Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 14043–14061.
DOI: 10.3934/math.2024683
Received: 03 February 2024
Revised: 01 April 2024
Accepted: 10 April 2024
Published: 18 April 2024

Research article

Rational interpolative contractions with applications in extended b-metric spaces

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Abstract: In this manuscript, utilizing interpolative contractions with fractional forms, some unique fixed-point results were studied in the context of extended b-metric spaces. For the validity of the presented results some examples are given. In the last section an existence theorem is provided to study the existence of a solution for the Fredholm integral equation.

Keywords: fixed point; metric space; interpolative contractions; extended b-metric space; integral equation
Mathematics Subject Classification: 45E99, 47H10, 54H25, 55M20

1. Introduction and preliminaries

The concept of distance was axiomatically formulated in the beginning of the 19th century with the introduction of metric spaces , by Frechet and Haussdorff. Since then, many authors have developed this concept, with several results available in the literature. For the generalization of this concept, the axioms of the metric space have been relaxed in several ways (see [1]), among which the notion of a b-metric space takes great importance. Bakhtin [2] (and, independently, Czerwik [3]) presented the idea of b-metric spaces and showed different results based on the existence of fixed points. For the sake of understanding, we present here the definition of a b-metric, also called a quasi-metric (see [4]).

Definition 1.1. *(Czerwik [3]) Consider Q to be a non-empty set and b: $Q \times Q \to [0, +\infty)$ to be a self-map fulfilling the below prerequisites:*

(1) $b(s, p) = 0 \Leftrightarrow s = p$;
(2) $b(s, p) = b(p, s)$ *for all* $s, p \in Q$;
(3) $b(s, u) \leq q[b(s, p) + b(p, u)]$ *for all* $s, p, u \in Q$, *where* $q \geq 1$.

The function b: $Q \times Q \to [0, +\infty)$ is called a b-metric, while the pair (Q, b) is known as a b-metric space.

Example 1.1. *[4] The space $M^p[0,1]$ (where $p \in (0, 1)$) of all real functions $k(s)$, $s \in [0, 1]$ such that*

$$\int_0^1 |k(s)|^p ds < +\infty$$

together with the functional

$$b(k, u) = \left(\int_0^1 |k(s) - u(s)|^p ds \right)^{\frac{1}{p}}, \quad \text{for each} \quad k, u \in M^p[0, 1]$$

is a b-metric space. Here, $q = 2^{\frac{1}{p}-1}$.

Example 1.2. *[5] Let*

$$Q = \{t_k : 1 \leq k \leq J\}$$

for some $J \in N$ and $a \geq 2$. Define a function b: $Q \times Q \to [0, +\infty)$ by

$$b(t_k, t_l) = \begin{cases} 0, & \text{if } k = l; \\ a, & \text{if } |k - l| = 1; \\ 2, & \text{if } |k - l| = 2; \\ 1, & \text{otherwise.} \end{cases}$$

Accordingly, we obtain

$$b(t_i, t_j) \leq \frac{a}{2}[b(t_i, t_k) + b(t_k, t_j)]$$

for all $i, j, k \in \{1, 2, \ldots, J\}$. The pair (Q, b) forms a b-metric space for $a > 2$. We can observe that the standard triangular inequality does not hold in this case.

The b-metric space shares many topological properties with traditional metric spaces but does not require continuity. Recently, Kamran et al. [6] presented a new generalization of metric spaces and proved some important fixed-point results in the newly defined space. Further more, Alqahtani et al. [7] studied common fixed point results on extended b-metric space.

Definition 1.2. *[6] Consider Q to be a non-empty set and ϑ: $Q \times Q \to [1, +\infty)$. A function b_ϑ: $Q \times Q \to [0, +\infty)$ is said to be an extended b-metric if for all $s, t, u \in Q$ the given axioms are satisfied*

(1) $b_\vartheta(s, t) = 0$ *implies* $s = t$;

(2) $b_\vartheta(s,t) = b_\vartheta(t,s)$;

(3) $b_\vartheta(s,u) \le \vartheta(s,u)[b_\vartheta(s,t) + b_\vartheta(t,u)]$.

The pair (Q, b_ϑ) *is known as extended b-metric space.*

Remark 1.1. *[5] Suppose* $\vartheta(s,t) = a$, *for* $a \ge 1$, *then it is obvious that the b-metric and extended b-metric spaces (bMS) will coincide. Note that either the b-metric or the extended b-metric need to be continuous like metric spaces.*

Example 1.3. *[5] Suppose* $p \in (0,1)$, $q > 1$ *and* $Q = l_p(R) \cup l_q(R)$ *equipped with the metric*

$$b(s,v) = \begin{cases} b_p(s,v), & \text{if } s,v \in l_p(R); \\ b_q(s,v), & \text{if } s,v \in l_q(R); \\ 0, & \text{otherwise.} \end{cases}$$

Where

$$l_r(R) = \{s = \{s_n\} \subset R : \sum_{n=1}^{+\infty} |s_n|^r < +\infty\}$$

for $r = p, q$, *and*

$$b_r(s,v) = (\sum_{n=1}^{+\infty} |s_n - v_n|^r)^{1/r}$$

for $r = p, q$, *we can observe that* (Q, b_ϑ) *forms an extended bMS with*

$$\vartheta(s,v) = \begin{cases} 2^{1/p}, & \text{if } s,v \in l_p(R); \\ 2^{1/q}, & \text{if } s,v \in l_q(R); \\ 1, & \text{otherwise.} \end{cases}$$

Example 1.4. *[6] Let* $G = \{1,2,3\}$, ϑ*:* $G\times G \to [1,+\infty)$ *and* b_ϑ*:* $G\times G \to [0,+\infty)$ *as* $\vartheta(s,t) = 1+s+t$ *and*

$$b_\vartheta(1,1) = b_\vartheta(2,2) = b_\vartheta(3,3) = 0 \text{ and } b_\vartheta(1,2) = b_\vartheta(2,1) = 80,$$
$$b_\vartheta(1,3) = b_\vartheta(3,1) = 1000, \text{ and } b_\vartheta(2,3) = b_\vartheta(3,2) = 600.$$

Example 1.5. *[7] Let* $G = [0,1]$, ϑ*:* $G \times G \to [1,+\infty)$ *and* b_ϑ*:* $G \times G \to [0,+\infty)$ *be defined by*

$$\vartheta(s,e) = \frac{1+s+e}{s+e}, \quad b_\vartheta(s,e) = \frac{1}{se}, \quad s,e \in (0,1], \ s \ne e;$$
$$b_\vartheta(s,e) = 0, \quad s,e \in [0,1] \ s = e;$$
$$b_\vartheta(s,0) = b_\vartheta(0,s) = \frac{1}{s}, \quad s \in (0,1].$$

Now we are going to discuss some basic notions like convergence, completeness, and Cauchy sequence in extended bMS that are defined as:

Definition 1.3. *[6] Suppose* (Q, b_ϑ) *be an extended b-metric space.*

(1) A sequence $\{s_j\}_{j\in\mathbb{K}}$ *in* Q *will converge to* $t \in Q$*, if for every* $\zeta > 0$ *there exists* $K = K(\zeta) \in \mathbb{K}$ *such that* $b_\vartheta(s_j, s) < \zeta$ *for all* $j \ge K$*. In this case, we write*

$$\lim_{j\to+\infty} s_j = s.$$

(2) A sequence $\{s_j\}_{j\in\mathbb{K}}$ *in* Q *is known as Cauchy sequence if for every* $\zeta > 0$ *exists* $K = K(\zeta) \in \mathbb{K}$ *such that* $b_\vartheta(s_j, s_m) < \zeta$*, for all* $m, j \geq K$.

Definition 1.4. *[5] Suppose every Cauchy sequence in* Q *is convergent, then the extended-bMS* (Q, b_ϑ) *is said to be complete.*

Definition 1.5. *[5] Let* (Q, b_ϑ) *be an extended-bMS and* $\urcorner\colon Q \to Q$ *be a self-map. For* $t_o \in Q$*, the orbit of* $\urcorner$ *at* t_o *is the set*

$$O(t_o, \urcorner) = \{t_o, \urcorner t_o, \urcorner^2 t_o, \ldots\}.$$

The function $\urcorner$ *is known as orbitally continuous at a given point* $e \in Q$ *if*

$$\lim_{j\to+\infty} \urcorner^j t_o = e \ \text{ implies } \ \urcorner\urcorner^j t_o = \urcorner e.$$

Besides that, suppose every Cauchy sequence $\{\urcorner^j t_o\}$ *in* Q *is convergent, then the extended-bMS* (Q, b_ϑ) *is called* $\urcorner$*-orbitally complete.*

Definition 1.6. *[8] Suppose* (R, b_ϑ) *be an extended b-metric. The mapping* $\urcorner\colon R \to R$ *is known as m-continuous, where* $m = 1, 2, \ldots,$ *if*

$$\lim_{n\to+\infty} \urcorner^m t_n = \urcorner e,$$

whenever t_n *is a sequence in* R *such that*

$$\lim_{n\to+\infty} \urcorner^{m-1} t_n = e.$$

Remark 1.2. *[1] It is notable that every continuous function is orbitally continuous in* Q *and every complete extended-bMS is* $\urcorner$*-orbitally complete for any* $\urcorner\colon Q \to Q$*, but the converse is not necessarily true.*

Besides that, it is obvious that 1-continuity results in 2-continuity, which in turn will result in 3-continuity, and so on; but the converse of this is not true. This might be clearer from this example: consider the self-mapping $\urcorner\colon Q \to Q$, where $Q = [0, +\infty)$, defined by

$$Ts = \begin{cases} 5, & \text{if } s \in [0, 5], \\ 1, & \text{if } s \in (5, +\infty), \end{cases}$$

we can clearly see that $\urcorner$ is discontinuous (at $s = 5$), while it is 2-continuous because $T^2 s = 5$.

Definition 1.7. *[4] A self-mapping* ¥*:* $[0, +\infty) \to [0, +\infty)$ *is said to be a comparison function if it is increasing and* $¥^n(s) \to 0$ *as* $n \to +\infty$ *for every* $s \in [0, +\infty)$*, where* $¥^n$ *is the* n^{th} *iterate of* ¥.

Lemma 1.1. *[4] Suppose* ¥*:* $[0, +\infty) \to [0, +\infty)$ *is a comparison function, then*

(1) ¥ *is continuous at 0;*
(2) every iterate $¥^k$ *of* ¥, $k \geq 1$ *is also a comparison function;*
(3) $¥(s) < s$ *for all* $s > 0$.

Definition 1.8. *[4] Suppose $t \geq 1$ be a real number. A self-mapping ¥: $[0, +\infty) \to [0, +\infty)$ is said to be a (b)-comparison function if it is increasing and if there exist $k_o \in N, a \in [0, 1)$ and a convergent non-negative series $\sum_{k=1}^{+\infty} w_k$ such that*

$$t^{k+1}¥^{k+1}(s) \leq at^k¥^k(s) + w_k$$

for $k \geq k_o$ and any $s \geq 0$.

The set of all (b)-comparison functions is denoted by Θ. The (b)-comparison function is said to be a (c)-comparison function if we take $t = 1$. It is easy to show that every (c)-comparison function is a (b)-comparison function, but the converse is not true. Another important property of (b)-comparison functions is presented by Berinde [4].

Lemma 1.2. *[4] Suppose Ξ: $[0, +\infty) \to [0, +\infty)$ be a (b)-comparison function. Then:*

(1) The series $\sum_{t=0}^{\infty} h^t \Xi^t(s)$ converges for any $s \in [0, +\infty)$;
(2) The function b_s: $[0, +\infty) \to [0, +\infty)$ defined as

$$b_s = \sum_{t=0}^{+\infty} h^t \Xi^t(s)$$

is increasing and is continuous at $s = 0$.

Remark 1.3. *[5] Each (b)-comparison function Ξ satisfies $\Xi(s) < s$ and*

$$\lim_{n \to +\infty} \Xi^n(t) = 0$$

for each $s > 0$.

Definition 1.9. *[9] Let α: $Q \times Q \to [0, +\infty)$ be a mapping and $Q \neq \emptyset$. A self-mapping $\urcorner$: $Q \to Q$ is called α-orbital admissible if for all $a \in Q$, we have*

$$\alpha(a, \urcorner(a)) \geq 1 \textit{ implies } \alpha(\urcorner(a), \urcorner^2(a)) \geq 1.$$

Besides, the α-orbital admissible function $\urcorner$ is said to be triangular α-orbital admissible if

$$(\urcorner O)\ \alpha(a, t) \geq 1 \text{ and } \alpha(a, \urcorner(t)) \geq 1 \text{ implies } \alpha(a, \urcorner(t)) \geq 1, \text{ for all } a, t \in Q.$$

Besides that, we say that the extended-bMS (Q, b_ϑ) is α-regular if for any sequence t_n in Q such that

$$\lim_{n \to \infty} t_n = t \quad \text{and} \quad \alpha(t_n, t_{n+1}) \geq 1,$$

we have $\alpha(t_n, t) \geq 1$ (for more details and examples, see [9]). Popescu [9] redefined the concept of α-admissible mapping and triangular α-admissible mapping. Qawagneha et al. [10] investigated common fixed points for pairs of triangular α -admissible mappings. The idea of interpolative contractions was very recently introduced by [11], and the well-known Kannan-type contractions were revisited in the context of interpolation. Subsequently, most famous contractions (Rus [4], Ćirić [12], Reich [13], Hardy and Rogers [14], Kannan [15], Bianchini and Grandolf [16]) have been revisited in this newly introduced context-(see [11, 17, 18]). Following this trend and using the idea of fractional interpolative contraction, Fulga [1] established some fixed-point results in the framework of *bMS*. Additionally Debnath et al. studied interpolative Hardy-Rogers and Reich-Rus-Ćirić- type contractions in *b*-metric and rectangular *bMS* [19].

Non-linear integral equations have emerged in various fields of science and engineering, offering powerful tools for modeling physical phenomena and solving problems in diverse areas such as physics, engineering, and economics. Various researchers have studied these equations using different approaches, some of which can be found in [20–22].

Motivated by the above contributions using fractional interpolative contractions some fixed-point results are studied in the setting of extended-*bMS*. The work here presented generalizes some well-known results from the existing literature. For the authenticity of the present work a key theorems is used to establish the existence of solutions for the Fredholm integral equations. The results obtained can be extended to investigate the existence of solutions for other integral equations (see [20, 21, 23]).

2. Main results

We initiate with the following definition of contractive mapping to prove the main results.

Definition 2.1. *Let* (Q, b_ϑ) *be an extended-bMS. A mapping* $\urcorner\colon Q \to Q$ *is known as* $A^l_\urcorner$*-admissible interpolative contraction* $(l = 1, 2)$ *if* $\exists\, \psi \in \Theta$ *and* $\Omega\colon Q \times Q \to [0, +\infty)$ *such that*

$$\frac{1}{2} b_\vartheta(s, \urcorner s) \leq b_\vartheta(s, a) \text{ implies } \Omega(s, a) b_\vartheta(\urcorner s, \urcorner a) \leq \psi(A^l_\urcorner(s, a)), \tag{2.1}$$

where $p_j \geq 0,\ j = 1, 2, 3, 4, 5$, *are such that* $\sum_{j=1}^{5} p_j = 1$ *and*

$$A^1_\urcorner(s, a) = [b_\vartheta(s, a)]^{p_1} \cdot [b_\vartheta(s, \urcorner s)]^{p_2} \cdot [b_\vartheta(a, \urcorner a)]^{p_3} \cdot [\frac{b_\vartheta(a, \urcorner a)(1 + b_\vartheta(s, \urcorner s))}{1 + b_\vartheta(s, a)}]^{p_4} \cdot [\frac{b_\vartheta(s, \urcorner a) + b_\vartheta(a, \urcorner s)}{2\vartheta(s, \urcorner a)}]^{p_5}, \tag{2.2}$$

and

$$A^2_\urcorner(s, a) = \begin{cases} [b_\vartheta(s, a)]^{p_1} \cdot [b_\vartheta(s, \urcorner s)]^{p_2} \cdot [b_\vartheta(a, \urcorner a)]^{p_3} \cdot [\frac{b_\vartheta(s, \urcorner s) b_\vartheta(a, \urcorner a) + b_\vartheta(s, \urcorner a) b_\vartheta(a, \urcorner s)}{\max\{b_\vartheta(a, \urcorner a), b_\vartheta(a, \urcorner s)\}}]^{p_4} & \\ \cdot [\frac{b_\vartheta(s, \urcorner s) b_\vartheta(s, \urcorner a) + b_\vartheta(a, \urcorner a) b_\vartheta(a, \urcorner s)}{\max\{b_\vartheta(s, \urcorner a), b_\vartheta(a, \urcorner s)\}}]^{p_5}, & \textit{if} \max\{b_\vartheta(s, \urcorner a), b_\vartheta(a, \urcorner s)\} \neq 0; \\ 0, & \textit{otherwise}, \end{cases} \tag{2.3}$$

for any $s, a \in Q \backslash Fix_\urcorner(Q), (Fix_\urcorner(Q) = \{s \in Q | \urcorner s = s\})$.

Theorem 2.1. *Let* (Q, b_ϑ) *be an extended-bMS and* $\urcorner$ *be an* $A^1_\urcorner$*-admissible interpolative contraction, assume that* $\exists$ *a sequence* $\{q_j\}_{j \in \mathbb{N}}, q_j > 1$, *for all* $j \in \mathbb{N}$, *such that* $\vartheta(a_j, a_m) < q_j$ *for all* $m > j$, *and* $\urcorner$ *also satisfies:*

i) There exists $a_o \in Q$ *such that* $\alpha(a_o, \urcorner a_o) \geq 1$;

ii) $\urcorner$ *is* α*-orbital admissible;*

iii_1) $\urcorner$ *is orbitally continuous; or*

iii_2) $\urcorner$ *is m-continuous for* $m \geq 1$.

Then, $\urcorner$ possesses a fixed point $\varpi \in Q$ and the sequence $\{\urcorner^m a_o\}$ converges to ϖ.

Proof. Suppose $a_o \in Q$ and the sequence $\{a_j\}$ be defined as $a_j = \urcorner^j a_o$, $\forall\ j \in \mathbb{N}$. Suppose there exists $k \in \mathbb{N}$ such that

$$a_k = a_{k+1} = \urcorner a_k,$$

then, we have that a_k is a fixed point of $\urcorner$ and the proof is complete. Therefore, we suppose that $a_j \neq a_{j+1}$ for any $j \in \mathbb{N}$. Using assumption (*ii*), we obtain that $\urcorner$ is α-orbital admissible, so consider that we have

$$\alpha(a_o, a_1) = \alpha(a_o, \urcorner a_o) \geq 1 \Rightarrow \alpha(a_1, a_2) = \alpha(\urcorner a_o, \urcorner(\urcorner a_o)) \geq 1 \Rightarrow \cdots \Rightarrow \alpha(a_{j-1}, a_j) \geq 1.$$

On the other hand, we have that

$$\frac{1}{2} b_\vartheta(a_{j-1}, \urcorner a_{j-1}) = \frac{1}{2} b_\vartheta(a_{j-1}, a_j) \leq b_\vartheta(a_{j-1}, a_j).$$

We mention in the beginning that $\urcorner$ is an $A^{\frac{1}{\urcorner}}$-admissible interpolative contraction, so from (2.1) we get

$$\begin{aligned} b_\vartheta(\urcorner a_{j-1}, \urcorner a_j) &\leq \alpha(a_{j-1}, a_j) b_\vartheta(\urcorner a_{j-1}, a_j) \leq \psi(A^{\frac{1}{\urcorner}}(a_{j-1}, a_j)) \\ &= \psi\Big([b_\vartheta(a_{j-1}, a_j)]^{p_1} \cdot [b_\vartheta(a_{j-1}, \urcorner a_{j-1})]^{p_2} \cdot [b_\vartheta(a_j, \urcorner a_j)]^{p_3} \\ &\cdot [\frac{b_\vartheta(a_j, \urcorner a_j)(1 + b_\vartheta(a_{j-1}, \urcorner a_{j_1}))}{1 + b_\vartheta(a_{j-1}, a_j)}]^{p_4} \cdot [\frac{b_\vartheta(a_{j-1}, \urcorner a_j) + b_\vartheta(a_j, \urcorner a_{j-1})}{2\vartheta(a_{j-1}, \urcorner a_j)}]^{p_5} \Big) \\ &= \psi\Big([b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2)} \cdot [b_\vartheta(a_j, a_{j+1})]^{(p_3+p_4)} \\ &\cdot [\frac{\vartheta(a_{j-1}, a_{j+1})[b_\vartheta(a_{j-1}, a_j) + b_\vartheta(a_j, a_{j+1})]}{2\vartheta(a_{j-1}, a_{j+1})}]^{p_5} \Big) \\ &= \psi\Big([b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2)} . [b_\vartheta(a_j, a_{j+1})]^{(p_3+p_4)} \cdot [\frac{b_\vartheta(a_{j-1}, a_j) + b_\vartheta(a_j, a_{j+1})}{2}]^{P_5} \Big). \end{aligned}$$

So,

$$b_\vartheta(a_j, a_{j+1}) = \psi\Big([b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2)} \cdot [b_\vartheta(a_j, a_{j+1})]^{(p_3+p_4)} \cdot [\frac{b_\vartheta(a_{j-1}, a_j) + b_\vartheta(a_j, a_{j+1})}{2}]^{P_5} \Big). \tag{2.4}$$

Therefore,

$$b_\vartheta(a_j, a_{j+1}) < [b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2)} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_3+p_4} \cdot [\frac{b_\vartheta(a_{j-1}, a_j) + b_\vartheta(a_j, a_{j+1})}{2}]^{p_5},$$

i.e.,

$$[b_\vartheta(a_j, a_{j+1})]^{(1-p_3-p_4)} < [b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2)} \cdot [\frac{b_\vartheta(a_{j-1}, a_j) + b_\vartheta(a_j, a_{j+1})}{2}]^{p_5}.$$

If exists $m_o \in \mathbb{N}$ such that

$$b_\vartheta(a_{m_o-1}, a_{m_o}) \leq b_\vartheta(a_{m_o}, a_{m_o+1}),$$

then the above inequality becomes

$$b_\vartheta(a_{m_o}, a_{m_o+1}) < [b_\vartheta(a_{m_o-1}, a_{m_o})]^{(p_1+p_2)} . [b_\vartheta(a_{m_o}, a_{m_o+1})]^{(p_5+p_3+p_4)},$$

i.e.,

$$[b_\vartheta(a_{m_o}, a_{m_o+1})]^{(p_1+p_2)} < [b_\vartheta(a_{m_o-1}, a_{m_o})]^{(p_1+p_2)},$$

so,

$$b_\vartheta(a_{m_o}, a_{m_o+1}) < b_\vartheta(a_{m_o-1}, a_{m_o}),$$

but it is a contradiction, so for any $j \in \mathbb{N}$,

$$b_\vartheta(a_j, a_{j+1}) < b_\vartheta(a_{j-1}, a_j).$$

Furthermore, returning to inequality (2.4), we have

$$b_\vartheta(a_j, a_{j+1}) \leq \psi(b_\vartheta(a_{j-1}, a_j)) \leq \cdots \leq \psi^j(b_\vartheta(a_o, a_1)). \tag{2.5}$$

Let $r \in \mathbb{N}$ and $j < m$, then by (2.5) together with the condition (iii) of extended-bMS, we obtain

$$\begin{aligned}
&b_\vartheta(a_j, a_m) \leq \vartheta(a_j, a_m)[b_\vartheta(a_j, a_{j+1}) + b_\vartheta(a_{j+1}, a_m)]\\
&\leq \vartheta(a_j, a_m)[b_\vartheta(a_j, a_{j+1})] + \vartheta(a_j, a_m)[\vartheta(a_{j+1}, a_m)[b_\vartheta(a_{j+1}, a_{j+2}) + b_\vartheta(a_{j+2}, a_m)]]\\
&\vdots\\
&\leq \vartheta(a_j, a_m)[b_\vartheta(a_j, a_{j+1})] + \vartheta(a_j, a_m)\vartheta(a_{j+}, a_m)b_\vartheta(a_{j+1}, a_{j+2})\\
&+ \cdots + \vartheta(a_j, a_m)\vartheta(a_{j+1}, a_m)\vartheta(a_{j+2}, a_m)\cdots\vartheta(a_{m-1}, a_m)b_\vartheta(a_o, a_1)\\
&\leq \vartheta(a_j, a_m)\psi^j(b_\vartheta(a_o, a_1)) + \vartheta(a_j, a_m)\vartheta(a_{j+1}, a_m)\psi^{j+1}(b_\vartheta(a_o, a_1))\\
&+ \cdots + [\vartheta(a_j, a_m)\cdots\vartheta(a_{m-1}, a_m)]\psi^{m-1}(b_\vartheta(a_o, a_1))\\
&\leq \vartheta(a_1, a_m)\vartheta(a_2, a_m)\cdots\vartheta(a_{m-1}, a_m)\psi^j(b_\vartheta(a_o, a_1)) + \vartheta(a_1, a_m)\vartheta(a_2, a_m)\\
&\cdots\vartheta(a_{m-1}, a_m)\psi^{j+1}(b_\vartheta(a_o, a_1)) + \cdots + \vartheta(a_1, a_m)\vartheta(a_2, a_m)\cdots\vartheta(a_{m-1}, a_m)\psi^{m-1}(b_\vartheta(a_o, a_1)).
\end{aligned}$$

Let

$$S_j = \sum_{e=1}^{j} \psi^e(b_\vartheta(a_o, a_1)) \prod_{k=1}^{j} \vartheta(a_k, a_m), \quad S_{m_1} = \sum_{e=1}^{m-1} \psi^e(b_\vartheta(a_o, a_1)),$$

we deduce

$$b_\vartheta(a_j, a_m) \leq S_{m-1} - S_{j-1} \text{ for all } m > j.$$

Consider the series

$$\sum_{j=1}^{\infty} \psi^j(b_\vartheta(a_o, a_1)) \prod_{e=1}^{j} \vartheta(a_e, a_m).$$

Let

$$q = \max\{q_1, q_2, \ldots, q_j\},$$

we have

$$u_j = \psi^j(b_\vartheta(a_o, a_1)) \prod_{j=1}^{k} \vartheta(a_j, a_m) \leq \psi^j(b_\vartheta(a_o, a_1))q^j = v_j.$$

From Lemma 1.2, we have that $\sum_{k=0}^{\infty}\psi^k(b_\vartheta(a_o,a_1))q^k$ converges. For the convergence of series using comparison criteria, we get that

$$\sum_{j=1}^{\infty}\psi^j(b_\vartheta(a_o,a_1))\prod_{e=1}^{j}\vartheta(a_e,a_m)$$

converges, and hence

$$\lim_{j,m\to\infty} b_\vartheta(a_j,a_m)=0.$$

As a result, we say that $\{a_j\}_{j\in N}$ is a Cauchy sequence in a $\urcorner$-orbitally complete extended-bMS. Hence, there exists a point $\varpi\in Q$, such that

$$\lim_{j\to\infty}\urcorner^j a_o=\varpi.$$

We can declare that ϖ is a fixed point of the self-mapping $\urcorner$ under of any hypothesis, (iii_1) or (iii_2). Indeed,

$$\varpi=\lim_{j\to\infty}a_j=\lim_{j\to\infty}\urcorner a_{j-1}.$$

Moreover,

$$\lim_{j\to\infty}\urcorner^m a_j=\varpi \tag{2.6}$$

for every $m\geq 1$.

If $\urcorner$ is m-continuous, then

$$\lim_{j\to\infty}\urcorner^m a_j=\urcorner\varpi,$$

and by (2.6), it follows that $\urcorner\varpi=\varpi$. Suppose $\urcorner$ is considered to be orbitally continuous on Q, then

$$\varpi=\lim_{j\to\infty}a_j=\lim_{j\to\infty}\urcorner a_{j-1}=\lim_{j\to\infty}\urcorner(\urcorner^{j-1}a_o)=\urcorner\varpi.$$

Therefore, $\varpi\in Fix_\urcorner(Q)$. □

Theorem 2.2. *Let (Q,b_ϑ) be an extended bMS. Suppose there exists a sequence $\{q_j\}, q_j>1$, for all $j\in\mathbb{N}$ such that $\vartheta(a_j,a_m)<q_j$, for all $m>j$, and $\urcorner$ is $A_\urcorner^2$-admissible interpolative contraction, and $\urcorner$ also satisfies:*

i) There exists $a_o\in Q$ such that $\alpha(a_o,\urcorner a_o)\geq 1$;

ii) $\urcorner$ is α-orbital admissible;

iii_1) $\urcorner$ is orbitally continuous; or

iii_2) $\urcorner$ is m-continuous for $m\geq 1$.

Then $\urcorner$ has a fixed point $\varpi\in Q$.

Proof. From the proof of the above theorem, for $a_o\in Q$, we construct the sequence $\{a_j\}$, where

$$a_j=\urcorner a_{j-1}=\urcorner^j a_o$$

for any $j \in N$. Since $a_{j-1} \neq a_j$ for any $j \in N$, keeping in mind that $\urcorner$ is assumed to be $A^2_\urcorner$-admissible interpolative contraction, we have

$$\begin{aligned}\frac{1}{2}b_\vartheta(a_{j-1}, \urcorner a_{j-1}) &= \frac{1}{2}b_\vartheta(a_{j-1}, a_j)\\ &\leq b_\vartheta(a_{j-1}, a_j),\\ \alpha(a_{j-1}, a_j)b_\vartheta(\urcorner a_{j-1}, \urcorner a_j) &\leq \psi(A^2_\urcorner(a_{j-1}, a_j)),\end{aligned}$$

where

$$\begin{aligned}A^2_\urcorner &= [b_\vartheta(a_{j-1}, a_j)]^{p_1} \cdot [b_\vartheta(a_{j-1}, \urcorner a_{j-1})]^{p_2} \cdot [b_\vartheta(a_j, \urcorner a_j)]^{p_3}\\ &\cdot [\frac{b_\vartheta(a_{j-1}, \urcorner a_{j-1})b_\vartheta(a_j, \urcorner a_j) + b_\vartheta(a_{j-1}, \urcorner a_j)b_\vartheta(a_j, \urcorner a_{j-1})}{\max\{b_\vartheta(a_j, \urcorner a_j), b_\vartheta(a_j, \urcorner a_{j-1})\}}]^{p_4}\\ &\cdot [\frac{b_\vartheta(a_{j-1}, \urcorner a_{j-1})b_\vartheta(a_{j-1}, \urcorner a_j) + b_\vartheta(a_j, \urcorner a_j)b_\vartheta(a_j, \urcorner a_{j-1})}{\max\{b_\vartheta(a_{j-1}, \urcorner a_j), b_\vartheta(a_j, \urcorner a_{j-1})\}}]^{p_5}\\ &= [b_\vartheta(a_{j-1}, a_j)]^{p_1} \cdot [b_\vartheta(a_{j-1}, a_j)]^{p_2} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_3}\\ &\cdot [\frac{b_\vartheta(a_{j-1}, a_j)b_\vartheta(a_j, a_{j+1}) + b_\vartheta(a_{j-1}, a_j)b_\vartheta(a_j, a_j)}{\max\{b_\vartheta(a_j, a_{j+1}), b_\vartheta(a_j, a_j)\}}]^{p_4}\\ &\cdot [\frac{b_\vartheta(a_{j-1}, a_j)b_(a_{j-1}, a_{j+1}) + b_\vartheta(a_j, a_{j+1})b_\vartheta(a_j, a_j)}{\max\{b_\vartheta(a_{j-1}, a_{j+1}), b_\vartheta(a_j, a_j)\}}]^{p_5}\\ &= [b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2+p_5+p_4)} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_3}.\end{aligned}$$

Since, by assumption, it follows that $\alpha(a_{j-1}, a_j) \geq 1$ for all $j \in \mathbb{N}$, we have

$$\begin{aligned}b_\vartheta(a_j, a_{j+1}) &\leq \alpha(a_{j-1}, a_j)b_\vartheta(\urcorner a_{j-1}, \urcorner a_j)\\ &\leq \psi(A^2_\urcorner(a_{j-1}, a_j))\\ &= \psi([b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2+p_4+p_5)} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_3})\\ &< [b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2+p_4+p_5)} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_3}.\end{aligned}$$

Therefore,

$$[b_\vartheta(a_j, a_{j+1})]^{(1-p_3)} < [b_\vartheta(a_{j-1}, a_j)]^{(p_1+p_2+p_4+p_5)},$$

i.e.,

$$b_\vartheta(a_j, a_{j+1}) < b_\vartheta(a_{j-1}, a_j), \text{ for any } j \in N.$$

Furthermore, keeping in mind ψ_2, we obtain

$$b_\vartheta(a_j, a_{j+1}) < \psi(b_\vartheta(a_{j-1}, a_j)) < \psi^2(b_\vartheta(a_{j-2}, a_{j-1})) < \cdots < \psi^j(b_\vartheta(a_o, a_1)),$$

and using the same method as in the proof of Theorem 2.1, we can see that the sequence $\{a_j\}$ is Cauchy. Furthermore, since (Q, b_ϑ) is considered to be $\urcorner$-orbitally complete, we can find a point $\varpi \in Q$ such that

$$\lim_{j\to\infty} \urcorner^j a_o = \varpi.$$

Consider that $\daleth$ is m-continuous, we have

$$\daleth\varpi = \lim_{j\to\infty} \daleth^m a_j = \lim_{j\to\infty} a_{j+m} = \varpi,$$

and suppose that $\daleth$ is orbitally continuous,we obtain

$$\daleth\varpi = \lim_{j\to\infty} \daleth(\daleth^j a_o) = \lim_{j\to\infty} \daleth a_j = \lim_{j\to\infty} a_{j+1} = \varpi,$$

it means that ϖ is a fixed point of $\daleth$. □

The following corollaries are observed from the above results.

Corollary 2.1. *Suppose (Q, b_ϑ) be a complete extended b-metric space. Suppose that there exists a sequence $\{p_j\}_{j\in N}$, $p_j > 1$ for all $j \in \mathbb{N}$ such that $\vartheta(s_j, s_m) < q_j$ for all $m > j$ and $\daleth$: $Q \to Q$ be a mapping such that*

$$\alpha(s, v)b_\vartheta(\daleth s, \daleth v) \le \psi(A^l_\daleth(s, v)).$$

For any $s, v \in Q\backslash Fix(Q)$, where $A^l_\daleth, l = 1, 2$ is defined by (2.2) and (2.3), and $\psi \in \Theta$. Then, $\daleth$ has a fixed point $\varpi \in Q$ provided that:

i) There exists $u_o \in Q$ such that $\alpha(u_o, \daleth u_o) \ge 1$;

ii) $\daleth$ is α-orbital admissible;

iii_1) $\daleth$ is orbitally continuous; or

iii_2) $\daleth$ is m-continuous for $m \ge 1$.

Corollary 2.2. *Suppose (Q, b_ϑ) be a complete extended b-metric space. Suppose that there exists a sequence $\{p_j\}_{j\in\mathbb{N}}$, $p_j > 1$, for all $j \in \mathbb{N}$ such that $\vartheta(s_j, s_m) < p_j$, for all $m > j$ and $\daleth$: $Q \to Q$ be a mapping such that*

$$1/2b_\vartheta(s, \daleth s) \le b_\vartheta(s, v) \quad implies \quad b_\vartheta(\daleth s, \daleth v) \le \psi(A^l_\daleth(s, v)).$$

For any $s, v \in Q\backslash Fix(Q)$, where $A^l_\daleth, l = 1, 2$, are defined by (2.2) and (2.3), and $\psi \in \Theta$. Then, $\daleth$ has a fixed point $\varpi \in Q$, provided that either $\daleth$ is orbitally continuous or $\daleth$ is m-continuous for $m \ge 1$.

Proof. Plug $\alpha(s, v) = 1$ in Theorems 2.1 and 2.2, respectively. □

By replacing the continuity of the function $\daleth$ with the continuity of b_ϑ, we will have the following result.

Theorem 2.3. *Suppose (Q, b_ϑ) be a complete, α-regular extended-bMS, where b_ϑ is continuous, and $\daleth$: $Q \to Q$ is such that*

$$\frac{1}{2\vartheta(a, v)} b_\vartheta(a, \daleth a) \le b_\vartheta(a, v) \quad implies \quad \alpha(a, v)b_\vartheta(\beth a, \daleth v) \le \psi(A^l_\daleth(a, v)),$$

where $\psi \in \Theta$ and $A^l_\daleth$, for $l = 1, 2$ are given by (2.2) and (2.3). Consider that:

(1) There exists $a_o \in Q$ such that $\alpha(a_o, \daleth a_o) \ge 1$;

(2) $\urcorner$ is α-orbital admissible.

Then, $\urcorner$ contains a fixed point $\varpi \in Q$, and the sequence $\{\urcorner^m a_o\}$ converges to this point ϖ.

Proof. As we know from the proof of Theorem 2.1, the sequence $\{a_j\}$ where

$$a_j = \urcorner a_{j-1} = \urcorner^j a_o$$

converges to a point $\varpi \in Q$, and this point ϖ is claimed to be a fixed point of the mapping $\urcorner$. For this reason, we can declare that

$$\frac{1}{2\vartheta(a,v)} b_\vartheta(a_j, \urcorner a_j) \leq b_\vartheta(a_j, \varpi) \tag{2.7}$$

or

$$\frac{1}{2\vartheta(a,v)} b_\vartheta(\urcorner a_j, \urcorner(\urcorner a_j)) \leq b_\vartheta(\urcorner a_j, \varpi). \tag{2.8}$$

Indeed, supposing on contrary

$$\frac{1}{2\vartheta(a,v)} b_\vartheta(a_j, \urcorner a_j) > b_\vartheta(a_j, \varpi)$$

and

$$\frac{1}{2\vartheta(a,v)} b_\vartheta(\urcorner a_j, \urcorner(\urcorner a_j)) > b_\vartheta(\urcorner a_j, \varpi),$$

we get that

$$\begin{aligned}
b_\vartheta(a_j, a_{j+1}) &= b_\vartheta(a_j, \urcorner a_j) \\
&\leq \vartheta(a,v)[b_\vartheta(a_j, \varpi) + b_\vartheta(\varpi, \urcorner a_j)] \\
&< \vartheta(a,v)[\frac{1}{2\vartheta(a,v)} b_\vartheta(a_j, \urcorner a_j) + \frac{1}{2\vartheta(a,v)} b_\vartheta(\urcorner a_j, \urcorner(\urcorner a_j))] \\
&= \frac{1}{2}[b_\vartheta(a_j, a_{j+1}) + b_\vartheta(a_{j+1}, a_{j+2})] \\
&\leq b_\vartheta(a_j, a_{j+1}),
\end{aligned}$$

$$\because \quad b_\vartheta(a_j, a_{j+1}) \geq b_\vartheta(a_{j+1}, a_{j+2}) \quad \Rightarrow \quad b_\vartheta(a_j, a_{j+1}) < b_\vartheta(a_j, a_{j+1}),$$

which leads to contradiction and then (2.7) and (2.8) holds. Keeping the regularity condition of the space (Q, b_ϑ) in mind, we have that $\alpha(a_j, \varpi) \geq 1$ for any $j \in N$.

Case 1. When $l = 1$, if (2.7) holds, we get

$$\begin{aligned}
b_\vartheta(a_{j+1}, \varpi) \leq \alpha(a_j, \varpi) b_\vartheta(\urcorner a_j, \urcorner\varpi) &\leq \psi(A^1_\urcorner(a_j, \varpi)) \leq A^1_\urcorner(a_j, \varpi) \\
&= [b_\vartheta(a_j, \varpi)]^{p_1} \cdot [b_\vartheta(a_j, \urcorner\varpi)]^{p_2} \cdot [b_\vartheta(\varpi, \urcorner\varpi)]^{p_3} \\
&\cdot [\frac{b_\vartheta(\varpi, \urcorner\varpi)(1 + b_\vartheta(a_j, a_{j+1}))}{1 + b_\vartheta(a_j, \varpi)}]^{p_4} \cdot [\frac{b_\vartheta(a_j, \urcorner\varpi) + b_\vartheta(\varpi, a_{j+1})}{2\vartheta(a_j, \urcorner\varpi)}]^{p_5},
\end{aligned}$$

we can distinguish the following two situations:

(1) $p_1 + p_2 > 0$, letting $j \to +\infty$ above, we obtain $b_\vartheta(\varpi, \urcorner\varpi) = 0$, thus $\urcorner\varpi = \varpi$.

(2) $p_1 = p_2 = 0$, when $j \to \infty$ above, and keeping in mind the continuity of extended-bMS we obtain

$$b_\vartheta(\varpi, \urcorner\varpi) < [b_\vartheta(\varpi, \urcorner\varpi)]^{(p_3+p_4+p_5)} = b_\vartheta(\varpi, \urcorner\varpi),$$

which is a contradiction. So, we have $\urcorner\varpi = \varpi$, i.e., ϖ is a fixed point of the mapping $\urcorner$.

Case 2. When $l = 2$. If (2.7) holds, we obtain

$$\begin{aligned} b_\vartheta(a_{j+1}, \urcorner\varpi) &\le \alpha(a_m, \varpi) b_\vartheta(\urcorner a_j, \urcorner\varpi) \le \psi(A^2_\urcorner(a_j, \varpi)) < A^2_\urcorner(a_j, \varpi) \\ &= [b_\vartheta(a_j, \varpi)]^{p_1} \cdot [b_\vartheta(a_j, a_{j+1})]^{p_2} \cdot [b_\vartheta(\varpi, \urcorner\varpi)]^{p_3} \\ &\cdot [\frac{b_\vartheta(\varpi, \urcorner\varpi) b_\vartheta(a_j, a_{j+1}) + b_\vartheta(\varpi, a_{j+1}) b_\vartheta(a_j, \urcorner\varpi)}{\max\{b_\vartheta(a_j, a_{j+1}), b_\vartheta(a_{j+1}, \urcorner\varpi)\}}]^{p_4} \\ &\cdot [\frac{b_\vartheta(\varpi, \urcorner\varpi) b_\vartheta(\varpi, a_{j+1}) + b_\vartheta(a_j, a_{j+1}) b_\vartheta(a_j, \urcorner\varpi)}{\max\{b_\vartheta(\varpi, a_{j+1}), b_\vartheta(a_{j+1}, \urcorner\varpi)\}}]^{p_5}, \end{aligned}$$

if (2.8) holds,

$$\begin{aligned} b_\vartheta(a_{j+2}, \urcorner\varpi) &\le \alpha(a_{j+1}, \varpi) b_\vartheta(\urcorner^2 a_j, \urcorner\varpi) \le \psi(A^2_\urcorner(\urcorner a_j, \varpi)) < A^2_\urcorner(\urcorner a_j, \varpi) \\ &= [b_\vartheta(a_{j+1}, \varpi)]^{p_1} \cdot [b_\vartheta(a_{j+1}, a_{j+2})]^{p_2} \cdot [b_\vartheta(\varpi, \urcorner\varpi)]^{p_3} \\ &\cdot [\frac{b_\vartheta(\varpi, \urcorner\varpi) b_\vartheta(a_{j+1}, a_{j+2}) + b_\vartheta(\varpi, a_{j+2}) b_\vartheta(a_{j+1}, \urcorner\varpi)}{\max\{b_\vartheta(a_{j+1}, a_{j+2}), b_\vartheta(a_{j+2}, \urcorner\varpi)\}}]^{p_4} \\ &\cdot [\frac{b_\vartheta(\varpi, \urcorner\varpi) b_\vartheta(\varpi, a_{j+2}) + b_\vartheta(a_{j+1}, a_{j+2}) b_\vartheta(a_{j+1}, \urcorner\varpi)}{\max\{b_\vartheta(\varpi, a_{j+2}), b_\vartheta(a_{j+2}, \urcorner\varpi)\}}]^{p_5}, \end{aligned}$$

we can distinguish the following two situations:

(1) $p_1 + p_2 + p_4 + p_5 > 0$, letting $j \to \infty$ above, we obtain $b_\vartheta(\varpi, \urcorner\varpi) = 0$, thus $\urcorner\varpi = \varpi$.
(2) $p_1 = p_2 = p_4 = p_5 = 0$, in this case, when $j \to \infty$ above, we get

$$b_\vartheta(\varpi, \urcorner\varpi) < [b_\vartheta(\varpi, \urcorner\varpi)]^{p_3} = b_\vartheta(\varpi, \urcorner\varpi),$$

which is a contradiction.

So, we get $\urcorner\varpi = \varpi$, i.e., ϖ is a fixed point of the mapping $\urcorner$. □

This result possesses the below corollaries.

Corollary 2.3. *Let* (Q, b_ϑ) *be a complete extended bMS. Suppose* $\{p_j\}_{j\in\mathbb{N}}$ *be a sequence,* $p_j > 1$ *for all* $j \in \mathbb{N}$ *such that* $\vartheta(s_j, s_m) < p_j$ *for all* $m > j$ *and* $\urcorner$*:* $Q \to Q$ *be a mapping such that* $\exists\ k \in [0, 1)$ *such that*

$$1/2 b_\vartheta(s, \urcorner s) \le b_\vartheta(s, v) \quad \textit{implies} \quad b_\vartheta(\urcorner s, \urcorner v) \le k A^l_\urcorner(s, v),$$

for any $s, v \in Q_{-Fie(Q)}$ *where* $A^l_\urcorner, l = 1, 2$ *are defined by* (2.2) *and* (2.3)*. Then,* $\urcorner$ *contains a fixed point* $\varpi \in Q$*, provided that either* $\urcorner$ *is orbitally continuous or* $\urcorner$ *is m-continuous for* $m \ge 1$.

Proof. Plug $\psi(t) = kt$ in the above corollary. □

Corollary 2.4. *Suppose (Q, b_ϑ) be a complete extended-bMS such that b_ϑ is continuous. Suppose there exist a sequence $\{p_j\}_{j\in\mathbb{N}}, p_j > 1$ for all $j \in N$ such that $\vartheta(s_j, s_m) < p_j$ for all $m > j$, and $\urcorner\colon Q \to Q$ be a self-mapping. Then $\urcorner$ has a fixed point provided that*

$$\frac{1}{2\vartheta(s,v)} b_\vartheta(s, \urcorner s) \le b_\vartheta(s, v) \quad \textit{implies} \quad b_\vartheta(\urcorner s, \urcorner v) \le \psi(A^l_\urcorner(s, v)),$$

where $\psi \in F$ and $A^l_\urcorner, l = 1, 2$ are given by (2.2) *and* (2.3).

Proof. Put $\alpha(s, v) = 1$ in Theorem 2.3. □

Corollary 2.5. *Consider (Q, b_ϑ) be a complete extended-bMS such that b_ϑ is continuous. Suppose that exists $\{p_j\}_{j\in\mathbb{N}}, p_j > 1$ for all $j \in \mathbb{N}$ such that $\vartheta(s_j, s_m) < p_j$ for all $m > j$ and $\urcorner\colon Q \to Q$, self-mapping. Then $\urcorner$ will have a fixed point in Q provided that there exist $k \in [0, 1)$ such that*

$$\frac{1}{2\vartheta(s,v)} b_\vartheta(s, \urcorner s) \le b_\vartheta(s, v) \quad \textit{implies} \quad b_\vartheta(\urcorner s, \urcorner v) \le kA^l_\urcorner(s, v),$$

where $A^l_\urcorner, l = 1, 2$ are given by (2.2) *and* (2.3).

Proof. Substituted $\psi(t) = kt$ in the above corollary. □

Now, we are going to present some examples of the above results.

Example 2.1. *Let $Q = [0, +\infty)$ and $b_\vartheta\colon Q \times Q \to [0, +\infty)$ be an extended-bMS defined as*

$$b_\vartheta(s, v) = \begin{cases} s+v, & \textit{if } s \ne v \textit{ for all } s, v \in Q; \\ 0, & \textit{if } s = v; \end{cases}$$

and $\vartheta\colon Q \times Q \to [1, +\infty)$ be defined as $\vartheta(s, v) = 1 + s + v$ for all $s, v \in Q$. Let the mapping $\urcorner\colon Q \to Q$ be defined by

$$\urcorner(s) = \begin{cases} \frac{1}{5}, & \textit{if } s \in [0, 1); \\ \frac{s+1}{4}, & \textit{if } s \in [1, 2]; \\ \frac{\sqrt{s}}{s^2+9} + \frac{In(s^2+1)}{s^2+7}, & \textit{if } s \in (2, +\infty); \end{cases}$$

and a function $\alpha : Q \times Q \to [0, +\infty)$, where

$$\alpha(s, v) = \begin{cases} \sqrt{s + v + 1}, & \textit{if } s, v \in [0, 1); \\ 5, & \textit{if } s = 0\ v = 2; \\ s^4 + \frac{v}{3}, & \textit{if } s = \frac{1}{4}\ , v \in \{3, 9\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Let also the comparison function $\psi\colon [0, \infty) \to [0, \infty)$, $\psi(s) = s/3$, and we choose $p_1 = p_5 = 1/5$, $p_2 = p_4 = 1/10$, and $p_3 = 2/5$. Therefore, we can clearly see that conditions (i) and (ii) are verified, and since $\urcorner^2(s) = 1/5$ is continuous, condition (iv) is also satisfied.

Case (1). *For $s, v \in [0, 1]$, we have $b_\vartheta(\urcorner s, \urcorner v) = 0$, so inequality* (2.1) *holds.*

Case (2). *For $s = 0$ and $v = 2$, we have $\frac{1}{2} b_\vartheta(0, 1/2) = 1/4 \le 2 = b_\vartheta(0, 2)$ and $b_\vartheta(\urcorner s, \urcorner v) = 0$. Thus, the* *inequality* (2.1) *holds.*

Case(3). *For* $s = 1/4$ *and* $v = 3$, *we have*

$$\begin{aligned} 1/2b_\vartheta(1/4, \daleth 1/4) &= 0.25 \leq 3.25 = b_\vartheta(1/4, 3) \\ &\Rightarrow \quad \alpha(1/4, 3)b_\vartheta(\daleth 1/4, T3) \\ &= 0.441716 < 0.8207 \\ &= A^1_\daleth(1/4, 3), \end{aligned}$$

hence (2.1) *holds.*

Case(4). *For* $s = 1/4$ *and* $v = 9$, *we have*

$$\begin{aligned} 1/2b_\vartheta(1/4, \daleth 1/4) &= 0.225 \leq 3.25 = b_\vartheta(1/4, 9) \\ &\Rightarrow \quad \alpha(1/4, 9)b_\vartheta(\daleth 1/4, \daleth 9) \\ &= 0.8513 < 2.0433 \\ &= A^1_\daleth(1/4, 9). \end{aligned}$$

All other cases are true because $\alpha(s, v) = 0$. *Hence, the mapping* $\daleth$ *is an* $A^1_\daleth$*-admissible interpolative contraction. So, as all the conditions of Theorem 2.1 are verified, we obtained that there exists a fixed point of the mapping* $\daleth$, *that is* $u = 1/5$.

Example 2.2. *Let* $Q = \{1, 2, 3, 5\}$ *and the extended-bMS defined* b_ϑ: $Q \times Q \to R^+$ *as* $b_\vartheta(s, v) = |s - v|^4$ *with* $\vartheta(s, v) = 1 + x + y$ *and* $\daleth$: $Q \to Q$ *such that* $\daleth(1) = \daleth(5) = 1$ *and* $\daleth(2) = \daleth(3) = 2$. *Taking* α: $Q \times Q \to R^+, \alpha(s, v) = 3$ *for all* $s, v \in Q$, *and* $\psi(t) = t/2$. *The constants here are all equal, i.e.,* $p_i = 1/5 \ \forall \ i = \{1, 2, 3, 4, 5\}$, *we have*

$$\frac{1}{2\vartheta(3, 5)} b_\vartheta(3, \daleth 3) = 1/18 < 16 = b_\vartheta(3, 5),$$

which implies

$$\alpha(3, 5)b_\vartheta(\daleth 3, \daleth 5) = 3 < 8.2 = \psi(A^2_\daleth(3, 5)).$$

Therefore, all the requirements of Theorem 2.3 are satisfied and it is clear that $\daleth$ *has (at least) a fixed point.*

3. Application

In this segment, we apply one of the observed results to study the existence of a solution for the Fredholm integral equation. Suppose $Q = C([a, b], R)$ be the space of all continuous real-valued functions defined on [a,b]. Note that the space Q is complete by considering the extended-bMS

$$b_\vartheta(s(e), v(e)) = \sup_{e \in [a,b]} |s(e) - v(e)|^2$$

with

$$\vartheta(s, v) = |s(e)| + |v(e)| + 2,$$

where $\vartheta(s, v)$: $Q \times Q \to [1, +\infty)$ and $\psi \in \Theta$ be the b-comparison function defined as $\psi(e) = e/2$. Consider the Fredholm integral equation as:

$$s(e) = f(e) + \int_a^b M(e, i, s(i))di \quad \text{for all} \quad i, e \in [a, b]. \tag{3.1}$$

Define a mapping $\urcorner$: $Q \to Q$, as

$$\urcorner(s(e)) = f(e) + \int_a^b M(e, i, s(i))di, \quad i, e \in [a, b].$$

Theorem 3.1. *Consider that the following conditions hold:*

(1) Suppose M: $[p, q] \times [p, q] \times R \to R$ and g: $[p, q] \to R$ be continuous.

(2) $\urcorner$ is $A^l_\urcorner$-admissible interpolative contraction, $A^l_\urcorner$, $l = 1, 2$ is defined in (2.2) *and* (2.3)*, respectively.*

(3)

$$sup_{e \in [p,q]} |M(e, i, s(i)) - M(e, i, v(e))| \leq \frac{\sqrt{A^l_\urcorner(s(e), v(e))}}{\sqrt{2}(q - p)}$$

for each $e, i \in [p, q]$ and $s, v \in Q$.

Then, the integral Eq (3.1) *has a solution.*

Proof. Suppose (Q, b_ϑ) be a complete extended-bMS and $\alpha(s, v) = 1$. Then as

$$\begin{aligned}
\frac{1}{2\vartheta(s(e), v(e))} b_\vartheta(s(e), \urcorner(s(e))) &\leq b_\vartheta(s(e), \urcorner s(e)) \\
&= \sup_{e \in [p,q]} \left| s(e) - \urcorner s(e) \right|^2 \\
&= \sup_{e \in [p,q]} \left| f(e) + \int_p^q M(e, i, s(i))di - f(e) - \int_p^q M(e, i, s(i))di \right|^2 \\
&\leq \sup_{e \in [p,q]} \left| f(e) + \int_p^q M(e, i, s(i))di - f(e) - \int_p^q M(e, i, v(i))di \right|^2 \\
&= b_\vartheta(s(e), v(e)),
\end{aligned}$$

we have

$$\begin{aligned}
\alpha(s(e), v(e)) b_\vartheta(\urcorner s(e), \urcorner v(e)) &= b_\vartheta(\urcorner s(e), \urcorner v(e)) \\
&= \sup_{e \in [p,q]} \left| \urcorner s(e) - \urcorner v(e) \right|^2 \\
&= \sup_{e \in [p,q]} \left| f(e) + \int_p^q M(e, i, s(i))di - f(e) - \int_p^q M(e, i, v(i))di \right|^2 \\
&= \sup_{e \in [p,q]} \left| \int_p^q (M(e, i, s(i)) - M(e, i, v(i)))di \right|^2 \\
&\leq \left(\int_p^q \sup_{e \in [p,q]} \left| M(e, i, s(i)) - M(e, i, v(i)) \right| di \right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_p^q \frac{\sqrt{A^l_{\urcorner}(s(e), v(e))}}{\sqrt{2}(q-p)} di \right)^2 \\
&= \frac{A^l_{\urcorner}(s(e), v(e))}{2} \\
&= \psi(A^l_{\urcorner}(s(e), v(e))).
\end{aligned}
$$

All the conditions of Theorem 2.3 are fulfilled. Therefore, the integral Eq (3.1) has a solution. □

4. Conclusions and future directions

Many physical problems can be described by various Fredholm integral equations. There are several methods available in the literature for the establishment of solutions to these equations. One powerful method is the fixed-point method. Therefore, in the current work, some new fractional interpolative contractions were introduced. With the help of these fractional interpolative contractions, some fixed-point results were studied in extended bMS. For the validity of the presented results, certain examples were given. Lastly, as a practical application, an existence theorem for the solution of the Fredholm integral equation was provided. This work generalizes some well-known results from the existing literature. In the future, one can explore the established work for multi-valued mapping and investigating the existence of solutions for integral inclusions.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

Authors are thankful to Prince Sultan University for the support of this work through TAS research lab.

This work is funded by Natural Science Basic Research Plan in Shaanxi Province of China (No. 2022JQ-040).

Conflicts of interest

The authors declare no conflicts of interest.

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