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Research article

Fixed point results for generalized almost contractions and application to a nonlinear matrix equation

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Abstract: The goal of this paper was to improve some known results of fixed points by using w-distances and properties of locally symmetric $\mathcal{H}$-transitivity of binary relations. Also, we gave the application of the obtained results for finding the solution of nonlinear matrix equations. Finally, we gave a numerical example to demonstrate the applicability of our results.

Keywords: generalized almost contraction; w-distance; symmetric locally $\mathcal{H}$-transitive binary relations; nonlinear matrix equation; relational metric space
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1. Introduction

In 2004, Berinde introduced a novel class of contractive type mappings initially referred to as "weak contractions" in [1]. However, Berinde later renamed this class as "almost contractions" in [2, 3].

Definition 1. A self-map $\mathcal{H}$ defined on a metric space (X, d) is termed as an almost contraction when there exists a constant δ within the interval $(0, 1)$ and a nonnegative constant $\mathcal{L}$, such that for all $\varrho, \varsigma \in X$

$$d(\mathcal{H}\varrho, \mathcal{H}\varsigma) \leqslant \delta d(\varrho, \varsigma) + \mathcal{L}d(\mathcal{H}\varrho, \varsigma). \tag{1.1}$$

Berinde [1] proves that an almost contraction mapping $\mathcal{H}$ defined on a complete metric space has at least one fixed point. This class of mappings not only unifies and generalizes renowned fixed point theorems such as Banach, Kannan, Chatterjea, Zamfirescu, Reich, Bianchini, Ćirić, Hardy and Rogers, Rus, Rhoades, and others, but also extends their scope significantly.

Note that the almost contraction condition (1.1) does not guarantee a unique fixed point, although a sequence of Picard iterations converges to a fixed point. In response to this issue, Berinde [1] posed an open question to identify a contractive condition distinct from (1.1) that ensures the uniqueness of fixed points for weak contractions, under the conditions stated in their result. This question was addressed by Babu et al. [4]. To establish a uniqueness theorem, they introduced a slightly stronger category of almost contraction conditions referred to as condition (B).

Definition 2. A self-map T defined on metric space (X, d) is said to satisfy condition (B) if there exits $\delta \in (0, 1)$ and a nonnegative constant $\mathcal{L}$ such that for all $\varrho, \varsigma \in X$,

$$d(\mathcal{H}\varrho, \mathcal{H}\varsigma) \leqslant \delta d(\varrho, \varsigma) + \mathcal{L} \min\{d(\varrho, \mathcal{H}\varrho), d(\varsigma, \mathcal{H}\varsigma), d(\varrho, \mathcal{H}\varsigma), d(\varsigma, \mathcal{H}\varrho)\}. \tag{1.2}$$

On the other hand, Kada et al. [5] introduced the concept of w-distance within metric spaces and utilized it to establish a generalized fixed point theorem, extending the results of Subrahmanyam, Kannan, Ćirić, and others. This concept influenced the improvement of several well-known results, such as Caristi's fixed point theorem, Ekeland's ϵ-variation principle, and Takahashi's non-convex minimization theorem. Some recent results in this area can be found in [6–12]. Moreover, a comprehensive collection of existence results concerning fixed points for contractive type mappings can be found in the work of Reich and Zaslavski [13].

In the paper [14], Ran and Reuring generalized Banach's theorem to partially ordered metric spaces. On the other hand, in the paper [15], Alam and Imdad obtained a relational-theoretical version of the Banach contraction principle. That result influenced the obtaining of new results, see [16–23].

The goal of this paper is to improve some known results of fixed points by using w-distances and properties of locally symmetric $\mathcal{H}$-transitivity of binary relations. Also, we give the application of the obtained results for finding the solution of nonlinear matrix equations. Finally, we give a numerical example to demonstrate the applicability of our results.

2. Preliminaries

In this paper, we consider that X represents a nonempty set, $\mathcal{R}$ denotes a nonempty binary relation on X, $\mathbb{N}$ denotes a set of natural numbers, and $\mathbb{N}_0$ denotes the set of nonnegative integers.

Alam and Imdad [15] gave the following definition.

Definition 3. Let $\mathcal{R}$ be a nonempty binary relation defined on the set X, and let $\mathcal{H}$ be a self-map on X.

(i) Any two elements $\varrho, \varsigma \in X$ are $\mathcal{R}$-comparative if $(\varrho, \varsigma) \in \mathcal{R}$ or $(\varsigma, \varrho) \in \mathcal{R}$. This relationship is symbolically represented as $[\varrho, \varsigma] \in \mathcal{R}$.

(ii) A sequence $\{\varrho_k\} \subset X$ that satisfies the condition $(\varrho_k, \varrho_{k+1}) \in \mathcal{R}$ for all $k \in \mathbb{N}_0$, is referred to as an $\mathcal{R}$-preserving sequence.

(iii) $\mathcal{R}$ is designated as $\mathcal{H}$-closed when it satisfies the condition that if (ϱ, ς) belongs to $\mathcal{R}$, then $(\mathcal{H}\varrho, \mathcal{H}\varsigma)$ also belongs to $\mathcal{R}$, for any $\varrho, \varsigma \in X$.

(iv) $\mathcal{R}$ is referred to as d-self-closed under the condition that whenever there exists an $\mathcal{R}$-preserving sequence $\{\varrho_k\}$ such that $\varrho_k \xrightarrow{d} \varrho$, we can always find a subsequence $\{\varrho_{k_n}\}$ of $\{\varrho_k\}$ such that $[\varrho_{k_n}, \varrho]$ belongs to $\mathcal{R}$ for all $n \in \mathbb{N}_0$.

The terms in the following definition were introduced by Maddux in [24].

Definition 4. (i) The completeness of $\mathcal{R}$ is defined by the property that every pair of elements in X is $\mathcal{R}$-comparative i.e., $[\varrho, \varsigma] \in \mathcal{R}$, for all $\varrho, \varsigma \in X$.

(ii) If $E \subseteq X$, then the set $\mathcal{R}|_E$ defined as $\mathcal{R} \cap E^2$ remains a relation on E, which is induced by $\mathcal{R}$.

In the following definition, we give the terms introduced by Alama, Kocev, and Imdad in [25].

Definition 5. (i) The $\mathcal{R}$-completeness of the metric space (X, d) is defined as the property where every sequence in X, which is both $\mathcal{R}$-preserving and Cauchy, converges.

(ii) A self-map $\mathcal{H}$ defined on X is termed $\mathcal{R}$-continuous at $\varrho \in X$, if any $\mathcal{R}$-preserving sequence $\varrho_k \xrightarrow{d} \varrho$, implies $\mathcal{H}\varrho_k \xrightarrow{d} \mathcal{H}\varrho$.

Furthermore, if $\mathcal{H}$ exhibits this behavior at every point in X, it is simply categorized as $\mathcal{R}$-continuous.

Definition 6. (Alam and Imdad, [16]) Let a self-mapping $\mathcal{H}$ be defined on X. If for every $\mathcal{R}$-preserving sequence $\{\varrho_n\} \subset \mathcal{H}(X)$, with a range denoted as $E = \{\varrho_n : n \in \mathbb{N}\}$, $\mathcal{R}|_E$ is transitive, then $\mathcal{H}$ is locally $\mathcal{H}$-transitive.

Definition 7. (Khan, Swaleh, and Sessa, [26]) A function $\Psi : [0, +\infty) \to [0, +\infty)$ is classified as an altering distance function if it adheres to the properties of being both continuous and nondecreasing, and, further, $\Psi(\nu)$ is equal to 0 if, and only if, ν is equal to 0.

The notion of $\mathcal{R}$-lower semi-continuity, often denoted as $\mathcal{R}$-LSC, has been introduced by Senapati and Dey in the paper [27]. The authors have provided elucidating examples that demonstrate the relatively weaker nature of $\mathcal{R}$-LSC compared to both $\mathcal{R}$-continuity and lower semi-continuity. Moreover, they have introduced modifications to the definition of the w-distance [5].

Definition 8. (Senapati and Dey, [27]) Let (X, d) be a metric space. A w-distance $q : X \times X \to [0, +\infty)$ is defined as a function that adheres to the following properties for all $\varrho, \varsigma, \xi \in X$:

(i) $q(\varrho, \varsigma) \leqslant q(\varrho, \xi) + q(\xi, \varsigma)$.

(ii) $q(\varrho, \cdot) : X \to [0, +\infty)$ is $\mathcal{R}$-LSC.

(iii) For every positive $\varepsilon > 0$, there exists a positive $\delta > 0$ such that, when $q(\varrho, \varsigma) \leqslant \delta$ and $q(\varrho, \xi) \leqslant \delta$ are satisfied, it implies that $d(\varsigma, \xi) \leqslant \varepsilon$.

It is a widely acknowledged fact that any metric defined on X also constitutes a w-distance on X. The following is another example of w-distance.

Example 9. A function $q : X \times X \to [0, +\infty)$ defined as $q(\varrho, \varsigma) = c$, for all $\varrho, \varsigma \in X$, where c is a positive real number, on a metric space (X, d) qualifies as a w-distance on X. However, it is important to note that q does not constitute a metric, as it does not satisfy the property $q(\varrho, \varrho) = 0$ for any $\varrho \in X$, which is one of the essential conditions for a function to be considered a metric.

Example 10. (Wongyat and Sintunavarat, [28]) Let X be the set of real numbers equipped with the metric $d : X \times X \to \mathbb{R}^+$, defined as $d(\varrho, \varsigma) = |\varrho - \varsigma|$ for all $\varrho, \varsigma \in X$. Let μ and υ be real numbers greater than 1. Define $q : X \times X \to [0, +\infty)$ as follows:

$$q(\varrho, \varsigma) = \max\{\mu(\varsigma - \varrho), \upsilon(\varrho - \varsigma)\}, \text{ for all } \varrho, \varsigma \in X,$$

then q is a w-distance for d. Additionally, the lack of symmetry in q prevents it from satisfying the requirements of a metric.

The following lemma was obtained in Babu and Sailaja [29]. See also Turinici [30].

Lemma 11. *Let (X, d) be a metric space with a w-distance q defined on it. If $\{\varrho_n\}$ in X is not a Cauchy sequence, then there exist a positive $\varepsilon > 0$, along with two subsequences $\{\varrho_{n_k}\}, \{\varrho_{m_k}\}$ of $\{\varrho_n\}$ such that:*

(i) *$k \leqslant n_k \leqslant m_k$, for all $k \in \mathbb{N}$.*
(ii) *$q(\varrho_{n_k}, \varrho_{m_{k-1}}) \leqslant \varepsilon$, for all $k \in \mathbb{N}$.*
(iii) *$q(\varrho_{n_k}, \varrho_{m_k}) > \varepsilon$, for all $k \in \mathbb{N}$.*

Moreover, suppose that $\lim_{k \to +\infty} q(\varrho_n, \varrho_{n+1}) = 0$, then

(iv) $\lim_{k \to +\infty} q(\varrho_{n_k}, \varrho_{m_k}) = \varepsilon$.
(v) $\lim_{k \to +\infty} q(\varrho_{n_k}, \varrho_{m_{k-1}}) = \varepsilon$.

Lemma 12. *(Senapati and Dey, [27]) Let $\mathcal{R}$ be a binary relation and let q be a w-distance defined on a metric space (X, d). Suppose that there are $\mathcal{R}$-preserving sequences $\{\varrho_r\}$ and $\{\varsigma_r\}$ in X, composed with positive real numbers that converge to 0. Under these conditions, for ϱ, ς, and $\xi \in X$, the following results hold:*

(i) *If $q(\varrho_n, \varsigma) \leqslant u_n$ and $q(\varrho_n, \xi) \leqslant v_n$, for all $n \in \mathbb{N}$, then $\varsigma = \xi$. Moreover, if $p(\varrho, \varsigma) = 0$ and $p(\varrho, \xi) = 0$, then $\varsigma = \xi$.*
(ii) *If $q(\varrho_n, \varsigma_n) \leqslant u_n$ and $q(\varrho_n, \xi) \leqslant v_n$, for all $n \in \mathbb{N}$, then $\varsigma_n \to \xi$.*
(iii) *If $q(\varrho_n, \varsigma) \leqslant u_n$, for all $n \in \mathbb{N}$, then $\{\varrho_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in X.*
(iv) *If $q(\varrho_n, \varrho_m) \leqslant u_n$, for all $m > n$, then $\{\varrho_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in X.*

Throughout this paper, we denote:

(a) $F(\mathcal{H})$ as the set of all fixed points of the self-mapping $\mathcal{H}$.
(b) $X(\mathcal{H}, \mathcal{R})$ as the subset of X consisting of all elements ϱ such that $(\varrho, \mathcal{H}\varrho) \in \mathcal{R}$.
(c) $\Gamma(\varrho, \varsigma) = \max\left\{q(\varrho, \varsigma), q(\varrho, \mathcal{H}\varrho), q(\varsigma, \mathcal{H}\varsigma), \frac{1}{2}\left[q(\varrho, \mathcal{H}\varsigma) + q(\mathcal{H}\varrho, \varsigma)\right]\right\}$.
(d) $\Delta(\varrho, \varsigma) = \min\{q(\varrho, \mathcal{H}\varrho), q(\varsigma, \mathcal{H}\varsigma), q(\mathcal{H}\varrho, \varsigma), q(\varrho, \mathcal{H}\varsigma)\}$.

3. Fixed point results for generalized almost contractions

In this section, we focus on establishing conditions under which the self-mapping $\mathcal{H}$ possesses a fixed point. In particular, condition $(c2)$ imposes requirements on the completeness of certain subsets of the metric space rather than the entire metric space, and condition $(c3)$ allows us to obtain fixed point results for discontinuous mappings.

Theorem 13. *Let $\mathcal{H}$ be a self-mapping defined on a metric space (X,d) endowed with a symmetric and locally $\mathcal{H}$-transitive binary relation $\mathcal{R}$, and let q be a w-distance on X, satisfying the condition $q(\varrho,\varrho)=0$. Suppose the following conditions are met:*

$(c1)$ *$X(\mathcal{H},\mathcal{R})$ is nonempty.*
$(c2)$ *There exists $Y\subseteq X$ with $\mathcal{H}(X)\subseteq Y$, such that (Y,d) is $\mathcal{R}$-complete.*
$(c3)$ *Either $\mathcal{R}|_Y$ is d-self-closed or $\mathcal{H}$ is $\mathcal{R}$-continuous.*
$(c4)$ *$\mathcal{R}$ is $\mathcal{H}$-closed.*
$(c5)$ *There exists Φ and Ψ such that*

$$\Psi(q(\mathcal{H}\varrho,\mathcal{H}\varsigma))\leqslant\Psi(\Gamma(\varrho,\varsigma))-\Phi(\Gamma(\varrho,\varsigma))+\mathcal{L}\Delta(\varrho,\varsigma), \tag{3.1}$$

for all $\varrho,\varsigma\in X$ and $(\varrho,\varsigma)\in\mathcal{R}$.
$(c6)$ *$y\mapsto q(\cdot,y)$ is $\mathcal{R}$-lower semi-continuous,*

where $\mathcal{L}$ is a nonnegative constant, $\Phi:[0,+\infty)\to[0,+\infty)$ is an LSC satisfying $\Phi(\nu)=0$ if, and only if, $\nu=0$, and Ψ represents an altering distance function, then $\mathcal{H}$ possesses a fixed point in X.

Proof. From assumption $(c1)$, we conclude that there exists $\varrho_0\in X(\mathcal{H},\mathcal{R})$. Define the sequence ϱ_r of Picard iterates with initial point ϱ_0, i.e., $\varrho_r=\mathcal{H}^r\varrho_0$ for all $r\in\mathbb{N}_0$. Since $(\varrho_0,\mathcal{H}\varrho_0)\in\mathcal{R}$, from $\mathcal{H}$-closedness of $\mathcal{R}$, we have

$$(\mathcal{H}^r\varrho_0,\mathcal{H}^{r+1}\varrho_0)\in\mathcal{R},\ \text{for all } r\in\mathbb{N}_0,$$

so that

$$(\varrho_r,\varrho_{r+1})\in\mathcal{R},\ \text{for all } r\in\mathbb{N}_0. \tag{3.2}$$

Therefore, it can be asserted that $\{\varrho_r\}$ is an $\mathcal{R}$-preserving sequence. By employing the condition $(c5)$, we obtain

$$\begin{aligned}\Psi(q(\varrho_r,\varrho_{r+1}))&=\Psi(q(\mathcal{H}\varrho_{r-1},\mathcal{H}\varrho_r))\\&\leqslant\Psi(\Gamma(\varrho_{r-1},\varrho_r))-\Phi(\Gamma(\varrho_{r-1},\varrho_r))+\mathcal{L}\Delta(\varrho_{r-1},\varrho_r),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\Gamma(\varrho_{r-1},\varrho_r)&=\max\Big\{q(\varrho_{r-1},\varrho_r),q(\varrho_{r-1},\mathcal{H}\varrho_{r-1}),q(\varrho_r,\mathcal{H}\varrho_r),\\&\qquad\frac{1}{2}[q(\varrho_{r-1},\mathcal{H}\varrho_r)+q(\mathcal{H}\varrho_{r-1},\varrho_r)]\Big\}\\&=\max\Big\{q(\varrho_{r-1},\varrho_r),q(\varrho_{r-1},\varrho_r),q(\varrho_r,\varrho_{r+1}),\\&\qquad\frac{1}{2}[q(\varrho_{r-1},\varrho_{r+1})+q(\varrho_r,\varrho_r)]\Big\}\\&=\max\{q(\varrho_{r-1},\varrho_r),q(\varrho_r,\varrho_{r+1})\},\end{aligned}$$

and

$$\Delta(\varrho_{r-1},\varrho_r)=\min\{q(\varrho_{r-1},\mathcal{H}\varrho_{r-1}),q(\varrho_r,\mathcal{H}\varrho_r),q(\mathcal{H}\varrho_{r-1},\varrho_r),q(\varrho_{r-1},\mathcal{H}\varrho_r)\}$$

$$= \min\left\{q(\varrho_{r-1},\varrho_r), q(\varrho_r,\varrho_{r+1}), q(\varrho_r,\varrho_r), q(\varrho_{r-1},\varrho_{r+1})\right\}.$$

Now, if $q(\varrho_{r-1},\varrho_r) \leqslant q(\varrho_r,\varrho_{r+1})$, then from (3.3), we have

$$\begin{aligned}\Psi(q(\varrho_r,\varrho_{r+1})) = \Psi(q(\mathcal{H}\varrho_{r-1},\mathcal{H}\varrho_r)) &\leqslant \Psi(q(\varrho_r,\varrho_{r+1})) - \Phi(q(\varrho_r,\varrho_{r+1}))\\ &< \Psi(q(\varrho_r,\varrho_{r+1})),\end{aligned}$$

which is a contradiction.

Thus, $q(\varrho_{r-1},\varrho_r) \geqslant q(\varrho_r,\varrho_{r+1})$. So, we obtain that there exists a nonnegative ν such that $q(\varrho_r,\varrho_{r+1}) \to \nu$ as $r \to +\infty$. Again, from Eq (3.3), we get

$$\begin{aligned}\Psi(\nu) &\leqslant \limsup_{r\to+\infty} \Psi(q(\varrho_{r-1},\varrho_r)) - \limsup_{r\to+\infty} \Phi(q(\varrho_{r-1},\varrho_r))\\ &\leqslant \limsup_{r\to+\infty} \Psi(q(\varrho_{r-1},\varrho_r)) - \liminf_{r\to+\infty} \Phi(q(\varrho_{r-1},\varrho_r))\\ &\leqslant \Psi(\nu) - \Phi(\nu),\end{aligned}$$

consequently, $\Phi(\nu) = 0$, which in turn leads to the conclusion that ν must equal 0. Hence $\lim_{r\to+\infty} q(\varrho_r,\varrho_{r+1}) = 0$.

Now, with the symmetric property of $\mathcal{R}$, it is evident that if $(\varrho_r,\varrho_{r+1}) \in \mathcal{R}$ so $(\varrho_{r+1},\varrho_r) \in \mathcal{R}$ for all $r \in \mathbb{N}_0$, then from (3.1), we get

$$\begin{aligned}\Psi(q(\varrho_{r+1},\varrho_r)) &= \Psi(\mathcal{H}\varrho_r,\mathcal{H}\varrho_{r-1})\\ &\leqslant \Psi(\Gamma(\varrho_r,\varrho_{r-1})) - \Phi(\Gamma(\varrho_r,\varrho_{r-1})) + \mathcal{L}\Delta(\varrho_r,\varrho_{r-1}),\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\Gamma(\varrho_r,\varrho_{r-1})) &= \max\Big\{q(\varrho_r,\varrho_{r-1}), q(\varrho_r,\mathcal{H}\varrho_r), q(\varrho_{r-1},\mathcal{H}\varrho_{r-1}),\\ &\qquad \frac{1}{2}[q(\varrho_r,\mathcal{H}\varrho_{r-1}) + q(\mathcal{H}\varrho_r,\varrho_{r-1})]\Big\}\\ &= \max\Big\{q(\varrho_r,\varrho_{r-1}), q(\varrho_r,\varrho_{r+1}), q(\varrho_{r-1},\varrho_r),\\ &\qquad \frac{1}{2}[q(\varrho_r,\varrho_r) + q(\varrho_{r+1},\varrho_{r-1})]\Big\}\\ &= \max\left\{q(\varrho_r,\varrho_{r-1}), q(\varrho_r,\varrho_{r+1}), q(\varrho_{r-1},\varrho_r), q(\varrho_{r+1},\varrho_r)\right\}\end{aligned}$$

and

$$\begin{aligned}\Delta(\varrho_r,\varrho_{r-1}) &= \min\{q(\varrho_r,\mathcal{H}\varrho_r), q(\varrho_{r-1},\mathcal{H}\varrho_{r-1}), q(\mathcal{H}\varrho_r,\varrho_{r-1}), q(\varrho_r,\mathcal{H}\varrho_{r-1})\}\\ &= \min\{q(\varrho_r,\varrho_{r+1}), q(\varrho_{r-1},\varrho_r), q(\varrho_{r+1},\varrho_{r-1}), q(\varrho_r,\varrho_r)\}.\end{aligned}$$

If $\Gamma(\varrho_r,\varrho_{r-1}) = q(\varrho_{r+1},\varrho_r)$, then from (3.4), we obtain,

$$\Psi(q(\varrho_{r+1},\varrho_r)) \leqslant \Psi(q(\varrho_{r+1},\varrho_r)) - \Phi(q(\varrho_{r+1},\varrho_r)),$$

which is a contradiction.

Now, suppose $\Gamma(\varrho_r, \varrho_{r-1}) = q(\varrho_r, \varrho_{r+1})$, then from (3.4), we obtain,

$$\Psi(q(\varrho_{r+1}, \varrho_r)) \leqslant \Psi(q(\varrho_r, \varrho_{r+1})) - \Phi(q(\varrho_r, \varrho_{r+1})),$$

and $\Psi\left(\lim_{r\to+\infty} q(\varrho_{r+1}, \varrho_r)\right) = 0$, which implies that $\lim_{r\to+\infty} q(\varrho_{r+1}, \varrho_r) = 0$. Likewise, if $\Gamma(\varrho_r, \varrho_{r-1}) = q(\varrho_{r-1}, \varrho_r)$, we can arrive at the same conclusion.

Now, suppose $\Gamma(\varrho_r, \varrho_{r-1}) = q(\varrho_r, \varrho_{r+1})$, then from (3.4), and using the property of monotonicity of Ψ we can obtain the nonincreasing sequence $\{q(\varrho_{r+1}, \varrho_r)\}$ converging to some $\nu \geqslant 0$. Continuing in this process, we obtain that $\lim_{r\to\infty} q(\varrho_{r+1}, \varrho_r) = 0$.

Thus, in any case, we can conclude that $\lim_{r\to+\infty} q(\varrho_{r+1}, \varrho_r) = 0$.

Now, to prove that $\{\varrho_r\}$ is a Cauchy sequence, suppose there does not exist $\varepsilon > 0$ and subsequences $\{\varrho_{n_k}\}$ and $\{\varrho_{m_k}\}$ of $\{\varrho_r\}$, such that $k \leqslant m_k < n_k$, $q(\varrho_{m_k}, \varrho_{n_k}) > \varepsilon \geqslant q(\varrho_{m_k}, \varrho_{n_{k-1}})$ for all $k \in \mathbb{N}$. Using Lemma 11, we have

$$\lim_{k\to+\infty} q(\varrho_{m_k}, \varrho_{n_k}) = \lim_{k\to+\infty} q(\varrho_{m_{k+1}}, \varrho_{n_{k+1}}) = \varepsilon. \tag{3.5}$$

Now, using (3.5) and the triangular inequality, we get

$$\begin{aligned} q(\varrho_{m_k}, \varrho_{n_k}) &\leqslant q(\varrho_{m_k}, \varrho_{n_{k+1}}) + q(\varrho_{n_{k+1}}, \varrho_{n_k}) \\ &\leqslant q(\varrho_{m_k}, \varrho_{n_k}) + q(\varrho_{n_k}, \varrho_{n_{k+1}}) + q(\varrho_{n_{k+1}}, \varrho_{n_k}). \end{aligned}$$

Also,

$$\begin{aligned} q(\varrho_{m_k}, \varrho_{n_k}) &\leqslant q(\varrho_{m_k}, \varrho_{m_{k+1}}) + q(\varrho_{m_{k+1}}, \varrho_{n_k}) \\ &\leqslant q(\varrho_{m_k}, \varrho_{m_{k+1}}) + q(\varrho_{m_{k+1}}, \varrho_{m_k}) + q(\varrho_{m_k}, \varrho_{n_k}). \end{aligned}$$

Letting $k \to +\infty$, we obtain that $q(\varrho_{m_k}, \varrho_{n_{k+1}}) = \varepsilon$, and $q(\varrho_{m_{k+1}}, \varrho_{n_k}) = \varepsilon$.

As $\{\varrho_r\}$ is $\mathcal{R}$-preserving and $\{\varrho_r\} \subset \mathcal{H}(X)$, by locally $\mathcal{H}$-transitivity of $\mathcal{R}$, we have $(\varrho_{m_k}, \varrho_{n_k}) \in \mathcal{R}$.

Hence, by (3.1), we obtain

$$\begin{aligned} \Psi(q(\varrho_{m_{k+1}}, \varrho_{n_{k+1}})) &= \Psi(q(\mathcal{H}\varrho_{m_k}, \mathcal{H}\varrho_{n_k})) \\ &\leqslant \Psi(\Gamma(\varrho_{m_k}, \varrho_{n_k})) - \Phi(\Gamma(\varrho_{m_k}, \varrho_{n_k}) + \mathcal{L}\Delta(\varrho_{m_k}, \varrho_{n_k}), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &\Gamma(\varrho_{m_k}, \varrho_{n_k}) = \max\{q(\varrho_{m_k}, \varrho_{n_k}), q(\varrho_{m_k}, \mathcal{H}\varrho_{m_k}), q(\varrho_{n_k}, \mathcal{H}\varrho_{n_k}), \\ &\frac{1}{2}[q(\varrho_{m_k}, \mathcal{H}\varrho_{n_k}) + q(\mathcal{H}\varrho_{m_k}, \varrho_{n_k})]\Big\} \\ &= \max\left\{q(\varrho_{m_k}, \varrho_{n_k}), q(\varrho_{m_k}, \varrho_{m_{k+1}}), q(\varrho_{n_k}, \varrho_{n_{k+1}}), \frac{1}{2}\left[q(\varrho_{m_k}, \varrho_{n_{k+1}}) + q(\varrho_{m_{k+1}}, \varrho_{n_k})\right]\right\}, \end{aligned}$$

and

$$\Delta(\varrho_{m_k}, \varrho_{n_k}) = \min\{q(\varrho_{m_k}, \mathcal{H}\varrho_{m_k}), q(\varrho_{n_k}, \mathcal{H}\varrho_{n_k}), q(\mathcal{H}\varrho_{m_k}, \varrho_{n_k}), q(\varrho_{m_k}, \mathcal{H}\varrho_{n_k})\}$$

$$= \min\{q(\varrho_{m_k}, \varrho_{m_{k+1}}), q(\varrho_{n_k}, \varrho_{n_{k+1}}), q(\varrho_{m_{k+1}}, \varrho_{n_k}), q(\varrho_{m_k}, \varrho_{n_{k+1}})\}.$$

Letting $k \to +\infty$ in (3.6) and leveraging the properties of Ψ and Φ, one arrives at the contradictory statement $\Psi(\varepsilon) \leqslant \Psi(\varepsilon) - \Phi(\varepsilon)$. Therefore, by Lemma 12, $\{\varrho_r\}$ is the $\mathcal{R}$-preserving Cauchy sequence in Y, and the $\mathcal{R}$-completeness of (Y, d) guarantees that there exists $\varrho^* \in Y$ with $\varrho_r \xrightarrow{d} \varrho^*$, as $r \to +\infty$. From $(c3)$, we first suppose that $\mathcal{H}$ is $\mathcal{R}$-continuous. Since $\{\varrho_r\}$ is $\mathcal{R}$-preserving with $\varrho_r \xrightarrow{d} \varrho^*$, which implies that $\varrho_{r+1} = \mathcal{H}\varrho_r \xrightarrow{d} \mathcal{H}\varrho^*$, and due to the uniqueness of limits, we obtain that $\mathcal{H}\varrho^* = \varrho^*$. Hence, ϱ^* is a fixed point of $\mathcal{H}$.

Now, if $\mathcal{R}|_Y$ is d-self-closed, and as $\{\varrho_r\}$ is $\mathcal{R}$-preserving such that $\{\varrho_r\} \to \varrho^*$, there exists subsequence $\{\varrho_{n_k}\}$ of $\{\varrho_r\}$ with $[\varrho_{n_k}, \varrho^*] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. On using $[\varrho_{n_k}, \varrho^*] \in \mathcal{R}$, symmetry of binary relation $\mathcal{R}$, and assumption $(c5)$, we have

$$\Psi(q(\varrho_{n_{k+1}}, \mathcal{H}\varrho^*)) = \Psi(q(\mathcal{H}\varrho_{n_k}, \mathcal{H}\varrho^*)) \leqslant \Psi(\Gamma(\varrho_{n_k}, \varrho^*)) - \Phi(\Gamma(\varrho_{n_k}, \varrho^*)) + \mathcal{L}\Delta(\varrho_{n_k}, \varrho^*), \tag{3.7}$$

where

$$\Gamma(\varrho_{n_k}, \varrho^*) = \max\left\{q(\varrho_{n_k}, \varrho^*), q(\varrho_{n_k}, \varrho_{n_{k+1}}), q(\varrho^*, \mathcal{H}\varrho^*), \frac{1}{2}\left[q(\varrho_{n_k}, \mathcal{H}\varrho^*) + q(\varrho_{n_{k+1}}, \varrho^*)\right]\right\}$$

and

$$\Delta(\varrho_{n_k}, \varrho^*) = \min\{q(\varrho_{n_k}, \varrho_{n_{k+1}}), q(\varrho^*, \mathcal{H}\varrho^*), q(\varrho_{n_{k+1}}, \varrho^*), q(\varrho_{n_k}, \mathcal{H}\varrho^*)\}.$$

Taking the limit $k \to +\infty$, in (3.7), we obtain

$$\Psi(q(\varrho^*, \mathcal{H}\varrho^*)) \leqslant \Psi(q(\varrho^*, \mathcal{H}\varrho^*)) - \Phi(q(\varrho^*, \mathcal{H}\varrho^*)),$$

which is possible only if $q(\varrho^*, \mathcal{H}\varrho^*) = 0$. Therefore, by (i) of Lemma 12, we obtain $\varrho^* = \mathcal{H}\varrho^*$. □

Theorem 14. *If, in conjunction with the assumptions in the statement of Theorem 13, the following condition holds:*

$(c7)$ $\mathcal{R}|_{F(\mathcal{H})}$ *is complete.*

Then $\mathcal{H}$ possesses a unique fixed point.

Proof. By Theorem 13, it can be established that $\mathcal{H}$ possesses a fixed point. The next task is to demonstrate that $\mathcal{H}$ has a unique fixed point. Suppose $\varrho, \varsigma \in F$, then we have $\mathcal{H}(\varrho) = \varrho$, $\mathcal{H}(\varsigma) = \varsigma$. As F is $\mathcal{R}$-complete $[\varrho, \varsigma] \in \mathcal{R}$, by applying condition $(c5)$ to these particular points, we obtain

$$\Psi(q(\varrho, \varsigma)) = \Psi(q(\mathcal{H}\varrho, \mathcal{H}\varsigma)) \leqslant \Psi(\Gamma(\varrho, \varsigma)) - \Phi(\Gamma(\varrho, \varsigma)) + \mathcal{L}\Delta(\varrho, \varsigma),$$

where,
$\Gamma(\varrho, \varsigma) = \max\left\{q(\varrho, \varsigma), q(\varrho, \mathcal{H}\varrho), q(\varsigma, \mathcal{H}\varsigma), \frac{1}{2}[q(\varrho, \mathcal{H}\varsigma) + q(\mathcal{H}\varrho, \varsigma)]\right\} = q(\varrho, \varsigma)$, and $\Delta(\varrho, \varsigma) = \min\{q(\varrho, \mathcal{H}\varrho), q(\varsigma, \mathcal{H}\varsigma), q(\mathcal{H}\varrho, \varsigma), q(\varrho, \mathcal{H}\varsigma)\} = 0$.

Thus, $\Psi(q(\varrho, \varsigma)) \leqslant \Psi(q(\varrho, \varsigma)) - \Phi(q(\varrho, \varsigma))$, which implies that $\Phi(q(\varrho, \varsigma)) = 0$. Consequently, $q(\varrho, \varsigma) = 0$, and using condition (i) of Lemma 12, leads to $\varrho = \varsigma$. It can be concluded that $\mathcal{H}$ possesses a unique fixed point. □

Example 15. Let $X = [0, +\infty)$, and consider the function $d : X \times X \to \mathbb{R}^+$ defined by $d(\varrho, \varsigma) = |\varrho - \varsigma|$ for all $\varrho, \varsigma \in X$. Additionally, let $\mathcal{H} : X \to X$ be such that

$$\mathcal{H}\varrho = \begin{cases} \frac{\varrho^2}{2}, & \text{if } \varrho \in [0, 2] \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

and $\psi, \phi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \frac{5t}{12}$, and $\phi(t) = \frac{t}{4}$.

It is evident that ϕ is a lower semi-continuous function and satisfies the condition $\phi(t) = 0$, if and only if, $t = 0$. Additionally, it can be observed that ψ is an altering distance function.

Define $q : X \times X \to [0, +\infty)$ by $q(\varrho, \varsigma) = \max\{2(\varsigma - \varrho), 4(\varrho - \varsigma)\}$ for all $\varrho, \varsigma \in X$, then, it can be observed that q is a w-distance for d. Also, define the relation $\mathcal{R}$ by

$$\mathcal{R} = \left\{ (\varrho, \varsigma) \in X^2 : \max\{\varrho, \varsigma\} \leqslant \frac{1}{2} \right\}.$$

This relation $\mathcal{R}$ exhibits the property of being symmetric locally $\mathcal{H}$-transitive, and $\mathcal{H}$ is $\mathcal{R}$-continuous. It can also be observed that $\mathcal{R}$ is $\mathcal{H}$-closed. Moreover, the set $X(\mathcal{H}, \mathcal{R})$ is nonempty, and there exists a subset $Y = [0, 2]$ of X such that $\mathcal{H}(X) \subseteq Y$ and (Y, d) is $\mathcal{R}$-complete.

We will now demonstrate that condition (3.1) holds for all $(\varrho, \varsigma) \in \mathcal{R}$. Here,

$$\psi(q(\mathcal{H}\varrho, \mathcal{H}\varsigma)) = \frac{5}{12} \left(\max \left\{ 2 \left(\frac{\varsigma^2}{2} - \frac{\varrho^2}{2} \right), 4 \left(\frac{\varrho^2}{2} - \frac{\varsigma^2}{2} \right) \right\} \right). \tag{3.8}$$

Suppose $\varrho, \varsigma \in X$ such that $(\varrho, \varsigma) \in \mathcal{R}$ and $\varrho \leqslant \varsigma$, then we have

$$\begin{aligned} \psi(q(\mathcal{H}\varrho, \mathcal{H}\varsigma)) &= \frac{5}{12} \left(2 \left(\frac{\varsigma^2}{2} - \frac{\varrho^2}{2} \right) \right) \\ &\leqslant \frac{5}{12} (\varsigma - \varrho) \qquad\qquad (\text{since } \varsigma + \varrho \leqslant 1) \\ &\leqslant \frac{1}{3} (2\varsigma - \varsigma^2) \\ &= \frac{1}{6} \left(\max \left\{ 2 \left(\frac{\varsigma^2}{2} - \varsigma \right), 4 \left(\varsigma - \frac{\varsigma^2}{2} \right) \right\} \right) \\ &\leqslant \frac{5}{12} (\Gamma(\varrho, \varsigma)) - \frac{1}{4} (\Gamma(\varrho, \varsigma)) + \mathfrak{L}\Delta(\varrho, \varsigma). \end{aligned}$$

Now, suppose that $\varrho, \varsigma \in X$ such that $(\varrho, \varsigma) \in \mathcal{R}$ and $\varrho \geqslant \varsigma$, then we have

$$\begin{aligned} \psi(q(\mathcal{H}\varrho, \mathcal{H}\varsigma)) &= \frac{5}{12} \left(4 \left(\frac{\varrho^2}{2} - \frac{\varsigma^2}{2} \right) \right) \\ &\leqslant \frac{1}{3} (2\varrho - \varrho^2) \\ &= \frac{1}{6} \left(\max \left\{ 2 \left(\frac{\varrho^2}{2} - \varrho \right), 4 \left(\varrho - \frac{\varrho^2}{2} \right) \right\} \right) \\ &\leqslant \frac{5}{12} (\Gamma(\varrho, \varsigma)) - \frac{1}{6} (\Gamma(\varrho, \varsigma)) + \mathfrak{L}\Delta(\varrho, \varsigma). \end{aligned}$$

Here, $\mathfrak{L} \geqslant 0$,

$$\Gamma(\varrho,\varsigma) = \max\Bigg\{\max\{2(\varsigma-\varrho),4(\varrho-\varsigma)\},\max\left\{2\left(\frac{\varrho^2}{2}-\varrho\right),4\left(\varrho-\frac{\varrho^2}{2}\right)\right\},$$
$$\max\left\{2\left(\frac{\varsigma^2}{2}-\varsigma\right),4\left(\varsigma-\frac{\varsigma^2}{2}\right)\right\},\max\frac{1}{2}\Bigg(\left\{2\left(\frac{\varsigma^2}{2}-\varrho\right),4\left(\varrho-\frac{\varsigma^2}{2}\right)\right\}$$
$$+\max\left\{2\left(\varsigma-\frac{\varrho^2}{2}\right),4\left(\frac{\varrho^2}{2}-\varsigma\right)\right\}\Bigg)\Bigg\}$$

and

$$\Delta(\varrho,\varsigma) = \min\Bigg\{\max\left\{2\left(\frac{\varrho^2}{2}-\varrho\right),4\left(\varrho-\frac{\varrho^2}{2}\right)\right\},\max\left\{2\left(\frac{\varsigma^2}{2}-\varsigma\right),4\left(\varsigma-\frac{\varsigma^2}{2}\right)\right\},$$
$$\max\left\{2\left(\varsigma-\frac{\varrho^2}{2}\right),4\left(\frac{\varrho^2}{2}-\varsigma\right)\right\},\max\left\{2\left(\frac{\varsigma^2}{2}-\varrho\right),4\left(\varrho-\frac{\varsigma^2}{2}\right)\right\}\Bigg\}.$$

Therefore, all the hypotheses of Theorem 13 are met and consequently, $\mathcal{H}$ possesses a fixed point in X.

Remark 16. (i) It is worth noting that the binary relation $\mathcal{R}$ considered in our example lacks reflexivity, irreflexivity, and orthogonality. Instead, $\mathcal{R}$ satisfies only the symmetry condition.

(ii) It is interesting to observe that the mapping $\mathcal{H}$ in the above example is not continuous.

(iii) The mapping $\mathcal{H}$ in the above example neither meets the contractive condition outlined in Theorem 3.1 by Lakziana and Rhoades [11], nor does it adhere to the contractive condition specified in Theorem 3.6 by Wongyat and Sintunavarat [28]. This can be verified by considering the values $\varrho = \frac{9}{5}, \varsigma = \frac{2}{5}$ and $\varrho = 1, \varsigma = 0$, respectively. Additionally, several other results in [31–33] are not applicable in the presented example.

(iv) Furthermore, it is worth mentioning that even though $(\frac{9}{20}, \frac{2}{5}) \in \mathcal{R}$, a recent finding in this direction by Antal, Khantwal, Negi, and Gairola (Theorem 3.1 in [23]) is not applicable to the presented example when considering $\varrho = \frac{9}{20}, \varsigma = \frac{2}{5}$.

The assumption $q = d$ added to the hypotheses of Theorem 13 yields the following result without relying on the symmetric property of the binary relation.

Theorem 17. *Let $\mathcal{H}$ be a self-mapping defined on a metric space (X, d) equipped with a locally $\mathcal{H}$ transitive binary relation $\mathcal{R}$. Assume that conditions (c1)–(c4) hold, and there exists $\Phi,\Psi : [0,+\infty) \to [0,+\infty)$ such that*

$$\Psi(d(\mathcal{H}\varrho,\mathcal{H}\varsigma)) \leqslant \Psi(\Gamma(\varrho,\varsigma)) - \Phi(\Gamma(\varrho,\varsigma)) + \mathcal{L}\Delta(\varrho,\varsigma), \tag{3.9}$$

for all $\varrho,\varsigma \in X$ with $(\varrho,\varsigma) \in \mathcal{R}$, then $\mathcal{H}$ has a fixed point.

Remark 18. By giving the precise definitions of the functions Φ and Ψ, along with the nonnegative constant $\mathcal{L}$, it becomes evident that we can draw the following conclusions, underscoring the extensive applicability and versatility of Theorem 13.

$(T1)$ When we substitute $\mathcal{L} = 0$ into Eq (3.1) of Theorem 13, we derive fixed-point results for Relational (Ψ,Φ)-weak contraction, i.e., for mapping $\mathcal{H}$ satisfying condition

$$\Psi(d(\mathcal{H}\varrho,\mathcal{H}\varsigma)) \leqslant \Psi(\Gamma(\varrho,\varsigma)) - \Phi(\Gamma(\varrho,\varsigma)), \text{ for all } \varrho,\varsigma \in X \text{ with } (\varrho,\varsigma) \in \mathcal{R}.$$

$(T2)$ If we set $\Psi(\nu) = \nu$ and $\Phi(\nu) = (1-k)\nu$ for all $\nu \in [0, +\infty)$ in Theorem 17, we attain a fixed-point result for Relational almost contractions, i.e., mapping $\mathcal{H}$ that satisfies

$$d(\mathcal{H}\varrho, \mathcal{H}\varsigma) \leqslant k\Gamma(\varrho, \varsigma) + \mathcal{L}\Delta(\varrho, \varsigma), \text{ for all } \varrho, \varsigma \in X, k < 1 \text{ with } (\varrho, \varsigma) \in \mathcal{R}.$$

It is worth highlighting that when we consider R as the universal relation in $(T1)$ and $(T2)$, we can deduce the classical forms of weak contraction and almost contraction, respectively. Consequently, all the corollaries associated with these classical definitions also transform into corollaries of the results we have demonstrated.

Corollary 19. *(Radenović and Kadelburg, [34]) Let $(X, \preceq, d)$ be the ordered complete metric space and let $\mathcal{H} : X \to X$ be a nondecreasing mapping such that there exists an element $\varrho_0 \in X$ satisfying $\varrho_0 \preceq f\varrho_0$. Furthermore, suppose there exist Φ and Ψ such that,*

$$\Psi(d(\mathcal{H}\varrho, \mathcal{H}\varsigma)) \leqslant \Psi(\Gamma(\varrho, \varsigma)) - \Phi(\Gamma(\varrho, \varsigma)), \textit{ for all } \varrho, \varsigma \in X \textit{ with } \varrho \preceq \varsigma,$$

where $\Gamma(\varrho, \varsigma) = \max\left\{d(\varrho, \varsigma), d(\varrho, \mathcal{H}\varrho), d(\varsigma, \mathcal{H}\varsigma), \frac{1}{2}(d(\varrho, \mathcal{H}\varsigma) + d(\varsigma, \mathcal{H}\varrho))\right\}$, then $\mathcal{H}$ has at least one fixed point in X if at least one of the following conditions is fulfilled

(i) *Either, whenever a nondecreasing sequence $\{\varrho_r\}$ converges to $\varrho \in X$, then $\varrho_r \preceq \varrho$ for all r.*
(ii) *$\mathcal{H}$ is continuous.*

Corollary 20. *(Alam and Imdad, [15]) Let (X, d) be the complete metric space endowed with $\mathcal{R}$, and $\mathcal{H}$ be a self-mapping defined on X. Let (c2)–(c4) hold, and there exists $\alpha \in [0, 1)$ such that*

$$d(\mathcal{H}\varrho, \mathcal{H}\varsigma) \leqslant \alpha d(\varrho, \varsigma), \textit{ for all } \varrho, \varsigma \in X \textit{ with } (\varrho, \varsigma) \in \mathcal{R}.$$

Thus, $\mathcal{H}$ has a fixed point.

4. Application

In this section, we have applied our research findings to derive a result concerning the existence of solutions for a nonlinear matrix equation, which is associated with an arbitrary binary relation. In this context, let the set denoted as $\mathcal{M}(n)$ encompass all square matrices with dimensions of $n \times n$, while $\mathcal{H}(n)$, $\mathcal{K}(n)$, and $\mathcal{P}(n)$, respectively, represent the sets of Hermitian matrices, positive semi-definite matrices, and positive definite matrices. When we have a matrix C from $\mathcal{H}(n)$, we use the notation $\|C\|_{tr}$ to refer to its trace norm, which is the sum of all its singular values. If we have matrices $\mathcal{P}$ and Q from $\mathcal{H}(n)$, the notation $\mathcal{P} \succeq Q$ signifies that the matrix $\mathcal{P} - Q$ is an element of the set $\mathcal{K}(n)$, while $\mathcal{P} \succ Q$ indicates that $\mathcal{P} - Q$ belongs to the set $\mathcal{P}(n)$. The upcoming discussion relies on the following lemmas of Ran and Reurings, [35].

Lemma 21. *If $\mathfrak{I} \in \mathcal{H}(n)$ satisfies $\mathfrak{I} \prec \mathcal{I}_n$, then $\|\mathfrak{I}\| < 1$.*

Lemma 22. *For $n \times n$ matrices $\mathfrak{I} \succeq O$ and $\eth \succeq O$, the following inequalities hold:*

$$0 \leqslant tr(\mathfrak{I}\eth) \leqslant \|\mathfrak{I}\| tr(\eth).$$

We shall now examine the following nonlinear matrix equation,

$$\mathfrak{I} = \mathcal{A} + \sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}) C_i. \tag{4.1}$$

In the above equation, $\mathcal{A}$ is defined as a Hermitian and positive definite matrix. Additionally, the notation C_i^* refers to the conjugate transpose of a square matrix C_i of size $n \times n$. Furthermore, Υ_j represents continuous functions that preserve order, mapping from $\mathcal{H}(n)$ to $\mathcal{P}(n)$. It is noteworthy that $\Upsilon(O) = O$, where O represents a zero matrix.

Theorem 23. *Let the following conditions apply:*

(H_1) *There exists* $\mathcal{A} \in \mathcal{P}(n)$ *with* $\sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathcal{A}) C_i \succ 0$.
(H_2) *For all* $\mathfrak{I}, \eth \in \mathcal{P}(n)$, $\mathfrak{I} \preceq \eth$ *implies* $\sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}) C_i \preceq \sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\eth) C_i$.
(H_3) *There exists a positive number* N *for which* $\sum_{i=1}^{u} C_i C_i^* \prec N\mathcal{I}_n$.
(H_4) *For all* $\mathfrak{I}, \eth \in \mathcal{P}(n)$ *with* $\mathfrak{I} \preceq \eth$, *the following inequality holds*

$$\max_j (tr(\Upsilon_j(\eth) - \Upsilon_j(\mathfrak{I}))) \leqslant \frac{tr(\eth - \mathfrak{I})}{2Nv},$$

then the nonlinear matrix equation (4.1) *has at least one solution. Moreover, the iteration*

$$\mathfrak{I}_r = \mathcal{A} + \sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}_{r-1}) C_i, \tag{4.2}$$

where $\mathfrak{I}_0 \in \mathcal{P}(n)$ *satisfies*

$$\mathfrak{I}_0 \preceq \mathcal{A} + \sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}_0) C_i,$$

converges to the solution of the matrix equation, in the context of the trace norm $\|\cdot\|_{tr}$.

Proof. Let $\Omega : \mathcal{P}(n) \to \mathcal{P}(n)$ be a mapping defined by,

$$\Omega(\mathfrak{I}) = \mathcal{A} + \sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}) C_i,$$

for all $\mathfrak{I} \in \mathcal{P}(n)$.

Consider $\mathcal{R} = \{(\mathfrak{I}, \eth) \in \mathcal{P}(n) \times \mathcal{P}(n) : \mathfrak{I} \preceq \eth\}$. Consequently, the fixed point of Ω serves as a solution to the nonlinear matrix equation (4.1). It is pertinent to mention that $\mathcal{R}$ is Ω-closed, and Ω is well-defined as well as $\mathcal{R}$-continuous. Form condition (H_1), we have $\sum_{i=1}^{u}\sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}) C_i \succ 0$ for some $\mathfrak{I} \in \mathcal{P}(n)$. Thus, $(\mathfrak{I}, \Omega(\mathfrak{I})) \in \mathcal{R}$ and, consequently, $\mathcal{P}(n)(\Omega, \mathcal{R})$ is nonempty.

Define $d : \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}^+$ by

$$d(\mathfrak{I}, \eth) = \|\mathfrak{I} - \eth\|_{tr}, \text{ for all } \mathfrak{I}, \eth \in \mathcal{P}(n).$$

Thus, $(\mathcal{P}(n), d)$ is an $\mathcal{R}$-complete relational metric space, then

$$\|\Omega(\eth) - \Omega(\mathfrak{I})\|_{tr} = tr(\Omega(\eth) - \Omega(\mathfrak{I}))$$

$$
\begin{aligned}
&= tr\left(\sum_{i=1}^{u}\sum_{j=1}^{v} C_i^*(\Upsilon_j(\eth) - \Upsilon_j(\Im))C_i\right) \\
&= \sum_{i=1}^{u}\sum_{j=1}^{v} tr(C_i^*(\Upsilon_j(\eth) - \Upsilon_j(\Im))C_i) \\
&= \sum_{i=1}^{u}\sum_{j=1}^{v} tr(C_iC_i^*(\Upsilon_j(\eth) - \Upsilon_j(\Im))) \\
&= tr\left(\left(\sum_{i=1}^{u} C_iC_i^*\right)\sum_{j=1}^{v}(\Upsilon_j(\eth) - \Upsilon_j(\Im))\right) \\
&\leqslant \left\|\sum_{i=1}^{u} C_iC_i^*\right\| \times t \times \max \|(\Upsilon_j(\eth) - \Upsilon_j(\Im))\|_{tr} \\
&\leqslant \frac{\|\eth - \Im\|_{tr}}{2} \\
&\leqslant \frac{\Gamma(\eth, \Im)}{2} \\
&= \Gamma(\eth, \Im) - \left[\Gamma(\eth, \Im) - \frac{\Gamma(\eth, \Im)}{2}\right].
\end{aligned}
$$

Now, consider $\Psi(v) = v$ and $\Phi(v) = \frac{v}{2}$

$$
\begin{aligned}
\Psi(\|\Omega(\eth) - \Omega(\Im)\|) &= \Psi(\Gamma(\eth, \Im) - \Phi(\Gamma(\eth, \Im)) \\
&\leqslant \Psi(\Gamma(\eth, \Im) - \Phi(\Gamma(\eth, \Im)) + \mathcal{L}\Delta(\eth, \Im),
\end{aligned}
$$

where,

$$
\begin{aligned}
\Gamma(\eth, \Im) = \max \Bigg\{ &\|\eth - \Im\|_{tr}, \|\eth - \Omega\eth\|_{tr}, \|\Im - \Omega\Im\|_{tr}, \\
&\frac{1}{2}\left[\|\eth - \Omega\Im\|_{tr} + \|\Omega\eth - \Im\|_{tr}\right] \Bigg\},
\end{aligned}
$$

$$
\Delta(\eth, \Im) = \min\left\{\|\eth - \Omega\eth\|_{tr}, \|\Im - \Omega\Im\|_{tr}, \|\Omega\eth - \Im\|_{tr}, \|\eth - \Omega\Im\|_{tr}\right\}.
$$

Consequently, upon fulfilling all the hypotheses stated in Theorem 17, it can be deduced that there exists an element $\Im^* \in \mathcal{P}(n)$ for which $\Omega(\Im^*) = \Im^*$ holds good. As a result, the matrix equation (4.1) is guaranteed to possess a solution within the set $\mathcal{P}(n)$. □

Remark 24. Notably, when v is fixed to 1 in (4.1), the matrix equation (4.1) reduces to the form

$$
\Im = \mathcal{A} + \sum_{i=1}^{u} C_i^*\Upsilon(\Im)C_i, \tag{4.3}
$$

which have been used in some recent works [22, 36]. Now, we provide a numerical example for $v = 2$, a case that cannot be addressed using the matrix equation (4.3). This example underscores the significance of Theorem 23 in addressing such cases.

Example 25. Consider the nonlinear matrix equation (4.1) for $u = v = 2$ and $n = 3$, with $\Upsilon_1(\mathfrak{I}) = \frac{\mathfrak{I}}{5}$, $\Upsilon_2(\mathfrak{I}) = \frac{\mathfrak{I}}{7}$, i.e.,

$$\mathfrak{I} = \mathcal{A} + C_1^* \frac{\mathfrak{I}}{5} C_1 + C_1^* \frac{\mathfrak{I}}{7} C_1 + C_2^* \frac{\mathfrak{I}}{5} C_2 + C_2^* \frac{\mathfrak{I}}{7} C_2, \tag{4.4}$$

where

$$\mathcal{A} = \begin{bmatrix} 0.176674562512000 & 0.001431425364758 & 0.143907569548500 \\ 0.001431425364758 & 0.175004512364000 & 0.136284455213600 \\ 0.143907569548500 & 0.136284455213600 & 0.265964578250000 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.229334125100201 & 0.104014127456352 & 0.071531425367845 \\ 0.276191245786598 & 0.127467845958641 & 0.069995456525158 \\ 0.149324142435689 & 0.118387539512365 & 0.241474567891235 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.255967414256354 & 0.173174561254879 & 0.270564568574965 \\ 0.074041245659835 & 0.223551425456895 & 0.102704562136980 \\ 0.152197485963250 & 0.136274152638570 & 0.190331452689300 \end{bmatrix}.$$

All the conditions specified in Theorem 23 are met with $N = 1$. To ascertain the convergence of $\{\mathfrak{I}_n\}$ defined in (4.2), we commence with three distinct initial values.

$$U_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$V_0 = \begin{bmatrix} 1.237284512365845 & 0.250614562315200 & 0.299494715230000 \\ 0.125324512360000 & 0.151844125986500 & 0.131377596845688 \\ 0.053747485629586 & 0.067587485963625 & 0.234187182930000 \end{bmatrix},$$

$$W_0 = \begin{bmatrix} 0.094417946131700 & 0.067327946131728 & 0.005448596215000 \\ 0.257846251849500 & 0.210305195620000 & 0.289837586420000 \\ 0.112469865423650 & 0.080324512369875 & 0.276111425369854 \end{bmatrix}.$$

After conducting 14 iterations, the subsequent approximation of the positive definite solution for the system presented in (4.1) is as follows,

$$\hat{U} \approx U_{14} = \begin{bmatrix} 0.211336629947799 & 0.026371497053666 & 0.174441185376292 \\ 0.026371497053666 & 0.195945887514472 & 0.160168100290036 \\ 0.174441185376292 & 0.160168100290036 & 0.295063523472589 \end{bmatrix},$$

with error 2.5095×10^{-10},

$$\hat{V} \approx V_{14} = \begin{bmatrix} 0.211336629947578 & 0.026371497053505 & 0.174441185376116 \\ 0.026371497053505 & 0.195945887514342 & 0.160168100289893 \\ 0.174441185376116 & 0.160168100289893 & 0.295063523472432 \end{bmatrix},$$

with error 2.45107×10^{-12},

$$\hat{W} \approx W_{14} = \begin{bmatrix} 0.211336629947578 & 0.026371497053505 & 0.174441185376116 \\ 0.026371497053505 & 0.195945887514342 & 0.160168100289893 \\ 0.174441185376116 & 0.160168100289893 & 0.295063523472432 \end{bmatrix},$$

and with error 7.65452×10^{-12}.

In Figure 1, we present a graphical depiction illustrating the convergence phenomenon.

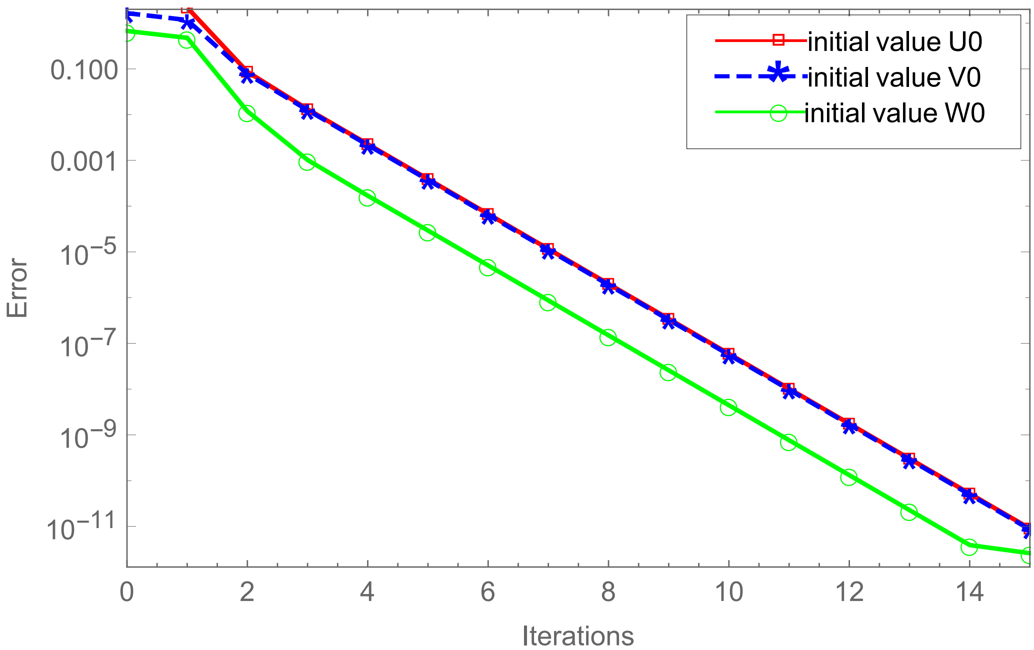

Figure 1. Convergence phenomenon.

5. Conclusions

This research establishes some fixed-point results for generalized almost contractions in a metric space equipped with a binary relation $\mathcal{R}$. We have improved some known fixed point results by using w-distance and locally symmetric $\mathcal{H}$-transitivity properties of binary relations. Also, we give the application of the obtained results for finding the solution of nonlinear matrix equations of the following type

$$\mathfrak{I} = \mathcal{A} + \sum_{i=1}^{u} \sum_{j=1}^{v} C_i^* \Upsilon_j(\mathfrak{I}) C_i,$$

where, $\mathcal{A}$ represents a Hermitian positive definite matrix, while C_i^* corresponds to the conjugate transpose operation applied to the $n \times n$ matrix C_i^*. Furthermore, Υ_j symbolizes order-preserving continuous functions, mapping from the set of all Hermitian matrices to the set of positive definite matrices. Finally, we provide a numerical example that demonstrates the applicability of our results.

Overall, this paper extends some results from the existing literature, opening new possibilities for the study of almost contractions and their applications in mathematics and related fields.

Use of AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9** (2004), 43–54.
2. V. Berinde, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces, *Carpathian J. Math.*, **24** (2008), 10–19. https://www.jstor.org/stable/43996855
3. V. Berinde, M. Pacurar, Fixed points and continuity of almost contractions, *Fixed Point Theory*, **9** (2008), 23–34.
4. G. V. R. Babu, M. L. Sandhya, M. V. R. Kameswari, A note on a fixed point theorem of Berinde on weak contractions, *Carpathian J. Math.*, **24** (2008), 8–12. https://www.jstor.org/stable/43996834
5. O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Mathematica Japonicae*, **44** (1996), 381–391.
6. B. S. Choudhury, P. Chakraborty, Fixed point problem of a multi-valued Kannan-Geraghty type contraction via w-distance, *J. Anal.*, **31** (2023), 439–458. https://doi.org/10.1007/s41478-022-00457-3
7. E. Karapınar, S. Romaguera, P. Tirado, Characterizations of quasi-metric and G-metric completeness involving w-distances and fixed points, *Demonstr. Math.*, **55** (2022), 939–951. https://doi.org/10.1515/dema-2022-0177
8. S. Barootkoob, E. Karapınar, H. Lakzian, A. Chanda, Extensions of Meir-Keeler contraction via w-distances with an application, *Kragujev. J. Math.*, **46** (2022), 533–547. https://doi.org/10.46793/KgJMat2204.533B
9. H. K. Nashine, R. Pant, Feng-Liu type fixed point theorems for w-distance spaces and applications, *Filomat*, **36** (2022), 3899–3917. https://doi.org/10.2298/FIL2211899N
10. R. Jain, H. K. Nashine, Z. Kadelburg, Positive solutions of nonlinear matrix equations via fixed point results in relational metric spaces with w-distance, *Filomat*, **36** (2022), 4811–4829. https://doi.org/10.2298/FIL2214811J
11. H. Lakzian, B. E. Rhoades, Some fixed point theorems using weaker Meir-Keeler function in metric spaces with w-distance, *Appl. Math. Comput.*, **342** (2019), 18–25. https://doi.org/10.1016/j.amc.2018.06.048

12. H. Lakzian, D. Kocev, V. Rakočević, Ćirić-generalized contraction via *wt*-distance, *Appl. Gen. Topol.*, **24** (2023), 267–280. http://dx.doi.org/10.4995/agt.2023.19268

13. S. Reich, A. J. Zaslavski, *Genericity in nonlinear analysis*, Springer Science & Business Media, 2013.

14. A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132** (2004), 1435–1443. https://doi.org/10.1090/S0002-9939-03-07220-4

15. A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17** (2015), 693–702. https://doi.org/10.1007/s11784-015-0247-y

16. A. Alam, M. Imdad, Nonlinear contractions in metric spaces under locally *t*-transitive binary relations, *Fixed Point Theory*, **19** (2018), 13–24. https://doi.org/10.24193/fpt-ro.2018.1.02

17. G. Prasad, Fixed point theorems via *w*-distance in relational metric spaces with an application, *Filomat*, **3**4 (2020), 1889–1898. https://doi.org/10.2298/FIL2006889P

18. G. Prasad, R. C. Dimri, Fixed point theorems for weakly contractive mappings in relational metric spaces with an application, *J. Anal.*, **26** (2018), 151–162. https://doi.org/10.1007/s41478-018-0076-7

19. W. Shatanawi, T. A. M. Shatnawi, New fixed point results in controlled metric type spaces based on new contractive conditions, *AIMS Math.*, **8** (2023), 9314–9330. https://doi.org/10.3934/math.2023468

20. A. Z. Rezazgui, A. A. Tallafha, W. Shatanawi, Common fixed point results via $A\nu$-α-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space, *AIMS Math.*, **8** (2023), 7225–7241. https://doi.org/10.3934/math.2023363

21. N. H. Altaweel, N. H. E. Eljaneid, H. I. A. Mohammed, I. M. Alanazi, F. A. Khan, Coincidence theorems under generalized nonlinear relational contractions, *Symmetry*, **15** (2023), 1–17. https://doi.org/10.3390/sym15020434

22. S. Shil, H. K. Nashine, Unique positive definite solution of non-linear matrix equation on relational metric spaces, *Fixed Point Theory*, **24**, (2023), 367–382. https://doi.org/10.24193/fpt-ro.2023.1.20

23. S. Antal, D. Khantwal, S. Negi, U. C. Gairola, Fixed points theorems for (ϕ, ψ, p)-weakly contractive mappings via *w*-distance in relational metric spaces with applications, *Filomat*, **37** (2023), 7319–7328.

24. R. D. Maddux, *Relation Algebras*, Amsterdam: Elsevier Science Limited, 2006.

25. A. Alama, M. Imdad, Relation-theoretic metrical coincidence theorems, *Filomat*, **31** (2017), 4421–4439. https://doi.org/10.2298/FIL1714421A

26. M. S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, *B. Aust. Math. Soc.*, **30** (1984), 1–9. https://doi.org/10.1017/S0004972700001659

27. T. Senapati, L. K. Dey, Relation-theoretic metrical fixed-point results via *w*-distance with applications, *J. Fixed Point Theory Appl.*, **19** (2017), 2945–2961. https://doi.org/10.1007/s11784-017-0462-9

28. T. Wongyat, W. Sintunavarat, The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fractional differential equations via w-distances, *Adv. Differ. Equ.*, **2017** (2017), 1–15. https://doi.org/10.1186/s13662-017-1267-2

29. G. V. R. Babu, P. D. Sailaja, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. Math.*, **9** (2012), 1–10.

30. M. R. Turinici, Contractive operators in relational metric spaces, *Handbook of Functional Equations: Functional Inequalities*, **95** (2014), 419–458. https://doi.org/10.1007/978-1-4939-1246-9

31. B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal. Theor.*, **47** (2001), 2683–2693.

32. P. N. Dutta, B. S. Choudhury, A generalisation of contraction principle in metric spaces, *Fixed Point Theory A.*, **2008** (2008), 1–8.

33. Q. Zhang, Y. Song, Fixed point theory for generalized ψ-weak contractions, *Appl. Math. Lett.*, **22** (2009), 75–78. https://doi.org/10.1016/j.aml.2008.02.007

34. S. Radenović, Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, *Comput. Math. Appl.*, **60** (2010), 1776–1783. https://doi.org/10.1016/j.camwa.2010.07.008

35. A. C. M. Ran, M. C. B. Reurings, On the nonlinear matrix equation $X + A^*F(X)A = Q$: Solutions and perturbation theory, *Linear Algebra Appl.*, **346** (2002), 15–26. https://doi.org/10.1016/S0024-3795(01)00508-0

36. S. Shil, H. K. Nashine, Positive definite solution of non-linear matrix equations through fixed point technique, *AIMS Math.*, **7** (2022), 6259–6281. https://doi.org/10.3934/math.2022348

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