AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14062–14074.
DOI: 10.3934/math.2024684
Received: 25 February 2024
Revised: 29 March 2024
Accepted: 08 April 2024
Published: 18 April 2024

Research article

A new approach to Leonardo number sequences with the dual vector and dual angle representation

Faik Babadağ[1,*] **and Ali Atasoy**[2]

[1] Department of Mathematics, Kırıkkale University, 71450, Yahşihan, Kırıkkale, Turkey
[2] Keskin Vocational School, Kırıkkale University, 71800, Keskin, Kırıkkale, Turkey

* **Correspondence:** Email: faik.babadag@kku.edu.tr.

Abstract: In this paper, we introduce dual numbers with components including Leonardo number sequences. This novel approach facilitates our understanding of dual numbers and properties of Leonardo sequences. We also investigate fundamental properties and identities associated with Leonardo number sequences, such as Binet's formula and Catalan's, Cassini's and D'ocagne's identities. Furthermore, we also introduce a dual vector with components including Leonardo number sequences and dual angles. This extension not only deepens our understanding of dual numbers, it also highlights the interconnectedness between numerical sequences and geometric concepts. In the future it would be valuable to replicate a similar exploration and development of our findings on dual numbers with Leonardo number sequences.

Keywords: Leonardo number sequences; recurrences; dual vector; dual angle
Mathematics Subject Classification: 05A15, 11B37, 11B39

1. Introduction

The concept of dual numbers [1], which is an extension of real numbers, was initially introduced by Clifford in 1873. It is defined as follows:

$$\mathbb{D} = \{\mathbb{A} = \gamma_1 + \varepsilon\gamma_2 \mid \gamma_1, \gamma_2 \in \mathbb{R}, \varepsilon \neq 0, \varepsilon^2 = 0\}.$$

These numbers form a commutative and associative algebra over the real numbers. Kotelnikov was the first to explore the practical applications of dual numbers, laying the groundwork for their utilization in various fields [2]. Notably, Study incorporated dual numbers and associated vectors into line geometry and kinematics, pioneering their use in these domains [3]. In the context of vectors, the set of dual vectors can be represented as $\mathbb{D}^3$, indicating all possible combinations of dual number pairs. Additionally, the collection of all unit dual vectors forms what is known as the unit dual sphere, denoted by

$$\mathbb{S}^2 = \left\{ \vec{\mathbb{A}} \in \mathbb{D}^3 | \quad \|\vec{\mathbb{A}}\| = 1 \right\},$$

which plays a significant role in geometric interpretations and calculations involving dual vectors. This rich framework of dual numbers and vectors offers a versatile toolset for mathematical and geometric analyses, with applications ranging from kinematics to computer graphics and beyond.

Study proved that there exists an isomorphism between the points of the dual unit sphere in $\mathbb{D}^3$ and the directed lines of Euclidean 3-space, which is known as Study mapping [3]. Dual numbers have been applied in most fields including research on rigid body motion, displacement analysis of spatial mechanisms, surface shape analysis and computer graphics, kinematic synthesis, human body motion analysis, and others [4–10].

The algebra of dual numbers is represented by a ring with the following addition and multiplication operations:

$$(\gamma_1 + \varepsilon\gamma_2) + (\delta_1 + \varepsilon\delta_2) = (\gamma_1 + \delta_1) + \varepsilon(\gamma_2 + \delta_2),$$

$$(\gamma_1 + \varepsilon\gamma_2)(\delta_1 + \varepsilon\delta_2) = \gamma_1\delta_1 + \varepsilon(\gamma_1\delta_2 + \gamma_2\delta_1).$$

The multiplicative inverse of $\mathbb{A} = \gamma_1 + \varepsilon\gamma_2$ is given by

$$\mathbb{A}^{-1} = \frac{1}{\gamma_1} - \varepsilon\frac{\gamma_2}{\gamma_1^2}, \ \gamma_1 \neq 0$$

and there is no inverse for pure dual numbers; hence, this algebra of numbers is not a field over real numbers. By using the inverse of dual numbers we can define the division operation of two dual numbers $\mathbb{A}$ and $\mathbb{B}$ as $\mathbb{A}\mathbb{B}^{-1}$ where $\mathbb{B}$ is not a pure dual number and $\mathbb{B} \neq 0$. The set given by

$$\mathbb{D}^3 = \{\vec{\mathbb{A}} = \vec{\gamma_1} + \varepsilon\vec{\gamma_2} \mid \vec{\gamma_1}, \vec{\gamma_2} \in \mathbb{R}^3\}$$

is amodule over the ring of dual numbers $\mathbb{D}$, the elements of this module are called dual vectors.

Proposition 1. *Let $\vec{\mathbb{A}}$ and $\vec{\mathbb{B}}$ be dual vectors; then, their scalar and cross products are respectively given by*

$$\langle\vec{\mathbb{A}}, \vec{\mathbb{B}}\rangle = \langle\vec{\gamma_1}, \vec{\delta_1}\rangle + \varepsilon\left(\langle\vec{\gamma_1}, \vec{\delta_2}\rangle + \langle\vec{\gamma_2}, \vec{\delta_1}\rangle\right),$$

$$\vec{\mathbb{A}} \times \vec{\mathbb{B}} = \vec{\gamma_1} \times \vec{\delta_1} + \varepsilon\left(\vec{\gamma_1} \times \vec{\delta_2} + \vec{\gamma_2} \times \vec{\delta_1}\right).$$

If $\vec{\mathbb{A}}$ and $\vec{\mathbb{B}}$ are unit dual vectors then

$$\langle\vec{\mathbb{A}}, \vec{\mathbb{B}}\rangle = cos\Phi = cos\phi - \varepsilon\phi^* sin\phi,$$

where $\Phi = \phi + \varepsilon\phi^*$ is a dual angle between them [1].

Proposition 2. *The norm of the dual vector $\vec{\mathbb{A}}$ is given by*

$$\left\|\vec{\mathbb{A}}\right\| = \left\|\vec{\gamma_1}\right\| + \varepsilon\frac{\langle\vec{\gamma_1}, \vec{\gamma_2}\rangle}{\left\|\vec{\gamma_1}\right\|}. \tag{1.1}$$

If $\vec{\mathbb{A}}$ is a unit dual vector, then $\|\vec{\gamma_1}\| = 1$ and $\langle\vec{\gamma_1}, \vec{\gamma_2}\rangle = 0$ [1].

Sequences of integers hold an important place in mathematical literature with many famous sequences playing crucial roles across various branches of mathematics. These sequences often arise naturally in mathematical investigations and have been extensively studied due to their properties and profound connections to different areas of mathematics.

The Fibonacci and Lucas number sequences [11–16] are defined by

$$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2};\ \mathrm{F}_0 = 0, \mathrm{F}_1 = 1,$$

$$\mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2}; \mathrm{L}_0 = 2, \mathrm{L}_1 = 1.$$

The characteristic equation of recurrences F_n and L_n is given by $x^2 - x - 1 = 0$. The Binet formulas for the F_n and L_n sequences are respectively given by

$$\mathrm{F}_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} \text{ and } \mathrm{L}_n = \varphi^n + \psi^n,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation. The following identities are hold for Fibonacci and Lucas number sequences

$$\mathrm{L}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n+1} \text{ and } \mathrm{L}_{n-1} + \mathrm{L}_{n+1} = 5\mathrm{F}_n. \tag{1.2}$$

The Leonardo number sequence [17–24] is defined as follows

$$\mathrm{Le}_n = \mathrm{Le}_{n-1} + \mathrm{Le}_{n-2} + 1$$

with the initial condition the $\mathrm{Le}_0 = \mathrm{Le}_1 = 1$. The Leonardo numbers are related to the Fibonacci numbers through the below relation:

$$\mathrm{Le}_n = 2\mathrm{F}_{n+1} - 1.$$

This Binet's formula for the Leonardo number sequence is given by

$$\mathrm{Le}_n = \frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1, \tag{1.3}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are roots of the characteristic equation $x^3 - 2x^2 + 1 = 0$ and the other root of this equation is $\lambda = 1$.

Here, our goal was to study this a new number system that can by generated by using dual numbers and well known Leonardo numbers as dual Leonardo number sequences and we describe dual Leonardo vectors and angles to apply these dual vectors and angles in the geometry of dual space.

2. Dual Leonardo number sequences

In this section, we demonstrate dual Leonardo number sequences, presenting their fundamental identities and properties. Some fundamental properties and identities of these number sequences are discussed, such as Binet's formula, Catalan's, Cassini's and D'ocagne's identities. These identities play crucial roles in obtaining on understanding of the properties of dual Leonardo number sequences, and we aim to establish and prove them by using Binet's formulas.

The dual Leonardo number is defined by the following relation:

$$\mathcal{L}_n = \mathrm{Le}_n + \varepsilon \mathrm{Le}_{n+1}, \tag{2.1}$$

where Le_n is the Leonardo number, $\varepsilon^2 = 0$ and $\varepsilon \neq 0$. The following recurrence relation can be obtained directly by using the definition of dual Leonardo numbers

$$\mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} + \mathcal{L}_0, \;\; n \geq 2,$$

where $\mathcal{L}_0 = 1 + \varepsilon, \mathcal{L}_1 = 1 + 3\varepsilon$.

Theorem 1. *The Binet formula for dual Leonardo numbers is given by*

$$\mathcal{L}_n = \frac{2\underline{\varphi}\varphi^{n+1} - 2\underline{\psi}\psi^{n+1}}{\varphi - \psi} - \mathcal{L}_0, \tag{2.2}$$

where $\underline{\varphi} = 1 + \varepsilon\varphi$, $\underline{\psi} = 1 + \varepsilon\psi$ *and* $\underline{\varphi}\,\underline{\psi} = 1 + \varepsilon = \mathcal{L}_0$.

Proof. From (1.3) and (2.1), we find that

$$\begin{aligned}
\mathcal{L}_n &= \mathrm{Le}_n + \varepsilon \mathrm{Le}_{n+1} \\
&= \frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1 + \varepsilon\left(\frac{2\varphi^{n+2} - 2\psi^{n+2}}{\varphi - \psi} - 1\right) \\
&= \frac{2\underline{\varphi}\varphi^{n+1} - 2\underline{\psi}\psi^{n+1}}{\varphi - \psi} - (1 + \varepsilon) \\
&= \frac{2\underline{\varphi}\varphi^{n+1} - 2\underline{\psi}\psi^{n+1}}{\varphi - \psi} - \mathcal{L}_0.
\end{aligned}$$

For the next proof we have taken into account the following equalities:

$$\varphi\psi = -1, \; \varphi + \psi = 1 \text{ and } \frac{(\varphi^r\psi^{-r} + \varphi^{-r}\psi^r - 2)}{(\varphi - \psi)^2} = (-1)^{-r}(\frac{\varphi^r - \psi^r}{\varphi - \psi})^2. \tag{2.3}$$

Theorem 2. *(Catalan's identity) For any integers n and r such that n ≥ r, we have*

$$\mathcal{L}_{n+r}\mathcal{L}_{n-r} - \mathcal{L}_n^2 = (-1)^{n-r}\left(\mathrm{Le}_{r-1} + 1\right)^2\mathcal{L}_0 - \left(\mathcal{L}_{n+r} + \mathcal{L}_{n-r} - 2\mathcal{L}_n\right)\mathcal{L}_0.$$

Proof. Using (1.2), (1.3), (2.3) and the Binet formula for dual Leonardo numbers in (2.2), we get

$$\begin{aligned}
\mathcal{L}_{n+r}\mathcal{L}_{n-r} - \mathcal{L}_n^2 =& \left(\frac{2\underline{\varphi}\varphi^{n+r+1} - 2\underline{\psi}\psi^{n+r+1}}{\varphi - \psi} - \mathcal{L}_0\right)\left(\frac{2\underline{\varphi}\varphi^{n-r+1} - 2\underline{\psi}\psi^{n-r+1}}{\varphi - \psi} - \mathcal{L}_0\right) \\
&- \left(\frac{2\underline{\varphi}\varphi^{n+1} - 2\underline{\psi}\psi^{n+1}}{\varphi - \psi} - \mathcal{L}_0\right)^2 \\
=& \frac{4}{(\varphi - \psi)^2}\,\underline{\varphi}\,\underline{\psi}(-1)^n\left(\varphi^r\psi^{-r} + \varphi^{-r}\psi^r - 2\right) - \frac{2\underline{\varphi}\varphi^{n+r+1} - 2\underline{\psi}\psi^{n+r+1}}{\varphi - \psi}\mathcal{L}_0
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2\underline{\varphi}\varphi^{n-r+1}-2\underline{\psi}\psi^{n-r+1}}{\varphi-\psi}\mathcal{L}_0+\frac{4\underline{\psi}\psi^{n+1}-4\underline{\varphi}\varphi^{n+1}}{\varphi-\psi}\mathcal{L}_0\\
=&\mathcal{L}_0(-1)^{n-r}\left(\frac{2\varphi^r-2\psi^r}{\varphi-\psi}\right)^2-\left(\mathcal{L}_{n+r}+\mathcal{L}_{n-r}-2\mathcal{L}_n\right)\mathcal{L}_0\\
=&(-1)^{n-r}\left(\mathrm{Le}_{r-1}+1\right)^2\mathcal{L}_0-\left(\mathcal{L}_{n+r}+\mathcal{L}_{n-r}-2\mathcal{L}_n\right)\mathcal{L}_0.
\end{aligned}
$$

Corollary 1. *(Cassini's identity) For any integer n and $n \geq 1$, the following identity holds:*

$$\mathcal{L}_{n+1}\mathcal{L}_{n-1}-\mathcal{L}_n^2=4\left(-1\right)^{n-1}\mathcal{L}_0-\left(\mathcal{L}_{n+1}+\mathcal{L}_{n-1}-2\mathcal{L}_n\right)\mathcal{L}_0.$$

Proof. This identity can be obtained from Catalan's identity by taking $r = 1$.

Theorem 3. *(D'ocagne's identity) For any integers m and n, we have*

$$\mathcal{L}_{m+1}\mathcal{L}_n-\mathcal{L}_m\mathcal{L}_{n+1}=\mathcal{L}_0(\mathcal{L}_{n-1}-\mathcal{L}_{m-1})+2(-1)^m\mathcal{L}_0(\mathrm{Le}_{n-m-1}+1).$$

Proof. Using (2.2) and (2.3), we will have

$$
\begin{aligned}
\mathcal{L}_{m+1}\mathcal{L}_n-\mathcal{L}_m\mathcal{L}_{n+1}=&\left(\frac{2\underline{\varphi}\varphi^{m+2}-2\underline{\psi}\psi^{m+2}}{\varphi-\psi}-\mathcal{L}_0\right)\left(\frac{2\underline{\varphi}\varphi^{n+1}-2\underline{\psi}\psi^{n+1}}{\varphi-\psi}-\mathcal{L}_0\right)\\
&-\left(\frac{2\underline{\varphi}\varphi^{m+1}-2\underline{\psi}\psi^{m+1}}{\varphi-\psi}-\mathcal{L}_0\right)\left(\frac{2\underline{\varphi}\varphi^{n+2}-2\underline{\psi}\psi^{n+2}}{\varphi-\psi}-\mathcal{L}_0\right)\\
=&\frac{4}{(\varphi-\psi)^2}\underline{\varphi}\,\underline{\psi}(\varphi^{m+1}\psi^{n+2}-\varphi^{m+2}\psi^{n+1}+\varphi^{n+2}\psi^{m+1}-\varphi^{n+1}\psi^{m+2})\\
&+\frac{2\mathcal{L}_0}{(\varphi-\psi)}(\underline{\varphi}\varphi^{n+2}-\underline{\psi}\psi^{n+2}+\underline{\varphi}\varphi^{m+1}-\underline{\psi}\psi^{m+1}\\
&-\underline{\varphi}\varphi^{m+2}+\underline{\psi}\psi^{m+2}-\underline{\varphi}\varphi^{n+1}+\underline{\psi}\psi^{n+1}).
\end{aligned}
$$

By taking (1.3), we get

$$\mathcal{L}_{m+1}\mathcal{L}_n-\mathcal{L}_m\mathcal{L}_{n+1}=\mathcal{L}_0(\mathcal{L}_{n-1}-\mathcal{L}_{m-1})+2(-1)^m\mathcal{L}_0(\mathrm{Le}_{n-m-1}+1).$$

The proof is completed.

3. Dual Leonardo vector and dual angle

In this section we introduce the dual Leonardo vector $\vec{\mathcal{L}}_n$ and dual angle Φ, respectively. Moreover, we will give the main identities for this vector.

Definition 1. *The dual Leonardo vector $\vec{\mathcal{L}}_n$ in $\mathbb{D}^3$ is defined by*

$$\vec{\mathcal{L}}_n=\vec{\mathrm{L}}e_n+\varepsilon\vec{\mathrm{L}}e_{n+1}, \tag{3.1}$$

where $\vec{\mathrm{L}}e_n=(\mathrm{Le}_n,\mathrm{Le}_{n+1},\mathrm{Le}_{n+2})$ *and* $\vec{\mathrm{L}}e_{n+1}=(\mathrm{Le}_{n+1},\mathrm{Le}_{n+2},\mathrm{Le}_{n+3})$ *are real vectors.*

Theorem 4. *If $\vec{\mathcal{L}}_n$ is a unit dual vector, then*

$$2\mathrm{Le}_{n+1} = 2\mathrm{Le}_{2n+2} + \mathrm{Le}_{n+1}^2 + 1,$$

and

$$2Le_{2n+3} + 1 = (Le_{n+2} - 1)(1 - Le_{n+3}).$$

Proof. From (1.1), (1.3) and (3.1), we get

$$\begin{aligned}
\left\|\vec{\mathrm{L}}e_n\right\|^2 &= \mathrm{L}e_n^2 + \mathrm{Le}_{n+1}^2 + \mathrm{L}e_{n+2}^2 \\
&= (2\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} - 1)^2 + (2\frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi} - 1)^2 + (2\frac{\varphi^{n+3} - \psi^{n+3}}{\varphi - \psi} - 1)^2 \\
&= \frac{4}{(\varphi - \psi)^2}\left(\varphi^{2n+2} + \psi^{2n+2} + \varphi^{2n+4} + \psi^{2n+4} + \varphi^{2n+6} + \psi^{n+6} + 2\,(-1)^n\right) \\
&- 2\left(\frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1 + \frac{2\varphi^{n+2} - 2\psi^{n+2}}{\varphi - \psi} - 1 + \frac{2\varphi^{n+3} - 2\psi^{n+3}}{\varphi - \psi} - 1 + 3\right) + 3 \\
&= \frac{4}{5}\,(\mathrm{L}_{2n+2} + \mathrm{L}_{2n+4} + \mathrm{L}_{2n+6} + 2\,(-1)^n) - 2\,(\mathrm{Le}_n + \mathrm{Le}_{n+1} + \mathrm{Le}_{n+2}) - 3.
\end{aligned}$$

Using (1.2) and (1.3), we have

$$\begin{aligned}
\left\|\vec{\mathrm{L}}e_n\right\| &= \sqrt{\frac{4}{5}(5\mathrm{F}_{2n+3} + 5\mathrm{F}_{n+3}^2) - 2(2\mathrm{Le}_{n+2} - 1) - 3} \\
&= \sqrt{2(\mathrm{Le}_{2n+2} + 1) + (\mathrm{Le}_{n+2} + 1)^2 - 4\mathrm{F}_{n+2} - 1} \\
&= \sqrt{2\mathrm{Le}_{2n+2} + \mathrm{Le}_{n+2}^2 - 2\mathrm{Le}_{n+2} + 2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\langle\vec{\mathrm{L}}e_n, \vec{\mathrm{L}}e_{n+1}\rangle &= \mathrm{Le}_n\mathrm{L}e_{n+1} + \mathrm{L}e_{n+1}\mathrm{L}e_{n+2} + \mathrm{L}e_{n+2}\mathrm{L}e_{n+3} \\
&= \frac{4}{(\varphi - \psi)^2}\left(\varphi^{2n+3} + \psi^{2n+3} - \varphi^{n+1}\psi^{n+2} - \varphi^{n+2}\psi^{n+1}\right) \\
&- \frac{2}{\varphi - \psi}\left(\varphi^{n+1} - \psi^{n+1} + \varphi^{n+2} - \psi^{n+2}\right) + 1 \\
&+ \frac{4}{(\varphi - \psi)^2}\left(\varphi^{2n+3} + \psi^{2n+3} - \varphi^{n+1}\psi^{n+2} - \varphi^{n+2}\psi^{n+1}\right) \\
&- \frac{2}{\varphi - \psi}\left(\varphi^{n+1} - \psi^{n+1} + \varphi^{n+2} - \psi^{n+2}\right) + 1 \\
&+ \frac{4}{(\varphi - \psi)^2}\left(\varphi^{2n+3} + \psi^{2n+3} - \varphi^{n+1}\psi^{n+2} - \varphi^{n+2}\psi^{n+1}\right) \\
&- \frac{2}{\varphi - \psi}\left(\varphi^{n+1} - \psi^{n+1} + \varphi^{n+2} - \psi^{n+2}\right) + 1.
\end{aligned}$$

Using (2.3) and performing necessary calculations, we obtain

$$\langle\vec{\mathrm{L}}e_n, \vec{\mathrm{L}}e_{n+1}\rangle = 2Le_{2n+3} + (Le_{n+2} - 1)(Le_{n+3} - 1) + 1.$$

Then, by using (1.1), the proof is completed.

Theorem 5. *For dual Leonardo vectors $\vec{\mathcal{L}}_n$ and $\vec{\mathcal{L}}_m$, we have*

$$\left\langle \vec{\mathcal{L}}_n, \vec{\mathcal{L}}_m \right\rangle = 2\mathcal{L}_{m+n+2} + (\mathcal{L}_{n+2} - \mathcal{L}_0)(\mathcal{L}_{m+2} - \mathcal{L}_0) + 2\varepsilon(\mathcal{L}_{n+m+3} + \mathcal{L}_0) + 1,$$

$$\begin{aligned}\vec{\mathcal{L}}_n \times \vec{\mathcal{L}}_m &= 2\,(-1)^m\,(\mathrm{Le}_{n-m-1}\,(1+\varepsilon)+1)\,(\vec{e_1} - \vec{e_2} - \vec{e_3}) \\ &+ (\mathcal{L}_n - \mathcal{L}_m + \varepsilon\,(\mathrm{Le}_n - \mathrm{Le}_m))\vec{e_1} + (\mathcal{L}_{n+1} - \mathcal{L}_{m+1})\vec{e_2} \\ &+ (\mathcal{L}_{n-1} - \mathcal{L}_{m-1} + \varepsilon\,(\mathrm{Le}_{n-1} - \mathrm{Le}_{m-1}))\vec{e_3}.\end{aligned}$$

Proof. Using (1.3), (2.2) and (2.3), we obtain

$$\begin{aligned}\left\langle \vec{\mathrm{L}}e_n, \vec{\mathrm{L}}e_m \right\rangle &= \mathrm{Le}_n\mathrm{Le}_m + \mathrm{Le}_{n+1}\mathrm{Le}_{m+1} + \mathrm{Le}_{n+2}\mathrm{Le}_{m+2} \\ &= \left(\frac{2\varphi^{n+1} - 2\psi^{n+1}}{\varphi - \psi} - 1\right)\left(\frac{2\varphi^{m+1} - 2\psi^{m+1}}{\varphi - \psi} - 1\right) \\ &+ \left(\frac{2\varphi^{n+2} - 2\psi^{n+2}}{\varphi - \psi} - 1\right)\left(\frac{2\varphi^{m+2} - 2\psi^{m+2}}{\varphi - \psi} - 1\right) \\ &+ \left(\frac{2\varphi^{n+3} - 2\psi^{n+3}}{\varphi - \psi} - 1\right)\left(\frac{2\varphi^{m+3} - 2\psi^{m+3}}{\varphi - \psi} - 1\right) \\ &= 2(\mathrm{Le}_{n+m+2} - \mathrm{Le}_{n+2} - \mathrm{Le}_{m+2}) + (\mathrm{Le}_{n+2} + 1)(\mathrm{Le}_{m+2}) + 1 \\ &= 2\mathrm{Le}_{n+m+2} + (\mathrm{Le}_{n+2} - 1)(Le_{m+2} - 1) + 1,\end{aligned}$$

and by using the properties of determinants, we get

$$\vec{\mathrm{L}}e_n \times \vec{\mathrm{L}}e_m = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \mathrm{Le}_n & \mathrm{Le}_{n+1} & \mathrm{Le}_{n+2} \\ \mathrm{Le}_m & Le_{m+1} & Le_{m+2} \end{vmatrix}$$

$$= (\mathrm{Le}_{n+1}\mathrm{Le}_{m+2} - \mathrm{Le}_{n+2}\mathrm{Le}_{m+1})\vec{e_1} - (\mathrm{Le}_n\mathrm{Le}_{m+2} - \mathrm{Le}_{n+2}\mathrm{Le}_m)\vec{e_2} + (\mathrm{Le}_n\mathrm{Le}_{m+1} - \mathrm{Le}_{n+1}\mathrm{Le}_m)\vec{e_3}$$

$$= 2\,(-1)^m\,(\mathrm{Le}_{n-m-1} + 1)\,(\vec{e_1} - \vec{e_2} - \vec{e_3}) + (\mathrm{Le}_n - \mathrm{Le}_m)\vec{e_1} + (\mathrm{Le}_{n+1} - \mathrm{Le}_{m+1})\vec{e_2} + (\mathrm{Le}_{n-1} - \mathrm{Le}_{m-1})\vec{e_3},$$

where $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ are unit direction vectors. By Proposition (1) and the above calculations we have

$$\begin{aligned}\left\langle \vec{\mathcal{L}}_n, \vec{\mathcal{L}}_m \right\rangle &= \left\langle \vec{\mathrm{L}}e_n, \vec{\mathrm{L}}e_m \right\rangle + \varepsilon\left(\left\langle \vec{\mathrm{L}}e_n, \vec{\mathrm{L}}e_{m+1} \right\rangle + \left\langle \vec{\mathrm{L}}e_{n+1}, \vec{\mathrm{L}}e_m \right\rangle\right) \\ &= 2\mathrm{Le}_{n+m+2} + (\mathrm{Le}_{m+2} - 1)(\mathrm{Le}_{n+2} - 1) + 1 \\ &+ \varepsilon\,(-\mathrm{Le}_{n+4} - \mathrm{Le}_{m+4} + 4\mathrm{Le}_{n+m+3} + \mathrm{Le}_{n+2}\mathrm{Le}_{m+3} + \mathrm{Le}_{n+3}\mathrm{Le}_{m+2} + 6) \\ &= 2\mathrm{Le}_{n+m+2} + (\mathrm{Le}_{m+2} - 1)(\mathrm{Le}_{n+2} - 1) + 1 \\ &+ \varepsilon\,(4\mathrm{Le}_{n+m+3} + (\mathrm{Le}_{m+3} - 1)(\mathrm{Le}_{n+2} - 1) + (\mathrm{Le}_{m+2} - 1)(\mathrm{Le}_{n+3} - 1) + 2) \\ &= 2\mathcal{L}_{m+n+2} + (\mathrm{Le}_{m+2} - 1)(\mathcal{L}_{n+2} - \mathcal{L}_0) \\ &+ \varepsilon(2\mathrm{Le}_{m+n+3} + \mathrm{Le}_{m+3}\mathrm{Le}_{n+2} - \mathrm{Le}_{m+3} - \mathrm{Le}_{n+2} + 3) + 1 \\ &= 2\mathcal{L}e_{m+n+2} + (\mathcal{L}_{n+2} - \mathcal{L}_0)(\mathrm{Le}_{m+2} - 1 + \varepsilon\mathrm{Le}_{n+3} - \varepsilon) + 2\varepsilon(\mathrm{Le}_{n+m+3} + 1) + 1 \\ &= 2\mathcal{L}_{m+n+2} + (\mathcal{L}_{n+2} - \mathcal{L}_0)(\mathcal{L}_{m+2} - \mathcal{L}_0) + 2\varepsilon(\mathcal{L}_{n+m+3} + \mathcal{L}_0) + 1,\end{aligned}$$

and

$$\begin{aligned}\vec{\mathcal{L}}_n \times \vec{\mathcal{L}}_m &= \vec{\mathrm{L}e}_n \times \vec{\mathrm{L}e}_m + \varepsilon(\vec{\mathrm{L}e}_n \times \vec{\mathrm{L}e}_{m+1} + \vec{\mathrm{L}e}_{n+1} \times \vec{\mathrm{L}e}_m) \\ &= 2\,(-1)^m\,(\mathrm{Le}_{n-m-1}\,(1+\varepsilon)+1)\,(\vec{e_1}-\vec{e_2}-\vec{e_3}) \\ &+ (\mathcal{L}_n - \mathcal{L}_m + \varepsilon\,(\mathrm{Le}_n - \mathrm{Le}_m))\vec{e_1} + (\mathcal{L}_{n+1} - \mathcal{L}_{m+1})\vec{e_2} \\ &+ (\mathcal{L}_{n-1} - \mathcal{L}_{m-1} + \varepsilon\,(\mathrm{Le}_{n-1} - \mathrm{Le}_{m-1}))\vec{e_3}.\end{aligned}$$

Proposition 3. *Let $\vec{\mathbb{A}} = \vec{\gamma_1} + \varepsilon\vec{\gamma_2}$ be a unit dual vector in $\mathbb{D}^3$; then the directed line that corresponds to $\vec{\mathbb{A}}$ has an equation of the form*

$$\vec{x} = \vec{\gamma_1} \times \vec{\gamma_2} + \mu\,\vec{\gamma_1},$$

where $0 \le \mu \le 1$.

Proof. Let $\vec{\mathbb{A}} = \vec{\gamma_1} + \varepsilon\vec{\gamma_2}$ be the unit dual vector and if T and X are points on the corresponding line d and O is the origin (see Figure 1), then

$$\overrightarrow{OX} = \overrightarrow{OT} + \overrightarrow{TX},$$
$$\vec{x} = \vec{t} + \lambda\,\vec{\gamma_1},$$

where λ is a real parameter. A point X is on the line of vectors $\vec{\gamma_1}$ and $\vec{\gamma_2}$ if and only if

$$\vec{\gamma_1} \times \vec{\gamma_2} = \vec{\gamma_1} \times (\vec{t} \times \vec{\gamma_1}).$$

Then,

$$\vec{t} = \vec{\gamma_1} \times \vec{\gamma_2} + \left\langle \vec{\gamma_1}, \vec{t} \right\rangle \vec{\gamma_1},$$

and

$$\vec{x} = \vec{\gamma_1} \times \vec{\gamma_2} + \left\langle \vec{\gamma_1}, \vec{t} \right\rangle \vec{\gamma_1} + \lambda\,\vec{\gamma_1}.$$

By taking $\mu = \left\langle \vec{\gamma_1}, \vec{t} \right\rangle + \lambda$, we get the result as follows

$$\vec{x} = \vec{\gamma_1} \times \vec{\gamma_2} + \mu\,\vec{\gamma_1}.$$

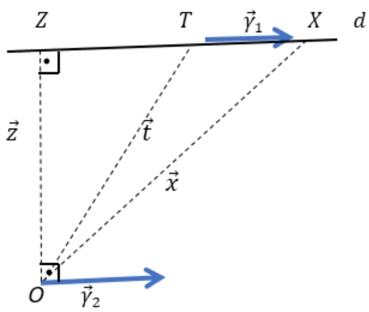

Figure 1. E. Study mapping.

Theorem 6. *Suppose that $\vec{\mathcal{L}}_{0_n}$ is a dual Leonardo vector and let $\vec{\mathcal{L}}_n = \vec{\mathrm{L}}e_n + \varepsilon\vec{\mathrm{L}}e_{n+1}$ be its unitized vector, that is $\vec{\mathcal{L}}_n \in \mathbb{S}^2$; then, the equation for the corresponding line is given by*

$$\begin{aligned}\vec{x_n} &= (2\,(-1)^{n+1}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_n-\mathrm{Le}_{n-1}-1)\vec{e_1}\\ &+(2\,(-1)^{n}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_{n+1}-\mathrm{Le}_{n}-1)\vec{e_2}\\ &+(2\,(-1)^{n}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_{n+2}+\mathrm{Le}_{n-3}+1)\vec{e_3}.\end{aligned}$$

Proof. By using (1.3) and Proposition (3), we obtain

$$\begin{aligned}\vec{x_n} &= \vec{\mathrm{L}}e_n\times\vec{\mathrm{L}}e_{n+1}+\mu\,\vec{\mathrm{L}}e_n,\\ &= \begin{bmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ \mathrm{Le}_n & \mathrm{Le}_{n+1} & \mathrm{Le}_{n+2}\\ \mathrm{Le}_{n+1} & \mathrm{Le}_{n+2} & \mathrm{Le}_{n+3}\end{bmatrix}+\mu\,(\mathrm{Le}_n\vec{e}_1+\mathrm{Le}_{n+1}\vec{e}_2+\mathrm{Le}_{n+2}\vec{e}_3)\\ &= (2\,(-1)^{n+1}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_n-\mathrm{Le}_{n-1}-1)\vec{e_1}\\ &+(2\,(-1)^{n}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_{n+1}-\mathrm{Le}_{n}-1)\vec{e_2}\\ &+(2\,(-1)^{n}\,(\mathrm{Le}_{n-2}+1)+\mu\,\mathrm{Le}_{n+2}+\mathrm{Le}_{n-3}+1)\vec{e_3}.\end{aligned}$$

The dual number $\Phi = \phi + \varepsilon\phi^*$ was defined by E. Study in 1903 as a dual angle, where ϕ is the angle between oriented lines l_1 and l_2 in $\mathbb{R}^3$ and ϕ^* is the vertical distance between these lines [3].

Corollary 2. *For unit vectors $\vec{\mathcal{L}}_n$ and $\vec{\mathcal{L}}_m$, we have*

$$\begin{aligned}\left\langle\vec{\mathcal{L}}_n,\vec{\mathcal{L}}_m\right\rangle &= 2\mathrm{Le}_{n+m+2}+(\mathrm{Le}_{m+2}-1)(\mathrm{Le}_{n+2}-1)+1\\ &+\varepsilon\,[-\mathrm{Le}_{n+4}-\mathrm{Le}_{m+4}+4\mathrm{Le}_{n+m+3}+\mathrm{Le}_{n+2}\mathrm{Le}_{m+3}+\mathrm{Le}_{n+3}\mathrm{Le}_{m+2}+6]\\ &= cos\phi-\varepsilon\phi^* sin\phi.\end{aligned}$$

The following cases can be given for a dual angle Φ satisfying that $cos\Phi = cos\phi - \varepsilon\phi^* sin\phi$, as wellas the conditions for the corresponding lines l_1 and l_2.

Case 1. *Assume that $\phi = \frac{\pi}{2}$ and $\phi^* \neq 0$;then,*

$$(1-\mathrm{Le}_{n+2})(\mathrm{Le}_{m+2}-1) = 1+2\mathrm{Le}_{n+m+2}$$

and

$$\phi^* = \mathrm{Le}_{n+4}+\mathrm{Le}_{m+4}-4\mathrm{Le}_{n+m+3}-\mathrm{Le}_{n+2}\mathrm{Le}_{m+3}-\mathrm{Le}_{n+3}\mathrm{Le}_{m+2}-6.$$

Additionally the corresponding lines l_1 and l_2 are perpendicular such that not intersect each other (see Figure 2).

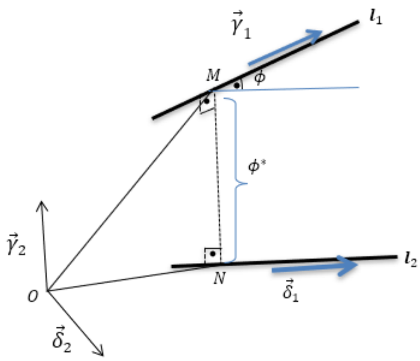

Figure 2. Geometric representation of a dual angle between the directed lines l_1 and l_2.

Case 2. *Assume that* $\phi^* = 0$ *and* $\phi \neq 0$*; then, we obtain*

$$\phi = \arccos\left(2\mathrm{Le}_{n+m+2} + (\mathrm{Le}_{n+2} - 1)(\mathrm{Le}_{m+2} - 1) + 1\right)$$

and

$$4\mathrm{Le}_{n+m+3} + \mathrm{Le}_{n+2}\mathrm{Le}_{m+3} + \mathrm{Le}_{n+3}\mathrm{Le}_{m+2} + 6 = \mathrm{Le}_{n+4} + \mathrm{Le}_{m+4},$$

in this case lines l_1 *and* l_2 *intersect each other (see Figure 3).*

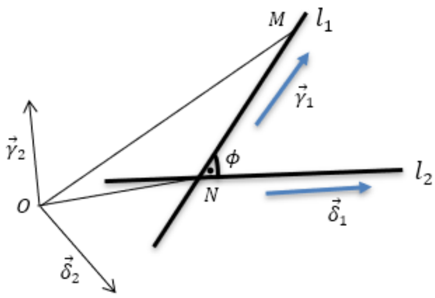

Figure 3. Intersection of lines.

Case 3. *Assume that* $\phi = \frac{\pi}{2}$ *and* $\phi^* = 0$*; then,*

$$(1 - \mathrm{Le}_{n+2})(\mathrm{Le}_{m+2} - 1) = 1 + 2\mathrm{Le}_{n+m+2}$$

and

$$4\mathrm{Le}_{n+m+3} + \mathrm{Le}_{n+2}\mathrm{Le}_{m+3} + \mathrm{Le}_{n+3}\mathrm{Le}_{m+2} + 6 = \mathrm{Le}_{n+4} + \mathrm{Le}_{m+4},$$

in this case lines l_1 *and* l_2 *intersect each other at a right angle (see Figure 4).*

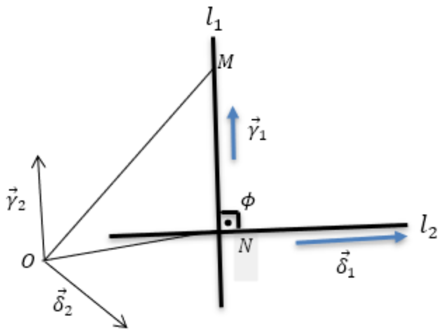

Figure 4. Perpendicular intersection of lines.

Case 4. *Assume that* $\phi = 0$ *and* $\phi^* \neq 0$*; then,*

$$2\text{Le}_{n+m+2} = (1 - \text{Le}_{n+2})(\text{Le}_{m+2} - 1)$$

and

$$4\text{Le}_{n+m+3} + \text{Le}_{n+2}\text{Le}_{m+3} + \text{Le}_{n+3}\text{Le}_{m+2} + 6 = \text{Le}_{n+4} + \text{Le}_{m+4},$$

in this case corresponding lines l_1 *and* l_2 *are parallel (see Figure 5).*

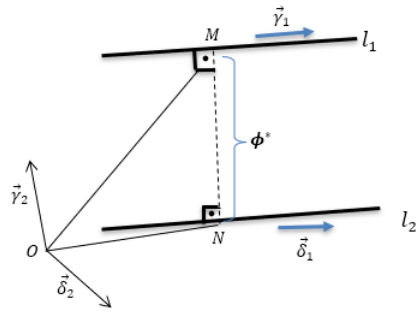

Figure 5. The lines are parallel.

4. Conclusions

In this paper, we have introduced the concept of dual Leonardo number sequences and derived the fundamental identities associated with them. Then, we introduced the notion of dual Leonardo vectors and investigated their scalar and cross products. Additionally, we have provided the properties of dual Leonardo vectors, including Study mapping and relationship with dual angles. These findings provide valuable insights into the interference between dual Leonardo number sequences, vectors and geometric concepts within the framework of dual space. Moreover, our explanation of the fundamental identities and properties opens the door further research on the geometry of dual space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. H. W. Guggenheimer, *Differential geometry*, New York: McGraw-Hill Book Company, 1963.
2. A. P. Kotelnikov, *Screw calculus and some applications to geometry and mechanics*, Annals of the Imperial University of Kazan, 1895.
3. E. Study, *Geometrie der dynamen*, Leipzig, 1903.
4. G. R. Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous, spatial kinematics, *Mech. Mach. Theory*, **11** (1976), 141–156. https://doi.org/10.1016/0094-114X(76)90006-9
5. J. Wittenburg, Dual quaternions in kinematics of spatial mechanisms, In: *Computer aided analysis and optimization of mechanical system dynamics*, **9** (1984), 129–145. https://doi.org/10.1007/978-3-642-52465-3_4
6. A. T. Yang, F. Freudenstein, Application of dual number quaternions algebra to the analysis of spatial mechanisms, *J. Appl. Mech.*, **31** (1964), 300–308. https://doi.org/10.1115/1.3629601
7. P. Azariadis, N. Aspragathos, Computer graphics representation and transformation of geometric entities using dual unit vectors and line transformations, *Comput. Graph.*, **25** (2001), 195–209. https://doi.org/10.1016/S0097-8493(00)00124-2
8. J. M. McCarthy, G. S. Soh, *Geometric design of linkages*, New York: Springer, 2011. https://doi.org/10.1007/978-1-4419-7892-9
9. Y. M. Moon, S. Kota, Automated synthesis of mechanisms using dual-vector algebra, *Mech. Mach. Theory*, **37** (2002), 143–166. https://doi.org/10.1016/S0094-114X(01)00073-8
10. K. Teu, W. Kim, Estimation of the axis of a screw motion from noisy data new method based on Plucker lines, *J. Biomech.*, **39** (2006), 2857–2862. https://doi.org/10.1016/j.jbiomech.2005.09.013
11. A. F. Horadam, A generalized Fibonacci sequence, *Am. Math. Mon.*, **68** (1961), 455–459. https://doi.org/10.1080/00029890.1961.11989696
12. A. F. Horadam, Complex Fibonacci numbers and Fibonacci quaternions, *Am. Math. Mon.*, **70** (1963), 289–291. https://doi.org/10.2307/2313129
13. T. Koshy, *Fibonacci and Lucas numbers with applications*, A Wiley-Interscience Publication, 2001. https://doi.org/10.1002/9781118033067
14. S. Vajda, *Fibonacci and Lucas numbers, and the golden section: theory and applications*, 1989.
15. V. E. Hoggatt, *Fibonacci and Lucas numbers*, Houghton Mifflin, 1969.

16. I. A. Guven, S. K. Nurkan, A new approach to Fibonacci, Lucas numbers and dual vectors, *Adv. Appl. Clifford Algebras*, **25** (2015), 577–590. https://doi.org/10.1007/s00006-014-0516-7
17. P. Catarino, A. Borges, On Leonardo numbers, *Acta Math. Univ. Comen.*, **89** (2019), 75–86.
18. Y. Alp, E. G. Kocer, Some properties of Leonardo numbers, *Konuralp J. Math.*, **9** (2021), 183–189.
19. A. G. Shannon, O. Deveci, A note on generalized and extended Leonardo sequences, *Notes Number Theory*, **28** (2022), 109–114. https://doi.org/10.7546/nntdm.2022.28.1.109-114
20. A. Karataş, On complex Leonardo numbers, *Notes Number Theory*, **28** (2022), 458–465. https://doi.org/10.7546/nntdm.2022.28.3.458-465
21. M. Shattuck, Combinatorial proofs of identities for the generalized Leonardo numbers, *Notes Number Theory*, **28** (2022), 778–790. https://doi.org/10.7546/nntdm.2022.28.4.778-790
22. E. Tan, D. Savin, S. Yılmaz, A new class of Leonardo hybrid numbers and some remarks on Leonardo quaternions over finite fields, *Mathematics*, **11** (2023), 4701. https://doi.org/10.3390/math11224701
23. S. K. Nurkan, İ. A. Güven, Ordered Leonardo quadruple numbers, *Symmetry*, **15** (2023), 149. https://doi.org/10.3390/sym15010149
24. K. Kuhapatanakul, J. Chobsorn, On the generalized Leonardo numbers, *Integers Electron. J. Comb. Number Theory*, **22** (2022), 48.

AIMS Press