((y(s) + \psi_1(s) y(\lambda_1(s)))'\right)^\alpha\right)' + \eta_2(s) y^\alpha(\delta_2(s)) = 0. \tag{2.31}$$

Moreover, if we consider the equation

$$\left(s^9 \left(\left(y(s) + \frac{1}{20} y\left(\frac{s}{4}\right) \right)' \right)^3 \right)' + \eta_0 s^5 y^3 (4s) = 0, \tag{2.32}$$

as a special case of Eq (2.31), we can apply Theorem 1.2 and conclude that

$$B^*(s) = 1 - \frac{16}{20}, \quad B_*(s) = 1 - \frac{1}{20},$$

$$H^*(s) = \eta_0 s^5 \left(1 - \frac{1}{20}\right)^3 \text{ and } G^*(s) = \eta_0 s^5 \left(1 - \frac{16}{20}\right)^3,$$

and so we see that $B_*(s) \geq B^*(s) > 0$ and $H^*(s) \geq G^*(s) > 0$.

Now, condition (1.9) becomes

$$\limsup_{s\to\infty} \pi^\alpha (\delta_2(s), \infty) \int_{s_1}^s G^*(\zeta) d\zeta = \limsup_{s\to\infty} \left(\frac{1}{2(4s)^2} \right)^3 \int_{s_1}^s \eta_0 \zeta^5 \left(1 - \frac{16}{20}\right)^3 d\zeta > 1.$$

Therefore, Eq (2.32) is oscillatory if $\eta_0 > 24.576 \times 10^6$.

Now, we can apply Theorem 1.1 and conclude that conditions (1.6) is satisfied, and condition (1.7) becomes

$$\limsup_{s\to\infty} \pi^{\alpha}(s) \int_{s_2}^{s} \eta(\zeta) \left(1 - \psi_1(\delta(\zeta)) \frac{\pi(\lambda_1(\delta(\zeta)))}{\pi(\delta(\zeta))} - \psi_2(\delta(\zeta))\right)^{\alpha} \frac{\pi^{\alpha}(\delta(\zeta))}{\pi^{\alpha}(\zeta)} \mathrm{d}\zeta$$
$$= \limsup_{s\to\infty} \left(\frac{1}{2s^2}\right)^3 \int_{s_2}^{s} \eta_0 \zeta^5 \left(1 - \frac{16}{20}\right)^3 \left(\frac{1}{2(4\zeta)^2}\right)^3 \left(2\zeta^2\right)^3 \mathrm{d}\zeta > 1.$$

Therefore, Eq (2.32) is oscillatory if $\eta_0 > 24.576 \times 10^6$.

Finally, by using Theorem 2.2, we see that condition (2.20) becomes

$$\int_{s_0}^{\infty} \left(k\pi^{\alpha}(\delta(\zeta, a_2)) \int_{a_1}^{a_2} \eta(\zeta, \varrho) \left(1 - \psi(\delta(\zeta, \varrho)) \frac{\pi(\lambda(\delta(\zeta, \varrho)))}{\pi(\delta(\zeta, \varrho))}\right)^{\alpha} \mathrm{d}\varrho - \frac{\alpha^{\alpha+1} r^{-1/\alpha}(\zeta)}{(\alpha+1)^{\alpha+1} \pi(\zeta)}\right) \mathrm{d}\zeta$$
$$= \int_{s_0}^{\infty} \left(\frac{1}{2^3 4^6 \zeta^6} \eta_0 \zeta^5 \left(1 - \frac{1}{20} \frac{2(4\zeta)^2}{2(\zeta)^2}\right)^3 - \frac{3^4\left(2\zeta^2\right)}{4^4(\zeta^9)^{1/3}}\right) \mathrm{d}\zeta$$
$$= \int_{s_0}^{\infty} \left(\frac{(0.2)^3 \eta_0}{2^3(4^6)\zeta} - \frac{3^4(2)}{(4^4)\zeta}\right) \mathrm{d}\zeta = \infty.$$

Therefore, Eq (2.32) is oscillatory if $\eta_0 > 2.592 \times 10^6$.

Thus, our results provide a better criterion for oscillation as it guarantees the oscillation of Eq (2.32) if $\eta_0 > 2.592 \times 10^6$.

3. Conclusions

This study aimed to investigate the oscillatory properties of solutions to second-order neutral differential equations of mixed type. The results of this paper contribute to the understanding of the asymptotic and oscillatory behavior of such equations. Taking into account conditions (M1)–(M5), new oscillation criteria are presented that improve and extend some results in previous studies by employing the Riccati transformation method.

A further extension of this study is to use our results to study a class of fourth-order neutral differential equations.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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