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Research article

On the exponential decay of a Balakrishnan-Taylor plate with strong damping

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Abstract: In this manuscript, we study a thin and narrow plate equation that models the deck of a suspension bridge that is subject to a Balakrishnan-Taylor damping and a strong damping. First, by using the Faedo Galerkin method, we prove the existence of both global weak and regular solutions. Second, we prove the exponential stability of the energy for regular solutions by combining the multiplier method and a well-known result of Komornik.

Keywords: plate; Balakrishnan-Taylor damping; strong damping; exponential decay
Mathematics Subject Classification: 35B40, 35L75, 74K10

1. Introduction

A rectangular thin and narrow plate that models the deck of a suspension bridge is considered in the domain $\Omega = (0,\pi)\times(-d,d)$, where $d << \pi$. The nonlocal evolution equations that describe how the plate is deformed are as follows:

$$\begin{cases} p_{tt}(x,y,t) + \Delta^2 p(x,y,t) - (\phi(p) + \delta\langle p_x, p_{xt}\rangle)\, p_{xx} + \alpha\Delta^2 p_t = 0 & \text{in } \Omega\times(0,+\infty), \\ p(0,y,t) = p_{xx}(0,y,t) = p(\pi,y,t) = p_{xx}(\pi,y,t) = 0 & (y,t)\in(-d,d)\times(0,+\infty), \\ p_{yy}(x,\pm d,t) + \mu p_{xx}(x,\pm d,t) = 0 & (x,t)\in(0,\pi)\times(0,+\infty), \\ p_{yyy}(x,\pm d,t) + (2-\mu)p_{xxy}(x,\pm d,t) = 0 & (x,t)\in(0,\pi)\times(0,+\infty), \\ p(x,y,0) = p_0(x,y),\ p_t(x,y,0) = p_1(x,y) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\delta,\ \alpha > 0$ and ϕ which introduces a nonlocal effect is given by

$$\phi(p) = -a + b \int_{\Omega} p_x^2 \, dxdy.$$

The constant μ is the Poisson ratio which is generally in the range of $(-1, \frac{1}{2})$ due to physical reasons (see [1] for more details). It has a value of about 0.3 for metals and between 0.1 and 0.2 for concrete. Due to this, we suppose that $0 < \mu < \frac{1}{2}$. The constant $b > 0$ is determined by the elasticity of the deck's material, $b \int_{\Omega} p_x^2 \, dxdy$ determines the plate's geometric nonlinearity as a result of its stretching, and $a > 0$ is the constant for prestressing. Specifically, if the plate is compressed, we have that $a > 0$ and if the plate is stretched, one has that $a < 0$.

The model (1.1) describes the vibrations of the deck of a suspension bridge in the presence of a Balakrishnan-Taylor damping (the term $\delta\langle p_x, p_{xt}\rangle$) and a strong damping (the term $\alpha\Delta^2 p_t$).

Note that $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2(\Omega)$.

Let us recall some works in the literature that are related to our problem. For one dimensional problems, in [2], the author considered the following equation

$$\begin{aligned} &p_{tt} + \alpha p_{xxxx} - \left(\beta + k \int_0^l \left[\frac{\partial p(\xi,t)}{\partial \xi}\right]^2 d\xi\right) p_{xx} + \gamma p_{xxxxt} \\ &-\mu\langle p_x, p_{tx}\rangle p_{xx} + \delta p_t = 0, \quad in\ (0,l)\times(0,+\infty), \end{aligned} \tag{1.2}$$

where the constants α, k, γ, μ are positive, and the constants β and δ have no restrictions on their sign. Here, l denotes the beam's length. The author established existence, uniqueness, and regularity theorems for the situations in which the beam's ends are clamped or hinged. Regarding higher dimensions, consider the work of Emmrich and Thalhammer [3], who provided a general model for describing nonlinear extensible beams with weak, viscous, strong, and Balakrishnan-Taylor damping, as follows:

$$\begin{aligned} &p_{tt} + \alpha\Delta^2 p + \xi p + \kappa p_t - \lambda\Delta p_t + \mu\Delta^2 p_t \\ &-\Big[\beta + \gamma \int_{\Omega} |\nabla p|^2 dx + \delta \left|\int_{\Omega} \nabla p\cdot\nabla p_t dx\right|^{q-2} \int_{\Omega} \nabla p\cdot\nabla p_t dx\Big]\Delta p = h \end{aligned} \tag{1.3}$$

in $\Omega\times(0,T)$, where Ω is a bounded domain and $T > 0$, the constants α and γ are positive, and λ, μ, and δ are nonnegative, whereas $\beta, \kappa, \xi \in \mathbb{R}$ and $q \geq 2$.

The authors proved the existence of a weak solution for (1.3) under hinged or clamped boundary conditions by using time discretization in both cases, i.e., when $\lambda, \mu > 0$ and $q \geq 2$ or when $\lambda = \mu = 0$ and $q = 2$. Under the conditions of applying $\kappa = \lambda = 0$, $\mu > 0$ and $q = 2$ in (1.2), Clark [4] established the existence, uniqueness, and asymptotic behavior of the solutions in N-dimensional bounded and unbounded domains. In [5], You proved that there are global solutions in the cases in which $\kappa = \mu = 0$, $\lambda > 0$, $q > 2$ and $\Omega = (0,1)$. Also, he gave results on the existence of inertial manifolds and the finite-dimensional stabilization. Subsequently Tavares et al. [6] studied the problem (1.3) for $\lambda = \mu = 0, \kappa \in \mathbb{R}$ and $q \geq 2$; they established the existence of a unique mild (and strong) solution and analyzed the long-time dynamics of solutions (in the mentioned case) when $\kappa > 0$ and β is bounded from below by a negative expression, as well as with the existence of nonlinear source.

Now, let us mention some works on suspension bridges. In [7, 8], the existence of nonlinear oscillations was proved. The deck of a suspension bridge has been modeled in a simple form in [9]. See also Gazzola's book [10] and recent results [11–13] for additional details. The bending and stretching energies of the model presented in [9] were examined by Al-Gwaiz et al. in [14]. We mention also the recent work of [15] in which the authors provide a new model for a suspension bridge.

Recently, many researchers have been interested in studying the stability of a plate model for the deck of a suspension bridge. Messaoudi and Mukiawa [16] showed an exponential decay in the presence of both a global frictional damping and a nonlinear term. In [17] (resp. in [18]), the authors studied the same problem as in [16] but with linear (resp. nonlinear) local damping distributed around a neighborhood of the boundary, and they proved an exponential decay estimate of the associated energy.

Liu and Zhuang [19] expanded the work of [20] and proved, without considering the relation between m and r, that the solutions of the equation, i.e.,

$$p_{tt} + \Delta^2 p + ap + |p_t|^{m-2} p_t = |p|^{r-2} p, \quad m \geq 2, \ r > 2,$$

exist globally if and only if there exists a real number $t_0 \in [0, T_{max})$ such that $p(t_0) \in W$ and the energy at the time t_0 is less than such a constant that depends on r and C_r (C_r is defined in (2.3)), where

$$T_{max} = \sup\{T > 0 : p = p(t) \text{ exists on } [0, T]\}$$

and

$$W = \{p \in N_+ : J(p) < d\},$$

with

$$N_+ = \{p \in V : I(p) > 0\} \cup \{0\}, I(p) = \|p\|_V^2 + (ap, p) - \|p\|_r^r,$$

$$J(p) = \frac{1}{2}\|p\|_V^2 + \frac{1}{2}(ap, p) - \frac{1}{r}\|p\|_r^r \text{ and } d = \inf_{p \in V \setminus \{0\}} \max_{\lambda > 0} J(\lambda p).$$

Moreover, the energy decay results were obtained, and when $r > m$ a blow-up result was established. Later, in [21], the authors established the existence of a global weak solution and proved a stability result under the conditions of an external force f and a nonlinear frictional damping. Finally, we cite the work [22], in which the author studied the same problem as described here, but it was subject to different types of damping, i.e., one of memory type (of the form $\int_0^t g(s)\Delta^2 p(s)ds$) and a nonlinear localized frictional damping (of the form $a(x, y)|p_t|^m p_t$). The author proved the existence of global solutions as well as a general stability result. For other results concerning partially hinged plate equations, we refer the reader to the recent papers [23–25].

Motivated by all mentioned works, our current paper investigates the exponential stability of solutions to system (1.1) with a strong damping and a Balakrishnan-Taylor damping. As mentioned at the end of the paper, the Balakrishnan-Taylor damping (alone) is insufficient to deduce exponential stability. For this reason, we chose to add another damping to obtain the uniform stability.

The structure of the paper is as follows. In the next section, we present some fundamental preliminaries that will be used to prove our main results. In the third section, the well-posedness of the problem (1.1) is proved. We show the exponential stability of system (1.1) in the last section.

2. Preliminaries

Here and in the sequel, we use $\|\cdot\|$ to denote the usual norm in $L^2(\Omega)$.

We define the space

$$V = \{w \in H^2(\Omega) : w = 0 \text{ on } \{0, \pi\} \times (-d, d)\},$$

with the scalar product

$$(p, q) = \int_\Omega \Big[\Delta p \Delta q + (1 - \mu)(2 p_{xy} q_{xy} - p_{xx} q_{yy} - p_{yy} q_{xx}) \Big] \, dxdy.$$

We note that $(V, (\cdot, \cdot))$ is a Hilbert space, and that the norm $\|.\|_V$ is equivalent to the H^2 norm (see [9, Lemma 4.1]).

Moreover, we denote by $\mathcal{H}(\Omega)$ the dual space of V, and we indicate by $\langle ., . \rangle_{2,-2}$ the associated duality. We have the following:

Lemma 2.1. *[9] If $0 < \mu < \frac{1}{2}$ and $f \in L^2(\Omega)$, then there is a unique $p \in V$ such that, for all $q \in V$, we have*

$$(p, q) = \int_\Omega f q. \tag{2.1}$$

The function $p \in V$ satisfying (2.1) is known as the weak solution to the following stationary problem

$$\begin{cases} \Delta^2 p = f, \\ p(0, y) = p(\pi, y) = p_{xx}(0, y) = p_{xx}(\pi, y) = 0, \\ p_{yy}(x, \pm d) + \mu p_{xx}(x, \pm d) = p_{yyy}(x, \pm d) + (2 - \mu) p_{xxy}(x, \pm d) = 0. \end{cases} \tag{2.2}$$

Lemma 2.2. *[20] Let $p \in V$ and $1 \leq r < +\infty$. Then, we have*

$$\|p\|_r^r \leq C_r \|p\|_V^r, \tag{2.3}$$

for some positive constant $C_r = C_r(\Omega, r)$.

Remark 2.3. *Let $f = \lambda p$ in* (2.2). *Then, Theorem 3.4 in [9] asserts that the set of eigenvalues of* (2.2) *may be ordered in an increasing sequence $\{\lambda_j\}_{j \geq 1}$ of strictly positive numbers that diverge to $+\infty$, and that the set of eigenfunctions $\{w_j\}_{j \geq 1}$ of* (2.2) *is a complete system in V.*

The energy related to (1.1) is given as follows

$$\mathcal{E}(t) = \frac{1}{2}\|p_t(t)\|^2 + \frac{1}{2}\|p(t)\|_V^2 - \frac{a}{2}\|p_x(t)\|^2 + \frac{b}{4}\|p_x(t)\|^4, \tag{2.4}$$

which satisfies the following identity

$$\mathcal{E}'(t) = -\alpha \|p_t\|_V^2 - \delta \left(\frac{1}{2} \frac{d}{dt} \|p_x\|^2 \right)^2 \leq 0, \tag{2.5}$$

This indicates that the energy decreases with time t and $\mathcal{E}(t) \leq \mathcal{E}(0), \ \forall \, t \geq 0$.

We recall the following theorem (see [26, Theorem 8.1]) that will be useful in the proof of the main result.

Theorem 2.4. *Let $E : [0,\infty) \to [0,\infty)$ be a non-increasing function and assume that there exists a constant $C > 0$ such that*

$$\int_t^{\infty} E(s)ds \le CE(t), \ \forall\ t \ge 0.$$

Then

$$E(t) \le E(0)e^{1-\frac{t}{C}}, \ \forall t \ge C.$$

Remark 2.5. *We remark that the energy is nonnegative if $a < 0$, and this case is equivalent to a stretched plate. However, this scenario is not applicable to real-world bridges [27]. When $a > 0$, which is the utmost likely situation for bridges, the energy $\mathcal{E}(t)$ may be negative. This issue can be solved by following some ideas from [14, Section 3]. To do this, we define*

$$\begin{aligned} W &:= \{w \in H^1(\Omega) : w = 0 \text{ on } \{0,\pi\} \times (-d,d)\}, \\ C_*^{\infty}(\Omega) &:= \{w \in C^{\infty}(\overline{\Omega}) : \exists\, \varepsilon > 0, w(x,y) = 0 \text{ if } x \in [0,\varepsilon] \cup [\pi-\varepsilon,\pi]\}. \end{aligned}$$

endowed with the following norm

$$\|p\|_W := \left(\int_{\Omega} |\nabla p|^2 \, dx\, dy \right)^{1/2}, \tag{2.6}$$

where W is a normed space.

Remark 2.6. *[14] W is defined as the completion of $C_*^{\infty}(\Omega)$ according to the norm $\|\cdot\|_W$. It is clear that the embedding $V \hookrightarrow W$ is compact and the optimal embedding constant satisfies*

$$\Lambda_1 := \min_{w \in V} \frac{\|w\|_V^2}{\|w\|_W^2}.$$

Lemma 2.7. *[17] Assume that $0 \le a \le \Lambda_1$; then, $\mathcal{E}(t) \ge 0$.*

Proof. Using Remark 2.6, we obtain the following inequality

$$\|w\|_W^2 \le \Lambda_1^{-1} \|w\|_V^2, \quad \text{for all } w \in V. \tag{2.7}$$

Since

$$\|p_x\|^2 \le \int_{\Omega} |\nabla p|^2 \, dx\, dy \le \Lambda_1^{-1} \, \|p\|_V^2,$$

then we have

$$-\frac{a}{2}\|p_x\|^2 \ge -\frac{a}{2}\Lambda_1^{-1} \, \|p\|_V^2, \quad \forall p \in V,$$

and consequently

$$\frac{1}{2}\|p\|_V^2 - \frac{a}{2}\|p_x\|^2 \ge \frac{1}{2}\|p\|_V^2 \left(1 - a\Lambda_1^{-1}\right).$$

So, if $0 \le a \le \Lambda_1$ we conclude that $\frac{1}{2}\|p\|_V^2 - \frac{a}{2}\|p_x\|^2 \ge 0$ and therefore $\mathcal{E}(t) \ge 0$. This is in agreement with the hypothesis of Theorem 4 in [27]. □

3. Well-posedness

Definition 3.1. *Let T be a positive number. The functions*

$$p \in L^\infty(0,T;V),\ p_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), \quad \text{and} \quad p_{tt} \in L^2(0,T;\mathcal{H}(\Omega)),$$

constitue a weak solution of (1.1) *when*

$$\begin{aligned} \langle p_{tt}, w\rangle_{2,-2} + (p,w) + \int_\Omega \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdy \\ +\alpha(p_t, w) = 0,\ \forall\ w \in V, \\ p(x,y,0) = p_0(x,y), \quad p_t(x,y,0) = p_1(x,y), \end{aligned} \tag{3.1}$$

for almost everywhere $t \in [0,T]$.

Theorem 3.2. *Suppose that $0 \le a \le \Lambda_1$ and let $(p_0, p_1) \in V \times L^2(\Omega)$. Then, the problem* (1.1) *has a unique global weak solution on $[0,T]$ for any $T > 0$.*

Proof. We divide our proof into 4 steps.

Step 1. In this step, we will prove some convergence results for the sequence $(p^k)_{k\ge 1}$ (defined below) and its derivative.

We start by applying the Faedo-Galerkin approach. By Remark 2.3, we may consider $\{w_j\}_{j=1}^\infty$ as a basis of V and let $V_k = span\{w_1, w_2, ..., w_k\}$ be subspace of V with finite dimensions, which is spanned by the first k vectors. Let

$$p_0^k(x,y) = \sum_{j=1}^k a_j w_j(x,y), \quad p_1^k = \sum_{j=1}^k b_j w_j(x,y),$$

such that $p_0^k, p_1^k \in V_k$ and

$$p_0^k \to p_0 \text{ in } V, \quad \text{and } p_1^k \to p_1 \text{ in } L^2(\Omega). \tag{3.2}$$

We are looking for a solution of the form

$$p^k(x,y,t) = \sum_{j=1}^k c_j(t) w_j(x,y), \tag{3.3}$$

that solves the following in V_k:

$$\begin{aligned} \langle p_{tt}^k, w_j\rangle_{2,-2} + (p^k, w_j) + \int_\Omega \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k (w_j)_x \, dxdy \\ +\alpha(p_t^k, w_j) = 0,\ \forall\ j = 1, ..., k, \\ p^k(x,y,0) = p_0^k(x,y), \quad p_t^k(x,y,0) = p_1^k(x,y). \end{aligned} \tag{3.4}$$

It is easy to check that, for any $k \ge 1$, the above problem (3.4) yields a solution p^k on $[0, t_k)$, where $0 < t_k \le T$. Now, we multiply (3.4) by $c_j'(t)$ and sum over $j = 1, ..., k$ to obtain

$$\frac{d}{dt}\mathcal{E}^k(t) = -\alpha\|p_t^k\|_V^2 - \delta\left(\frac{1}{2}\frac{d}{dt}\|p_x^k\|^2\right)^2 \le 0, \tag{3.5}$$

where

$$\mathcal{E}^k(t) = \frac{1}{2}\|p_t^k(t)\|^2 + \frac{1}{2}\|p^k(t)\|_V^2 - \frac{a}{2}\|p_x^k(t)\|^2 + \frac{b}{4}\|p_x^k(t)\|^4. \tag{3.6}$$

Now, we integrate (3.5) over $(0,t)$, where $0 < t < t_k$; we also note, from (3.2), that (p_0^k) and (p_1^k) are respectively bounded in V and $L^2(\Omega)$; we then obtain

$$\mathcal{E}^k(t) + \alpha \int_0^t \|p_t^k\|_V^2 ds + \delta \int_0^t \left(\frac{1}{2}\frac{d}{dt}\|p_x^k\|^2\right)^2 ds \leq \mathcal{E}^k(0) \leq C, \tag{3.7}$$

where C is a positive constant that does not depend on t and k, and that may vary from line to line.

Hence, we get the following bounds:

$$\|p^k\|_V^2,\ \|p_t^k(t)\|^2,\ \int_0^t \|p_t^k\|_V^2 \leq C. \tag{3.8}$$

As a result, one obtains that $t_k = T$ and we have the following:

$$\begin{cases} (p^k) \quad is \ \ bounded \ in \ \ L^\infty(0,T;V), \\ (p_t^k) \quad is \ bounded \ in \ \ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V). \end{cases} \tag{3.9}$$

Hence, there exists a subsequence of (p^k), still denoted by (p^k), that verifies the following:

$$\begin{cases} p^k \rightharpoonup p \quad weakly\ star\ in \ \ L^\infty(0,T;V), \\ p_t^k \rightharpoonup p_t \quad weakly\ star\ in \ \ L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), \\ p^k \longrightarrow p \quad in\ L^2(Q) \ \ strongly\ and\ a.e, \\ \|p_x^k\|^2 p_{xx}^k \rightharpoonup \mathcal{X}_1 \quad weakly\ star\ in \ \ L^\infty(0,T;L^2(\Omega)), \\ \langle p_x^k, p_{xt}^k\rangle p_{xx}^k \rightharpoonup \mathcal{X}_2 \quad weakly\ star\ in \ \ L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{3.10}$$

where $Q = \Omega \times (0,T)$.

Step 2. Here, we will prove that $\mathcal{X}_1 = \|p_x\|^2 p_{xx}$ and $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$ by following the same arguments as in [2, 28].

For the first one, the following lemma is required.

Lemma 3.3. *Suppose that $p, q \in V$. We have*

$$\langle \|p_x\|^2 p_{xx} - \|q_x\|^2 q_{xx}, p - q\rangle \leq 0.$$

Proof. One has

$$\langle \|p_x\|^2 p_{xx} - \|q_x\|^2 q_{xx}, p - q\rangle$$

$$
\begin{aligned}
&= \|p_x\|^2\left(\langle p_x, q_x\rangle - \|p_x\|^2\right) + \|q_x\|^2\left(\langle p_x, q_x\rangle - \|q_x\|^2\right) \\
&\le \|p_x\|^2\left(\|p_x\|\|q_x\| - \|p_x\|^2\right) + \|q_x\|^2\left(\|p_x\|\|q_x\| - \|q_x\|^2\right) \\
&= -\left(\|p_x\| - \|q_x\|\right)\left(\|p_x\|^3 - \|q_x\|^3\right) \le 0.
\end{aligned}
$$

□

Now, let $q \in L^2(0,T;V)$. From Lemma 3.3, one obtains that

$$\int_0^T \langle \|p_x^k\|^2 p_{xx}^k - \|q_x\|^2 q_{xx}, p^k - q\rangle dt \le 0.$$

By following the same steps as in [28], we derive that

$$\mathcal{X}_1 = \|p_x\|^2 p_{xx}.$$

Next, to prove that $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$, we note first note that

$$\langle p_x^k, p_{xt}^k\rangle = -\langle p_{xx}, p_t^k\rangle - \langle p^k - p, p_{xxt}^k\rangle.$$

It is clear that $\langle p_{xx}, p_t^k\rangle \longrightarrow \langle p_{xx}, p_t\rangle$ in $L^\infty(0,T)$. Besides, from (3.8), we have

$$
\begin{aligned}
\int_0^T |\langle p^k - p, p_{xxt}^k\rangle| dt &\le \left(\int_0^T \|p^k - p\|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \|p_{xxt}^k\|^2 dt\right)^{\frac{1}{2}} \\
&\le C\left(\int_0^T \|p^k - p\|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence $\langle p^k - p, p_{xxt}^k\rangle \longrightarrow 0$ in $L^1(0,T)$; thus, we deduce that

$$\langle p_x^k, p_{xt}^k\rangle \longrightarrow \langle p_x, p_{xt}\rangle \ \text{ in } L^1(0,T). \tag{3.11}$$

Now, let $\varphi \in L^1(0,T;L^2(\Omega))$. Then

$$
\begin{aligned}
\int_0^T \langle p_x, p_{xt}\rangle\langle p_{xx}, \varphi\rangle dt &= \int_0^T \langle p_x^k, p_{xt}^k\rangle\langle p_{xx}^k, \varphi\rangle dt \\
&+ \int_0^T [\langle p_x, p_{xt}\rangle - \langle p_x^k, p_{xt}^k\rangle]\langle p_{xx}^k, \varphi\rangle dt \\
&+ \int_0^T \langle p_{xx} - p_{xx}^k, \langle p_x, p_{xt}\rangle\varphi\rangle dt.
\end{aligned} \tag{3.12}
$$

The last two terms on the right side of (3.12) go to zero as $k \to +\infty$ when applying (3.8) and (3.11). Since φ is arbitrary, we conclude that $\mathcal{X}_2 = \langle p_x, p_{xt}\rangle p_{xx}$.

Step 3. In this step, we will show that p is the unique solution of the system (3.1).

By integrating (3.4) over $(0,t)$, we obtain

$$
\begin{aligned}
\int_\Omega p_t^k w \, dxdy + \int_0^t (p^k, w) ds + \int_0^t \int_\Omega \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k w_x \, dxdyds \\
+\alpha \int_0^t (p_t^k, w) ds = \int_\Omega p_1^k w, \ \forall \ w \in V.
\end{aligned} \tag{3.13}
$$

Recall (3.10) and let $k \to \infty$; we get

$$\int_{\Omega} p_t w \, dxdy - \int_{\Omega} p_1 w = -\int_0^t (p, w) ds - \int_0^t \int_{\Omega} \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdyds - \int_0^t \alpha(p_t, w) ds. \quad (3.14)$$

This means that (3.14) holds true for any $w \in V$. Since the terms on the right hand side of (3.14) are absolutely continuous, then (3.14) is differentiable for almost everywhere $t \geq 0$. It holds that

$$\langle p_{tt}, w\rangle_{2,-2} + (p, w) + \int_{\Omega} \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \, dxdy + \alpha(p_t, w) = 0, \ \forall \ w \in V. \quad (3.15)$$

Regarding the initial conditions, from (3.10), and by using Lions' lemma [29], we can simply get

$$p^k \to p \quad in \ \ C([0, T), L^2(\Omega)). \quad (3.16)$$

$p^k(x, y, 0)$ then makes sense and $p^k(x, y, 0) \to p(x, y, 0)$ in $L^2(\Omega)$. Noting that

$$p^k(x, y, 0) = p_0^k(x, y) \to p_0(x, y) \quad in \ \ V,$$

we get

$$p(x, y, 0) = p_0(x, y). \quad (3.17)$$

Besides, as in [30], we multiply (3.4) by $\phi \in C_0^\infty(0, T)$ and integrate over $(0, T)$; we get the following for any $j \leq k$:

$$-\int_0^T \int_{\Omega} p_t^k(t) w \phi'(t) \, dx \, dy \, dt = -\int_0^T (p^k, w)\phi(t) dt - \int_0^T \int_{\Omega} \left(-a + b\|p_x^k\|^2 + \delta\langle p_x^k, p_{xt}^k\rangle\right) p_x^k w_x \phi(t) \, dx \, dy \, dt - \alpha \int_0^T (p_t^k, w)\phi(t) dt. \quad (3.18)$$

As $k \to +\infty$, we have the following for any $w \in V$ and any $\phi \in C_0^\infty(0, T)$

$$-\int_0^T \int_{\Omega} p_t(t) w \phi'(t) \, dx \, dy \, dt = -\int_0^T (p, w)\phi(t) dt - \int_0^T \int_{\Omega} \left(-a + b\|p_x\|^2 + \delta\langle p_x, p_{xt}\rangle\right) p_x w_x \phi(t) \, dx \, dy \, dt - \alpha \int_0^T (p_t, w)\phi(t) dt, \quad (3.19)$$

which implies that (see [30])

$$p_{tt} \in L^2(0, T; \mathcal{H}(\Omega)).$$

Given that $p_t \in L^2(0,T;L^2(\Omega))$, we conclude that $p_t \in C(0,T;\mathcal{H}(\Omega))$.
$p_t^k(x,y,0)$ therefore makes sense and

$$p_t^k(x,y,0) \to p_t(x,y,0) \ \text{ in } \ \mathcal{H}(\Omega).$$

However

$$p_t^k(x,y,0) = p_1^k(x,y) \to p_1(x,y) \ \text{ in } \ L^2(\Omega).$$

So,

$$p_t(x,y,0) = p_1(x,y). \tag{3.20}$$

For the uniqueness, assume that p and $\overline{p}$ verify (3.15), (3.17), and (3.20). So, by integrating by parts, $q = p - \overline{p}$ satisfies

$$\begin{aligned}&\int_\Omega q_{tt}(x,t)w\,dx\,dy + (q,w) + a\int_\Omega q_{xx}w\,dx\,dy - b\int_\Omega \left(\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2\overline{p}_{xx}\right) w\,dx\,dy \\ &\quad - \int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) w\,dx\,dy + \alpha(q_t,w) = 0,\ \forall\, w \in V, \\ &\quad q(x,y,0) = q_t(x,y,0) = 0.\end{aligned} \tag{3.21}$$

Then, (3.21) holds true for any $w \in C_0^\infty(\Omega \times (0,T))$ by the density it is also valid for any $w \in L^2(\Omega \times (0,T))$.

If we test (3.21) with q_t, we get

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\|q_t\|^2 + \frac{1}{2}\|q\|_V^2\right\} + a\int_\Omega q_{xx}q_t\,dx\,dy - b\int_\Omega \left(\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2\overline{p}_{xx}\right) q_t\,dx\,dy \\ &\quad - \int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) q_t\,dx\,dy + \alpha\|q_t\|_V^2 = 0.\end{aligned} \tag{3.22}$$

By using Young's inequality, we get

$$\begin{aligned}-a\int_\Omega q_{xx}q_t\,dx\,dy &\le \frac{a}{2}\|q_t\|^2 + \frac{a}{2}\|q_{xx}\|^2 \\ &\le C\left(\|q_t\|^2 + \|q\|_V^2\right).\end{aligned} \tag{3.23}$$

Next, it is easy to see that

$$\begin{aligned}&\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2\overline{p}_{xx} \\ &= \|p_x\|^2 q_{xx} - \langle p + \overline{p}, q_{xx}\rangle \overline{p}_{xx},\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}&\int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) q_t\,dx\,dy \\ &= -\delta(\langle q_t, \overline{p}_{xx}\rangle)^2 - \delta\langle q_t, \overline{p}_{xx}\rangle\langle p_t, q_{xx}\rangle - \delta\langle p_t, p_{xx}\rangle\langle q_t, q_{xx}\rangle.\end{aligned} \tag{3.25}$$

Then, using (3.24), we infer that

$$b\int_\Omega \left(\|p_x\|^2 p_{xx} - \|\overline{p}_x\|^2\overline{p}_{xx}\right) q_t\,dx\,dy$$

$$\leq C\|q_{xx}\|\|q_t\|$$
$$\leq C\left(\|q_t\|^2 + \|q\|_V^2\right). \tag{3.26}$$

Besides, from (3.25), one derives the following:

$$\int_\Omega \delta\left(\langle p_x, p_{xt}\rangle p_{xx} - \langle \overline{p}_x, \overline{p}_{xt}\rangle \overline{p}_{xx}\right) q_t \, dx\, dy$$
$$\leq -\delta(\langle q_t, \overline{p}_{xx}\rangle)^2 + C\|q_{xx}\|\|q_t\|$$
$$\leq -\delta(\langle q_t, \overline{p}_{xx}\rangle)^2 + C\left(\|q_t\|^2 + \|q\|_V^2\right). \tag{3.27}$$

By using (3.23), (3.26), and (3.27) we can deduce that

$$\frac{d}{dt}\left\{\frac{1}{2}\|q_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|_V^2\right\} + \delta(\langle q_t, \overline{p}_{xx}\rangle)^2 + \alpha\|q_t\|_V^2$$
$$\leq C\left(\|q_t\|^2 + \|q\|_V^2\right). \tag{3.28}$$

By Gronwall's inequality, we obtain

$$\|q_t\|^2 + \|q\|_V^2 \quad \leq \quad Ce^{Ct}\left(\|q_t(0)\|^2 + \|q(0)\|_V^2\right),$$

which gives that $q = 0$ and thus $p = \overline{p}$. □

The following theorem gives an additional regularity result.

Theorem 3.4. *Suppose that* $0 \leq a \leq \Lambda_1$ *holds true and let* $(p_0, p_1) \in X \times V$*, with* $X = H^4(\Omega) \cap V$*. Then there is a unique function* $p = p(x, y, t)$ *that satisfies the initial conditions* (3.17) *and* (3.20)*, and that it satisfies*

$$p \in L^\infty(0, T; X),\;\; p_t \in L^\infty(0, T; V) \cap L^2(0, T; X), \quad \textit{and} \;\; p_{tt} \in L^2(0, T; L^2(\Omega)),$$

and

$$p_{tt} + \Delta^2 p - (\phi(p) + \delta\langle p_x, p_{xt}\rangle)\, p_{xx} + \alpha\Delta^2 p_t = 0, \quad \textit{in } L^2(0, T; L^2(\Omega)). \tag{3.29}$$

Proof. Let $\{\varphi_j\}_{j=1}^\infty$ be a basis of X (this basis exists since X is a separable Hilbert space). The solutions p^k can be written in the form (3.3) and verify (3.4) as well as the following initial conditions

$$p_0^k \to p_0 \;\; \textit{in } X, \quad \text{and } p_1^k \to p_1 \;\; \textit{in } V.$$

It is easy to see that the bounds defined in (3.8) are satisfied. If we test (3.4) with $\Delta^2 p_t^k$ and integrate by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|p_t^k\|_V^2 + \|\Delta^2 p^k\|^2\right) + \alpha\|\Delta^2 p_t^k\|^2 \;\; = \;\; -a\langle p_{xx}^k, \Delta^2 p_t^k\rangle + b\|p_x^k\|^2\langle p_{xx}^k, \Delta^2 p_t^k\rangle$$
$$+ \;\; \delta\langle p_x^k, p_{xt}^k\rangle\langle p_{xx}^k, \Delta^2 p_t^k\rangle.$$

Therefore, by integrating by parts over $(0, t)$, it follows that

$$\|p_t^k\|_V^2 + \|\Delta^2 p^k\|^2 + 2\alpha\int_0^t \|\Delta^2 p_t^k\|^2 ds$$

$$\leq C_1 + C_2 \int_0^t |\langle p^k_{xx}, \Delta^2 p^k_t\rangle| ds$$
$$\leq C_1 + C_2 \int_0^t \|p^k_{xx}\|^2 ds + \alpha \int_0^t \|\Delta^2 p^k_t\|^2 ds.$$

By using (3.9), we derive that

$$\|p^k_t\|^2_V, \quad \|\Delta^2 p^k\|^2, \quad \textit{and} \quad \int_0^t \|\Delta^2 p^k_t\|^2 ds \leq C.$$

We proceed as in Theorem (3.2) to prove the existence of a unique $p \in L^\infty(0,T;X)$ that satisfies (3.17), (3.20), (3.29), and $p_t \in L^\infty(0,T;V) \cap L^2(0,T;X)$. It follows from (3.29) that $p_{tt} \in L^2(0,T;L^2(\Omega))$. □

4. Exponential stability

This section's major result is as follows:

Theorem 4.1. *Let* $0 \leq a < \Lambda_1$. *Then, there are two constants* $K > 0$ *and* $\lambda > 0$ *such that the energy defined in* (2.4) *verifies*

$$\mathcal{E}(t) \leq K e^{-\lambda t}, \quad \forall\, t \geq 0. \tag{4.1}$$

Proof. We will work with regular solutions and by using standard density arguments; the decay holds true even for weak solutions. Multiplying (1.1) by p and integrating over $\Omega \times (s,T)$, for $0 < s < T$, we get

$$\int_s^T \int_\Omega \Big(p_{tt} p + p\Delta^2 p - \phi(p) p_{xx} p - \delta\langle p_x, p_{xt}\rangle p_{xx} p + \alpha p \Delta^2 p_t \Big)\, dxdydt = 0. \tag{4.2}$$

Using Lemma 2.1 and integration by parts, we obtain

$$\int_s^T \int_\Omega (p_t p)_t\, dxdydt - \int_s^T \int_\Omega p_t^2\, dxdydt + \int_s^T \|p\|^2_V\, dt + \int_s^T \int_\Omega \phi(p) p_x^2\, dxdydt$$
$$+ \int_s^T \int_\Omega \delta \langle p_x, p_{xt}\rangle p_x^2\, dxdydt + \alpha \int_s^T (p, p_t)\, dt = 0. \tag{4.3}$$

This yields

$$\int_s^T \mathcal{E}(t)\, dt + \int_s^T \int_\Omega (p_t p)_t - \frac{3}{2} \int_s^T \int_\Omega p_t^2 + \frac{1}{2} \int_s^T \|p\|^2_V - \frac{a}{2} \int_s^T \|p_x\|^2$$
$$+ \frac{3b}{4} \int_s^T \|p_x\|^4 + \delta \int_s^T \int_\Omega \langle p_x, p_{xt}\rangle p_x^2\, dxdydt + \alpha \int_s^T (p, p_t)\, dt = 0. \tag{4.4}$$

Then, we obtain

$$\int_s^T \mathcal{E}(t)\, dt \;\leq\; -\int_s^T \int_\Omega (p_t p)_t + \frac{3}{2} \int_s^T \int_\Omega p_t^2 + \frac{a}{2} \int_s^T \|p_x\|^2$$
$$-\; \delta \int_s^T \int_\Omega \langle p_x, p_{xt}\rangle p_x^2\, dxdt - \alpha \int_s^T (p, p_t)\, dt. \tag{4.5}$$

The terms on the right hand side of (4.5) can be estimated as follows. Using Lemma 2.2 and Young's inequality, we infer that

$$\begin{aligned}\left|-\int_s^T\int_\Omega (p_t p)_t\right| &\le \left|\int_\Omega p_t(s)p(s)\right| + \left|\int_\Omega p_t(T)p(T)\right|\\ &\le \frac{1}{2}\int_\Omega p_t^2(s) + \frac{1}{2}\int_\Omega p_t^2(T) + \frac{1}{2}\int_\Omega p^2(s) + \frac{1}{2}\int_\Omega p^2(T)\\ &\le \mathcal{E}(s) + \mathcal{E}(T) + C\|p(s)\|_V^2 + C\|p(T)\|_V^2\\ &\le C\mathcal{E}(s), \end{aligned} \tag{4.6}$$

where C is a generic positive constant. For the second term, thanks to Lemma 2.2 we have

$$\frac{3}{2}\int_s^T\int_\Omega p_t^2 \le C\int_s^T \|p_t\|_V^2 \le \frac{C}{\alpha}\int_s^T (-\mathcal{E}'(t))\,dt \le \frac{C}{\alpha}\mathcal{E}(s). \tag{4.7}$$

The third term on the right hand side of (4.5) may be estimated as follows:

$$\frac{a}{2}\int_s^T \|p_x\|^2 \le \frac{a\Lambda_1^{-1}}{2}\int_s^T \|p\|_V^2 \le a\Lambda_1^{-1}\int_s^T \mathcal{E}(t)dt. \tag{4.8}$$

Thanks to Young's inequality and (2.5) , we deduce, for any $\varepsilon > 0$, that

$$\begin{aligned}\left|-\delta\int_s^T\int_\Omega \langle p_x, p_{xt}\rangle p_x^2\,dxdt\right| &\le C_\varepsilon\int_s^T\left(\frac{d}{dt}\|p_x\|^2\right)^2 dt + \frac{\delta\varepsilon}{4}\int_s^T \|p_x\|^4 dt\\ &\le C_\varepsilon\int_s^T (-\mathcal{E}'(t))\,dt + \frac{\delta\varepsilon}{b}\int_s^T \mathcal{E}(t)dt\\ &\le C_\varepsilon\mathcal{E}(s) + \frac{\delta\varepsilon}{b}\int_s^T \mathcal{E}(t)dt, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\left|-\alpha\int_s^T (p, p_t)\,dt\right| &\le \alpha\int_s^T \|p\|_V\|p_t\|_V\\ &\le C_\varepsilon\int_s^T \|p_t\|_V^2 dt + \frac{\alpha\varepsilon}{2}\int_s^T \|p\|_V^2 dt\\ &\le C_\varepsilon\mathcal{E}(s) + \alpha\varepsilon\int_s^T \mathcal{E}(t)dt. \end{aligned} \tag{4.10}$$

By inserting (4.6)–(4.10) into (4.5), and choosing ε such that $1 - a\Lambda_1^{-1} - (\frac{\delta}{b} + \alpha)\varepsilon > 0$, we conclude that there exists a positive constant C_1 satisfying

$$\int_s^T \mathcal{E}(t)\,dt \le C_1\mathcal{E}(s),\ \ \forall\ s > 0.$$

Let $T \to +\infty$, and thanks to Theorem 2.4, we get the desired inequality (4.1). □

Remark 4.2. *As remarked in [31] (for extensible beams), the Balakrishnan-Taylor damping does not seem sufficient to provide a "good" stability result to our problem* (1.1)*. In fact, if $a = b = \alpha = 0$ in* (1.1)*, we have the following equation:*

$$p_{tt} + \Delta^2 p - \delta\langle p_x, p_{xt}\rangle p_{xx} = 0, \quad \text{in } \Omega \times (0, +\infty). \tag{4.11}$$

The corresponding energy for system (4.11) *is given by*

$$\mathcal{E}(t) = \frac{1}{2}\|p_t\|^2 + \frac{1}{2}\|p\|_V^2,$$

which satisfies

$$\mathcal{E}'(t) = -\delta[\langle p_x, p_{xt}\rangle]^2 \leq 0, \ \forall \ t > 0. \tag{4.12}$$

Hence, the system is dissipative. One can ask about its stability. If the system is strongly stable, that is, $\mathcal{E}(t) \to 0$ as $t \to +\infty$, then, using the fact that $\langle p_x, p_{xt}\rangle = -\langle p_{xx}, p_t\rangle$, we get

$$|\langle p_x, p_{xt}\rangle| \leq C\mathcal{E}(t) \to 0, \quad \text{as } t \to +\infty. \tag{4.13}$$

This indicates that the Balakrishnan-Taylor damping gets less and less effective as $t \to +\infty$. In addition, it is clear, from (4.12) *and* (4.13)*, that*

$$\begin{aligned} \mathcal{E}(t) \quad &\geq \quad \frac{|\langle p_x, p_{xt}\rangle|}{C} \\ &\geq \quad \frac{\sqrt{-\mathcal{E}'(t)}}{C\sqrt{\delta}}. \end{aligned}$$

This subsequently gives

$$-\mathcal{E}'(t)\mathcal{E}^{-2}(t) \leq C^2\delta. \tag{4.14}$$

Integrating the inequality (4.14) *over $(0, t)$, it follows that*

$$\mathcal{E}(t) \geq \frac{1}{\delta C^2 t + \mathcal{E}(0)^{-1}}, \ t > 0, \tag{4.15}$$

which means that the energy is bounded from below polynomially, and consequently the Balakrishnan-Taylor damping term $-\delta\langle p_x, p_{xt}\rangle p_{xx}$ (alone) is no longer enough to ensure exponential stability. In conclusion, we need to add another damping term, like a strong damping of the form $\alpha\Delta^2 p_t$, to recover exponential decay for system (4.11)*.*

5. Conclusions

This paper describes the study of a plate equation that is subject to a Balakrishnan-Taylor damping and a strong damping. This equation models the deformation of the deck of a suspension bridge. First, we have proved the existence of weak solutions and regular ones by using the Faedo-Galerkin approach. Second, by using the multiplier techniques, we proved the exponential decay of energy in our model. We also showed that if the plate equation is subject only to Balakrishnan-Taylor damping, then the exponential stability of this model cannot be reached. In conclusion, the Balakrishnan-Taylor damping is not enough to stabilize (exponentially) the deck. Hence, the need to add another damping.

Regarding future works, we can change the type of damping by considering, for example, structural damping (of the form Δp_t). Also, we can study a coupled Balakrishnan-Taylor plate with only one strong damping.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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