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Research article

A Generalization of Lieb concavity theorem

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Abstract: Lieb concavity theorem, successfully solved the Wigner-Yanase-Dyson conjecture, which is a very important theorem, and there are many proofs of it. Generalization of the Lieb concavity theorem has been obtained by Huang, which implies that it is jointly concave for any nonnegative matrix monotone function $f(x)$ over $\left(\mathrm{Tr}\left[\wedge^k(A^{\frac{qs}{2}}K^*B^{sp}KA^{\frac{sq}{2}})^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}$. In this manuscript, we obtained $\left(\mathrm{Tr}\left[\wedge^k(f(A^{\frac{qs}{2}})K^*f(B^{sp})Kf(A^{\frac{sq}{2}}))^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}$ was jointly concave for any nonnegative matrix monotone function $f(x)$ by using Epstein's theorem, and some more general results were obtained.

Keywords: Lieb concavity; the Wigner-Yanase-Dyson conjecture; nonnegative matrix monotone function; Epstein theorem
Mathematics Subject Classification: 15A15

1. Introduction

In 1963, Wigner and Yanase introduced the Wigner-Yanase skew information $\mathrm{I}_{WY}(\rho)$ of a density matrix ρ in a quantum mechanical system [1] with the definition

$$\mathrm{I}_{WY}(\rho) = -\frac{1}{2}\mathrm{Tr}[[\sqrt{\rho}, H]^2],$$

where ρ is the density matrix ($\rho \geq 0, \mathrm{tr}\,\rho = 1$) and H is a Hermitian matrix. They raised a question: For a positive definite matrix, find whether the value of

$$\mathrm{Tr}[\rho^s K \rho^{1-s} K^*],$$

has convex or concave properties for the matrix function that satisfies the conditions. In fact, trace operator is a useful tool in the mechanical learning; see [2–4].

In 1973, Lieb proved the convexity of the function above for all $0 < p < 1$, known as the Lieb concavity theorem [5], which successfully solved the Wigner-Yanase-Dyson conjecture by using the fact

$$\Psi(e_i \otimes e_j^*) = e_i e_j^* = I_{ij},$$

and the concavity of $\rho^{1-s} \otimes \rho^s$ [6]. In fact, a more elegant proof of the Lieb concavity theorem appeared in [2] using a tensor product designed by Ando.

In 2009, Effros gave another proof of the Lieb concavity theorem based on the affine version of the Hansen-Pedersen-Jensen inequality and obtained some celebrated quantum inequalities [7]. After that, Aujla provided a simple proof of this well-known theorem in 2011 using some derived properties of positive semidefinite matrices [3]. Several years later, Nikoufar, Ebadian, and Gordji also gave a simple proof of the Lieb concavity theorem by showing that jointly convex and jointly concave functions hold for generalized perspectives of some elementary functions [8].

Recently, Huang [9] obtained the function

$$L(A,B) = \left(\mathrm{Tr}\left[\wedge^k (A^{\frac{qs}{2}} K^* B^{sp} K A^{\frac{sq}{2}})^{\frac{1}{s}}\right]\right)^{\frac{1}{k}},$$

as jointly concave for any $A, B \geq 0$, which is a generalization of the Lieb concavity theorem. In our manuscript, we will obtain that the following function is jointly concave for any $A, B \geq 0$, and the nonnegative matrix monotone function $f(x)$

$$G(A,B) = \mathrm{Tr}\left[(f(A^{\frac{qs}{2}}) K^* f(B^{sp}) K f(A^{\frac{sq}{2}}))^{\frac{1}{s}}\right],$$

by using Epstein's theorem and some corollary. The rest of the paper is organized as follows. In Section 2, we introduce some definitions and conclusions about matrix tensor product, convexity of matrix, and Epstein¡¯s theorem. With these preparations, we obtain some useful results in the following Section 3 such as the generalization of Lieb concavity theorem.

2. Preliminary

For an $m \times n$ matrix A and a $p \times q$ matrix B, the tensor product of A and B is defined by [10]

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix},$$

where $A = \left(a_{ij}\right)_{1\leq i \leq m, 1 \leq j \leq n}$, then exterior algebra [11], denoted by " $\wedge$ ", is a binary operation for any $A_{n\times n}$, and the definition is

$$\begin{aligned} &(\underbrace{A_1 \wedge A_2 \wedge \cdots \wedge A_k}_{k})(\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n} \\ &= (A_1 \xi_{i_1} \wedge A_2 \xi_{i_2} \cdots \wedge A_k \xi_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n}, \end{aligned}$$

where $\{\xi_j\}_{j=1}^n$ is an orthonormal basis of $\mathbb{C}^n$ and

$$\xi_{i_1} \wedge \xi_{i_2} \cdots \wedge \xi_{i_k} = \frac{1}{\sqrt{n!}} \sum_{\pi \in \sigma_n} (-1)^{\pi} \xi_{\pi(i_1)} \otimes \xi_{\pi(i_2)} \cdots \otimes \xi_{\pi(i_k)},$$

σ_n is the family of all permutations on $\{1, 2, \cdots, n\}$. From above, one can obtain the Brunn-Minkowski inequality [12].

2.1. Lemma

For any $A, B > 0$,
$$\{\mathrm{Tr}[\wedge^k (A+B)]\}^{\frac{1}{k}} \geq \{\mathrm{Tr}\left[\wedge^k A\right]\}^{\frac{1}{k}} + \{\mathrm{Tr}\left[\wedge^k B\right]\}^{\frac{1}{k}}.$$

Proof. Let $\{\xi_i\}_{i=1}^n$ be the eigenvectors of $A + B$ with the eigenvalue $\{\lambda_i\}_{i=1}^n$, then

$$\begin{aligned}
&\{\mathrm{Tr}[\wedge^k (A+B)]\}^{\frac{1}{k}} \\
&= \left[\sum_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} \lambda_{i_1} \cdots \lambda_{i_k} \right]^{\frac{1}{k}} \\
&= \left[\sum_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} \left(\det \left| P^*_{i_1,\cdots,i_k} (A+B) P_{i_1,\cdots,i_k} \right| \right) \right]^{\frac{1}{k}} \\
&\geq \left[\sum_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} \left(\det \left| P^*_{i_1,\cdots,i_k} A P_{i_1,\cdots,i_k} \right| + \det \left| P^*_{i_1,\cdots,i_k} B P_{i_1,\cdots,i_k} \right| \right) \right]^{\frac{1}{k}} \\
&\geq \left[\sum_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} \det \left| P^*_{i_1,\cdots,i_k} A P_{i_1,\cdots,i_k} \right| \right]^{\frac{1}{k}} + \left[\sum_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} \det \left| P^*_{i_1,\cdots,i_k} B P_{i_1,\cdots,i_k} \right| \right]^{\frac{1}{k}} \\
&= \{\mathrm{Tr}[\wedge^k A]\}^{\frac{1}{k}} + \{\mathrm{Tr}[\wedge^k B]\}^{\frac{1}{k}},
\end{aligned}$$

where $P_{i_1,\cdots,i_k} = (\xi_{i_1}, \cdots, \xi_{i_k})$, the first " $\geq$ " obtains $\det(A + B) \geq \det(A) + \det(B)$ [13], and the second " $\geq$ " obtains by using the that fact $S_k = \left[\sum\limits_{1 \leq \xi_{i_1} < \cdots < \xi_{i_k} \leq n} x_{i_1} \cdots x_{i_k} \right]^{\frac{1}{k}}$ is concave [14]. □

Associated with the function $f(x)$ ($x \in (0, +\infty)$), the matrix function $f(A)$ is defined as [15]

$$f(A) = P^* f(\Lambda_A) P = \sum_{i=1} f(\lambda_i) P_i,$$

where $f(\Lambda_A) := \mathrm{diag}\{f(\lambda_1), ..., f(\lambda_n)\}$ and $P_i^2 = P_i$. For any A, B are two nonnegative Hermitian matrices, we denote $A \leq B$ if $x^* A x \leq x^* B x$ for any $x \in \mathbb{R}^n$, then the matrix monotone function $f(x)$ is defined [4] as

$$f(A) \geq f(B) \ \text{ for all } \ A \geq B > 0.$$

Since the matrix-monotone function is a special kind of operator monotone function, we present the following general conclusions about the operator-monotone function, which can be found in [16].

2.2. Lemma

The following statements for a real valued continuous function f on $(0, +\infty)$ are equivalent:

(**i**) f is operator-monotone;

(**ii**) f admits an analytic continuation to the whole domain $\mathrm{Im}\, z \neq 0$ in such a way that $\mathrm{Im}\, z\, \mathrm{Im}\, f(z) > 0$. □

Using Lemma 2.1, one can obtain that $f(x^p)^{\frac{1}{p}}$ is a matrix monotone function and matrix-concave function for any nonnegative matrix monotone function $f(x)$ and $0 < p \leq 1$ [2].

The jointly matrix-concave function, is defined as follows [17]:

$$f(tA_1 + (1-t)A_2, tB_1 + (1-t)B_2) \geq tf(A_1, B_1) + (1-t)f(A_2, B_2), \tag{2.1}$$

for all $A_1, A_2, B_1, B_2 \in \mathrm{H}_n^+$, and all $t \in [0, 1]$. From (2.1) and associated with spectral theory, H. Epstein [18] obtained the following three lammes:

2.3. Lemma

If $\mathrm{Im}\, C = \frac{C-C^*}{2i} > 0$ and $0 < \alpha < 1$, then

$$\mathrm{Im}\, e^{-i\alpha\pi} C^\alpha < 0 < \mathrm{Im}\, C^\alpha. \tag{2.2}$$

This lemma can be obtained from the integral representation [19] of

$$C^\alpha = \int_0^{+\infty} (\frac{1}{t} - \frac{1}{C+t})\, \mathrm{d}\mu(t).$$

Setting $A_1, B_1 \in \mathrm{H}_n$, and $A_2, B_2 \in \mathrm{H}_n^+$, and $A = A_1 + iA_2, B = B_1 + iB_2$, if

$$\mathrm{Im}\, A^\alpha > 0, \mathrm{Im}\, B^\beta > 0, \mathrm{Im}\, e^{-i\alpha\pi} A^\alpha < 0, \mathrm{Im}\, e^{-i\beta\pi} B^\beta < 0,$$

where $0 < \alpha, \beta$, straightforward calculations show.

2.4. Lemma

Let A and B be defined as above, then

$$\mathrm{SP}(AB) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < \alpha + \beta\}. \tag{2.3}$$

Using Lemmas (2.3) and (2.4), set

$$\mathrm{D} = \bigcup_{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}} \bigcup_{0 < \varepsilon \in \mathbb{R}} \{A \in \mathrm{M}_n : \mathrm{Re}\, e^{-i\theta} A \geq \varepsilon\},$$

where $G(z) = f(A_1 + zA_2)$ and $F(z) = f(A_2 + zA_1)$. If $sgn\, \mathrm{Im}(A) = sgn\, \mathrm{Im}(f(A))$ and $f(sA) = s^p f(A)(0 < p \leq 1)$ hold for any A and $s > 0$, then we know $G(z)$ is a Herglotz function and we can obtain the following theorem.

2.5. Lemma

Let $f(z)$ be a complex valued holomorphic function on D, and let $sgn\, \mathrm{Im}\, f(A) = sgn\, \mathrm{Im}(A)$ and $f(sA) = s^p f(A)(0 < p \leq 1)$ hold for any A and $s > 0$, then $f(A)$ is concave for any $A > 0$.

3. The main results

Based on the preparation, in this section, we will obtain some useful theorems. To begin, let $f(x)$ be a matrix monotone function, then we can obtain the following theorem by using Lemma 2.5.

3.1. Theorem

For any $p, q, s > 0$ and $s, p + q \leq 1$, the function

$$G(A) = \mathrm{Tr}\left[(f(A^{\frac{qs}{2}})K^* f(A^{sp})K f(A^{\frac{sq}{2}}))^{\frac{1}{s}}\right], \tag{3.1}$$

is concave for any $A \geq 0$ and nonnegative matrix monotone function $f(x)$.

Proof. To prove this theorem, first, we can obtain that

$$\bar{G}(A) = \mathrm{Tr}\left[(A^{\frac{qs}{2}} K^* A^{sp} K A^{\frac{sq}{2}})^{\frac{1}{s}}\right],$$

satisfies Lemma 2.5.

From an expression of $\bar{G}(A)$, we know it is a holomorphic function and

$$\bar{G}(\rho A) = \mathrm{Tr}\left[((\rho^{\frac{qs}{2}} A^{\frac{qs}{2}})K^*(\rho A)^{sp} K(\rho^{\frac{qs}{2}} A^{\frac{sq}{2}}))^{\frac{1}{s}}\right] = \rho^{p+q} G(A).$$

Finally, setting $A = A_1 + iB_1$ where $B_1 > 0$, we know that

$$sgn\, \mathrm{Im}(A) = sgn\, \mathrm{Im}(A^{ps}) = sgn\, \mathrm{Im}(K^* A^{qs} K).$$

By using Lemmas 2.3 and 2.4, one can obtain

$$\mathrm{SP}(K^* A^{sp} K A^{sq}) \subset \{z = \rho e^{i\theta} : 0 < \rho, 0 < \theta < s(p+q)\pi\}.$$

This implies

$$\begin{aligned} sgn\, \mathrm{Im}(G(A)) &= sgn\, \mathrm{Im}\, \mathrm{Tr}\left[(A^{\frac{qs}{2}} K^* A^{sp} K A^{\frac{sq}{2}})^{\frac{1}{s}}\right] \\ &= sgn\, \mathrm{Im}\, \mathrm{Tr}\left[(K^* A^{sp} K A^{sq}))^{\frac{1}{s}}\right] \\ &= sgn\, \mathrm{Im}\, \mathrm{Tr}\left[(T^{-1} \Lambda T)^{\frac{1}{s}}\right], \Lambda = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{pmatrix} \\ &= sgn\, \mathrm{Im}\, \mathrm{Tr}\left[\Lambda^{\frac{1}{s}}\right] \\ &= sgn\, \mathrm{Im}\, \mathrm{Tr}[\int_0^{+\infty} \frac{\Lambda^{[\frac{1}{s}]}}{\Lambda + t}\, \mathrm{d}\mu(t)] \\ &= sgn\, \mathrm{Im} \sum_{i=1}^{n} \lambda_i^{\frac{1}{s}}, \end{aligned}$$

where $\lambda_i = \rho_i e^{\theta_i}$ and $0 < \theta_i < s(p+q)\pi$.

Hence, we know $sgn\, \mathrm{Im}(\bar{G}(A)) > 0$ if $sgn\, \mathrm{Im}(A) > 0$. So, from Lemma 2.5, we obtain that $\bar{G}(A)$ is concave for any $A \geq 0$. Specifically, setting $A = \begin{pmatrix} Z & 0 \\ 0 & B \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}$, we know

$$\mathrm{Tr}\left[(Z^{\frac{qs}{2}} K^* B^{sp} K Z^{\frac{sq}{2}})^{\frac{1}{s}}\right], \tag{3.2}$$

is jointly concave for any $Z, B \geq 0$.

Next, since $f(x^p)^{\frac{1}{p}}$ is a matrix concave function for any nonnegative matrix monotone function $f(x)$ $(0 < p \leq 1)$, we have

$$\begin{aligned}
&G(\frac{A+B}{2})\\
&= \mathrm{Tr}\left[(f((\frac{A+B}{2})^{\frac{qs}{2}})K^* f((\frac{A+B}{2})^{sp})Kf((\frac{A+B}{2})^{\frac{sq}{2}}))^{\frac{1}{s}}\right]\\
&\geq \mathrm{Tr}\left[(\frac{f^{\frac{2}{qs}}(A^{\frac{qs}{2}}) + f^{\frac{2}{qs}}(B^{\frac{qs}{2}})}{2})^{qs}K^*(\frac{f^{\frac{1}{ps}}(A^{ps}) + f^{\frac{1}{ps}}(B^{ps})}{2})^{ps}K)^{\frac{1}{s}}\right] \text{ (concavity of } f(x^p)^{\frac{1}{p}})\\
&\geq \frac{G(A)+G(B)}{2}.
\end{aligned}$$

□

Finishing this theorem, one can obtain the Lieb concavity theorem as the following corollary.

3.2. Corollary

Let $0 < p + q \leq 1$, then

$$L(Z, B) = \mathrm{Tr}[Z^q H^* B^p H], \tag{3.3}$$

is jointly concave for any $Z, B \in H_n^+$.

Proof. Set $A = \begin{pmatrix} Z & 0 \\ 0 & B \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}$. When $s = 1$ and $f(x) = x$ we know the function

$$\begin{aligned}
G(A) &= \mathrm{Tr}\left[(A^{\frac{q}{2}}K^*A^pKA^{\frac{q}{2}})\right]\\
&= \mathrm{Tr}\left[(K^*B^pKZ^q)\right],
\end{aligned}$$

is jointly concave for any $Z, B \in H_n^+$, which is the Lieb concavity theorem. □

In fact, we can obtain the expansion of the Lieb concavity theorem by Huang [20], which is the following corollary.

3.3. Corollary

Let $0 \leq p, q, s \leq 1$ and $p + q \leq 1$, then

$$H(A, B) = \left(\mathrm{Tr}\left[\wedge^k(A^{\frac{qs}{2}}K^*B^{sp}KA^{\frac{sq}{2}})^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}, \tag{3.4}$$

is jointly concave for any $A, B \geq 0$.

Proof. Set $Z = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $\overline{K} = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ and $f(x) = x$. Hence, $H(A, B)$ is jointly concave for any $A, B \geq 0$ equivalent to $\left(\mathrm{Tr}\left[\wedge^k(Z^{\frac{qs}{2}}\overline{K}^*Z^{sp}\overline{K}Z^{\frac{sq}{2}})^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}$, which is concave for any $Z \geq 0$.

So, we obtain

$$
\begin{aligned}
&H(\frac{A_1+A_2}{2},\frac{B_1+B_2}{2})\\
&=\left(\mathrm{Tr}\left[\wedge^k\left((\frac{Z_1+Z_2}{2})^{\frac{qs}{2}}\overline{K}^*(\frac{Z_1+Z_2}{2})^{sp}\overline{K}(\frac{Z_1+Z_2}{2})^{\frac{sq}{2}}\right)^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}\\
&=\left(\mathrm{Tr}\left[\wedge^k\left((\frac{Z_1+Z_2}{2})^{qs}\overline{K}^*(\frac{Z_1+Z_2}{2})^{sp}\overline{K}\right)^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}\\
&=\left(\mathrm{Tr}\left[\left(\frac{Z_1\wedge^{k-1}I+Z_2\wedge^{k-1}I}{2}\right)^{qs}\widetilde{K}^*\left(\frac{Z_1\wedge^{k-1}I+Z_2\wedge^{k-1}I}{2}\right)^{sp}\widetilde{K}\right]^{\frac{1}{s}}\right)^{\frac{1}{k}}\\
&\geq\left(\mathrm{Tr}\left[\frac{[(Z_1\wedge^{k-1}I)^{qs}\widetilde{K}^*\left(Z_1\wedge^{k-1}I\right)^{sp}\widetilde{K}]^{\frac{1}{s}}+[(Z_2\wedge^{k-1}I)^{qs}\widetilde{K}^*\left(Z_2\wedge^{k-1}I\right)^{sp}\widetilde{K}]^{\frac{1}{s}}}{2}\right]\right)^{\frac{1}{k}} \qquad (3.5)\\
&=\left(\mathrm{Tr}\left[\left(\frac{[Z_1^{qs}\overline{K}^*Z_1^{sp}\overline{K}]^{\frac{1}{s}}+[Z_2^{qs}\overline{K}^*Z_2^{sp}\overline{K}]^{\frac{1}{s}}}{2}\right)\wedge^{k-1}\left((\frac{Z_1+Z_2}{2})^{qs}\overline{K}^*(\frac{Z_1+Z_2}{2})^{sp}\overline{K}\right)^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}\\
&=\left(\mathrm{Tr}\left[\left(\left[\Lambda\left(\frac{z_1+z_2}{2}\right)^{sq}\Lambda^{k-2}\right]\right)\bar{k}_2^*\left(\left[\Lambda\left(\frac{z_1+z_2}{2}\right)^{sp}\Lambda^{k-2}\right]\right)\right]^{\frac{1}{s}}\bar{k}_2\right)^{\frac{1}{k}}\\
&\geq\left(\mathrm{Tr}\left[\wedge^k\left(\frac{[Z_1^{qs}\overline{K}^*Z_1^{sp}\overline{K}]^{\frac{1}{s}}+[Z_2^{qs}\overline{K}^*Z_2^{sp}\overline{K}]^{\frac{1}{s}}}{2}\right)\right]\right)^{\frac{1}{k}}\\
&\geq\frac{1}{2}\left(\mathrm{Tr}\left[\wedge^k[Z_1^{qs}\overline{K}^*Z_1^{sp}\overline{K}]^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}+\frac{1}{2}\left(\mathrm{Tr}\left[\wedge^k[Z_2^{qs}\overline{K}^*Z_2^{sp}\overline{K}]^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}\ \text{(Thm 2.1)}\\
&=\frac{H(A_1,B_1)+H(A_2,B_2)}{2},
\end{aligned}
$$

where $\widetilde{K}=\overline{K}\wedge^{k-1}\left((\frac{Z_1+Z_2}{2})^{qs}\overline{K}^*(\frac{Z_1+Z_2}{2})^{sp}\overline{K}\right)^{\frac{1}{s}}$, $\bar{k}_2=\left(\frac{\left[z_1{}^{sq}\bar{k}^*z_1{}^{sp}k\right]^{\frac{1}{s}}+\left[z_2{}^{sq}\bar{k}^*z_2{}^{sp}\bar{k}\right]^{\frac{1}{s}}}{2}\right)^{\frac{1}{2}}\Lambda^{k-1}\bar{k}$, and the first $\geq$ is obtained by using Theorem 3.1 and the second $\geq$ is obtained by using Theorem 3.1 recycling. □

Generally, the following result can be obtained.

3.4. Corollary

Let $0\leq p,q,s\leq 1$, and $p+q\leq 1$, then

$$
H(A,B)=\left(\mathrm{Tr}\left[\wedge^k(f(A^{\frac{qs}{2}})K^*f(B^{sp})Kf(A^{\frac{sq}{2}}))^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}, \qquad (3.6)
$$

is jointly concave for any $A,B\geq 0$, where f is a nonnegative matrix monotone function.

The proof is similar to Corollary 3.3, where it is omitted.

Setting $Z=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$, $K=\begin{pmatrix}0 & 0\\ e^{\frac{sA}{2}} & 0\end{pmatrix}$ and using the Lie-Trotter formula, when $p=1,q=0$, we obtain that

$$\lim_{t\to 0} \mathrm{Tr}\left[((Z)^{\frac{qs}{2}} K^*(Z)^{sp} K(Z)^{\frac{sq}{2}})^{\frac{1}{s}}\right]$$
$$= \mathrm{Tr}\left[e^{A+\ln B}\right],$$

is concave for any $B \geq 0$. So, we can obtain a useful result [9] from Corollary 3.3.

3.5. Corollary

Let $Z, B \in H_n^+$, then

$$H(B) = \left(\mathrm{Tr}\left[\wedge^k e^{A+\ln B}\right]\right)^{\frac{1}{k}}, \tag{3.7}$$

is concave for any $B \geq 0$.

4. Conclusions

This paper shows that $\left(\mathrm{Tr}\left[\wedge^k(f(A^{\frac{qs}{2}})K^* f(B^{sp}) K f(A^{\frac{sq}{2}}))^{\frac{1}{s}}\right]\right)^{\frac{1}{k}}$ is jointly concave for any nonnegative matrix monotone function f (x) and a generalization of the Lieb concavity theorem is given by using the properties of external algebras. In fact, we guess that the following function

$$F(A) = \frac{\left\{\mathrm{Tr}\left[\Lambda_n\left(f\left(A^{\frac{sq}{2}}\right)K^* f\left(A^{sp}\right)Kf\left(A^{\frac{sq}{2}}\right)\right)\right]\right\}^{\frac{1}{n}}}{\left\{\mathrm{Tr}\left[\Lambda_{n-1}\left(f\left(A^{\frac{sq}{2}}\right)K^* f\left(A^{sp}\right)Kf\left(A^{\frac{sq}{2}}\right)\right)\right]\right\}^{\frac{1}{n-1}}},$$

should also be concave.

For the time being, this theory has not been applied to mechanics, and the follow-up research is to apply the anti-information matrix to mechanics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Finally, all three authors contributed to the writing and revision of the paper, and offered some constructive comments and suggestions for improvement to ensure that the paper can accurately explain the rationality of theorem construction and proof.

Conflict of interest

The authors declare no conflict of interest.

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